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Types are weak ω-groupoids

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Abstract

We define a notion of weak ω-category internal to a model of Martin-Löf's type theory, and prove that each type bears a canonical weak ω-category structure obtained from the tower of iterated identity types over that type. We show that the ω-categories arising in this way are in fact ω-groupoids.

1. *Introduction*

It has long been understood that there is a close connection between algebraic topology and higher-dimensional category theory [**9**]. More recently, it has become apparent that both are in turn related to the intensional type theory of Martin-Löf [**17**]. While attempts to make this link precise have only borne fruit in the past few years [**1, 6, 8, 19**], the basic idea dates back to an observation made by Hofmann and Streicher [**10**]. Recall that in Martin-Löf's type theory, we may construct from a type A and elements $a, b \in A$, a new type $\mathrm{Id}(a, b)$ whose elements are to be thought of as proofs that a and b are propositionally equal. Hofmann and Streicher observe that the type-theoretic functions

$$1 \longrightarrow \mathrm{Id}(a,a), \quad \mathrm{Id}(b,c) \times \mathrm{Id}(a,b) \longrightarrow \mathrm{Id}(a,c) \quad \text{and} \quad \mathrm{Id}(a,b) \longrightarrow \mathrm{Id}(b,a)$$

expressing the reflexivity, transitivity and symmetry of propositional equality allow us to view the type A as a *groupoid* (a category whose every morphism is invertible) wherein objects are elements $a \in A$ and morphisms $a \to b$ are elements $p \in \mathrm{Id}(a, b)$. However, as is made clear in [**10**], this is not the end of the story. The groupoid axioms for A hold only 'up to propositional equality'; which is to say that, for example, the associativity diagram

$$\begin{tikzcd}
\mathrm{Id}(c,d) \times \mathrm{Id}(b,c) \times \mathrm{Id}(a,b) \arrow[r] \arrow[d] & \mathrm{Id}(c,d) \times \mathrm{Id}(a,c) \arrow[d] \\
\mathrm{Id}(b,d) \times \mathrm{Id}(a,b) \arrow[r] & \mathrm{Id}(a,d)
\end{tikzcd}$$

does not commute on the nose, but only up to suitable terms

$$\alpha_{p,q,r} \in \mathrm{Id}(r \circ (q \circ p),\ (r \circ q) \circ p) \quad (p \in \mathrm{Id}(a,b), q \in \mathrm{Id}(b,c), r \in \mathrm{Id}(c,d)).$$

Thus, if we wish to view A as an honest groupoid, we must first quotient out the sets of elements $p \in \mathrm{Id}(a, b)$ by propositional equality. A more familiar instance of the same phenomenon occurs in constructing the fundamental groupoid of a space, where we must identify paths up to homotopy, and this suggests the following analogy: that types are like topological spaces, and propositional equality is like the homotopy relation. Using the machinery of abstract homotopy theory, this analogy has been given a precise form in [**1**], which constructs type-theoretic structures from homotopy-theoretic ones, and in [**6**], which does the converse.

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The connection with algebraic topology in turn suggests the one with higher-dimensional category theory. A more sophisticated construction of the fundamental groupoid of a space (suggested in [**9**] and made rigorous in [**2**]) does not quotient out paths by the homotopy relation; but instead incorporates these homotopies, and all higher homotopies between them, into an infinite-dimensional categorical structure known as a *weak ω-groupoid*, whose various identities, compositions and inverses satisfy coherence laws, not strictly, but 'up to all higher homotopies'. This leads us to ask whether the construction of the type-theoretic 'fundamental groupoid' admits a similar refinement, which constructs a weak ω-groupoid from a type by considering not just elements of the type, and proofs of their equality, but also proofs of equality between such proofs, and so on. The principal aim of this paper is to show this to be the case.

In order to give the proof, we must first choose an appropriate notion of weak ω-groupoid to work with; and since, in the literature, weak ω-groupoids are studied in the broader context of *weak ω-categories*, which are 'weak ω-groupoids without the inverses', this is tantamount to choosing an appropriate notion of weak ω-category. There are a number of definitions to pick from, and these differ from each other both in their general approach and in the details; see [**13**] for an overview. Of these, it is the definition of Batanin [**2**] which matches the type theory most closely, for the following two reasons. Firstly, its basic cellular data are *globular*: which is to say that an n-cell $\alpha : x \to y$ can only exist between a pair of parallel $(n-1)$-cells $x, y : f \to g$. A corresponding property holds for proofs of equality in type theory: to know that $\alpha \in \mathrm{Id}(x, y)$, we must first know that x and y inhabit the same type $\mathrm{Id}(f, g)$. Secondly, Batanin's definition is *algebraic*, which is to say that composition operations are explicitly specified, rather than merely asserted to exist. This accords with the constructivist notion, central to the spirit of intensional type theory, that to know something to exist is nothing less than to be provided with a witness to that fact. On these grounds, it is Batanin's definition which we will adopt here; or rather, a mild reformulation of his definition given by Leinster [**13**].

The paper is arranged as follows. In Section 2, we recall Batanin's theory of weak ω-categories, the appropriate specialization to weak ω-groupoids, and the necessary background from intensional type theory. Then in Section 3 we give the proof of our main result. We begin in Subsection 3.1 with an explicitly type-theoretic, but informal, account. When we come to make this precise, it turns out to be convenient to isolate just those categorical properties of the type theory that make the proof go through, and then to work in an axiomatic setting assuming only these. We describe this setting in Subsection 3.2, and then in Subsections 3.3 and 3.4 use it to give a formal proof that every type is a weak ω-groupoid.

2. *Preparatory material*

In this section, we review the material necessary for our main result; firstly, from higher category theory, and secondly, from Martin-Löf type theory.

2.1. *Weak ω-categories and weak ω-groupoids*

As mentioned in Section 1, the most appropriate definition of weak ω-category for our purposes is that of [**2**], which describes them as globular sets equipped with algebraic structure. A *globular set* is a diagram of sets and functions

$$X_0 \underset{t}{\overset{s}{\leftleftarrows}} X_1 \underset{t}{\overset{s}{\leftleftarrows}} X_2 \underset{t}{\overset{s}{\leftleftarrows}} X_3 \underset{t}{\overset{s}{\leftleftarrows}} \cdots$$

satisfying the globularity equations $ss = st$ and $ts = tt$. We refer to elements $x \in X_n$ as *n-cells* of X, and write them as $x : sx \to tx$. In this terminology, the globularity equations express that any $(n+2)$-cell $f \to g$ must mediate between $(n+1)$-cells f and g which are *parallel*, in the sense of having the same source and target. Globular sets also have a coinductive characterization: to give a globular set X is to give a set $\mathrm{ob}\, X$ of objects, together with for each $x, y \in \mathrm{ob}\, X$, a globular set $X(x, y)$.

The algebraic structure required to make a globular set into a weak ω-category is encoded by any one of a certain class of monads on the category of globular sets: those arising from *normalized, contractible, globular operads*. Informally, such monads are obtained by 'deforming' the monad T whose algebras are strict ω-categories. To make this precise, we must first recall some details concerning strict ω-categories.

If $\mathcal{V}$ is any category with finite products, then one can speak of categories enriched in $\mathcal{V}$, and of $\mathcal{V}$-enriched functors between them [**12**]. The category $\mathcal{V}$-Cat of small $\mathcal{V}$-categories is then itself a category with finite products, so that we can iterate the process; and when we do so starting from $\mathcal{V} = 1$, we obtain the sequence 1, Set, Cat, 2-Cat, ..., whose nth term is the category of small strict $(n-1)$-categories. Now, because any finite-product preserving functor $\mathcal{V} \to \mathcal{W}$ induces a finite-product preserving functor $\mathcal{V}\text{-Cat} \to \mathcal{W}\text{-Cat}$, we obtain, by iteration on the unique functor $\mathrm{Set} \to 1$, a chain

$$\cdots \longrightarrow n\text{-Cat} \cdots \longrightarrow 2\text{-Cat} \longrightarrow \mathrm{Cat} \longrightarrow \mathrm{Set} \longrightarrow 1;$$

and ω-Cat, the category of small strict ω-categories, is the limit of this sequence. Unfolding this definition, we find that a strict ω-category is given by, first, an underlying globular set; next, operations of identity and composition: thus for each n-cell x, an $(n+1)$-cell $\mathrm{id}_x : x \to x$, and for each pair of n-cells f and g sharing a k-cell boundary (for $k < n$), a composite n-cell $g \circ_k f$; and finally, axioms expressing that any two ways of composing a diagram of n-cells using the above operations yield the same result.

There is an evident forgetful functor $U : \omega\text{-Cat} \to \mathrm{GSet}$, where GSet denotes the category of globular sets; and it is shown in [**14**, Appendix B] that this has a left adjoint and is finitarily monadic. The corresponding monad T on the category of globular sets may be described as follows. First we give an inductive characterization of $T1$, its value at the globular set with one cell in every dimension. We have that:

(i) $(T1)_0 = \{\star\}$;
(ii) $(T1)_{n+1} = \{\, (\pi_1, \ldots, \pi_k) \mid k \in \mathbb{N}, \pi_1, \ldots, \pi_k \in (T1)_n \,\}$.

The source and target maps $s, t : (T1)_{n+1} \to (T1)_n$ coincide and we follow [**14**] in writing ∂ for the common value. This too has an inductive description as follows:

(i) $\partial(\pi) = \star$ for $\pi \in (T1)_1$;
(ii) $\partial(\pi_1, \ldots, \pi_k) = (\partial(\pi_1), \ldots, \partial(\pi_k))$ otherwise.

We regard elements of $(T1)_n$ as indexing possible shapes for pasting diagrams of n-cells. For example, $((*), (*, *)) \in (T1)_2$ corresponds to the shape

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left] \arrow[r] \arrow[r, bend right] \arrow[r, phantom, shift left=2, "\Downarrow"] \arrow[r, phantom, shift right=2, "\Downarrow"] & \bullet \end{tikzcd} . \tag{1}$$

We formalize this correspondence by associating to each element $\pi \in (T1)_n$ a globular set $\hat{\pi}$ which is the 'shape indexed by π'.

(i) If $\pi = \star$, then $\hat{\pi}$ is the globular set with $\mathrm{ob}\, \hat{\pi} = \{\bullet\}$ and $\hat{\pi}(\bullet, \bullet) = \emptyset$.

(ii) If $\pi = (\pi_1, \ldots, \pi_k)$, then $\hat{\pi}$ is the globular set with $\mathrm{ob}\, \hat{\pi} = \{0, \ldots, k\}$, $\hat{\pi}(i-1, i) = \widehat{\pi_i}$ (for $1 \leqslant i \leqslant k$), and $\hat{\pi}(i, j) = \emptyset$ otherwise.

By a further induction, we define source and target embeddings $\sigma, \tau : \widehat{\partial \pi} \to \hat{\pi}$.

(i) For $\pi \in (T1)_1$, the maps $\sigma, \tau : \hat{\star} \to \hat{\pi}$ send the unique object of $\hat{\star}$ to the smallest and largest elements of $\mathrm{ob}\, \hat{\pi}$, respectively.

(ii) Otherwise, for $\pi = (\pi_1, \ldots, \pi_k)$ the morphisms σ and τ are the identity on objects and map $\widehat{\partial \pi}(i-1, i)$ into $\hat{\pi}(i-1, i)$ via $\sigma, \tau : \widehat{\partial \pi_i} \to \widehat{\pi_i}$.

Taken together, the globular set $T1$, the globular sets $\hat{\pi}$, and the maps σ and τ, completely determine the functor T; this by virtue of it being *familially representable* in the sense of [**15**, Definition C.3.1] (though see also [**4**]). Explicitly, TX is the globular set whose cells are pasting diagrams labelled with cells of X:

$$(TX)_n = \sum_{\pi \in (T1)_n} \mathrm{GSet}(\hat{\pi}, X),$$

and whose source and target maps are induced in an obvious way by the maps σ and τ. The unit and multiplication of the monad T are *cartesian* natural transformations, which is to say that all of their naturality squares are pullbacks; from which it follows that these are in turn determined by their components $\eta_1 : 1 \to T1$ and $\mu_1 : TT1 \to T1$. The former map associates to the unique n-cell of 1 the pasting diagram $\iota_n := (\ldots(\star)\ldots) \in (T1)_n$, while the latter sends a typical element

$$(\pi \in (T1)_n,\ \phi : \hat{\pi} \longrightarrow T1)$$

of $(TT1)_n$ to the element $\phi \circ \pi \in (T1)_n$ obtained by substituting into π the pasting diagrams that ϕ indexes (see [**14**, Section 4.2] for a pictorial account of this process).

A *globular operad* can now be defined rather succinctly: it is a monad P on GSet equipped with a cartesian monad morphism $\rho : P \Rightarrow T$. The cartesianness of ρ implies that the functor part of P is determined by its component at 1 together with the map $\rho_1 : P1 \to T1$, and it will be convenient to have a description of P in these terms. Given $\pi \in (T1)_n$, we write P_π for the set of those $\theta \in (P1)_n$ that are mapped to π by ρ_1, and write $s, t : P_\pi \to P_{\partial\pi}$ for the corresponding restriction of the source and target maps of $P1$. The value of P at an arbitrary globular set X is now given (up to isomorphism) by

$$(PX)_n = \sum_{\pi \in (T1)_n} P_\pi \times \mathrm{GSet}(\hat{\pi}, X), \tag{2}$$

with the source and target maps determined in the obvious way. Thus, if we think of a T-algebra structure on X as providing a unique way of composing each X-labelled pasting diagram of shape π, then a P-algebra structure provides a set of possible ways of composing such diagrams, indexed by the elements of P_π.

It follows from the cartesianness of ρ that the unit and the multiplication of P are themselves cartesian natural transformations, and hence determined by their components $\eta_1 : 1 \to P1$ and $\mu_1 : PP1 \to P1$. The former sends the unique n-cell of 1 to an element $\iota_n \in P_{\iota_n}$, which we think of as the trivial composition operation of dimension n; while the latter assigns to the element

$$(\pi \in (T1)_n,\ \theta \in P_\pi,\ \psi : \hat{\pi} \longrightarrow P1)$$

of $(PP1)_n$ an element $\theta \circ \psi \in P_{\phi\circ\pi}$ (where ϕ is the composite $\rho_1\psi : \hat{\pi} \to T1$), which we think of as the composition operation obtained by substituting into θ the collection of operations indexed by ψ.

Not every globular operad embodies a sensible theory of weak ω-categories (since, for example, the identity monad on GSet is a globular operad), but [**2**] provides two conditions which together distinguish those which do: *normalization* and *contractibility*. Normalization is straightforward: it asserts that the monad P is bijective on objects in the sense that $(PX)_0 \cong X_0$, naturally in X; or equivalently, that the set $P_\star$ is a singleton. The second condition is a little more subtle. A globular operad P is said to be *contractible* if:

(a) given $\pi \in (T1)_1$ and $\theta_1, \theta_2 \in P_\star$, there exists an element $\phi \in P_\pi$ with $s(\phi) = \theta_1$ and $t(\phi) = \theta_2$;

(b) given $\pi \in (T1)_n$ (for $n > 1$) and $\theta_1, \theta_2 \in P_{\partial\pi}$ satisfying $s(\theta_1) = s(\theta_2)$ and $t(\theta_1) = t(\theta_2)$, there exists an element $\phi \in P_\pi$ such that $s(\phi) = \theta_1$ and $t(\phi) = \theta_2$.

Contractibility expresses a globular operad has 'enough' ways of composing to yield a theory of weak ω-categories. In homotopy-theoretic terms, a contractible globular operad is a 'deformation' of the monad T; an idea which can be made precise using the language of weak factorization systems (see [**7**]).

DEFINITION 2.1. A *weak ω-category* is an algebra for a contractible, normalized, globular operad: more formally, it is a pair (P, X), where P is a contractible, normalized, globular operad and X is an algebra for it.

REMARK 2.2. Some consideration must be paid to the exact force of the term *contractible*, which has been used in different ways by different authors; our usage accords with that given in [**15**, Definition 9.1.3]. In particular, the reader should carefully distinguish between the *property* of being contractible described above, and the corresponding *structure* of being equipped with a contraction.

We now turn from the definition of weak ω-category to that of weak ω-groupoid. For this we will require the coinductive notion of *equivalence* in a weak ω-category.

DEFINITION 2.3. Let (P, X) be a weak ω-category. An *equivalence* $x \simeq y$ between parallel n-cells x, y is given by the following:

(i) $n+1$-cells $f : x \to y$ and $g : y \to x$;
(ii) equivalences $\eta\colon g \circ f \simeq \mathrm{id}_x$ and $\epsilon\colon f \circ g \simeq \mathrm{id}_y$.

We say that an $(n+1)$-cell $f : x \to y$ is *weakly invertible* if it participates in an equivalence (f, g, η, ϵ).

In order for this definition to make sense, we must determine what is meant by the expressions 'id_x', 'id_y', '$g \circ f$' and '$f \circ g$' appearing in it. We do this as follows. First, for each $n \geqslant 1$, we define the pasting diagrams 0_n and $2_n \in (T1)_n$ to be given by

$$0_n := \underbrace{(\ldots(}_{n\text{ times}})\ldots) \quad \text{and} \quad 2_n := \underbrace{(\ldots(}_{n\text{ times}}\star,\star)\ldots).$$

Next, if P is a normalized, contractible, globular operad, then we define a *system of compositions* for P to be a choice, for each $n \geqslant 1$, of operations $i_n \in P_{0_n}$ and $m_n \in P_{2_n}$. Note that the contractibility of P ensures that it will possess at least one system of compositions. Finally, if we are given a system of compositions and a P-algebra X, then we define the functions

$$\mathrm{id}_{(-)} : X_{n-1} \longrightarrow X_n \quad \text{and} \quad \circ : X_{n+1}\ {}_s\!\times_t X_{n+1} \longrightarrow X_n$$

to be the interpretations of the operations i_n and m_n, respectively. This allows us to give meaning to the undefined expressions appearing in Definition 2.3.

DEFINITION 2.4. A weak ω-category (P, X) is a *weak ω-groupoid* if every cell of X is weakly invertible with respect to every system of compositions on P.

It will be convenient to give a more elementary reformulation of the notion of weak ω-groupoid due to Cheng [**5**]. This is given in terms of *duals*. If $f : x \to y$ is an n-cell (for $n \geqslant 1$) in a weak ω-category, then a *dual* for f is an n-cell $f^* : y \to x$ together with $(n+1)$-cells

$\eta : \mathrm{id}_x \to f^* \circ f$ and $\epsilon : f \circ f^* \to \mathrm{id}_y$, subject to no axioms. Again, this definition is to be interpreted with respect to some given system of compositions.

PROPOSITION 2.5. *A weak ω-category is a weak ω-groupoid if and only if, with respect to every system of compositions, every cell has a dual.*

Proof. By coinduction. □

2.2. *Martin-Löf type theory*

By *intensional Martin-Löf type theory*, we mean the logical calculus set out in [**17**, Part II]. We now summarize this calculus. It has four basic forms of judgement: $A\ \mathsf{type}$ ('A is a type'); $a \in A$ ('a is an element of the type A'); $A = B\ \mathsf{type}$ ('A and B are definitionally equal types'); and $a = b \in A$ ('a and b are definitionally equal elements of the type A'). These judgements may be made either absolutely, or relative to a context Γ of assumptions, in which case we write them as

$$(\Gamma)\ A\ \mathsf{type}, \quad (\Gamma)\ a \in A, \quad (\Gamma)\ A = B\ \mathsf{type} \quad \text{and} \quad (\Gamma)\ a = b \in A,$$

respectively. Here, a *context* is a list $\Gamma = (x_1 \in A_1,\ x_2 \in A_2,\ \ldots,\ x_n \in A_{n-1})$, wherein each A_i is a type relative to the context $(x_1 \in A_1,\ \ldots,\ x_{i-1} \in A_{i-1})$. There are now some rather natural requirements for well-formed judgements: in order to assert that $a \in A$, we must first know that $A\ \mathsf{type}$; to assert that $A = B\ \mathsf{type}$, we must first know that $A\ \mathsf{type}$ and $B\ \mathsf{type}$; and so on. We specify intensional Martin-Löf type theory as a collection of inference rules over these forms of judgement. Firstly we have the *equality rules*, which assert that the two judgement forms $A = B\ \mathsf{type}$ and $a = b \in A$ are congruences with respect to all the other operations of the theory; then we have the *structural rules*, which deal with weakening, contraction, exchange and substitution; and finally, the *logical rules*, which specify the type-formers of our theory, together with their introduction, elimination and computation rules. For the purposes of this paper, we require only the rules for the identity types, which we list in Table 1. We commit the usual abuse of notation in leaving implicit an ambient context Γ common to the premises and conclusions of each rule, and omitting the rules expressing stability under substitution in this ambient context. Let us remark also that in the rules Id-ELIM and Id-COMP, we allow the type C over which elimination is occurring to depend upon an additional contextual parameter Δ. Were we to add Π-types (dependent products) to our calculus, then these rules would be equivalent to the usual identity type rules. However, in the absence of Π-types, this extra parameter is essential to derive all but the most basic properties of the identity type.

TABLE 1. Identity type rules.

Identity types

$$\frac{A\ \mathsf{type} \qquad a, b \in A}{\mathrm{Id}_A(a,b)\ \mathsf{type}}\ \text{Id-FORM}; \qquad \frac{A\ \mathsf{type} \qquad a \in A}{\mathrm{r}(a) \in \mathrm{Id}_A(a,a)}\ \text{Id-INTRO}$$

$$\frac{\begin{array}{c}(x, y \in A,\ p \in \mathrm{Id}_A(x,y),\ \Delta(x,y,p))\ C(x,y,p)\ \mathsf{type} \\ (x \in A,\ \Delta(x,x,\mathrm{r}(x)))\ d(x) \in C(x,x,\mathrm{r}(x)) \qquad a, b \in A \qquad p \in \mathrm{Id}_A(a,b)\end{array}}{(\Delta(a,b,p))\ \mathrm{J}_d(a,b,p) \in C(a,b,p)}\ \text{Id-ELIM}$$

$$\frac{\begin{array}{c}(x, y \in A,\ p \in \mathrm{Id}_A(x,y),\ \Delta(x,y,p))\ C(x,y,p)\ \mathsf{type} \\ (x \in A,\ \Delta(x,x,\mathrm{r}(x)))\ d(x) \in C(x,x,\mathrm{r}(x)) \qquad a \in A\end{array}}{(\Delta(a,a,\mathrm{r}(a)))\ \mathrm{J}_d(a,a,\mathrm{r}(a)) = d(a) \in C(a,a,\mathrm{r}(a))}\ \text{Id-COMP}$$

We now establish some further notational conventions. Where it improves clarity we may omit brackets in function applications, writing $hgfx$ in place of $h(g(f(x)))$, for example. We may drop the subscript A in an identity type $\mathrm{Id}_A(a,b)$ where no confusion seems likely to occur. Given $a, b \in A$, we may say that a and b are *propositionally equal* to indicate that the type $\mathrm{Id}(a,b)$ is inhabited. We shall also make use of *vector notation* in the style of [**3**]. Given a context $\Gamma = (x_1 \in A_1, \ldots, x_n \in A_n)$, we may abbreviate a series of judgements:

$$a_1 \in A_1, \quad a_2 \in A_2(a_1), \ldots, a_n \in A_n(a_1, \ldots, a_{n-1}),$$

as $a \in \Gamma$, where $a := (a_1, \ldots, a_n)$. We may also use this notation to abbreviate sequences of hypothetical elements; so, for example, we may specify a dependent type in context Γ as $(x \in \Gamma)\ A(x)$ type. We will also make use of the notion of *telescope* defined in [**3**]. Given Γ a context as before, this allows us to abbreviate the series of judgements

$$\begin{gathered} (x \in \Gamma)\ B_1(x)\ \mathsf{type}, \\ (x \in \Gamma,\ y_1 \in B_1)\ B_2(x, y_1)\ \mathsf{type}, \\ \vdots \\ (x \in \Gamma,\ y_1 \in B_1,\ \ldots,\ y_{m-1} \in B_{m-1})\ B_m(x, y_1, \ldots y_{m-1})\ \mathsf{type} \end{gathered}$$

as $(x \in \Gamma)\ \Delta(x)$ ctxt, where $\Delta(x) := (y_1 \in B_1(x),\ y_2 \in B_2(x, y_1),\ \ldots)$. We say that Δ is a *context dependent upon* Γ. Given such a dependent context, we may abbreviate the series of judgements

$$\begin{gathered} (x \in \Gamma)\ f_1(x) \in B_1(x), \\ \vdots \\ (x \in \Gamma)\ f_m(x) \in B_m(x, f_1(x), \ldots, f_{m-1}(x)) \end{gathered}$$

as $(x \in \Gamma)\ f(x) \in \Delta(x)$, and say that f is a *dependent element* of Δ. We can similarly assign a meaning to the judgements $(x \in \Gamma)\ \Delta(x) = \Theta(x)$ ctxt and $(x \in \Gamma)\ f(x) = g(x) \in \Delta(x)$, expressing the definitional equality of two dependent contexts, and the definitional equality of two dependent elements of a dependent context.

Let us now recall some basic facts about categorical models of type theory. For a more detailed treatment the reader could refer to [**11, 18**], for example. If $\mathbb{T}$ is a dependently typed calculus admitting each of the rules described above, then we may construct from it a category $\mathcal{C}_{\mathbb{T}}$ known as the *classifying category* of $\mathbb{T}$. Its objects are contexts Γ, Δ, ..., in $\mathbb{T}$, considered modulo definitional equality (so we identify Γ and Δ whenever $\Gamma = \Delta$ ctxt is derivable); and its maps $\Gamma \to \Delta$ are *context morphisms*, which are judgements $(x \in \Gamma)\ f(x) \in \Delta$ considered modulo definitional equality. The identity map on Γ is given by $(x \in \Gamma)\ x \in \Gamma$, while composition is given by substitution of terms. Now, for any judgement $(x \in \Gamma)\ A(x)$ type of $\mathbb{T}$, there is a distinguished context morphism

$$(x \in \Gamma,\ y \in A(x)) \longrightarrow (x \in \Gamma)$$

which sends (x, y) to x. We call morphisms of $\mathcal{C}_{\mathbb{T}}$ of this form *basic dependent projections*. By a *dependent projection*, we mean any composite of zero or more basic dependent projections. An important property of dependent projections is that they are stable under pullback, in the sense that, for every $(x \in \Gamma)\ A(x)$ type and context morphism $f \colon \Delta \to \Gamma$, we may show the square

$$\begin{tikzcd} (w \in \Delta,\ y \in A(f(w))) \arrow[r] \arrow[d, "p'"'] & (x \in \Gamma,\ y \in A(x)) \arrow[d, "p"] \\ \Delta \arrow[r, "f"'] & \Gamma, \end{tikzcd}$$

wherein the uppermost arrow sends (w, y) to (fw, y), to be a pullback in $\mathcal{C}_{\mathbb{T}}$. Let us now recall from [6] a second class of maps in $\mathcal{C}_{\mathbb{T}}$ which will play an important role in this paper. A context morphism $f : \Gamma \to \Delta$ is said to be an *injective equivalence* if it validates type-theoretic rules:

$$\frac{(y \in \Delta)\ \Lambda(y)\ \mathsf{ctxt} \qquad (x \in \Gamma)\ d(x) \in \Lambda(f(x)) \qquad b \in \Delta}{\mathrm{E}_d(b) \in \Lambda(b)}$$

and

$$\frac{(y \in \Delta)\ \Lambda(y)\ \mathsf{ctxt} \qquad (x \in \Gamma)\ d(x) \in \Lambda(f(x)) \qquad a \in \Gamma}{\mathrm{E}_d(f(a)) = d(a) \in \Lambda(f(a))}.$$

The name is motivated by the groupoid model of type theory, wherein the injective equivalences are precisely the injective groupoid equivalences. Intuitively, a morphism $f : \Gamma \to \Delta$ is an injective equivalence just when every (dependent) function out of Δ is determined, up to propositional equality, by its restriction to Γ. The leading example of an injective equivalence is given by the context morphism $A \to (x, y \in A, p \in \mathrm{Id}(x, y))$ sending x to $(x, x, \mathrm{r}x)$. That this map is an injective equivalence is precisely the content of the Id-elimination and computation rules. Diagramatically, a map f is an injective equivalence if, for every commutative square of the form

$$\begin{tikzcd}
\Gamma \arrow[r, "d"] \arrow[d, "h"'] & (\Delta, \Lambda) \arrow[d, "p"] \\
\Delta \arrow[r, equal] & \Delta
\end{tikzcd}$$

with p a dependent projection, we may find a diagonal filler $E_d : \Delta \to (\Delta, \Lambda)$ making both induced triangles commute. By the stability of dependent projections under pullback, this is equivalent with the property that we should be able to find fillers for all commutative squares of the form

$$\begin{tikzcd}
\Gamma \arrow[r, "d"] \arrow[d, "h"'] & (\Phi, \Lambda) \arrow[d, "p"] \\
\Delta \arrow[r, "k"'] & \Phi
\end{tikzcd} \tag{3}$$

again with p a dependent projection. See [6, Section 5] for an elementary characterization of the class of injective equivalences.

3. *The main result*

3.1. *An overview of the proof*

We are now ready to begin the proof of our main result, which says that if $\mathbb{T}$ is a dependently typed calculus admitting each of the rules described in Subsection 2.2, then each type A therein gives rise to a weak ω-groupoid whose objects are elements of A, and whose higher cells are elements of the iterated identity types on A. In fact, we will be able to prove the stronger result that A provides the 'type of objects' for a weak ω-groupoid which is, in a suitable sense, *internal* to $\mathbb{T}$.

As explained in Section 1, we will give our proof twice: once informally, using a type-theoretic language, and once formally, using an axiomatic categorical framework which captures just those aspects of the type theory that allow the proof to go through. In this section, we give the informal proof. We shall concentrate in the first instance on constructing a weak ω-category, and defer the question of whether or not it is a weak ω-groupoid until the formal proof.

We begin by defining what we mean by a weak ω-category internal to a type theory $\mathbb{T}$. More specifically, given some globular operad P, we define a notion of P-algebra internal to $\mathbb{T}$. The underlying data for such a P-algebra is a *globular context* $(\Delta)\ \Gamma \in \mathbb{T}$; which is a sequence of judgements as follows:

$$\begin{aligned} (\Delta)\ &\Gamma_0\ \mathsf{ctxt},\\ (\Delta, x, y \in \Gamma_0)\ &\Gamma_1(x,y)\ \mathsf{ctxt},\\ (\Delta, x, y \in \Gamma_0,\ p, q \in \Gamma_1(x,y))\ &\Gamma_2(x,y,p,q)\ \mathsf{ctxt},\\ &\vdots \end{aligned}$$

Like globular sets, globular contexts also have a coinductive characterization: to give a globular context $(\Delta)\ \Gamma$ is to give a context $(\Delta)\ \Gamma_0$ and a globular context $(\Delta,\ x, y : \Gamma_0)\ \Gamma_{+1}(x,y)$. In order to define the operations making a globular context Γ (where henceforth we simplify the notation by omitting the precontext Δ) into a P-algebra, we first define for each pasting diagram $\pi \in (T1)_n$ the context Γ^π consisting of 'π-indexed elements of Γ'. This is done by induction on π:

(i) if $\pi = \star$, then $\Gamma^\pi := \Gamma_0$;
(ii) if $\pi = (\pi_1, \dots, \pi_k)$, then Γ^π is the context

$$(x_0, \dots, x_k \in \Gamma_0,\ y_1 \in \Gamma_{+1}(x_0,x_1)^{\pi_1}, \dots, y_k \in \Gamma_{+1}(x_{k-1},x_k)^{\pi_k}).$$

For example, if π is the pasting diagram (1), then the context Γ^π is given by

$$\begin{aligned}&(x_0, x_1, x_2 \in \Gamma_0,\ s, t \in \Gamma_1(x_0,x_1),\ \alpha \in \Gamma_2(x_0,x_1,s,t),\ u, v, w \in \Gamma_1(x_1,x_2),\\ &\quad \beta \in \Gamma_2(x_1,x_2,u,v),\ \gamma \in \Gamma_2(x_1,x_2,v,w))\end{aligned}$$

while if $\pi \in (T1)_n$ is the element $\iota_n = (\dots(\star)\dots)$, then Γ^{ι_n} is the context

$$(x_0, y_0 \in \Gamma_0,\ x_1, y_1 \in \Gamma_1(x_0,y_0),\ \dots,\ x_n \in \Gamma_n(x_0,y_0,\dots,x_{n-1},y_{n-1}))$$

indexing the totality of the n-cells of Γ. Now to give a P-algebra structure on the globular context Γ will be to give, for every $\pi \in (T1)_n$ and $\theta \in P_\pi$, a context morphism

$$[\theta] : \Gamma^\pi \longrightarrow \Gamma^{\iota_n}$$

interpreting the operation θ, subject to the following axioms. Firstly, the interpretations should be compatible with source and target, which is to say that diagrams of the form

$$\begin{tikzcd} \Gamma^\pi \arrow[r, "{[\theta]}"] \arrow[d, "\sigma"'] & \Gamma^{\iota_n} \arrow[d, "\sigma"] \\ \Gamma^{\partial\pi} \arrow[r, "{[s\theta]}"'] & \Gamma^{\iota_{n-1}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Gamma^\pi \arrow[r, "{[\theta]}"] \arrow[d, "\tau"'] & \Gamma^{\iota_n} \arrow[d, "\tau"] \\ \Gamma^{\partial\pi} \arrow[r, "{[t\theta]}"'] & \Gamma^{\iota_{n-1}} \end{tikzcd}$$

should commute; here $\sigma, \tau : \Gamma^\pi \to \Gamma^{\partial\pi}$ are source and target projections, respectively, defined by a further straightforward induction over π. Secondly, the trivial pasting operations should have a trivial interpretation; which is to say that

$$[\iota_n] = \mathrm{id}_{\Gamma^{\iota_n}} : \Gamma^{\iota_n} \longrightarrow \Gamma^{\iota_n}.$$

Thirdly, the interpretation of a composite $[\theta \circ \psi]$ should be 'given by the composite of $[\theta]$ with $[\psi]$', in the sense that the following diagram commutes:

$$\begin{tikzcd} \Gamma^{\pi\circ\phi} \arrow[r, "{[\psi]}"] \arrow[dr, "{[\theta\circ\psi]}"'] & \Gamma^{\pi} \arrow[d, "{[\theta]}"] \\ & \Gamma^{\iota_n}. \end{tikzcd}$$

This is not yet entirely formal, because we have not indicated how the map $[\psi] : \Gamma^{\pi\circ\phi} \to \Gamma^{\pi}$ should be defined. Intuitively, it is the morphism that applies simultaneously the interpretations of the operations indexed by $\psi : \hat{\pi} \to P1$; but it is not immediately clear how to make this precise. We shall do so in Subsection 3.3 below, using Michael Batanin's machinery of *monoidal globular categories* [**2**]. A general result from this theory allows us to associate to the globular context Γ a particular globular operad $[\Gamma, \Gamma]$, the *endomorphism operad* of Γ, which is such that we may *define* P-algebra structures on Γ to be globular operad morphisms $P \to [\Gamma, \Gamma]$. This operad $[\Gamma, \Gamma]$ has as operations of shape π, all serially commutative diagrams

$$\begin{tikzcd} \Gamma^{\pi} \arrow[r, "f_n"] \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & \Gamma^{\iota_n} \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] \\ \Gamma^{\partial\pi} \arrow[r, shift left, "f_{n-1}"] \arrow[r, shift right, "g_{n-1}"'] \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & \Gamma^{\iota_{n-1}} \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] \\ \vdots \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & \vdots \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] \\ \Gamma^{\star} \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] & \Gamma^{\star} \end{tikzcd} \tag{4}$$

of context morphisms. The source and target functions $[\Gamma, \Gamma]_{\pi} \to [\Gamma, \Gamma]_{\partial\pi}$ send such a diagram to its subdiagram headed by f_{n-1} and g_{n-1}, respectively; the identity operation $\iota_n \in [\Gamma, \Gamma]_{\iota_n}$ has each f_i and g_i given by an identity map; whilst describing substitution of operations in $[\Gamma, \Gamma]$ is precisely the problem that we encountered above, and which Batanin's machinery solves. It is easy to see that a map of globular operads $P \to [\Gamma, \Gamma]$ encodes exactly the structure of an internal P-algebra sketched above.

We may now give a precise statement of the main result. Given a type theory $\mathbb{T}$ admitting the rules of Subsection 2.2 and a type $A \in \mathbb{T}$, we construct a normalized, contractible, globular operad P such that the globular context $\underline{A}$ given by

$$\begin{array}{rl} & A\ \mathsf{ctxt}, \\ (x, y \in A) & \mathrm{Id}_A(x, y)\ \mathsf{ctxt}, \\ (x, y \in A,\ p, q \in \mathrm{Id}_A(x, y)) & \mathrm{Id}_{\mathrm{Id}_A(x,y)}(p, q)\ \mathsf{ctxt}, \\ & \vdots \end{array}$$

admits an internal P-algebra structure. Now, it is straightforward to find an operad for which $\underline{A}$ is an algebra, namely, the endomorphism operad $[\underline{A}, \underline{A}]$, with algebra structure given by the identity morphism $[\underline{A}, \underline{A}] \to [\underline{A}, \underline{A}]$, but this does not help us, since there is no reason to expect this operad to be either normalized or contractible. However, it comes rather close to being contractible, in a sense that we shall now explain. For $[\underline{A}, \underline{A}]$ to be contractible would be for us to ask that, for every serially commutative diagram

$$
\begin{tikzcd}
\underline{A}^{\pi} \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & \underline{A}^{\iota_n} \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] \\
\underline{A}^{\partial\pi} \arrow[r, shift left, "f_{n-1}"] \arrow[r, shift right, "g_{n-1}"'] \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & \underline{A}^{\iota_{n-1}} \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] \\
\vdots \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & \vdots \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] \\
\underline{A}^{\star} \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] & \underline{A}^{\star}
\end{tikzcd}
\tag{5}
$$

of context morphisms, we could find a map $\underline{A}^{\pi} \to \underline{A}^{\iota_n}$ completing it to a diagram like (4). Let us consider in particular the case where π is the pasting diagram of (1). Here, to give the data of (5) is to give the judgements

$$
\begin{gathered}
(x \in A)\ f_0(x) \in A, \\
(x \in A)\ g_0(x) \in A, \\
(x, y, z \in A,\ p \in \mathrm{Id}(x, y), q \in \mathrm{Id}(y, z))\ f_1(x, y, z, p, q) \in \mathrm{Id}(f_0(x), g_0(z)), \\
(x, y, z \in A,\ p \in \mathrm{Id}(x, y), q \in \mathrm{Id}(y, z))\ g_1(x, y, z, p, q) \in \mathrm{Id}(f_0(x), g_0(z)),
\end{gathered}
\tag{6}
$$

while to give its completion $f_2 : \underline{A}^{\pi} \to \underline{A}^{\iota_2}$ would be to give a judgement

$$
\begin{aligned}
&(x, y, z \in A,\ s, t \in \mathrm{Id}(x, y),\ \alpha \in \mathrm{Id}(s, t),\ u, v, w \in \mathrm{Id}(y, z),\ \beta \in \mathrm{Id}(u, v),\ \gamma \in \mathrm{Id}(v, w)) \\
&\qquad f_2(x, y, z, s, t, \alpha, u, v, w, \beta, \gamma) \in \mathrm{Id}(f_1(x, y, z, s, u), g_1(x, y, z, t, w)).
\end{aligned}
$$

We might attempt to obtain such a judgement by repeated application of the identity type elimination rule. Indeed, by Id-elimination on α, it suffices to consider the case where $s = t$ and $\alpha = \mathrm{r}(s)$; and by Id-elimination on γ and β, it suffices to consider the case where $u = v = w$ and $\gamma = \beta = \mathrm{r}(u)$. Thus it suffices to find a term

$$
(x, y, z \in A,\ s \in \mathrm{Id}(x, y),\ u \in \mathrm{Id}(y, z))\ f_2'(x, y, z, s, u) \in \mathrm{Id}(f_1(x, y, z, s, u), g_1(x, y, z, s, u)).
$$

But now by Id-elimination on s and on u, it suffices to consider the case where $x = y = z$ and $s = u = \mathrm{r}(x)$; so that it even suffices to find a term

$$
(x \in A)\ f_2''(x) \in \mathrm{Id}(f_1(x, x, x, \mathrm{r}x, \mathrm{r}x), g_1(x, x, x, \mathrm{r}x, \mathrm{r}x)). \tag{7}
$$

Yet here we encounter the problem that f_1 and g_1, being arbitrarily defined, need not agree at $(x, x, x, \mathrm{r}x, \mathrm{r}x)$, so that there is in general no reason for a term like (7) to exist. However, there is a straightforward way of removing this obstruction: we restrict attention to those operations of shape π that, when applied to a term consisting solely of reflexivity proofs, yield another reflexivity proof. We may formalize this as follows. For each $\pi \in (T1)_n$, we define, by induction on π, a pointing $r_\pi : A \to \underline{A}^{\pi}$ such that:

(i) if $\pi = \star$, then $r_\star := \mathrm{id} : A \to A$;
(ii) if $\pi = (\pi_1, \dots, \pi_k)$, then r_π is the context morphism

$$
(x \in A)\ (\underbrace{x, \dots, x}_{k \text{ times}}, r_{\pi_1}(\mathrm{r}x), \dots, r_{\pi_k}(\mathrm{r}x)) \in \underline{A}^{\pi}.
$$

In our example, if the judgements in (6) commuted with the A-pointings, then we would have that $f_1(x, x, x, \mathrm{r}x, \mathrm{r}x) = g_1(x, x, x, \mathrm{r}x, \mathrm{r}x) = \mathrm{r}(x) \in \mathrm{Id}(x, x)$, so that in (7) we could define

$$(x \in A)\ f_2''(x) := \mathrm{r}(\mathrm{r}x) \in \mathrm{Id}(\mathrm{r}x, \mathrm{r}x)$$

and in this way obtain by repeated Id-elimination the desired completion $f_2 : \underline{A}^{\pi} \to \underline{A}^{\iota_2}$.

Motivated by this, we define the suboperad $P \subset [\underline{A}, \underline{A}]$ to have as its operations of shape π those diagrams of the form (4) in which each f_i and g_i commutes with the A-pointings just defined. Again, it is intuitively clear that this defines a suboperad, which is to say that the operations with this property are closed under identities and substitution, but to prove this requires a second excursion into the theory of monoidal globular categories, one which for the purposes of the present section, we omit. However, we claim further that P is both normalized and contractible. This will then prove our main result, since the globular context $\underline{A}$ is a P-algebra, as witnessed by the map of globular operads $P \hookrightarrow [\underline{A}, \underline{A}]$, so that we will have shown the globular context $\underline{A}$ to be an algebra for a normalized, contractible, globular operad P, and hence a weak ω-category.

Now, to show P normalized is trivial, since its operations of shape $\star$ are those context morphisms $A \to A$ that commute with the pointing $\mathrm{id}_A : A \to A$, and there is of course only one such. On the other hand, we see that it is contractible through a generalization of the argument given in the example above. The only part requiring some thought is how to describe generically the process of repeatedly applying Id-elimination. The key to doing this is to prove by induction on π that each of the pointings $r_\pi : A \to \underline{A}^{\pi}$ is an injective equivalence in the sense defined in Subsection 2.2. The injective equivalence structure now encodes the process of repeated Id-elimination. Using this, we may show P contractible as follows. Suppose that we are given a diagram like (5), where each f_i and g_i commutes with the A-pointings. We let $B\underline{A}^{\iota_n}$ denote the context obtained from $\underline{A}^{\iota_n}$ by removing its final variable, and let $p : \underline{A}^{\iota_n} \to B\underline{A}^{\iota_n}$ denote the corresponding dependent projection. Then we have a commutative square

$$\begin{tikzcd}
A \arrow[r, "r_{\iota_n}"] \arrow[d, "r_\pi"'] & \underline{A}^{\iota_n} \arrow[d, "p"] \\
\underline{A}^{\pi} \arrow[r] & B\underline{A}^{\iota_n},
\end{tikzcd}$$

where the lower arrow is obtained by applying first the projection $\underline{A}^{\pi} \to B\underline{A}^{\pi}$, and then the maps f_{n-1} and g_{n-1}. Commutativity obtains by virtue of the fact that f_{n-1} and g_{n-1} commute with the pointings; and so, because r_π is an injective equivalence and p is a dependent projection, we can find a diagonal filler, which will be the required map $f_n : \underline{A}^{\pi} \to \underline{A}^{\iota_n}$.

3.2. *An axiomatic framework*

We now wish to make rigorous the above proof; and as we have already mentioned, we shall do so not in an explicitly type-theoretic manner, but rather within an axiomatic categorical framework. In this section, we describe this framework and give the intended type-theoretic interpretation.

DEFINITION 3.1. A category $\mathcal{C}$ is an *identity type category* if it comes equipped with two classes of maps $\mathcal{I}, \mathcal{P} \subset \mathrm{mor}\,\mathcal{C}$ satisfying the following axioms:

(i) *Empty*: $\mathcal{C}$ has a terminal object 1, and for all $A \in \mathcal{C}$, the unique map $A \to 1$ is a $\mathcal{P}$-map;
(ii) *Composition*: the classes of $\mathcal{P}$-maps and $\mathcal{I}$-maps contain the identities and are closed under composition;
(iii) *Stability*: pullbacks of $\mathcal{P}$-maps along arbitrary maps exist, and are again $\mathcal{P}$-maps;
(iv) *Frobenius*: the pullback of an $\mathcal{I}$-map along a $\mathcal{P}$-map is an $\mathcal{I}$-map;

(v) *Orthogonality*: for every commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "g"'] & D \end{tikzcd} \tag{8}$$

with $i \in \mathcal{I}$ and $p \in \mathcal{P}$, we can find a diagonal filler $j : B \to C$ such that $ji = f$ and $pj = g$;

(vi) *Identities*: for every $\mathcal{P}$-map $p : C \to D$, the diagonal map $\Delta : C \to C \times_D C$ has a factorization

$$\Delta = C \xrightarrow{\;\;r\;\;} \mathrm{Id}(C) \xrightarrow{\;\;e\;\;} C \times_D C$$

where $r \in \mathcal{I}$ and $e \in \mathcal{P}$.

We make two remarks concerning this definition. Firstly, by (Empty) and (Stability), any identity type category will have finite products, and product projections will be $\mathcal{P}$-maps. Secondly, in order to verify (Orthogonality), it suffices, by (Stability), to do so only in those cases where the map along the bottom of (8) is an identity.

Proposition 3.2. *Let $\mathbb{T}$ be a dependent type theory admitting each of the inference rules described in Subsection 2.2. Then the classifying category $\mathcal{C}_{\mathbb{T}}$ is an identity type category, where we take $\mathcal{P}$ to be the class of dependent projections and $\mathcal{I}$ to be the class of injective equivalences.*

Proof. The empty context ( ) provides a terminal object of $\mathcal{C}_{\mathbb{T}}$. (Composition) is immediate from the definitions. (Stability) corresponds to the possibility of performing type-theoretic substitution. (Frobenius) is shown to hold in [**6**, Proposition 14]; it is a categorical correlate of the fact that we allow an extra contextual parameter Δ in the statement of the Id-elimination rule. (Orthogonality) holds by the very definition of injective equivalence, together with the remark made above. Finally, (Identities) says something more than that identity types exist; it says that identity *contexts* exist: which is to say that, for every dependent context $(\Delta)\ \Gamma\ \mathsf{ctxt}$, we may find a context $(\Delta,\ x, y \in \Gamma)\ \mathrm{Id}_\Gamma(x, y)\ \mathsf{ctxt}$ such that the contextual analogues of the identity type rules are validated. That this is possible is proved in [**8**, Proposition 3.3.1]. □

We also require two stability properties of identity type categories.

Proposition 3.3. *Let $\mathcal{C}$ be an identity type category and let $X \in \mathcal{C}$. Then the coslice category $X/\mathcal{C}$ is also an identity type category, where we take the class of $\mathcal{I}$-maps and $\mathcal{P}$-maps to consist of those morphisms that become $\mathcal{I}$-maps and $\mathcal{P}$-maps, respectively, upon application of the forgetful functor $X/\mathcal{C} \to \mathcal{C}$.*

Proposition 3.4. *Let $\mathcal{C}$ be an identity type category and let $X \in \mathcal{C}$. Then the category $\mathcal{C}_X$, whose objects are $\mathcal{P}$-maps $A \to X$ and whose morphisms are commutative triangles, is also an identity type category, where we define the classes of $\mathcal{I}$-maps and $\mathcal{P}$-maps in a manner analogous to that of Proposition* 3.3.

The proofs are trivial; the only point of note is that, in the second instance, we could not take $\mathcal{C}_X$ to be the full slice category $\mathcal{C}/X$, as then (Empty) would not be satisfied.

3.3. *Internal weak ω-groupoids*

In this section, we describe the notion of weak ω-groupoid internal to an identity type category $\mathcal{C}$. We begin by defining internal P-algebras for a globular operad P.

DEFINITION 3.5. A *pre-globular context* in $\mathcal{C}$ is a diagram

$$\Gamma_0 \underset{t}{\overset{s}{\leftleftarrows}} \Gamma_1 \underset{t}{\overset{s}{\leftleftarrows}} \Gamma_2 \underset{t}{\overset{s}{\leftleftarrows}} \Gamma_3 \underset{t}{\overset{s}{\leftleftarrows}} \cdots$$

satisfying the globularity equations $ss = st$ and $ts = tt$. A pre-globular context is a *globular context* if, for each $n \geqslant 1$, the map

$$(s,t) : \Gamma_n \longrightarrow B_n\Gamma \tag{9}$$

is a $\mathcal{P}$-map, where $B_n\Gamma$ is defined as follows. We have $B_1\Gamma := \Gamma_0 \times \Gamma_0$, and have $B_{n+1}\Gamma$ given by the pullback

$$\begin{tikzcd} B_{n+1}\Gamma \arrow[r] \arrow[d] & \Gamma_n \arrow[d, "(s,t)"] \\ \Gamma_n \arrow[r, "(s,t)"'] & B_n\Gamma . \end{tikzcd} \tag{10}$$

Observe that requiring (9) to be a $\mathcal{P}$-map for $n = 1$ ensures the existence of the pullback (10) defining $B_2\Gamma$; which in turn allows us to require that (9) should be a $\mathcal{P}$-map for $n = 2$, and so on. Once again, we have a coinductive characterization of globular contexts: to give a globular context $\Gamma \in \mathcal{C}$ is to give an object Γ_0 together with a globular context $\Gamma_{+1} \in \mathcal{C}_{\Gamma_0 \times \Gamma_0}$.

The first step in defining P-algebra structure on a globular context Γ is to describe the object Γ^π of 'π-indexed elements of Γ'.

DEFINITION 3.6. Let Γ be a globular context in $\mathcal{C}$ and let $\pi \in (T1)_n$. We define the object $\Gamma^\pi \in \mathcal{C}$ by the following induction.

(i) If $\pi = \star$, then $\Gamma^\pi := \Gamma_0$.

(ii) If $\pi = (\pi_1, \ldots, \pi_k)$, then we first form the objects $(\Gamma_{+1})^{\pi_1}, \ldots, (\Gamma_{+1})^{\pi_k}$ of $\mathcal{C}_{\Gamma_0 \times \Gamma_0}$. This yields a diagram

$$\begin{tikzcd}[column sep=small] & (\Gamma_{+1})^{\pi_1} \arrow[dl, "s"'] \arrow[dr, "t"] & & \cdots \arrow[dl, "s"'] \arrow[dr, "t"] & & (\Gamma_{+1})^{\pi_k} \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \Gamma_0 & & \Gamma_0 & & \Gamma_0 & & \Gamma_0 \end{tikzcd}$$

in $\mathcal{C}$. Note that each s and t is a $\mathcal{P}$-map so that this diagram has a limit, which we define to be Γ^π.

We define the maps $\sigma, \tau : \Gamma^\pi \to \Gamma^{\partial\pi}$ by a further induction.

(i) For $\pi \in (T1)_1$, we have Γ^π given by the limit of a diagram

$$\begin{tikzcd}[column sep=small] & \Gamma_1 \arrow[dl, "s"'] \arrow[dr, "t"] & & \cdots \arrow[dl, "s"'] \arrow[dr, "t"] & & \Gamma_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \Gamma_0 & & \Gamma_0 & & \Gamma_0 & & \Gamma_0; \end{tikzcd}$$

and so we may take $\sigma, \tau : \Gamma^\pi \to \Gamma^\star = \Gamma_0$ to be given by the projections from this limit into the leftmost and rightmost copies, respectively, of Γ_0.

(ii) Otherwise, given $\pi = (\pi_1, \ldots, \pi_k)$, we first construct the maps $\sigma, \tau : (\Gamma_{+1})^{\pi_i} \to (\Gamma_{+1})^{\partial \pi_i}$. These give rise to a diagram

$$\begin{tikzcd}
& (\Gamma_{+1})^{\pi_1} \arrow[d, "\sigma"] \arrow[ddl, "s"'] \arrow[ddr, "t"] & & \cdots \arrow[d, "\sigma"] \arrow[ddl, "s"'] \arrow[ddr, "t"] & & (\Gamma_{+1})^{\pi_k} \arrow[d, "\sigma"] \arrow[ddl, "s"'] \arrow[ddr, "t"] & \\
& (\Gamma_{+1})^{\partial \pi_1} \arrow[dl, "s"'] \arrow[dr, "t"] & & \cdots \arrow[dl, "s"'] \arrow[dr, "t"] & & (\Gamma_{+1})^{\partial \pi_k} \arrow[dl, "s"'] \arrow[dr, "t"] & \\
\Gamma_0 & & \Gamma_0 & & \Gamma_0 & & \Gamma_0
\end{tikzcd}$$

and correspondingly for τ. We now take $\sigma, \tau : \Gamma^\pi \to \Gamma^{\partial\pi}$ to be the induced maps from the limit of the upper subdiagram (which is Γ^π) to the limit of the lower one (which is $\Gamma^{\partial\pi}$).

PROPOSITION 3.7. *Let $\Gamma \in \mathcal{C}$ be a globular context. Then there is a globular operad $[\Gamma, \Gamma]$ whose set of operations of shape π comprises all serially commutative diagrams of the form* (4).

We will prove this proposition using Michael Batanin's theory of *monoidal globular categories* [**2**]. The notion of monoidal globular category bears the same relationship to that of strict ω-category as the notion of monoidal category does to that of monoid; in both cases, the former notion is obtained from the latter by replacing everywhere sets with categories, functions with functors, and equalities with coherent natural isomorphisms.

DEFINITION 3.8. A *monoidal globular category* $\mathcal{E}$ is a sequence of categories and functors

$$\begin{tikzcd}
\mathcal{E}_0 & \mathcal{E}_1 \arrow[l, shift right, "S"'] \arrow[l, shift left, "T"] & \mathcal{E}_2 \arrow[l, shift right, "S"'] \arrow[l, shift left, "T"] & \mathcal{E}_3 \arrow[l, shift right, "S"'] \arrow[l, shift left, "T"] & \cdots \arrow[l, shift right, "S"'] \arrow[l, shift left, "T"]
\end{tikzcd}$$

satisfying the globularity equations $SS = ST$ and $TS = TT$, together with, for each natural number n, an identities functor

$$Z : \mathcal{E}_n \longrightarrow \mathcal{E}_{n+1}$$

and, for each pair of natural numbers $0 \leqslant k < n$, a composition functor

$$\otimes_k : \mathcal{E}_n \times_k \mathcal{E}_n \longrightarrow \mathcal{E}_n,$$

where $\mathcal{E}_n \times_k \mathcal{E}_n$ denotes the pullback

$$\begin{tikzcd}
\mathcal{E}_n \times_k \mathcal{E}_n \arrow[r] \arrow[d] & \mathcal{E}_n \arrow[d, "S^{n-k}"] \\
\mathcal{E}_n \arrow[r, "T^{n-k}"'] & \mathcal{E}_k\ .
\end{tikzcd}$$

In addition, there are given invertible natural transformations witnessing:

(i) Associativity:

$$\alpha_{n,k} : A \otimes_k (B \otimes_k C) \cong (A \otimes_k B) \otimes_k C;$$

(ii) Unitality:

$$\lambda_n : Z^{n-k} T^{n-k} A \otimes_k A \cong A \quad \text{and} \quad \rho_n : A \otimes_k Z^{n-k} S^{n-k} A \cong A;$$

(iii) Interchange:

$$\chi_{n,k,l} : (A \otimes_k B) \otimes_l (C \otimes_k D) \cong (A \otimes_l C) \otimes_k (B \otimes_l D) \quad \text{(for } k < l).$$

These data are required to satisfy a number of coherence axioms, which the reader may find in [**2**, Definition 2.3].

Just as monoidal categories provide a general environment within which we can speak of monoids, so too monoidal globular categories provide a general environment within which we can speak of algebras for a globular operad. The underlying data for an algebra in this general setting is given as follows.

Definition 3.9. A *globular object* X in a monoidal globular category $\mathcal{E}$ is a sequence of objects $X_i \in \mathcal{E}_i$, one for each natural number i, such that $S(X_{i+1}) = T(X_{i+1}) = X_i$ for all i.

To describe the additional structure required to make a globular object into a P-algebra, we employ one of the central constructions of [**2**]. This associates to each globular object $X \in \mathcal{E}$ an endomorphism operad $[X, X]$; which allows us to define a P-algebra in $\mathcal{E}$ to be a globular object X together with a globular operad morphism $P \to [X, X]$. We now describe the construction of $[X, X]$. First observe that if $\mathcal{E}$ is a monoidal globular category, then so too is $\mathcal{E}_{+1}$, where $(\mathcal{E}_{+1})_n = \mathcal{E}_{n+1}$ and the remaining data is defined in the obvious way. Moreover, if X is a globular object in $\mathcal{E}$, then X_{+1} is a globular object in $\mathcal{E}_{+1}$, where again we define $(X_{+1})_n = X_{n+1}$. Now, given a globular object $X \in \mathcal{E}$ and a pasting diagram $\pi \in (T1)_n$, we define, by induction on π, an object $X^{\otimes \pi} \in \mathcal{E}_n$ such that:

(i) if $\pi = \star$, then $X^{\otimes \pi} := X_0 \in \mathcal{E}_0$;
(ii) if $\pi = (\pi_1, \ldots, \pi_k)$, then $X^{\otimes \pi} := (X_{+1})^{\otimes \pi_1} \otimes_0 \ldots \otimes_0 (X_{+1})^{\otimes \pi_k}$.

Proposition 3.10. *Let $\mathcal{E}$ be a monoidal globular category and $X \in \mathcal{E}$ be a globular object. Then there is a globular operad $[X, X]$ with*

$$[X, X]_\pi := \mathcal{E}_n(X^{\otimes \pi}, X_n) \quad \textit{for all } \pi \in (T1)_n.$$

Proof. This is [**2**, Proposition 7.2]. □

We now use this result to prove Proposition 3.7. The first step is to construct, from our identity type category $\mathcal{C}$, a monoidal globular category $\mathcal{E}(\mathcal{C})$.

Definition 3.11. Let $\mathbb{G}$ denote the category

$$0 \underset{\tau}{\overset{\sigma}{\leftleftarrows}} 1 \underset{\tau}{\overset{\sigma}{\leftleftarrows}} 2 \underset{\tau}{\overset{\sigma}{\leftleftarrows}} \cdots$$

The *generic n-span* $\mathbb{S}_n$ is defined to be the coslice category $n/\mathbb{G}$. In low dimensions, we have that

$$\mathbb{S}_0 = \bullet, \quad \mathbb{S}_1 = \begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet \end{tikzcd}, \quad \mathbb{S}_2 = \begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[d] \arrow[drr] & & \bullet \arrow[d] \arrow[dll] \\ \bullet & & \bullet \end{tikzcd}, \ldots .$$

The monoidal globular category $\mathcal{E}(\mathcal{C})$ is defined by taking $\mathcal{E}(\mathcal{C})_n$ to be the full subcategory of the functor category $\mathcal{C}^{\mathbb{S}_n}$ on those functors that send every morphism of $\mathbb{S}_n$ to a $\mathcal{P}$-map. The remaining structure of $\mathcal{E}(\mathcal{C})$ may be found described in [2, Definition 3.2]. As a representative sample, we describe on objects the functor $Z : \mathcal{E}(\mathcal{C})_1 \to \mathcal{E}(\mathcal{C})_2$, which is given by

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} & C \arrow[dl, "1_C"'] \arrow[dr, "1_C"] & \\ C \arrow[d, "f"'] \arrow[drr, "g" near start] & & C \arrow[d, "g"] \arrow[dll, "f"' near start] \\ A & & B; \end{tikzcd}$$

and the functor $\otimes_0 : \mathcal{E}(\mathcal{C})_2 \times_0 \mathcal{E}(\mathcal{C})_2 \to \mathcal{E}(\mathcal{C})_2$, which sends the object

$$\left(\begin{tikzcd} & H \arrow[dl, "m"'] \arrow[dr, "n"] & \\ D \arrow[d, "f"'] \arrow[drr, "g" near start] & & E \arrow[d, "k"] \arrow[dll, "h"' near start] \\ A & & B \end{tikzcd} , \quad \begin{tikzcd} & K \arrow[dl, "u"'] \arrow[dr, "v"] & \\ F \arrow[d, "r"'] \arrow[drr, "q" near start] & & G \arrow[d, "t"] \arrow[dll, "s"' near start] \\ B & & C \end{tikzcd} \right)$$

of $\mathcal{E}(\mathcal{C})_2 \times_0 \mathcal{E}(\mathcal{C})_2$ to the object

$$\begin{tikzcd} & H \times_B K \arrow[dl, "m \times_B u"'] \arrow[dr, "n \times_B v"] & \\ D \times_B F \arrow[d, "f\pi_1"'] \arrow[drr, "q\pi_2" near start] & & E \times_B G \arrow[d, "t\pi_2"] \arrow[dll, "h\pi_1"' near start] \\ A & & C \end{tikzcd}$$

of $\mathcal{E}(\mathcal{C})_2$. Note that the requisite pullbacks exist by virtue of the requirement that every arrow in the above diagrams should be a $\mathcal{P}$-map.

We next observe that if Γ is a globular context in $\mathcal{C}$, then there is an associated globular object $X_\Gamma \in \mathcal{E}(\mathcal{C})$, where $(X_\Gamma)_n$ is the n-span

$$\begin{tikzcd}
& \Gamma_n \arrow[dl, "s"'] \arrow[dr, "t"] & \\
\Gamma_{n-1} \arrow[d, "s"'] \arrow[drr, "t" near start] & & \Gamma_{n-1} \arrow[d, "t"] \arrow[dll, "s"' near start] \\
\vdots \arrow[d, "s"'] \arrow[drr, "t" near start] & & \vdots \arrow[d, "t"] \arrow[dll, "s"' near start] \\
\Gamma_0 & & \Gamma_0.
\end{tikzcd}$$

By a straightforward induction on π, we may now prove that, for any $\pi \in (T1)_n$, $(X_\Gamma)^{\otimes \pi} \in \mathcal{E}_n$ is given by the n-span

$$\begin{tikzcd}
& \Gamma^{\pi} \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\
\Gamma^{\partial\pi} \arrow[d, "\sigma"'] \arrow[drr, "\tau" near start] & & \Gamma^{\partial\pi} \arrow[d, "\tau"] \arrow[dll, "\sigma"' near start] \\
\vdots \arrow[d, "\sigma"'] \arrow[drr, "\tau" near start] & & \vdots \arrow[d, "\tau"] \arrow[dll, "\sigma"' near start] \\
\Gamma^{\star} & & \Gamma^{\star};
\end{tikzcd}$$

from which it follows that the hom-set $\mathcal{E}(\mathcal{C})_n((X_\Gamma)^{\otimes\pi}, (X_\Gamma)_n)$ is precisely the set of commutative diagrams of the form (4). This allows us to complete the proof of Proposition 3.7: indeed, we may take the globular operad $[\Gamma, \Gamma]$ whose existence is asserted there to be the globular operad $[X_\Gamma, X_\Gamma]$ obtained from an application of Proposition 3.10.

DEFINITION 3.12. Let $\mathcal{C}$ be an identity type category. An *internal P-algebra* for a globular operad P is a pair (Γ, f), where Γ is a globular context in $\mathcal{C}$ and $f : P \to [\Gamma, \Gamma]$ a map of globular operads. By a *weak ω-category* in $\mathcal{C}$, we mean a triple (P, Γ, f), where P is a normalized, contractible, globular operad and (Γ, f) an internal algebra for it.

It remains to extend this to a definition of weak ω-groupoid in $\mathcal{C}$. To do this, we exploit the characterization of weak ω-groupoids given by Proposition 2.5.

DEFINITION 3.13. Let $f : P \to [\Gamma, \Gamma]$ be a weak ω-category in an identity type category $\mathcal{C}$. Now a *choice of duals* for Γ, with respect to some system of compositions (i_n, m_n) on P, is given by maps

$$\begin{aligned}
(-)^* : \Gamma_n &\longrightarrow \Gamma_n, \\
\eta : \Gamma_n &\longrightarrow \Gamma_{n+1}, \\
\epsilon : \Gamma_n &\longrightarrow \Gamma_{n+1},
\end{aligned}$$

for each $n \geqslant 1$, making the following diagrams commute:

$$\begin{tikzcd}
\Gamma_n \arrow[rr, "(-)^*"] \arrow[dr, "{(s,t)}"'] & & \Gamma_n \arrow[dl, "{(t,s)}"] \\
& \Gamma_{n-1} \times \Gamma_{n-1} &
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma_n \arrow[r, "\eta"] \arrow[d, "s"'] & \Gamma_{n+1} \arrow[d, "s"] & \Gamma_n \arrow[r, "\eta"] \arrow[d, "{((-)^*, \mathrm{id})}"'] & \Gamma_{n+1} \arrow[d, "t"] \\
\Gamma_{n-1} \arrow[r, "{[i_n]}"'] & \Gamma_n & \Gamma_n \, {}_s\!\times_t \Gamma_n \arrow[r, "{[m_n]}"'] & \Gamma_n \\
\Gamma_n \arrow[r, "\epsilon"] \arrow[d, "t"'] & \Gamma_{n+1} \arrow[d, "t"] & \Gamma_n \arrow[r, "\eta"] \arrow[d, "{(\mathrm{id}, (-)^*)}"'] & \Gamma_{n+1} \arrow[d, "s"] \\
\Gamma_{n-1} \arrow[r, "{[i_n]}"'] & \Gamma_n & \Gamma_n \, {}_s\!\times_t \Gamma_n \arrow[r, "{[m_n]}"'] & \Gamma_n .
\end{tikzcd} \tag{11}$$

We say that (Γ, f) is a *weak ω-groupoid* if it has a choice of duals with respect to every system of compositions on P.

3.4. *Types are weak ω-groupoids*

We are now ready to prove our main theorem. It follows from a general result that shows a particular class of globular contexts to admit a weak ω-groupoid structure.

DEFINITION 3.14. Let $\mathcal{C}$ be an identity type category. A globular context Γ is said to be *reflexive* if it comes equipped with morphisms

$$\begin{tikzcd}
\Gamma_0 \arrow[r, bend left, "r_0"] & \Gamma_1 \arrow[l, shift left, "s"'] \arrow[l, shift right, "t"] \arrow[r, bend left, "r_1"] & \Gamma_2 \arrow[l, shift left, "s"'] \arrow[l, shift right, "t"] \arrow[r, bend left, "r_2"] & \cdots \arrow[l, shift left, "s"'] \arrow[l, shift right, "t"]
\end{tikzcd},$$

where each r_i is an $\mathcal{I}$-map satisfying $sr_i = tr_i = \mathrm{id}_{\Gamma_i}$.

THEOREM 3.15. *Every reflexive globular context (Γ, r_i) admits the structure of a weak ω-groupoid.*

To prove the theorem, we first exhibit a weak ω-category structure, and then show this to be a weak ω-groupoid. To obtain the ω-category structure, we show the endomorphism operad $[\Gamma, \Gamma]$ of Proposition 3.7 to admit a normalized, contractible suboperad P, whereupon the inclusion of operads $P \hookrightarrow [\Gamma, \Gamma]$ exhibits Γ as a P-algebra, and hence a weak ω-category.

DEFINITION 3.16. Let (Γ, r_i) be a reflexive globular context. We define, for each $\pi \in (T1)_n$, a map $r_\pi : \Gamma_0 \to \Gamma^\pi$ by induction on π. If $\pi = \star$, then we take $r_\pi := \mathrm{id}_{\Gamma_0} : \Gamma_0 \to \Gamma_0$. Otherwise, if $\pi = (\pi_1, \ldots, \pi_k)$, then we first observe that (Γ_{+1}, r_{+1}) is a reflexive globular context in

$\mathcal{C}_{\Gamma_0\times\Gamma_0}$, where $(r_{+1})_n := r_{n+1}$. Hence by induction, we obtain, for each $1 \leqslant i \leqslant k$, maps

$$\begin{tikzcd}
\Gamma_1 \arrow[rr, "r'_{\pi_i}"] \arrow[dr, "(s,t)"'] & & (\Gamma_+)^{\pi_i} \arrow[dl, "(s,t)"] \\
& \Gamma_0 \times \Gamma_0 &
\end{tikzcd} \tag{12}$$

in $\mathcal{C}_{\Gamma_0\times\Gamma_0}$. These now give rise to a diagram

$$\begin{tikzcd}[column sep=small]
& & & \Gamma_0 \arrow[dll, "r'_{\pi_1}\circ r_0"'] \arrow[d] \arrow[drr, "r'_{\pi_k}\circ r_0"] & & & \\
& (\Gamma_{+1})^{\pi_1} \arrow[dl, "s"'] \arrow[dr, "t"] & & \cdots \arrow[dl, "s"'] \arrow[dr, "t"] & & (\Gamma_{+1})^{\pi_k} \arrow[dl, "s"'] \arrow[dr, "t"] & \\
\Gamma_0 & & \Gamma_0 & & \Gamma_0 & & \Gamma_0,
\end{tikzcd} \tag{13}$$

wherein, by a straightforward calculation, any map from Γ_0 at the top to some Γ_0 at the bottom is an identity. In particular, this means that Γ_0, together with the maps out of it, form a cone over the remainder of the diagram. However, Γ^π is, by definition, the limit of this subdiagram, and so we induce a map $r_\pi : \Gamma_0 \to \Gamma^\pi$ as required.

Proposition 3.17. *Let (Γ, r_i) be a reflexive globular context in $\mathcal{C}$. Then the globular operad $[\Gamma, \Gamma]$ has a suboperad P whose set of operations of shape π comprises all serially commutative diagrams of the form* (4) *in which the f_i and g_i commute with the pointings $r_\pi : \Gamma_0 \to \Gamma^\pi$ of Definition* 3.16.

Proof. Let us write Γ_* to denote the globular context

$$\begin{tikzcd}
& \Gamma_0 \arrow[dl, "\mathrm{id}"'] \arrow[d, "r_0"] \arrow[dr, "r_0 r_1"] & & \\
\Gamma_0 & \Gamma_1 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] & \Gamma_2 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] & \cdots \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"]
\end{tikzcd}$$

in the identity type category $\Gamma_0/\mathcal{C}$. We claim the object $(\Gamma_*)^\pi \in \Gamma_0/\mathcal{C}$ is given by $r_\pi : \Gamma_0 \to \Gamma^\pi$. Observe that this implies the result, because the endomorphism operad $[\Gamma_*, \Gamma_*]$ is then precisely the suboperad $P \subset [\Gamma, \Gamma]$ we require. We will prove the claim by induction on π. When $\pi = \star$, it is clear. So suppose now that $\pi = (\pi_1, \ldots, \pi_k)$. By the description given in Definition 3.6, and the inductive hypothesis, we see that $(\Gamma_*)^\pi$ is given by the unique map $\Gamma_0 \to \Gamma^\pi$ induced by the following cone:

$$\begin{tikzcd}[column sep=small]
& & & \Gamma_0 \arrow[dll, "r_{\pi_1}"'] \arrow[d] \arrow[drr, "r_{\pi_k}"] & & & \\
& (\Gamma_{+1})^{\pi_1} \arrow[dl, "s"'] \arrow[dr, "t"] & & \cdots \arrow[dl, "s"'] \arrow[dr, "t"] & & (\Gamma_{+1})^{\pi_k} \arrow[dl, "s"'] \arrow[dr, "t"] & \\
\Gamma_0 & & \Gamma_0 & & \Gamma_0 & & \Gamma_0.
\end{tikzcd}$$

Thus, it suffices to show that this cone coincides with (13); which is to show that, for each $1 \leqslant i \leqslant k$, we have $r_{\pi_i} = r'_{\pi_i} \circ r_0$. Now, observe that r_{π_i} is obtained as $((\Gamma_*)_{+1})^{\pi_i}$, where $(\Gamma_*)_{+1}$

is the globular context

$$\begin{tikzcd} & \Gamma_0 \arrow[dl, "r_0"'] \arrow[d, "r_0 r_1"] \arrow[dr, "r_0 r_1 r_2"] & & \\ \Gamma_1 \arrow[dr] & \Gamma_2 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \arrow[d] & \Gamma_3 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \arrow[dl] & \cdots \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \\ & \Gamma_0 \times \Gamma_0 & & \end{tikzcd}$$

in $\Gamma_0/\mathcal{C}_{\Gamma_0\times\Gamma_0}$. On the other hand, by a further application of the inductive hypothesis, r'_{π_i} is obtained as the map $((\Gamma_{+1})_*)^{\pi_i}$, where $(\Gamma_{+1})_*$ is the globular context

$$\begin{tikzcd} & \Gamma_1 \arrow[dl, "\mathrm{id}"'] \arrow[d, "r_1"] \arrow[dr, "r_1 r_2"] & & \\ \Gamma_1 \arrow[dr] & \Gamma_2 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \arrow[d] & \Gamma_3 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \arrow[dl] & \cdots \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \\ & \Gamma_0 \times \Gamma_0 & & \end{tikzcd}$$

in $\Gamma_1/\mathcal{C}_{\Gamma_0\times\Gamma_0}$. But the functor $(r_0)_! : \Gamma_1/\mathcal{C}_{\Gamma_0\times\Gamma_0} \to \Gamma_0/\mathcal{C}_{\Gamma_0\times\Gamma_0}$ given by precomposition with the map $r_0 : \Gamma_0 \to \Gamma_1$ of $\mathcal{C}_{\Gamma_0\times\Gamma_0}$ sends the latter of these globular contexts to the former; and thus, because $(r_0)_!$ preserves limits, it must also send $((\Gamma_{+1})_*)^{\pi_i}$ to $((\Gamma_*)_{+1})^{\pi_i}$, which is to say that $r_{\pi_i} = r'_{\pi_i} \circ r_0$, as required. □

Thus, for a reflexive globular context (Γ, r_i), we have now defined the suboperad $P \subset [\Gamma, \Gamma]$ required for the proof of Theorem 3.15. It remains only to show that P is normalized and contractible. To do this, we need the following result.

PROPOSITION 3.18. *Let (Γ, r_i) be a reflexive globular context in $\mathcal{C}$. Then each of the maps $r_\pi : \Gamma_0 \to \Gamma^\pi$ of Definition* 3.16 *is an $\mathcal{I}$-map.*

Proof. We proceed by induction on π. When $\pi = \star$, we have r_π an identity map, and hence an $\mathcal{I}$-map. So suppose now that $\pi = (\pi_1, \ldots, \pi_k)$, and consider the diagram (13) defining the map $r_\pi : \Gamma_0 \to \Gamma^\pi$. In it, each of the maps r'_{π_i} is an $\mathcal{I}$-map by induction, and so because r_0 is an $\mathcal{I}$-map by assumption, and $\mathcal{I}$-maps are closed under composition, $r'_{\pi_i} \circ r_0$ is also an $\mathcal{I}$-map. Repeated application of the following lemma now completes the proof. □

LEMMA 3.19. *Suppose that*

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dd, "\mathrm{id}_A"] \arrow[dr, "j"] & \\ B \arrow[dr, "p"'] & & C \arrow[dl, "q"] \\ & A & \end{tikzcd}$$

is a commutative diagram in an identity type category $\mathcal{C}$. Suppose further that i and j are $\mathcal{I}$-maps, and p and q are $\mathcal{P}$-maps. Then the induced map $(i, j) : A \to B \times_A C$ is also an $\mathcal{I}$-map.

Proof. We first form the pullback square

$$\begin{tikzcd} B \times_A C \arrow[r, "q'"] \arrow[d, "p'"'] & B \arrow[d, "p"] \\ C \arrow[r, "q"'] & A. \end{tikzcd}$$

Now the universal property of this pullback induces a factorization of the commutative square

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}_B"] \arrow[d, "jp"'] & B \arrow[d, "p"] \\ C \arrow[r, "q"'] & A \end{tikzcd}$$

as

$$\begin{tikzcd} B \arrow[r, "j'"] \arrow[d, "p"'] & B \times_A C \arrow[r, "q'"] \arrow[d, "p'"] & B \arrow[d, "p"] \\ A \arrow[r, "j"'] & C \arrow[r, "q"'] & A. \end{tikzcd}$$

Since the outer rectangle has identities along both horizontal edges, it is a pullback. However, the right-hand square is a pullback, and so we deduce that the left-hand square is too. Now p' is a $\mathcal{P}$-map by (Stability) and j is an $\mathcal{I}$-map by assumption, and so by (Frobenius), j' is also an $\mathcal{I}$-map. It follows, by (Composition) and the fact that i is an $\mathcal{I}$-map, that

$$A \xrightarrow{i} B \xrightarrow{j'} B \times_A C$$

is also an $\mathcal{I}$-map. But this map is the induced map $(i, j) : A \to B \times_A C$, since it has i as its projection onto B, and $jpi = j$ as its projection onto C. $\square$

PROPOSITION 3.20. *Let (Γ, r_i) be a reflexive globular context in $\mathcal{C}$. Then the suboperad $P \subset [\Gamma, \Gamma]$ of Proposition 3.17 is both normalized and contractible.*

Proof. Note first that $P_\star$ is the set of all maps $f_0 : \Gamma_0 \to \Gamma_0$ for which $f_0 \circ \mathrm{id}_{\Gamma_0} = \mathrm{id}_{\Gamma_0}$ and hence a singleton, which proves that P is normalized. To show it contractible, we must show that, given a serially commutative diagram of the form

$$\begin{tikzcd} \Gamma^\pi \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & \Gamma_n \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ \Gamma^{\partial\pi} \arrow[r, shift left, "f_{n-1}"] \arrow[r, shift right, "g_{n-1}"'] \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & \Gamma_{n-1} \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ \vdots \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & \vdots \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ \Gamma_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] & \Gamma_0 \end{tikzcd} \tag{14}$$

wherein each f_i and g_i commutes with the pointings, we can find a map $f_n : \Gamma^\pi \to \Gamma_n$ completing the diagram (and commuting with the pointings). First we note that the diagram

$$\begin{tikzcd} \Gamma^\pi \arrow[r, "f_{n-1}\circ\sigma"] \arrow[d, "g_{n-1}\circ\tau"'] & \Gamma_{n-1} \arrow[d, "(s,t)"] \\ \Gamma_{n-1} \arrow[r, "(s,t)"'] & B_{n-1}\Gamma \end{tikzcd}$$

commutes, as may be seen by postcomposing it with the two projections $B_{n-1}\Gamma \rightrightarrows \Gamma_{n-2}$, and observing that the resultant diagrams are commutative. Thus we induce a map $k : \Gamma^\pi \to B_n\Gamma$. We now consider the diagram

$$\begin{tikzcd} \Gamma_0 \arrow[r, "r_{n-1}\cdots r_0"] \arrow[d, "r_\pi"'] & \Gamma_n \arrow[d, "(s,t)"] \\ \Gamma^\pi \arrow[r, "k"'] & B_n\Gamma. \end{tikzcd}$$

That this is commutative once again follows from the fact that it is so upon postcomposition with the two projections $B_n\Gamma \rightrightarrows \Gamma_{n-1}$. Moreover, r_π is an $\mathcal{I}$-map by Proposition 3.18, and (s,t) is a $\mathcal{P}$-map by the definition of globular context, so that by (Orthogonality), we can find a map $f_n : \Gamma^\pi \to \Gamma_n$ making both induced triangles commute. The fact that the lower triangle commutes indicates that f_n renders the diagram (14) serially commutative; while the fact that the upper triangle commutes indicates that f_n commutes with the pointings. □

Thus we have shown the operad $P \subset [\Gamma,\Gamma]$ to be normalized and contractible, from which it follows that the inclusion $P \hookrightarrow [\Gamma,\Gamma]$ exhibits Γ as a weak ω-category. It remains to show that this weak ω-category is a weak ω-groupoid.

PROPOSITION 3.21. *Let (Γ, r_i) be a reflexive globular context in $\mathcal{C}$ and let $P \subset [\Gamma,\Gamma]$ be the operad defined above. Then the inclusion $P \hookrightarrow [\Gamma,\Gamma]$ exhibits Γ as a weak ω-groupoid.*

Proof. According to Definition 3.13, we must show that, for any given system of compositions (i_n, m_n) for P, there is a corresponding choice of duals for Γ. Now, for each $n \geqslant 1$ we have a commutative diagram

$$\begin{tikzcd} \Gamma_0 \arrow[r, "r_{n-1}\cdots r_0"] \arrow[d, "r_{n-1}\cdots r_0"'] & \Gamma_n \arrow[d, "(t,s)"] \\ \Gamma_n \arrow[r, "(s,t)"'] & B_n\Gamma. \end{tikzcd}$$

The left-hand morphism is an $\mathcal{I}$-map, and the right-hand one is a $\mathcal{P}$-map; and so by (Orthogonality) we have a diagonal filler $(-)^* : \Gamma_n \to \Gamma_n$. Commutativity of the lower triangle implies the commutativity of the first diagram in (11). We induce η and ϵ similarly, by considering the commutative squares

$$\begin{tikzcd} \Gamma_0 \arrow[r, "r_n r_{n-1}\cdots r_0"] \arrow[d, "r_{n-1}\cdots r_0"'] & \Gamma_{n+1} \arrow[d, "(s,t)"] \\ \Gamma_n \arrow[r, "([i_n]s,\ [m_n]\circ((-)^*{,}\mathrm{id}))"'] & B_{n+1}\Gamma \end{tikzcd}$$

$$\begin{tikzcd}
\Gamma_0 \arrow[rr, "r_n r_{n-1} \cdots r_0"] \arrow[d, "r_{n-1}\cdots r_0"'] & & \Gamma_{n+1} \arrow[d, "(s,t)"] \\
\Gamma_n \arrow[rr, "([m_n]\circ(\mathrm{id},(-)^*),\ [i_n]t)"'] & & B_{n+1}\Gamma.
\end{tikzcd}$$

Again, commutativity of the lower triangles entails the commutativity of the remaining four diagrams in (11). □

We have thus shown that every reflexive globular context in an identity type category $\mathcal{C}$ bears a structure of a weak ω-groupoid. Note that in giving this proof, we have nowhere used the axiom (Identities). In fact, the only reason we need it is to show that, from an object of $\mathcal{C}$, we can construct a reflexive globular context corresponding to its tower of identity types.

Definition 3.22. Let $\mathcal{C}$ be an identity type category and let $A \in \mathcal{C}$. We define a reflexive globular context $\underline{A} \in \mathcal{C}$ by the following induction. For the base case, we take $A_0 = A$. For the inductive step, suppose that we have defined $A_0, \ldots, A_n$. Then we may form the n-dimensional boundary $B_n\underline{A}$ of A, and by induction the map $(s,t) : A_n \to B_n\underline{A}$ is a $\mathcal{P}$-map. So by (Identities), we may factorize the diagonal morphism $A_n \to A_n \times_{B_n\underline{A}} A_n$ as

$$A_n \xrightarrow{r_{n+1}} \mathrm{Id}(A_n) \xrightarrow{e_{n+1}} A_n \times_{B_n\underline{A}} A_n \quad ,$$

with r_{n+1} an $\mathcal{I}$-map and e_{n+1} a $\mathcal{P}$-map. We now define A_{n+1} to be $\mathrm{Id}(A_n)$, and $s,t : A_{n+1} \to A_n$ to be the composites of e_{n+1} with the two projection morphisms $A_n \times_{B_n\underline{A}} A_n \to A_n$. It remains to show that the induced map $(s,t) : A_{n+1} \to B_{n+1}\underline{A}$ is a $\mathcal{P}$-map. But we recall that $B_{n+1}\underline{A}$ was defined by the pullback diagram (10), so that $B_{n+1}\underline{A} = A_n \times_{B_n\underline{A}} A_n$, and the induced map (s,t) is precisely e_{n+1}, which is, by assumption, a $\mathcal{P}$-map.

Taking this definition together with Theorem 3.15, we immediately obtain the following.

Theorem 3.23. *Let $\mathcal{C}$ be an identity type category and let $A \in \mathcal{C}$. Then the globular context $\underline{A}$ is a weak ω-groupoid in $\mathcal{C}$.*

In particular, taking $\mathcal{C}$ to be the identity type category $\mathcal{C}_{\mathbb{T}}$ associated with some dependent type theory $\mathbb{T}$ yields the following corollary.

Theorem 3.24. *Let $\mathbb{T}$ be a dependent type theorsy admitting each of the rules described in Subsection 2.2. Then, for each type A of $\mathbb{T}$, the tower of identity types over A is a weak ω-groupoid.*

Acknowledgements. It seems appropriate to say a few words about the history of this paper. The main result was described by the first-named author in 2006 in a presentation at the workshop ‘Identity Types—Topological and Categorical Structure’ held at Uppsala University. The details of the proof were then worked out by both authors during a 2008 visit by the first-named author to Uppsala; and it was at this stage that the axiomatic approach was introduced. While preparing this manuscript for publication, we become aware that, independently, Peter Lumsdaine had been considering the same question. His analysis may be found in [**16**]. Let us remark only that, where our argument is category-theoretic in nature, that given by Lumsdaine is essentially proof-theoretic. We gratefully acknowledge the support of Uppsala University's

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