On the Dividend Strategies with Non-Exponential Discounting

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Abstract

In this paper, we study the dividend strategies for a shareholder with non-constant discount rate in a diffusion risk model. We assume that the dividends can only be paid at a bounded rate and restrict ourselves to the Markov strategies. This is a time inconsistent control problem. The extended HJB equation is given and the verification theorem is proved for a general discount function. Considering the pseudo-exponential discount functions (Type I and Type II), we get the equilibrium dividend strategies and the equilibrium value functions by solving the extended HJB equations.

Keywords: Dividend strategies; Non-exponential discounting; Time inconsistence; Equilibrium strategies; Extended HJB equation

1 Introduction

Since it was proposed by De Finetti (1957), the optimization of dividend strategy has been investigated by many researchers under various risk models. This problem is usually phrased as the management's problem of determining the optimal timing and the size of dividend payments in the presence of bankruptcy risk. For more literature on this problem, we refer the reader to a recent survey paper Avanzi (2009).

In the very rich literature, a common assumption is that the discount rate is constant over time so the discount function is exponential. However, the empirical studies of human behavior suggest that the assumption of constant discount rate is unrealistic, see, e.g., Thaler (1981), Ainslie (1992) and Loewenstein and Prelec (1992). Indeed, there is experimental evidence that people are impatient about choices in the short term but are more patient when choosing between long-term alternatives. More precisely, events in the near future tend to be discounted at a higher rate than events that occur

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in the long run. Considering such effect, individual behavior is best described by the hyperbolic discounting (see Phelps and Pollak (1968)), which has been extensively studied in the areas of microeconomics, macroeconomics, and behavioral finance, such as Laibson (1997) and Barro (1999) among others.

However, difficulties arise when we solve an optimal control problem with a non-constant discount rate by the standard dynamic programming approach. In fact, the standard optimal control techniques give rise to time inconsistent strategies, i.e, a strategy that is optimal for the initial time may be not optimal later. This is the so-called time inconsistent control problem and the classical dynamic programming principle does not hold any more. Strotz (1955) studies the time inconsistent problem within a game theoretic framework by the using of Nash equilibrium points. They seek the equilibrium policy as the solution of a subgame-perfect equilibrium where the players are the agent and her future selves.

Recently, there is an increasing attention in the time inconsistent control problem due to the practical applications in economics and finance. A modified HJB equation is derived in Marín-Solano and Navas (2010) which solves the optimal consumption and investment problem with the non-constant discount rate for both naive and sophisticated agents. The similar problem is also considered by another approach in Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008), which provide the precise definition of the equilibrium concept in continuous time for the first time. They characterize the equilibrium policies through the solutions of a flow of BSDEs, and they show, with special form of the discount factor, this BSDE reduces to a system of two ODEs which has a solution. Considering the hyperbolic discounting, Ekeland et al. (2012) studies the portfolio management problem for an investor who is allowed to consume and take out life insurance, and they characterize the equilibrium strategy by an integral equation. Following their definition of the equilibrium strategy, Björk and Murgoci (2010) studied the time-inconsistent control problem in a general Markov framework, and derived the extended HJB equation together with the verification theorem. Björk et al. (2012) studied the Markowitz's problem with state-dependent risk aversion by utilizing the extended HJB equation obtained in Björk and Murgoci (2010).

In this paper, we study the dividend strategies for the shareholders with non-constant discount rate in a diffusion risk model. We assume that the dividends can only be paid at a bounded rate and restrict ourselves to the Markov strategies. We use the extended HJB equation to solve this problem. In contrast to the papers mentioned above which consider a fixed time horizon or an infinite time horizon, in the dividend problem the ruin risk should be taken into account and the time horizon is a random variable (the time of ruin). Thus, the extended HJB equation given in this paper looks different with the one obtained in Björk and Murgoci (2010). We first give the extended HJB equation and the verification theorem for a general discount function. Then we solve the extended HJB equation for two special non-exponential discount functions which are proposed by Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008). They called them the pseudo-exponential discount functions (Type I and Type II). For more details about these discount functions, we refer the reader to their papers (see also Section 4). Under the Type I discount function, our results show that if the bound of the dividend rate is small enough the equilibrium strategy is to always pay the maximal dividend rate; otherwise, the equilibrium strategy is to pay the maximal dividend rate when the surplus is above a barrier and pay nothing when the surplus is below the barrier. The results are similar under the Type II discount function, except that if the bound is large enough the equilibrium strategy is to pay nothing at all time. These features of the equilibrium dividend strategies are similar to the optimal strategies obtained in Asmussen and Taksar (1997) which considers the exponential discounting in the diffusion risk model.

The remainder of this paper is organized as follows. The dividend problem and the definition of equilibrium strategy are given in Section 2. The extended HJB equation and verification theorem are presented in Section 3. In Section 4, we study two cases with pseudo-exponential discount functions (Type I and Type II).

2 The model

In the case of no control, the surplus process is assumed to be

$$\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t, \qquad t \geq 0,$$

where μ, σ are positive constants and $\{W_t\}_{t\geq 0}$ is a 1-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathsf{P})$ satisfying the usual hypothesis. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is completed and generated by $\{W_t\}_{t\geq 0}$.

A dividend strategy is described by a stochastic process $\{l_t\}_{t\geq 0}$. Here, $l_t \geq 0$ is the rate of dividend payout at time t which is assumed to be bounded by a constant $M > 0$. We restrict ourselves to the feedback control strategies (Markov strategies), i.e. at time t, the control l_t is given by

$$l_t = \pi(t, x),$$

where x is the surplus level at time t and the control law $\pi : [0, \infty) \times [0, \infty) \to [0, M]$ is a Borel measurable function.

When applying the control law π, we denote by the controlled risk process $\{X_t^\pi\}_{t\geq 0}$. Considering the controlled system starting from the initial time $t \in [0, \infty)$, $\{X_s^\pi\}$ evolves according to

$$\begin{cases} \mathrm{d}X_s^\pi &= \mu \mathrm{d}s + \sigma \mathrm{d}W_s - \pi(s, X_s^\pi)\mathrm{d}s, \quad s \geq t, \\ X_t^\pi &= x. \end{cases}$$

Let

$$\tau_t^\pi := \inf\{s \geq t : X_s^\pi \leq 0\}$$

be the time of ruin under the control law π. Without loss of generality, we assume that $X_s^\pi \equiv 0$ for $s \geq \tau_t^\pi$.

Let $h:[0,\infty)\to\mathbb{R}$ be a discount function which satisfies $h(0)=1$, $h(s)\geq 0$ and $\int_0^\infty h(t)\mathrm{d}t<\infty$. Furthermore, h is assumed to be continuously differentiable on $[0,\infty)$.

Definition 2.1. A control law π is said to be admissible if it satisfies: $0\leq\pi(t,x)\leq M$ for all $(t,x)\in[0,\infty)\times[0,\infty)$, $\pi(t,0)\equiv 0$ for all $t\in[0,\infty)$. We denote by Π the set of all admissible control laws.

For a given admissible control law π and an initial state $(t,x)\in[0,\infty)\times[0,\infty)$, we define the return function V^π by

$$V^\pi(t,x)=\mathsf{E}_{t,x}\left[\int_t^{\tau_t^\pi} h(s-t)\pi(s,X_s^\pi)\mathrm{d}s\right],$$

where $\mathsf{E}_{t,x}[\cdot]$ is the expectation conditioned on the event $\{X_t^\pi=x\}$. Note that for any admissible strategy $\pi\in\Pi$, we have

$$\mathsf{E}_{t,x}\left[\int_t^{\tau_t^\pi}\left|h(s-t)\pi(s,X_s^\pi)\right|\mathrm{d}s\right]\leq M\int_0^\infty h(t)\mathrm{d}t<\infty,\quad\forall(t,x)\in[0,\infty)\times[0,\infty),\tag{2.1}$$

which means the performance function $V^\pi(t,x)$ are well-defined for all admissible strategy.

In classical risk theory, the optimal dividend strategy, denoted by π^*, is an admissible strategy such that

$$V^{\pi^*}(t,x)=\sup_{\pi\in\Pi}V^\pi(t,x).$$

However, in our settings, this optimization problem is time-inconsistent in the sense that the Bellman optimality principle fails.

Similar to Ekeland and Pirvu (2008) and Björk and Murgoci (2010), we view the entire problem as a non-cooperative game and look for Nash equilibria for the game. More specifically, we consider a game with one player for each time t, where player t can be regarded as the future incarnation of the decision maker at time t. Given state (t,x), player t will choose a control action $\pi(t,x)$, and she/he wants to maximize the functional $V^\pi(t,x)$. In the continuous-time model, Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008) give the precise definition of this equilibrium strategy for the first time. Intuitively, equilibrium strategies are the strategies such that, given that they will be implemented in the future, it is optimal to implement them right now.

Definition 2.2. Choose a control law $\hat{\pi}\in\Pi$, a fixed $l\in[0,M]$ and a fixed real number $\epsilon>0$. For any fixed initial point $(t,x)\in[0,\infty)\times[0,\infty)$, we define the control law π_ϵ by

$$\pi_\epsilon(s,y)=\begin{cases}0, & \text{for } s\in[t,\infty),\ y=0;\\ l, & \text{for } s\in[t,t+\epsilon],\ y\in(0,\infty);\\ \hat{\pi}(s,y), & \text{for } s\in[t+\epsilon,\infty),\ y\in(0,\infty).\end{cases}$$

If

$$\liminf_{\epsilon\to 0}\frac{V^{\hat{\pi}}(t,x)-V^{\pi_\epsilon}(t,x)}{\epsilon}\geq 0,$$

for all $l \in [0,M]$, we say that $\hat{\pi}$ is an equilibrium control law. And the equilibrium value function V is defined by

$$V(t,x) = V^{\hat{\pi}}(t,x). \tag{2.2}$$

In the following section, we will first give the extended HJB equation for the equilibrium value function V, and then prove a verification theorem.

3 The Extended Hamilton-Jacobi-Bellman Equation

In this section, we consider the objective function having the form

$$V^{\pi}(t,x) = \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\pi}} C\left(t,s,\pi(s,X_s^{\pi})\right)\mathrm{d}s\right], \tag{3.1}$$

where $C\left(t,s,\pi(s,X_s^{\pi})\right) = h(s-t)\pi(s,X_s^{\pi})$, for $s \geq t$.

For all $\pi \in \Pi$ and any real valued function $f(t,x) \in C^{1,2}\left([0,\infty)\times(0,\infty)\right)$, which means that the partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ exist and are continuous on $[0,\infty)\times(0,\infty)$, we define the infinitesimal generator $\mathcal{L}^{\pi}$ by

$$\mathcal{L}^{\pi} f(t,x) = \frac{\partial f}{\partial t}(t,x) + (\mu - \pi(t,x))\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x).$$

Motivated by Björk and Murgoci (2010), we assume that there exists an equilibrium strategy $\hat{\pi}$ and we consider the extended HJB equation given in the following definition. Since the ruin risk is considered in the dividend problem, the following extended HJB equation appears different with the one in Björk and Murgoci (2010).

Definition 3.1. Given the objective functional (3.1), the extended HJB equation for V is given by

$$\sup_{\pi\in\Pi}\left\{\mathcal{L}^{\pi}V(t,x) + C\left(t,t,\pi(t,x)\right) - \mathcal{L}^{\pi}c\left(t,t,x\right) + \mathcal{L}^{\pi}c^{t}(t,x)\right\} = 0, \quad t \geq 0, x > 0, \tag{3.2}$$

with the boundary condition

$$V(t,0) = 0, \quad t \geq 0 \tag{3.3}$$

Here,

$$c^{s}(t,x) = c(s,t,x), \quad 0 \leq s \leq t, \tag{3.4}$$

and for every fixed $0 \leq s \leq t$, the function $c^{s}(t,x)$ satisfies

$$\mathcal{L}^{\hat{\pi}}c^{s}(t,x) + C\left(s,t,\hat{\pi}(t,x)\right) = 0, \quad t \geq 0, x > 0, \tag{3.5}$$

and

$$c^{s}(t,0) = 0, \quad t \geq 0. \tag{3.6}$$

where $\hat{\pi}$ attains the supremum in (3.2).

Remark 3.2. (i) The difference between $c^s(t,x)$ and $c(s,t,x)$, is that we view c as a function of the three variables s,t and x, whereas c^s is, for a fixed s, viewed as a function of variables t and x. It is easy to simplify the Equation (3.2) to

$$\sup_{\pi\in\Pi}\{(\mathcal{L}^\pi V)(t,x)+C(t,t,\pi(t,x))\}-\frac{\partial c}{\partial s}(t,t,x)=0,\quad t\geq 0,\ x\geq 0, \tag{3.7}$$

where $\frac{\partial c}{\partial s}$ denotes the partial derivative with respective to the first variable of c.

In fact, for any $\pi\in\Pi$, one has

$$\begin{aligned}
&\mathcal{L}^\pi c(t,t,x)-\mathcal{L}^\pi c^t(t,x)\\
=\ &\frac{\partial c}{\partial s}(t,t,x)+\frac{\partial c}{\partial t}(t,t,x)+(\mu-\pi(t,x))\frac{\partial c}{\partial x}(t,t,x)+\frac{1}{2}\sigma^2\frac{\partial^2 c}{\partial x^2}(t,t,x)\\
&-\left[\frac{\partial c^t}{\partial t}(t,x)+(\mu-\pi(t,x))\frac{\partial c^t}{\partial x}(t,x)+\frac{1}{2}\sigma^2\frac{\partial^2 c^t}{\partial x^2}(t,x)\right].
\end{aligned}$$

Recalling that $c(t,t,x)=c^t(t,x)$, we get

$$\mathcal{L}^\pi c(t,t,x)-\mathcal{L}^\pi c^t(t,x)=\frac{\partial c}{\partial s}(t,t,x),$$

which is independent of π.

(ii) In order to solve V from Equation (3.7), we need to know c. But c is determined by the equilibrium control law $\hat{\pi}$ through (3.5) and (3.6), which in turn is determined by the sup-part of the equation satisfied by V. Thus, we have a system of recursion equation for the simultaneous determination of V and c.

Since the extended HJB system given in Definition 3.1 is informal, we are now giving a strict verification theorem.

Theorem 3.3. *(Verification Theorem) Assume that the supremum in Definition 3.1 is attained for each (t,x) given a control law $\hat{\pi}$, and there are bounded functions V and c, which are smooth enough ($C^{1,2}([0,\infty)\times(0,\infty))\cap C([0,\infty)\times[0,\infty))$), solve the extended HJB equation system in Definition 3.1, then $\hat{\pi}$ is the equilibrium control law, and V is the corresponding equilibrium value function.*

Proof. We give the proof by two steps:

1. First we show that V is the value function corresponding to $\hat{\pi}$, i.e., $V(t,x)=V^{\hat{\pi}}(t,x)$.
2. Then we prove that $\hat{\pi}$ is indeed the equilibrium control law which is defined by Definition 2.2.

Step 1.

The method is similar to Højgaard and Taksar (1999, Section 2.3). Recalling Definition 2.1 of admissible strategies (see also (2.1)), for given $s\leq t$, we have

$$\mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}}\left|C\left(s,z,\hat{\pi}(z,X_z^{\hat{\pi}})\right)\right|\mathrm{d}z\right]<\infty,\quad \forall (t,x)\in[0,\infty)\times[0,\infty).$$

Let

$$\tau_n = n \wedge \tau_t^{\hat{\pi}}, \quad n \geq t,\ n = 1, 2, \cdots.$$

Then, by (3.5), (3.6) and Dynkin's formula we have

$$\begin{aligned} c^s(t,x) &= \mathsf{E}_{t,x}\left[c^s(\tau_n, X^{\hat{\pi}}_{\tau_n})\right] - \mathsf{E}_{t,x}\left[\int_t^{\tau_n} \mathcal{L}^{\hat{\pi}} c^s(z, X^{\hat{\pi}}_z)\mathrm{d}z\right] \\ &= \mathsf{E}_{t,x}\left[c^s(\tau_t^{\hat{\pi}}, X^{\hat{\pi}}_{\tau_t^{\hat{\pi}}})\mathbf{1}_{\{n \geq \tau_t^{\hat{\pi}}\}}\right] + \mathsf{E}_{t,x}\left[c^s(n, X^{\hat{\pi}}_n)\mathbf{1}_{\{n<\tau_t^{\hat{\pi}}\}}\right] \\ &\quad + \mathsf{E}_{t,x}\left[\int_t^{\tau_n} C\left(s, z, \hat{\pi}(z, X^{\hat{\pi}}_z)\right)\mathrm{d}z\right] \\ &= \mathsf{E}_{t,x}\left[c^s(n, X^{\hat{\pi}}_n)\mathbf{1}_{\{n<\tau_t^{\hat{\pi}}\}}\right] + \mathsf{E}_{t,x}\left[\int_t^{\tau_n} C\left(s, z, \hat{\pi}(z, X^{\hat{\pi}}_z)\right)\mathrm{d}z\right]. \end{aligned}$$

Letting $n \to \infty$, it follows from dominated convergence theorem that

$$\mathsf{E}_{t,x}\left[c^s(n, X^{\hat{\pi}}_n)\mathbf{1}_{\{n<\tau_t^{\hat{\pi}}\}}\right] \to 0,$$

and

$$\mathsf{E}_{t,x}\left[\int_t^{\tau_n} C\left(s, z, \hat{\pi}(z, X^{\hat{\pi}}_z)\right)\mathrm{d}z\right] \to \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}} C\left(s, z, \hat{\pi}(z, X^{\hat{\pi}}_z)\right)\mathrm{d}z\right].$$

Thus,

$$c^s(t,x) = \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}} C\left(s, z, \hat{\pi}(z, X^{\hat{\pi}}_z)\right)\mathrm{d}z\right], \quad 0 \leq s \leq t < \infty. \tag{3.8}$$

Here (3.8) gives the probabilistic interpretation of $c^t(t,x)$. From (3.7) and (3.3) we have

$$\begin{aligned} \left(\mathcal{L}^{\hat{\pi}} V\right)(t,x) + C\left(t, t, \hat{\pi}(t,x)\right) - \frac{\partial c}{\partial s}(t,t,x) &= 0, \quad t \geq 0,\ x > 0, \\ V(t,0) &= 0, \quad t \geq 0. \end{aligned}$$

Similarly, by Dynkin's formula we have

$$V(t,x) = \mathsf{E}_{t,x}\left[V(\tau_n, X^{\hat{\pi}}_{\tau_n})\right] + \mathsf{E}_{t,x}\left[\int_t^{\tau_n}\left(C\left(z, z, \hat{\pi}(z, X^{\hat{\pi}}_z)\right) - \frac{\partial c}{\partial s}\left(z, z, X^{\hat{\pi}}_z\right)\right)\mathrm{d}z\right].$$

Noting that for any $t \leq z \leq \tau_n \leq \tau_t^{\hat{\pi}}$, it holds that $\tau_z^{\hat{\pi}} = \tau_t^{\hat{\pi}}$ a.s.. Thus, it follows from (3.8) that

$$\begin{aligned} \mathsf{E}_{t,x}\left[\int_t^{\tau_n} \frac{\partial c}{\partial s}\left(z, z, X^{\hat{\pi}}_z\right)\mathrm{d}z\right] &= \mathsf{E}_{t,x}\left[\int_t^{\tau_n} \mathsf{E}_{z, X^{\hat{\pi}}_z}\left[\int_z^{\tau_z^{\hat{\pi}}} \frac{\partial C}{\partial s}\left(z, v, \hat{\pi}(v, X^{\hat{\pi}}_v)\right)\mathrm{d}v\right]\mathrm{d}z\right] \\ &= \mathsf{E}_{t,x}\left[\int_t^{\tau_n}\int_z^{\tau_t^{\hat{\pi}}} \frac{\partial C}{\partial s}\left(z, v, \hat{\pi}(v, X^{\hat{\pi}}_v)\right)\mathrm{d}v\mathrm{d}z\right] \\ &= \mathsf{E}_{t,x}\left[\int_t^{\tau_n}\int_t^{v} \frac{\partial C}{\partial s}\left(z, v, \hat{\pi}(v, X^{\hat{\pi}}_v)\right)\mathrm{d}z\mathrm{d}v\right] \end{aligned}$$

$$
\begin{aligned}
&+\mathsf{E}_{t,x}\left[\int_{\tau_n}^{\tau_t^{\hat\pi}}\int_t^{\tau_n}\frac{\partial C}{\partial s}\Big(z,v,\hat\pi(v,X_v^{\hat\pi})\Big)\mathrm{d}z\mathrm{d}v\right]\\
=\ &\mathsf{E}_{t,x}\left[\int_t^{\tau_n}C\Big(v,v,\hat\pi(v,X_v^{\hat\pi})\Big)\mathrm{d}v\right]-\mathsf{E}_{t,x}\left[\int_t^{\tau_n}C\Big(t,v,\hat\pi(v,X_v^{\hat\pi})\Big)\mathrm{d}v\right]\\
&+\mathsf{E}_{t,x}\left[\int_{\tau_n}^{\tau_t^{\hat\pi}}C\Big(\tau_n,v,\hat\pi(v,X_v^{\hat\pi})\Big)\mathrm{d}v\right]-\mathsf{E}_{t,x}\left[\int_{\tau_n}^{\tau_t^{\hat\pi}}C\Big(t,v,\hat\pi(v,X_v^{\hat\pi})\Big)\mathrm{d}v\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
V(t,x)\ =\ &\mathsf{E}_{t,x}\Big[V(n,X_n^{\hat\pi})\mathbf{1}_{\{n<\tau_t^{\hat\pi}\}}\Big]+\mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat\pi}}C\Big(t,v,\hat\pi(v,X_v^{\hat\pi})\Big)\mathrm{d}v\right]\\
&-\mathsf{E}_{t,x}\left[\int_{\tau_n}^{\tau_t^{\hat\pi}}C\Big(\tau_n,v,\hat\pi(v,X_v^{\hat\pi})\Big)\mathrm{d}v\right].
\end{aligned}
$$

Note that

$$
\mathsf{E}_{t,x}\left[\int_{\tau_n}^{\tau_t^{\hat\pi}}C\Big(\tau_n,v,\hat\pi(v,X_v^{\hat\pi})\Big)\mathrm{d}v\right]\le M\int_0^\infty h(v)\mathrm{d}v<\infty.
$$

Letting $n\to\infty$ and applying dominated convergence theorem again, we obtain

$$
V(t,x)=\mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat\pi}}C\Big(t,v,\hat\pi(v,X_v^{\hat\pi})\Big)\mathrm{d}v\right]=V^{\hat\pi}(t,x).
$$

Step 2. For a given $l\in[0,M]$, and a fixed real number $\epsilon>0$, we define π_ϵ by Definition 2.2. For simplicity, we denote by X^ϵ the path under the control law π^ϵ. Without loss of generality, we consider the case where ϵ is sufficient small such that $t+\epsilon<\tau_t^{\pi^\epsilon}\wedge\tau_t^{\hat\pi}$. By the definition of V^π, we obtain

$$
\begin{aligned}
V^{\hat\pi}(t,x)-V^{\pi_\epsilon}(t,x)\ =\ &\mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat\pi}}C\Big(t,s,\hat\pi\Big(s,X_s^{\hat\pi}\Big)\Big)\mathrm{d}s-\int_t^{\tau_t^{\pi_\epsilon}}C\left(t,s,\pi^\epsilon\left(s,X_s^\epsilon\right)\right)\mathrm{d}s\right]\\
=\ &\mathsf{E}_{t,x}\left[\int_t^{t+\epsilon}h(s-t)\Big(\hat\pi\Big(s,X_s^{\hat\pi}\Big)-\pi^\epsilon\left(s,X_s^\epsilon\right)\Big)\mathrm{d}s\right]\\
&+\mathsf{E}_{t,x}\Big[V^{\hat\pi}\Big(t+\epsilon,X_{t+\epsilon}^{\hat\pi}\Big)-V^{\hat\pi}\left(t+\epsilon,X_{t+\epsilon}^\epsilon\right)\Big]\\
&+\mathsf{E}_{t,x}\left[\int_{t+\epsilon}^{\tau_t^{\hat\pi}}\left(h(s-t)-h(s-t-\epsilon)\right)\hat\pi\Big(s,X_s^{\hat\pi}\Big)\mathrm{d}s\right]\\
&-\mathsf{E}_{t,x}\left[\int_{t+\epsilon}^{\tau_t^{\pi_\epsilon}}\left(h(s-t)-h(s-t-\epsilon)\right)\hat\pi\left(s,X_s^\epsilon\right)\mathrm{d}s\right].
\end{aligned}\tag{3.9}
$$

Here $\hat\pi(s,X_s^\epsilon)$ and $\hat\pi(s,X_s^{\hat\pi})$ are the equilibrium control processes associated the path X^ϵ and $X^{\hat\pi}$, respectively.

According to the equation (3.9), we now consider the limitation $\lim_{\epsilon\to0}\frac{V^{\hat\pi}(t,x)-V^{\pi_\epsilon}(t,x)}{\epsilon}$ in four parts separately:

1. Noting that $\int_0^\infty h(t)\mathrm{d}t<\infty$, l and $\hat\pi$ are bounded and applying the dominated convergence

theorem, we get

$$\lim_{\epsilon\to 0}\frac{\mathsf{E}_{t,x}\left[\int_t^{t+\epsilon} h(s-t)\left(\hat{\pi}\left(s,X_s^{\hat{\pi}}\right)-\pi^{\epsilon}\left(s,X_s^{\epsilon}\right)\right)\mathrm{d}s\right]}{\epsilon}=\hat{\pi}(t,x)-\pi^{\epsilon}(t,x).$$

2. We rewrite the second part in the right-side of the equation (3.9) by

$$\begin{aligned}
&\mathsf{E}_{t,x}\left[V^{\hat{\pi}}\left(t+\epsilon,X_{t+\epsilon}^{\hat{\pi}}\right)-V^{\hat{\pi}}\left(t+\epsilon,X_{t+\epsilon}^{\epsilon}\right)\right]\\
=\;&\mathsf{E}_{t,x}\left[V^{\hat{\pi}}\left(t+\epsilon,X_{t+\epsilon}^{\hat{\pi}}\right)-V^{\hat{\pi}}(t,x)\right]-\mathsf{E}_{t,x}\left[V^{\hat{\pi}}\left(t+\epsilon,X_{t+\epsilon}^{\epsilon}\right)-V^{\hat{\pi}}(t,x)\right]\\
=\;&\mathsf{E}_{t,x}\left[\int_t^{t+\epsilon}\mathrm{d}V^{\hat{\pi}}\left(u,X_u^{\hat{\pi}}\right)\right]-\mathsf{E}_{t,x}\left[\int_t^{t+\epsilon}\mathrm{d}V^{\hat{\pi}}\left(u,X_u^{\epsilon}\right)\right].
\end{aligned}$$

Applying the Itô formula, we get

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{\mathsf{E}_{t,x}\left[\int_t^{t+\epsilon}\mathrm{d}V^{\hat{\pi}}\left(u,X_u^{\hat{\pi}}\right)\right]}{\epsilon}\\
=\;&\frac{\partial V^{\hat{\pi}}(t,x)}{\partial t}+(\mu-\hat{\pi}(t,x))\frac{\partial V^{\hat{\pi}}(t,x)}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 V^{\hat{\pi}}(t,x)}{\partial x^2}\\
=\;&\left(\mathcal{L}^{\hat{\pi}}V^{\hat{\pi}}\right)(t,x)\\
=\;&\left(\mathcal{L}^{\hat{\pi}}V\right)(t,x),
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{\mathsf{E}_{t,x}\left[\int_t^{t+\epsilon}\mathrm{d}V^{\hat{\pi}}\left(u,X_u^{\epsilon}\right)\right]}{\epsilon}\\
=\;&\frac{\partial V^{\hat{\pi}}(t,x)}{\partial t}+(\mu-l)\frac{\partial V^{\hat{\pi}}(t,x)}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 V^{\hat{\pi}}(t,x)}{\partial x^2}\\
=\;&\left(\mathcal{L}^{\pi^{\epsilon}}V^{\hat{\pi}}\right)(t,x)\\
=\;&\left(\mathcal{L}^{\pi^{\epsilon}}V\right)(t,x).
\end{aligned}$$

3. Considering the cases with $\tau_t^{\hat{\pi}}\geq\tau_t^{\pi^{\epsilon}}$ and $\tau_t^{\hat{\pi}}\leq\tau_t^{\pi^{\epsilon}}$ and noting that $\hat{\pi}\left(s,X_s^{\hat{\pi}}\right)\equiv 0$ for $s\geq\tau_t^{\hat{\pi}}$, we have

$$\begin{aligned}
&\mathsf{E}_{t,x}\left[\int_{t+\epsilon}^{\tau_t^{\hat{\pi}}}(h(s-t)-h(s-t-\epsilon))\hat{\pi}\left(s,X_s^{\hat{\pi}}\right)\mathrm{d}s\right]\\
&-\mathsf{E}_{t,x}\left[\int_{t+\epsilon}^{\tau_t^{\pi^{\epsilon}}}(h(s-t)-h(s-t-\epsilon))\hat{\pi}\left(s,X_s^{\epsilon}\right)\mathrm{d}s\right]\\
\geq&\mathsf{E}_{t,x}\left[\int_{t+\epsilon}^{\tau_t^{\pi^{\epsilon}}}(h(s-t)-h(s-t-\epsilon))\left[\hat{\pi}\left(s,X_s^{\hat{\pi}}\right)-\hat{\pi}\left(s,X_s^{\epsilon}\right)\right]\mathrm{d}s\right].
\end{aligned}$$

Noting that $\hat{\pi}$ is bounded and $\int_0^\infty h(s)\mathrm{d}s < \infty$, by the dominated convergence theorem, we get

$$\lim_{\epsilon\to 0}\frac{\mathsf{E}_{t,x}\left[\int_{t+\epsilon}^{\tau_t^{\pi^\epsilon}}[h(s-t)-h(s-t-\epsilon)]\left(\hat{\pi}\left(s,X_s^{\hat{\pi}}\right)-\hat{\pi}(s,X_s^\epsilon)\right)\mathrm{d}s\right]}{\epsilon}=0.$$

Therefore, we obtain

$$\lim_{\epsilon\to 0}\frac{V^{\hat{\pi}}(t,x)-V^{\pi_\epsilon}(t,x)}{\epsilon}\geq\left[\mathcal{L}^{\hat{\pi}}V(t,x)+C(t,t,\hat{\pi}(t,x))\right]-\left[\mathcal{L}^{\pi^\epsilon}V(t,x)+C(t,t,\pi^\epsilon(t,x))\right]. \tag{3.10}$$

It follows from (3.7) that

$$\begin{aligned}\left(\mathcal{L}^{\hat{\pi}}V\right)(t,x)+C(t,t,\hat{\pi}(t,x)) &= \sup_{\pi\in\Pi}\{(\mathcal{L}^{\pi}V)(t,x)+C(t,t,\pi(t,x))\} \\ &= \frac{\partial c}{\partial s}(t,t,x).\end{aligned} \tag{3.11}$$

Therefore, (3.10) and (3.11) imply that

$$\lim_{\epsilon\to 0}\frac{V^{\hat{\pi}}(t,x)-V^{\pi_\epsilon}(t,x)}{\epsilon}\quad\geq\quad 0.$$

This complete the proof. □

4 The Solutions

In this section, we try to find the solution of the HJB system in Definition 3.1 for specific discount functions. First of all, we make a conjecture of the equilibrium strategy for a general discount function. From (3.7), we can rewrite the HJB equation as

$$\begin{aligned}\sup_{\pi(t,x)\in[0,M]}\left\{\left(1-\frac{\partial V}{\partial x}(t,x)\right)\pi(t,x)\right\}+\frac{\partial V}{\partial t}(t,x)+\mu\frac{\partial V}{\partial x}(t,x)&\\ +\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}(t,x)-\frac{\partial c}{\partial s}(t,t,x)=0,&\quad t\geq 0,\ x>0, \qquad (4.1)\\ V(t,0)=0,&\quad t\geq 0; \qquad (4.2)\end{aligned}$$

and for every fixed s, the function $c^s(t,x)$ satisfies

$$\begin{aligned}\frac{\partial c^s}{\partial t}(t,x)+(\mu-\hat{\pi}(t,x))\frac{\partial c^s}{\partial x}(t,x)+\frac{1}{2}\sigma^2\frac{\partial^2 c^s}{\partial x^2}(t,x)+h(t-s)\hat{\pi}(t,x) &= 0,\quad t\geq 0,\ x>0, \qquad (4.3)\\ c^s(t,0) &= 0,\quad t\geq 0. \qquad (4.4)\end{aligned}$$

We assume that there exists a constant $b\geq 0$ such that $\frac{\partial V}{\partial x}(t,x)\geq 1$, if $0\leq x<b$, and $\frac{\partial V}{\partial x}(t,x)<1$, if $x \ge b$. It follows from (4.1) that the equilibrium strategy is given by

$$\hat{\pi}(t,x) = \begin{cases} 0, & \text{if } 0 \le x < b, \\ M, & \text{if } x \ge b. \end{cases}$$

Note that $\hat{\pi}$ given above is time homogeneous, i.e., $\hat{\pi}(t,x) = \hat{\pi}(s,x)$ for $t \neq s$. If the time homogeneous strategy $\hat{\pi}$ is indeed a equilibrium strategy, then

$$\begin{aligned} V(t,x) &= \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}} h(s-t)\hat{\pi}(s, X_s^{\hat{\pi}})\mathrm{d}s\right] \\ &= \mathsf{E}_{t,x}\left[\int_0^{\tau_t^{\hat{\pi}}-t} h(z)\hat{\pi}(z+t, X_{z+t}^{\hat{\pi}})\mathrm{d}z\right] \\ &= \mathsf{E}_{t,x}\left[\int_0^{\tau_t^{\hat{\pi}}-t} h(z)\hat{\pi}(z, X_{z+t}^{\hat{\pi}})\mathrm{d}z\right] \\ &= \mathsf{E}_{0,x}\left[\int_0^{\tau_0^{\hat{\pi}}} h(z-0)\hat{\pi}(z, X_z^{\hat{\pi}})\mathrm{d}z\right] \\ &= V(0,x), \end{aligned}$$

where the forth equation follows from the fact that $\left\{X_{z+t}^{\hat{\pi}}(t,x), \tau_t^{\hat{\pi}} - t\right\}_{z\ge 0}$ and $\left\{X_z^{\hat{\pi}}(0,x), \tau_0^{\hat{\pi}}\right\}_{z\ge 0}$ have the same distribution. Here $\left\{X_{z+t}^{\hat{\pi}}(t,x)\right\}_{z\ge 0}$ means it starts from the initial state (t,x).

Thus, we just want to find a time homogeneous function V, and the equations (4.1)-(4.4) can be represented as

$$\begin{cases} \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2}(x) + \mu \frac{\partial V}{\partial x}(x) - \frac{\partial c}{\partial s}(t,t,x) = 0, & 0 < x < b, \\ \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2}(x) + (\mu - M)\frac{\partial V}{\partial x}(x) - \frac{\partial c}{\partial s}(t,t,x) + M = 0, & x \ge b, \\ V(0) = 0, & \\ \frac{\partial c^s}{\partial t}(t,x) + \mu \frac{\partial c^s}{\partial x}(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2 c^s}{\partial x^2}(t,x) = 0, & 0 < x < b, \\ \frac{\partial c^s}{\partial t}(t,x) + (\mu - M)\frac{\partial c^s}{\partial x}(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2 c^s}{\partial x^2}(t,x) + h(t-s)M = 0, & x \ge b, \\ c^s(t,0) = 0, & \end{cases} \tag{4.5}$$

where $V(x) \equiv V(t,x)$ for all $t \ge 0$.

Then it follows from (3.8) that

$$\begin{aligned} c(s,t,x) &= c^s(t,x) \\ &= \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}} C\left(s, z, \hat{\pi}(z, X_z^{\hat{\pi}})\right)\mathrm{d}z\right] \\ &= \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}} h(z-s)\hat{\pi}(z, X_z^{\hat{\pi}})\mathrm{d}z\right]. \end{aligned} \tag{4.6}$$

Remark 4.1. (i) Note that for different discount functions h, the function c has different structures. From the first three equations of (4.5), we do not need the expression of $c(s,t,x)$ but only $c(t,t,x)$ to solve V. In fact, if we get the equilibrium strategy $\hat{\pi}$, then $c(s,t,x)$ defined by (4.6) always satisfies the HJB equation. Thus, in the following we focus on finding the equilibrium strategy $\hat{\pi}$ and the equilibrium value function V for specific discount functions.

(ii) If $h(t) = e^{-\delta t}$ where $\delta > 0$ is a constant, i.e., the exponential discounting, the problem reduces to the one studied by Asmussen and Taksar (1997, Section 2). It easy to check $\frac{\partial c}{\partial s}(t,t,x) = \delta V(x)$ and the first two equations of (4.5) becomes the equations (2.12) and (2.13) of Asmussen and Taksar (1997). This means that if the control problem is time consistent, the equilibrium strategy is consistent with the optimal strategy (see also Björk and Murgoci (2010)).

In the following subsections, we try to obtain the solutions of V under two special cases besides the exponential discounting, which are called pseudo-exponential discount functions (Type I and Type II). We refer the reader to Ekeland and Pirvu (2008) for explanations of these discount functions.

4.1 Type I

Let us consider a case where the dividends are proportionally paid to two inhomogenous shareholders. In terms of inhomogenous, we mean that the shareholders have different discount rates. Then given a control law π, the return function is

$$V^{\pi}(t,x) = \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\pi}} \omega e^{-\delta_1(s-t)}\pi(s,X_s^{\pi})\mathrm{d}s + \int_t^{\tau_t^{\pi}} (1-\omega)e^{-\delta_2(s-t)}\pi(s,X_s^{\pi})\mathrm{d}s\right],$$

where $\omega \in [0,1]$ is the proportion at which the dividends are paid to the shareholders, $\delta_1, \delta_2 > 0$ are the constant discount rates of the shareholders, respectively.

In fact, a mixture of exponential discount functions is used in the above example. This is the Type I pseudo-exponential discount function which is defined as

$$h(t) = \omega e^{-\delta_1 t} + (1-\omega)e^{-\delta_2 t}, \quad t \geq 0, \tag{4.7}$$

where $0 < \delta_1 \leq \delta_2$ and $\omega \in [0,1]$. It is obvious that when $\delta_1 = \delta_2$ or $\omega = 0$ or $\omega = 1$, h is the exponential discount function. However in our paper, we only discuss the non-exponential case with $0 < \delta_1 < \delta_2$ and $0 < \omega < 1$.

Recalling (4.6), we have

$$c(s,t,x) \quad = \quad \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}} \left[\omega e^{-\delta_1(z-s)} + (1-\omega)e^{-\delta_2(z-s)}\right]\mathbf{1}_{\{X_z^{\hat{\pi}} \geq b\}} M\mathrm{d}z\right],$$

which implies that

$$\frac{\partial c}{\partial s}(t,t,x) = \omega\delta_1 V_1(x) + (1-\omega)\delta_2 V_2(x),$$

where

$$\begin{cases} V_1(x) & = \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}} e^{-\delta_1(z-t)}\mathbf{1}_{\{X_z^{\hat{\pi}}\geq b\}}M\mathrm{d}z\right], \\ V_2(x) & = \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}} e^{-\delta_2(z-t)}\mathbf{1}_{\{X_z^{\hat{\pi}}\geq b\}}M\mathrm{d}z\right]. \end{cases} \tag{4.8}$$

Moreover, $V(x)$ can be expressed by

$$V(x) = \omega V_1(x) + (1-\omega)V_2(x). \tag{4.9}$$

To get the form of $V(x)$, we only need to find V_1 and V_2. By the standard techniques (see Gerber and Shiu (2004, 2006)), the equations satisfied by V_i, $i = 1,2$ are given by

$$\begin{cases} \frac{1}{2}\sigma^2\frac{\partial^2 V_i}{\partial x^2}(x) + \mu\frac{\partial V_i}{\partial x}(x) - \delta_i V_i(x) = 0, & 0 < x < b, \\ \frac{1}{2}\sigma^2\frac{\partial^2 V_i}{\partial x^2}(x) + (\mu - M)\frac{\partial V_i}{\partial x}(x) - \delta_i V_i(x) + M = 0, & x \geq b, \\ V_i(0) = 0. \end{cases} \tag{4.10}$$

Denote by $\theta_1(\eta,c)$ and $-\theta_2(\eta,c)$ the positive and negative roots of the equation $\frac{1}{2}\sigma^2y^2 + \eta y - c = 0$, respectively. Then

$$\begin{cases} \theta_1(\eta,c) & = \frac{-\eta+\sqrt{\eta^2+2\sigma^2c}}{\sigma^2}, \\ \theta_2(\eta,c) & = \frac{\eta+\sqrt{\eta^2+2\sigma^2c}}{\sigma^2}. \end{cases}$$

Thus a general solution of the equation (4.10) has the form

$$V_i(x) = \begin{cases} C_{i1}e^{\theta_1(\mu,\delta_i)x} + C_{i2}e^{-\theta_2(\mu,\delta_i)x}, & 0 \leq x < b, \\ \frac{M}{\delta_i} + C_{i3}e^{\theta_1(\mu-M,\delta_i)x} + C_{i4}e^{-\theta_2(\mu-M,\delta_i)x}, & x \geq b, \end{cases} \tag{4.11}$$

for $i = 1,2$.

Since $V_i(0) = 0$, and $V_i(x) > 0$, for all $x > 0$, we have $C_{i1} = -C_{i2} := C_i > 0$, $i = 1,2$. Note that for any barrier b in (4.8), $V_i(x)$ will not exceed $\int_0^\infty e^{-\delta_i x}M\mathrm{d}x = \frac{M}{\delta_i}$, for all $x \geq 0$, $i = 1,2$. Therefore $C_{i3} = 0$ and $C_{i4} := -d_i < 0$, $i = 1,2$.

Now to find the value of C_1, C_2, d_1, d_2 and b, we use "the principle of smooth fit" to get

$$\begin{cases} V_1(b+) & = V_1(b-), \\ V_1'(b+) & = V_1'(b-), \\ V_2(b+) & = V_2(b-), \\ V_2'(b+) & = V_2'(b-), \\ V'(b+) & = 1 \ \text{(or equivalently, } V'(b-) = 1\text{)}. \end{cases} \tag{4.12}$$

Therefore denoting

$$\theta_{i1} = \theta_1(\mu,\delta_i),\ \theta_{i2} = \theta_2(\mu,\delta_i),\ \theta_{i3} = \theta_2(\mu-M,\delta_i), \quad i = 1,2,$$

we can rewrite (4.12) as

$$
\begin{aligned}
C_1\left(e^{\theta_{11}b}-e^{-\theta_{12}b}\right) &= \frac{M}{\delta_1}-d_1e^{-\theta_{13}b}, && (4.13)\\
C_1\left(\theta_{11}e^{\theta_{11}b}+\theta_{12}e^{-\theta_{12}b}\right) &= d_1\theta_{13}e^{-\theta_{13}b}, && (4.14)\\
C_2\left(e^{\theta_{21}b}-e^{-\theta_{22}b}\right) &= \frac{M}{\delta_2}-d_2e^{-\theta_{23}b}, && (4.15)\\
C_2\left(\theta_{21}e^{\theta_{21}b}+\theta_{22}e^{-\theta_{22}b}\right) &= d_2\theta_{23}e^{-\theta_{23}b}, && (4.16)
\end{aligned}
$$

$$
\omega C_1\left(\theta_{11}e^{\theta_{11}b}+\theta_{12}e^{-\theta_{12}b}\right)+(1-\omega)C_2\left(\theta_{21}e^{\theta_{21}b}+\theta_{22}e^{-\theta_{22}b}\right)=1. \tag{4.17}
$$

From (4.13) - (4.16) we can get C_i and d_i in the expression of b:

$$
C_i=\frac{M\theta_{i3}}{\delta_i}\left[(\theta_{i1}+\theta_{i3})e^{\theta_{i1}b}+(\theta_{i2}-\theta_{i3})e^{-\theta_{i2}b}\right]^{-1}, \tag{4.18}
$$

$$
d_i=\frac{M}{\delta_i}e^{\theta_{i3}b}\frac{\theta_{i1}e^{\theta_{i1}b}+\theta_{i2}e^{-\theta_{i2}b}}{(\theta_{i1}+\theta_{i3})e^{\theta_{i1}b}+(\theta_{i2}-\theta_{i3})e^{-\theta_{i2}b}}, \tag{4.19}
$$

for $i=1,2$.

Substituting C_1 and C_2 into (4.17), we obtain

$$
A_1e^{(\theta_{11}+\theta_{12})b}+A_2e^{-(\theta_{22}-\theta_{12}+\theta_{21}-\theta_{11})b}+A_3e^{-(\theta_{21}+\theta_{22})b}+A_4=0, \tag{4.20}
$$

where

$$
\begin{aligned}
A_1 &= \omega\frac{M\theta_{13}}{\delta_1}\theta_{11}(\theta_{21}+\theta_{23})+(1-\omega)\frac{M\theta_{23}}{\delta_2}\theta_{21}(\theta_{11}+\theta_{13})-(\theta_{11}+\theta_{13})(\theta_{21}+\theta_{23})\\
&= [\omega P+(1-\omega)Q-1]\theta_{11}\theta_{21}-(1-\omega P)\theta_{11}\theta_{23}-[1-(1-\omega)Q]\theta_{13}\theta_{21}-\theta_{13}\theta_{23},\\
A_2 &= \omega\frac{M\theta_{13}}{\delta_1}\theta_{11}(\theta_{22}-\theta_{23})+(1-\omega)\frac{M\theta_{23}}{\delta_2}\theta_{22}(\theta_{11}+\theta_{13})-(\theta_{11}+\theta_{13})(\theta_{22}-\theta_{23})\\
&= [\omega P+(1-\omega)Q-1]\theta_{11}\theta_{22}+(1-\omega P)\theta_{11}\theta_{23}-[1-(1-\omega)Q]\theta_{13}\theta_{22}+\theta_{13}\theta_{23},\\
A_3 &= \omega\frac{M\theta_{13}}{\delta_1}\theta_{12}(\theta_{22}-\theta_{23})+(1-\omega)\frac{M\theta_{23}}{\delta_2}\theta_{22}(\theta_{12}-\theta_{13})-(\theta_{12}-\theta_{13})(\theta_{22}-\theta_{23})\\
&= [\omega P+(1-\omega)Q-1]\theta_{12}\theta_{22}+(1-\omega P)\theta_{12}\theta_{23}+[1-(1-\omega)Q]\theta_{13}\theta_{22}-\theta_{13}\theta_{23},\\
A_4 &= \omega\frac{M\theta_{13}}{\delta_1}\theta_{12}(\theta_{21}+\theta_{23})+(1-\omega)\frac{M\theta_{23}}{\delta_2}\theta_{21}(\theta_{12}-\theta_{13})-(\theta_{12}-\theta_{13})(\theta_{21}+\theta_{23})\\
&= [\omega P+(1-\omega)Q-1]\theta_{12}\theta_{21}-(1-\omega P)\theta_{12}\theta_{23}+[1-(1-\omega)Q]\theta_{13}\theta_{21}+\theta_{13}\theta_{23},
\end{aligned}
$$

and

$$
P=M\frac{\theta_{13}}{\delta_1},\quad Q=M\frac{\theta_{23}}{\delta_2}.
$$

Denoting

$$
F(b):=A_1e^{(\theta_{11}+\theta_{12})b}+A_2e^{-(\theta_{22}-\theta_{12}+\theta_{21}-\theta_{11})b}+A_3e^{-(\theta_{21}+\theta_{22})b}+A_4,
$$

then we have

$$F(0) = (\theta_{11}+\theta_{12})(\theta_{21}+\theta_{22})[\omega P + (1-\omega)Q - 1].$$

Lemma 4.2. *If* $M\left[\omega\frac{\theta_{13}}{\delta_1}+(1-\omega)\frac{\theta_{23}}{\delta_2}\right] > 1$*, then* $F(b) = 0$ *has a positive solution.*

Proof. The condition $M\left[\omega\frac{\theta_{13}}{\delta_1}+(1-\omega)\frac{\theta_{23}}{\delta_2}\right] > 1$ implies that $F(0) > 0$.

From Lemma 2.1 of Asmussen and Taksar (1997), we know that

$$\left(\frac{M}{\delta_i} - \frac{1}{\theta_{i3}}\right)\theta_{i1} < 1, \quad i = 1,2,$$

which implies that

$$(P-1)\theta_{11} < \theta_{13}, \quad \text{and} \quad (Q-1)\theta_{21} < \theta_{23}.$$

Thus,

$$\begin{aligned}
A_1 &= \omega(P-1)\theta_{11}\theta_{21} + (1-\omega)(Q-1)\theta_{11}\theta_{21} - (1-\omega P)\theta_{11}\theta_{23} - [1-(1-\omega)Q]\theta_{13}\theta_{21} - \theta_{13}\theta_{23} \\
&< \omega\theta_{13}\theta_{21} + (1-\omega)\theta_{11}\theta_{23} - (1-\omega P)\theta_{11}\theta_{23} - [1-(1-\omega)Q]\theta_{13}\theta_{21} - \theta_{13}\theta_{23} \\
&= (1-\omega)(Q-1)\theta_{13}\theta_{21} + \omega(P-1)\theta_{11}\theta_{23} - \theta_{13}\theta_{23} \\
&< (1-\omega)\theta_{13}\theta_{23} + \omega\theta_{13}\theta_{23} - \theta_{13}\theta_{23} \\
&= 0.
\end{aligned}$$

Also noting that $\theta_{22}-\theta_{12}+\theta_{21}-\theta_{11} > 0$, it follows that $F(+\infty) = -\infty$. Together with $F(0) > 0$, we know that there exists a positive solution b such that $F(b) = 0$. □

Theorem 4.3. *Given the discount function (4.7), there exists a twice continuously differentiable concave solution to (4.1) and (4.2).*

(i) If $M\left[\omega\frac{\theta_{13}}{\delta_1}+(1-\omega)\frac{\theta_{23}}{\delta_2}\right] \le 1$*, then* $b = 0$*, i.e., the equilibrium strategy is to always pay the maximal dividend rate, and the equilibrium value function is given by*

$$V(x) = \omega\frac{M}{\delta_1}\left(1-e^{-\theta_{13}x}\right) + (1-\omega)\frac{M}{\delta_2}\left(1-e^{-\theta_{23}x}\right), \quad x \ge 0. \tag{4.21}$$

(ii) If $M\left[\omega\frac{\theta_{13}}{\delta_1}+(1-\omega)\frac{\theta_{23}}{\delta_2}\right] > 1$*, then*

$$V(x) = \begin{cases} \omega C_1\left(e^{\theta_{11}x} - e^{-\theta_{12}x}\right) + (1-\omega)C_2\left(e^{\theta_{21}x} - e^{-\theta_{22}x}\right), & 0 \le x < b, \\ \omega\left(\frac{M}{\delta_1} - d_1 e^{-\theta_{13}x}\right) + (1-\omega)\left(\frac{M}{\delta_2} - d_2 e^{-\theta_{23}x}\right), & x \ge b, \end{cases} \tag{4.22}$$

where (C_1, C_2, d_1, d_2, b) *is the unique solution to (4.13)-(4.17).*

Proof. (i) Obviously, the function V defined by (4.21) is concave and satisfies $V(0) = 0$, and

$$V'(0) = \omega \frac{M}{\delta_1}\theta_{13} + (1-\omega)\frac{M}{\delta_2}\theta_{23} \le 1.$$

Therefore $V'(x) \le 1$ for all $x > 0$ and

$$(M-\pi)\left(V'(x)-1\right) \le 0, \quad \pi \in [0, M]. \tag{4.23}$$

Given $b = 0$, it is easy to see from the system of equations (4.10) that

$$V_i(x) = \frac{M}{\delta_i}\left(1 - e^{-\theta_{i3}x}\right), \quad i = 1, 2.$$

Thus, the function V given by (4.21) satisfies

$$\frac{1}{2}\sigma^2 V''(x) + (\mu - M)V'(x) - \frac{\partial c}{\partial s}(t,t,x) + M = 0. \tag{4.24}$$

Adding (4.23) to (4.24), we get (4.1).

(ii) Obviously, we have $V(0) = 0$. The first and second derivatives of (4.22) are given by

$$V'(x) = \begin{cases} \omega C_1\left(\theta_{11}e^{\theta_{11}x} + \theta_{12}e^{-\theta_{12}x}\right) + (1-\omega)C_2\left(\theta_{21}e^{\theta_{21}x} + \theta_{22}e^{-\theta_{22}x}\right), & 0 \le x < b, \\ \omega d_1\theta_{13}e^{-\theta_{13}x} + (1-\omega)d_2\theta_{23}e^{-\theta_{23}x}, & x \ge b, \end{cases}$$

and

$$V''(x) = \begin{cases} \omega C_1\left(\theta_{11}^2 e^{\theta_{11}x} - \theta_{12}^2 e^{-\theta_{12}x}\right) + (1-\omega)C_2\left(\theta_{21}^2 e^{\theta_{21}x} - \theta_{22}^2 e^{-\theta_{22}x}\right), & 0 \le x < b, \\ -\omega d_1\theta_{13}^2 e^{-\theta_{13}x} - (1-\omega)d_2\theta_{23}^2 e^{-\theta_{23}x}, & x \ge b, \end{cases}$$

respectively.

It is easy to check that $V'(x) > 0$, for all $x \ge 0$, which implies that V is strictly increasing. Next we show that V is a concave function on $[0,\infty)$, i.e. $V''(x) < 0$, for all $x \ge 0$.

At first we show that $V''(x)$ is continuous at $x = b$. Apparently, $V''(x) < 0$, for all $x \ge b$. Recalling (4.9), (4.10) and (4.12), we have

$$\begin{aligned} \frac{1}{2}\sigma^2 V''(b-) &= -\mu V'(b) + \omega\delta_1 V_1(b) + (1-\omega)\delta_2 V_2(b), \\ \frac{1}{2}\sigma^2 V''(b+) &= -(\mu - M)V'(b) + \omega\delta_1 V_1(b) + (1-\omega)\delta_2 V_2(b) - M. \end{aligned}$$

Since $V'(b) = 1$, we get $V''(b-) = V''(b+) = V''(b)$.

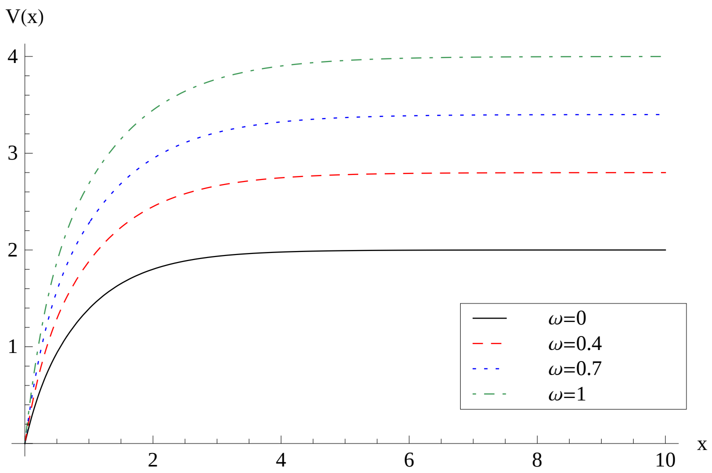

Figure 4.1: Equilibrium value functions with Type I pseudo-exponential discount function

Obviously, for all $0 \leq x \leq b$, we have

$$V'''(x) = \omega C_1 \left(\theta_{11}^3 e^{\theta_{11} x} + \theta_{12}^3 e^{-\theta_{12} x}\right) + (1-\omega) C_2 \left(\theta_{21}^3 e^{\theta_{21} x} + \theta_{22}^3 e^{-\theta_{22} x}\right) > 0.$$

Thus $V''(x)$ is an increasing function on $[0, b]$, which means that $V''(x) \leq V''(b) < 0$, for all $0 \leq x \leq b$.

Above all, we have shown that V is an increasing and concave function on $[0, \infty)$. Note that the uniqueness of b is assured by the strict concavity of V.

Now we verify that (4.22) satisfies (4.1).

If $x \leq b$, then $V'(x) > 1$. Adding the inequality $-\pi(V'(x)-1) \leq 0$ to the first equation in (4.5), we obtain (4.1). And similarly, if $x \geq b$, then $V'(x) \leq 1$. Adding $(M-\pi)(V'(x)-1) \leq 0$ to the second equation in (4.5), we obtain (4.1).

This completes the proof. □

Example 4.4. Let $\mu = 1$, $\sigma = 1$, $M = 0.8$, $\delta_1 = 0.2$, $\delta_2 = 0.4$. Figure 4.1 illustrates the equilibrium value functions for Type I pseudo-exponential discount functions with $\omega = 0$, 0.4, 0.7 and 1. The barrier b are 0.6525, 0.8781, 1.0207 and 1.1452, respectively. The cases with $\omega = 0$ and 1 are time consistent and the equilibrium strategies are optimal.

4.2 Type II

The Type II pseudo-exponential discount function is defined as

$$h(t) = (1 + \lambda t) e^{-\delta t}, \quad t \geq 0, \tag{4.25}$$

where $\lambda > 0,\ \delta > 0$ are parameters. In this case, we have

$$c(s,t,x) \quad = \quad \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}} [1+\lambda(z-s)]\, e^{-\delta(z-s)} \mathbf{1}_{\{X_z^{\hat{\pi}} \geq b\}} M \mathrm{d}z\right],$$

which implies that

$$\frac{\partial c}{\partial s}(t,t,x) = -\lambda V_3(x) + \delta V_4(x), \tag{4.26}$$

where

$$\begin{aligned} V_3(x) &= \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}} e^{-\delta(z-t)} \mathbf{1}_{\{X_z^{\hat{\pi}} \geq b\}} M \mathrm{d}z\right], \\ V_4(x) &= \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}} [1+\lambda(z-t)]\, e^{-\delta(z-t)} \mathbf{1}_{\{X_z^{\hat{\pi}} \geq b\}} M \mathrm{d}z\right] \\ &= V(x). \end{aligned}$$

The function for V_3 is given by

$$\begin{cases} \frac{1}{2}\sigma^2 \frac{\partial^2 V_3}{\partial x^2}(x) + \mu \frac{\partial V_3}{\partial x}(x) - \delta V_3(x) = 0, & 0 < x < b, \\ \frac{1}{2}\sigma^2 \frac{\partial^2 V_3}{\partial x^2}(x) + (\mu - M)\frac{\partial V_3}{\partial x}(x) - \delta V_3(x) + M = 0, & x \geq b, \\ V_3(0) = 0. \end{cases} \tag{4.27}$$

Recalling the situation we discussed in Subsection 5.1, the equation (4.27) has a general solution

$$V_3(x) = \begin{cases} C\left(e^{\theta_1(\mu)x} - e^{-\theta_2(\mu)x}\right), & 0 \leq x < b, \\ \frac{M}{\delta} - d e^{-\theta_2(\mu - M)x}, & x \geq b, \end{cases}$$

where $C > 0,\ d > 0$ are two unknown constants to be determined, $\theta_1(\eta)$ and $-\theta_2(\eta)$ are the positive and negative roots of the equation $\frac{1}{2}\sigma^2 y^2 + \eta y - \delta = 0$, respectively.

According to "the principle of smooth fit", we have

$$\begin{cases} V_3(b+) &= V_3(b-), \\ V_3'(b+) &= V_3'(b-), \end{cases} \tag{4.28}$$

which yields that

$$\begin{aligned} C &= \frac{M\theta_3}{\delta}\left[(\theta_1+\theta_3)\, e^{\theta_1 b} + (\theta_2 - \theta_3)\, e^{-\theta_2 b}\right]^{-1}, \\ d &= \frac{M}{\delta} e^{\theta_3 b} \frac{\theta_1 e^{\theta_1 b} + \theta_2 e^{-\theta_2 b}}{(\theta_1+\theta_3)\, e^{\theta_1 b} + (\theta_2-\theta_3)\, e^{-\theta_2 b}}, \end{aligned}$$

where

$$\theta_1 = \theta_1(\mu), \quad \theta_2 = \theta_2(\mu), \quad \theta_3 = \theta_2(\mu - M).$$

After obtaining V_3, we substitute (4.26) into the equations for V in (4.5), and then we have

$$\begin{cases} \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2}(x) + \mu \frac{\partial V}{\partial x}(x) - \delta V(x) + \lambda V_3(x) = 0, & 0 < x < b, \\ \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2}(x) + (\mu - M)\frac{\partial V}{\partial x}(x) - \delta V(x) + \lambda V_3(x) + M = 0, & x \geq b, \\ V(0) = 0, \end{cases} \tag{4.29}$$

which admits a general solution

$$V(x) = \begin{cases} \frac{\lambda V_3(x)}{\delta} + D_1 e^{\theta_1(\mu)x} + D_2 e^{-\theta_2(\mu)x}, & 0 \leq x < b, \\ \frac{\lambda V_3(x) + M}{\delta} + D_3 e^{\theta_1(\mu - M)x} + D_4 e^{-\theta_2(\mu - M)x}, & x \geq b. \end{cases}$$

Since $V(0) = 0$, $V_3(0) = 0$ and $V(x) > 0$, $V_3(x) > 0$, for all $x > 0$, we have $D_1 = -D_2 := \hat{C} > 0$. Noting that for any control, the discounted cumulative dividend will not exceed $\int_0^\infty (1 + \lambda t) e^{-\delta t} M \mathrm{d}t = \frac{M}{\delta}\left(\frac{\lambda}{\delta} + 1\right)$. Thus we need to find $V(x) \leq \frac{M}{\delta}\left(\frac{\lambda}{\delta} + 1\right)$, for all $x \geq 0$. Since $V(x)$ is bounded, it follows that $D_3 = 0$. For simplicity, denote by $D_4 := -\hat{d}$.

Applying "the principle of smooth fit", we obtain

$$\begin{cases} V(b+) &= V(b-), \\ V'(b+) &= V'(b-), \\ V'(b+) &= 1 \ \text{(or equavalently, } V'(b-) = 1\text{)}. \end{cases} \tag{4.30}$$

Recalling (4.28), we have

$$\hat{C} = C = \frac{M\theta_3}{\delta}\left[(\theta_1 + \theta_3) e^{\theta_1 b} + (\theta_2 - \theta_3) e^{-\theta_2 b}\right]^{-1},$$
$$\hat{d} = d = \frac{M}{\delta} e^{\theta_3 b} \frac{\theta_1 e^{\theta_1 b} + \theta_2 e^{-\theta_2 b}}{(\theta_1 + \theta_3) e^{\theta_1 b} + (\theta_2 - \theta_3) e^{-\theta_2 b}}.$$

Then using $V'(b+) = 1$, we have

$$\left[\left(\frac{\lambda}{\delta} + 1\right)\frac{M\theta_3}{\delta}\theta_1 - (\theta_1 + \theta_3)\right] e^{\theta_1 b} = \left[(\theta_2 - \theta_3) - \left(\frac{\lambda}{\delta} + 1\right)\frac{M\theta_3}{\delta}\theta_2\right] e^{-\theta_2 b}.$$

Lemma 4.5. *If* $1 < \left(\frac{\lambda}{\delta} + 1\right)\frac{M\theta_3}{\delta} < 1 + \frac{\theta_3}{\theta_1}$, *there exists a unique positive triple* $\left(\hat{C}, \hat{d}, b\right)$ *solving (4.30).*

Proof. If $1 < \left(\frac{\lambda}{\delta} + 1\right)\frac{M\theta_3}{\delta} < 1 + \frac{\theta_3}{\theta_1}$, then $A := \frac{(\theta_2 - \theta_3) - \left(\frac{\lambda}{\delta} + 1\right)\frac{M\theta_3}{\delta}\theta_2}{\left(\frac{\lambda}{\delta} + 1\right)\frac{M\theta_3}{\delta}\theta_1 - (\theta_1 + \theta_3)} > 1$, and we get a unique solution for b given by

$$b = \frac{1}{\theta_1 + \theta_2} \log A > 0. \tag{4.31}$$

Then it follows that

$$\hat{C} = C = \frac{M\theta_3}{\delta}\left[(\theta_1+\theta_3)A^{\frac{\theta_1}{\theta_1+\theta_2}} + (\theta_2-\theta_3)A^{-\frac{\theta_2}{\theta_1+\theta_2}}\right]^{-1}, \tag{4.32}$$

$$\hat{d} = d = \frac{M}{\delta}\frac{\theta_1 A^{\frac{\theta_1+\theta_3}{\theta_1+\theta_2}} + \theta_2 A^{-\frac{\theta_2-\theta_3}{\theta_1+\theta_2}}}{(\theta_1+\theta_3)A^{\frac{\theta_1}{\theta_1+\theta_2}} + (\theta_2-\theta_3)A^{-\frac{\theta_2}{\theta_1+\theta_2}}}. \tag{4.33}$$

□

Theorem 4.6. *Given the discount function (4.25), there exists a twice continuously differentiable concave solution to (4.1) and (4.2).*

(i) If $\left(\frac{\lambda}{\delta}+1\right)\frac{M\theta_3}{\delta} \le 1$, then $b=0$, i.e. the equilibrium strategy is to always pay the maximal dividend rate, and the equilibrium value function is given by

$$V(x) = \left(\frac{\lambda}{\delta}+1\right)\frac{M}{\delta}\left(1-e^{-\theta_3 x}\right), \quad x \ge 0. \tag{4.34}$$

(ii) If $1 < \left(\frac{\lambda}{\delta}+1\right)\frac{M\theta_3}{\delta} < 1+\frac{\theta_3}{\theta_1}$, then

$$V(x) = \begin{cases} C\left(\frac{\lambda}{\delta}+1\right)\left(e^{\theta_1 x}-e^{-\theta_2 x}\right), & 0 \le x < b, \\ \left(\frac{\lambda}{\delta}+1\right)\left(\frac{M}{\delta}-de^{-\theta_3 x}\right), & x \ge b, \end{cases} \tag{4.35}$$

where b, C, and d are given by (4.31), (4.32), and (4.33), respectively.

(iii) If $\left(\frac{\lambda}{\delta}+1\right)\frac{M\theta_3}{\delta} \ge 1+\frac{\theta_3}{\theta_1}$, then $b=+\infty$, i.e. the equilibrium strategy is never pay the dividend and the equilibrium value function $V(x)\equiv 0$.

Proof. (i) It is easy to see the function V defined by (4.34) is a concave function, $V(0)=0$ and $V'(0) = \left(\frac{\lambda}{\delta}+1\right)\frac{M}{\delta}\theta_3 \le 1$. Therefore $V'(x) \le 1$ for all $x>0$ and

$$(M-\pi)(V'(x)-1) \le 0, \quad \pi \in [0,M]. \tag{4.36}$$

Given $b=0$, it is easy to get

$$V_3(x) = \frac{M}{\delta}\left(1-e^{-\theta_3 x}\right).$$

Thus, the function V given by (4.34) satisfies

$$\frac{1}{2}\sigma^2 V''(x) + (\mu - M)V'(x) - \frac{\partial c}{\partial s}(t,t,x) + M = 0. \tag{4.37}$$

Adding (4.36) to (4.37), we get (4.1).

(ii) Obviously, V is an increasing function on $[0,\infty)$. Next we show that $V''(x)<0$, for all $x>0$. Similarly to the previous subsection, we first show that V'' is continuous at b. According to (4.29) and (4.30), we have

$$
\begin{aligned}
\frac{1}{2}\sigma^2 V''(b-) &= -\mu V'(b) + \delta V(b) - \lambda V_3(b), \\
\frac{1}{2}\sigma^2 V''(b+) &= -(\mu - M)V'(b) + \delta V(b) - \lambda V_3(b) - M \\
&= -\mu V'(b) + \delta V(b) - \lambda V_3(b),
\end{aligned}
$$

which yields that $V''(b+) = V''(b-) = V''(b)$.

The second derivative of V is given by

$$
V''(x) = \begin{cases} C\left(\frac{\lambda}{\delta}+1\right)\left(\theta_1^2 e^{\theta_1 x} - \theta_2^2 e^{-\theta_2 x}\right), & 0 \le x < b, \\ -d\left(\frac{\lambda}{\delta}+1\right)\theta_3^2 e^{-\theta_3 x}, & x \ge b. \end{cases}
$$

Obviously, $V''(x) < 0$, for all $x \ge b$.

Since $V'''(x) = C\left(\frac{\lambda}{\delta}+1\right)\left(\theta_1^3 e^{\theta_1 x} + \theta_2^3 e^{-\theta_2 x}\right) > 0$, for all $0 \le x \le b$, we know that $V''(x)$ is an increasing function on $[0, b]$. Thus the maximum of V'' on $[0, b]$ is $V''(b) < 0$, which implies that $V''(x) < 0$, for all $0 \le x \le b$. Therefore, V is concave on $[0, \infty)$.

Similar to the proof of Theorem 4.3, we know that (4.35) satisfies (4.1).

(iii) In this case, we check the limit of the first order derivative of $V(x)$ given by the first equation of (4.35). It is easy to see

$$
\lim_{b\to\infty, x\to\infty} \left(\frac{\lambda}{\delta}+1\right) C\left(\theta_1 e^{\theta_1 x} + \theta_2 e^{-\theta_2 x}\right) = \left(\frac{\lambda}{\delta}+1\right)\frac{M\theta_3}{\delta}\frac{\theta_1}{\theta_1+\theta_3} \ge 1,
$$

which means that $V'(x) \ge 1$ for all $x \in [0, \infty]$. Thus, the equilibrium strategy is never to pay the dividend and $V(x) \equiv 0$ which is obviously satisfies (4.1).

This completes the proof. □

Example 4.7. Let $\mu = 1$, $\sigma = 1$, $M = 0.8$, $\delta = 0.4$. Figure 4.2 shows the equilibrium value functions for Type II pseudo-exponential discount functions with $\lambda = 0$, 0.1 and 0.3. The barrier b are 0.6525, 0.8772 and 1.5522, respectively. The case with $\lambda = 0$ is time consistent and the equilibrium strategy is optimal.

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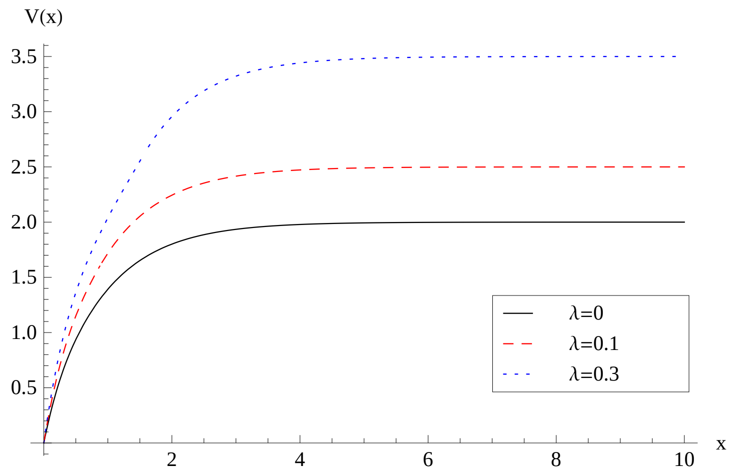

Figure 4.2: Equilibrium value functions with Type II pseudo-exponential discount function

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