Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks

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Abstract

This paper investigates the probability of ruin within finite horizon for a discrete time risk model, in which the reserve of an insurance business is currently invested in a risky asset. Under assumption that the risks are heavy tailed, some precise estimates for the finite time ruin probability are derived, which confirm a folklore that the ruin probability is mainly determined by whichever of insurance risk and financial risk is heavier than the other. In addition, some discussions on the heavy tails of the sum and product of independent random variables are involved, most of which have their own merits.
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1. Introduction

1.1. Background of the present study

Recently, a vast amount of papers has been published on the issue of ruin of an insurer who is exposed to a stochastic economic environment. Such the environment has two kinds of risk, which were called by Norberg (1999) as insurance risk and financial risk, respectively. The first kind of risk is the traditional liability risk related

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to the insurance portfolio, and the second is the asset risk related to the investment portfolio.

The aim of this paper is to derive precise estimates for the probability of ruin within finite time for a discrete time risk model as the initial capital tends to infinity, with emphasis on heavy-tailed insurance risk and financial risk. The stochastic economic environment is considered in the following way. First we denote by a random variable (r.v.) X_n the net payout of the insurer at year n, and by a positive r.v. Y_n the discount factor (from year n to year $n - 1$) related to the return on the investment, $n = 1, 2, \ldots$. Then the discounted value of the total risk amount accumulated till the end of year n can be modelled by a discrete time stochastic process

$$W_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \quad n = 1, 2, \ldots. \tag{1.1}$$

One sees that model (1.1) is only slightly different from the one proposed by Nyrhinen (1999), as commented by him on p. 320. Let the initial capital of the insurer be $x \geqslant 0$. We denote by $\psi(x) = \mathbb{P}(W_n > x$: for some $1 \leqslant n < \infty)$, respectively, $\psi(x, T) = \mathbb{P}(W_n > x$: for some $1 \leqslant n \leqslant T)$, the probabilities of the ultimate ruin and of the ruin within finite horizon T.

Nyrhinen (1999, 2001) investigated the asymptotic behavior of the ruin probabilities $\psi(x)$ and $\psi(x, T)$. Under a general assumption that both sequences $\{X_n: n = 1, 2, \ldots\}$ and $\{Y_n: n = 1, 2, \ldots\}$ are independent, Nyrhinen (1999) employed large deviations techniques in the discrete time model (1.1) and determined a rough (or crude) estimate for the ruin probability $\psi(x)$ in the form

$$\lim_{x \to \infty} (\log x)^{-1} \log \psi(x) = -w, \tag{1.2}$$

where w is a positive parameter which can explicitly be expressed by the distributions of $\{Y_n: n = 1, 2, \ldots\}$. What is really interesting is that, for the particular case where both $\{X_n: n = 1, 2, \ldots\}$ and $\{Y_n: n = 1, 2, \ldots\}$ are sequences of independent and identically distributed (i.i.d.) r.v.'s, the asymptotic relation (1.2), combining with a result by Goldie (1991), implies a stronger formula for the ruin probability $\psi(x)$ that

$$\lim_{x \to \infty} x^{-w} \psi(x) = C. \tag{1.3}$$

We call that relation (1.3) gives the ultimate ruin probability $\psi(x)$ a precise (or refined) estimate. Here, the words rough and precise are adopted from the study on large deviations; see, for instance, Mikosch and Nagaev (1998, p. 83). Unfortunately, the constant C in relation (1.3) is so involved and ambiguous that it is even not easy to infer directly from the representation given by Goldie (1991) whether or not it is positive. Lately, Nyrhinen (2001) further improved the results to a more general stochastic case by adding another sequence $\{L_n: n = 1, 2, \ldots\}$ to the above-mentioned stochastic model such that (X_n, Y_n, L_n), $n = 1, 2, \ldots$, constitute a sequence of i.i.d. random vectors. The advantage of the modelling in Nyrhinen (2001) is that with the help of the sequence $\{L_n: n = 1, 2, \ldots\}$ it is possible to treat continuous time models.

Kalashnikov and Norberg (2002) investigated the probability of ultimate ruin in the bivariate Lévy driven risk process. Applying the result in Goldie (1991), they showed

once again that the ultimate ruin probability decreases at a power rate as given in (1.3) as the reserve increases and is invested in a risky asset. They concluded that risky investments may impair the insurer's solvency just as severely as do large claims.

We mention that there are enormous papers which are devoted to the ultimate ruin of the continuous and discrete time risk models with risky assets since the pioneering work by Harrison (1977). We do not plan, it is also impossible for us, to cite here a complete list of references. In this connection we refer to the survey paper by Paulsen (1998).

We address in the present paper the asymptotic behavior of the finite time ruin probability of the risk model (1.1). Compared with the study on the probability of ultimate ruin, the research on the probability of ruin in finite time in the stochastic economic environment is quite scarce. Of course the ruin in finite time for the case without risky investment has been extensively investigated in the past. In this latter aspect we refer to Baltrünas (1999) and Malinovskii (2000), among others. Both references aimed at precise estimates for the finite time ruin probability in the renewal risk model, where Baltrünas (1999) handled the finite time ruin probability $\psi(x, n)$ for each fixed $n = 1, 2, \ldots$ in the discrete time version under the assumption that the claimsize is heavy tailed, and Malinovskii (2000) considered the case where the safety loading coefficient depends on the initial capital x and tends to 0 as $x \to \infty$, and derived some precise estimates for the finite time ruin probability $\psi(x, T)$ uniformly for $T \geqslant 0$ under the assumption that the claimsize is light tailed, i.e. satisfies the Cramér conditions. The most related reference on the finite time ruin corresponding to our case is still Nyrhinen (2001), which derived an asymptotic result for the ruin probability in finite time in the rough form that

$$\lim_{x \to \infty} (\log x)^{-1} \log \psi(x, t \log x) = -R(t) \tag{1.4}$$

for every large t, where $R(t)$ is an appropriate positive constant, mainly determined by the distribution of the financial risk Y_1. All the cited references above except Baltrünas (1999) did not pay special attention to the case of heavy-tailed risks in their models.

In the present paper, we will derive some precise estimates for the ruin probability $\psi(x, n)$, where the finite horizon $n = 1, 2, \ldots$ is fixed when we let the initial capital x tend to infinity. In doing so, we assume that the insurance risk X_1 and/or financial risk Y_1 are heavy tailed. Such the assumption is reasonable in view of the facts that, as remarked by Embrechts et al. (1997), the ruin is mainly due to one large claim, and that, corresponding to our model, the ruin is mainly due to one large insurance or financial risk. Researchers in mathematical finance usually have special interest on finite horizon models. They often fail, however, to find convenient numerical and analytical tools in their investigation. The advantage of our consideration is that we first derive a recurrence expression for the finite time ruin probability, which gives rise to the possibility of quantitative investigation and the convenience in calculation on the finite time ruin probability. Our method originates from the paper by Cline and Samorodnitsky (1994) and some related references, allowing us to derive precise estimates for the finite time ruin probability step by step. This differs from those applied in the papers cited above. Our results confirm the folklore that the ruin probability is mainly determined by whichever of insurance risk and financial risk is heavier than

the other. To a certain extent our work also shows that, for the case of heavy-tailed risks, the finite time ruin probability decreases approximately at a power rate as the initial capital tends to infinity. For the case of light-tailed risks, however, the continuing investigation in our next paper Tang and Tsitsiashvili (2003) will show that the finite time ruin probability may decrease at an exponential rate, which differs from those in the literature.

1.2. The outline of the paper

Section 2 describes the framework of the present investigation and defines the finite time ruin probability $\psi(x, n)$ with emphasis on the insurance risk X and financial risk Y. Specifically, an expression for $\psi(x, n)$ is derived, based on a backward recurrence formula. This result plays a fundamental role in the present work. Section 3 lists some preliminaries about heavy-tailed distributions and related important distribution classes. Special attention is paid to the tail equivalency of the sum and product of two independent random variables. Some discussions on the moment and Matuszewska indices are also given, most of which are of interests on their own right. The main results with their proofs are presented in the last three sections. In Section 4, we give a rough look at $\psi(x, n)$ via the moment and Matuszewska indices of X and Y, illustrating that $\psi(x, n)$ decreases approximately at a power rate as the initial capital x tends to infinity provided that X or Y has a dominatedly varying tail. Section 5 presents some precise estimates for $\psi(x, n)$ under the assumption that the insurance risk X is heavy tailed and dominates the financial risk Y in the sense that $\mathbb{P}(Y > x) = \mathrm{o}(\mathbb{P}(X > x))$. The other estimates are given in Section 6 corresponding to the inverse case, i.e. that Y is heavy tailed and dominates X. Regretfully, the study on the inverse case in Section 6 is not so complete as that in Section 5. Simple numerical results are added in Section 7.

1.3. Notational conventions

Throughout, for a given r.v. X concentrated on $(-\infty, \infty)$ with a distribution function (d.f.) F, we denote its right tail by $\bar{F}(x) = 1 - F(x) = \mathbb{P}(X > x)$, and denote its positive part by $X_+ = \max\{0, X\}$. For two d.f.'s F_1 and F_2 concentrated on $(-\infty, \infty)$, we write by $F_1 * F_2(x) = \int_{-\infty}^{\infty} F_1(x - t) F_2(\mathrm{d}t)$, $-\infty < x < \infty$, the convolution of F_1 and F_2, and write by $F_1^{*2} = F_1 * F_1$ the convolution of F_1 with itself. All limiting relationships, unless otherwise stated, are for $x \to \infty$. Let $a(x) \geqslant 0$ and $b(x) > 0$ be two infinitesimals, satisfying

$$l^- \leqslant \liminf_{x \to \infty} \frac{a(x)}{b(x)} \leqslant \limsup_{x \to \infty} \frac{a(x)}{b(x)} \leqslant l^+.$$

We write $a(x) = \mathrm{O}(b(x))$ if $l^+ < \infty$, $a(x) = \mathrm{o}(b(x))$ if $l^+ = 0$, and $a(x) \asymp b(x)$ if both $l^+ < \infty$ and $l^- > 0$; we write $a(x) \lesssim b(x)$ if $l^+ = 1$, $a(x) \gtrsim b(x)$ if $l^- = 1$, and $a(x) \sim b(x)$ if both. We say that $a(x)$ and $b(x)$ are weakly equivalent if $a(x) \asymp b(x)$, and that $a(x)$ and $b(x)$ are (strongly) equivalent if $a(x) \sim b(x)$.

2. Framework model

2.1. Ruin probabilities, insurance risk and financial risk

The basic assumptions of this paper are as follows, as applied by Nyrhinen (1999, 2001):

P_1. The successive net incomes A_n, $n=1,2,\dots$, constitute a sequence of i.i.d. r.v.'s with common d.f. concentrated on $(-\infty,\infty)$, where the net income A_n is understood as the total incoming premium minus the total claim amount within year n;

P_2. The reserve is currently invested into a risky asset which may earn negative interest r_n at year n, and r_n, $n=1,2,\dots$, also constitute a sequence of i.i.d. r.v.'s, with common d.f. concentrated on $(-1,\infty)$;

P_3. The two sequences $\{A_n: n=1,2,\dots\}$ and $\{r_n: n=1,2,\dots\}$ are mutually independent.

To save notation, we may say that the A_n, $n=1,2,\dots$, are independent replicates of a generic r.v. A. We will be using this device throughout, letting the symbols speak for themselves. In the literature, the r.v. $B_n = 1 + r_n$ is often called as the inflation coefficient from year $n-1$ to year n and the r.v. $Y_n = B_n^{-1}$ the discount factor from year n to year $n-1$, $n=1,2,\dots$. In the terminology of Norberg (1999), we call the r.v.'s $X = -A$ and Y as the insurance risk and financial risk, respectively. Clearly, $\mathbb{P}(0 < Y < \infty) = 1$.

Let the initial capital of the insurance company be $x \geqslant 0$. We tacitly assume that the income A_n is made or calculated at the end of year n, $n=1,2,\dots$. Hence, the surplus of the company accumulated till the end of year n can be characterized by S_n which satisfies the recurrence equation below:

$$S_0 = x, \quad S_n = B_n S_{n-1} + A_n, \quad n = 1,2,\dots, \tag{2.1}$$

where $B_n = 1 + r_n$, $n=1,2,\dots$. Clearly, if we assume that the income A_n is made or calculated at the beginning of year n, then this recurrence equation should be rewritten as

$$S_0 = x, \quad S_n = B_n(S_{n-1} + A_n), \quad n = 1,2,\dots. \tag{2.2}$$

Related discussions can be found in Cai (2002), where the author considered two nonstandard risk models where the interest rates r_n, $n=1,2,\dots$, follow a dependent autoregressive structure, and established some Lundberg bounds for the ultimate ruin probability under some Cramér conditions. In this paper, we shall primarily investigate model (2.1), and sometimes simply list some parallel results related to model (2.2) into remarks accordingly. The model we handle in the sequel, unless otherwise stated, will automatically be related to (2.1). By the recurrence equation (2.1), we immediately obtain

$$S_0 = x, \quad S_n = x \prod_{j=1}^{n} B_j + \sum_{i=1}^{n} A_i \prod_{j=i+1}^{n} B_j, \quad n = 1,2,\dots, \tag{2.3}$$

where $\prod_{j=n+1}^{n} = 1$ by convention.

The S_n in expression (2.3) is immediately recognized as the value of a perpetuity at the end of year n, $n = 0, 1, \ldots$; see Embrechts et al. (1997, Chapter 8.4) for a simple review, where we find that the limit behavior (as $n \to \infty$) of the process $\{S_n : n = 0, 1, \ldots\}$ in (2.3) has been extensively investigated. Kalashnikov and Norberg (2002, p. 214) pointed out that the process $\{S_n : n = 0, 1, \ldots\}$ in (2.3) coincides with the bivariate Lévy driven risk process when embedded at the occurrence times of the successive claims in their model.

We define, as usual, the time of ruin in the considered risk model with initial capital $x \geqslant 0$ by

$$\tau(x) = \inf\{n = 1, 2, \ldots : S_n < 0 | S_0 = x\}.$$

Hence, the probabilities of ruin within finite time, $\psi(x, n)$, and of ultimate ruin, $\psi(x)$, can be defined by

$$\psi(x, n) = \mathbb{P}(\tau(x) \leqslant n),$$

respectively,

$$\psi(x) = \psi(x, \infty) = \mathbb{P}(\tau(x) < \infty).$$

It is obvious that the function $\psi(x, n)$ is nonincreasing in $x \in [0, \infty)$ and nondecreasing in $n = 1, 2, \ldots$. This paper also gives some asymptotic results on the time of ruin. Clearly, the probability that the ruin occurs exactly at year n, which is naturally defined by $\phi(x, n) = \mathbb{P}(\tau(x) = n)$, satisfies

$$\phi(x, n) = \psi(x, n) - \psi(x, n - 1), \quad n = 1, 2, \ldots. \tag{2.4}$$

Remark 2.1. We have defined the ruin probabilities by these formulae mainly to be more compatible with related earlier studies in this field. Unfortunately, as remarked by our referee, these definitions are rather arguable, although this tradition has become embedded in the recent literature. A more relevant calculation might be $\mathbb{P}(\tau_y(x) \leqslant n)$ or $\mathbb{P}(\tau_y(x) < \infty)$ for $x > 0$ and $n = 1, 2, \ldots$, where $\tau_y(x)$ is a stopping time, defined by

$$\tau_y(x) = \inf\{n = 1, 2, \ldots : S_n < y | S_0 = x\}$$

for any regulatory or trigger boundary $y \geqslant 0$. This stopping time $\tau_y(x)$ may be interpreted as the first time at which there is a need to raise the capital in order to maintain solvency. The term 'ruin', however, is far too strong.

2.2. A backward recurrence formula

According to the notation above, we can rewrite the discounted value of the surplus S_n in (2.3) as

$$\tilde{S}_0 = x, \quad \tilde{S}_n = S_n \prod_{j=1}^{n} Y_j = x - \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j = x - W_n,$$

where W_n is given in (1.1), $n = 1, 2, \ldots$. Hence, we easily understand that, for each $n = 0, 1, \ldots$,

$$\psi(x, n) = \mathbb{P}(U_n > x), \tag{2.5}$$

where

$$U_n = \max\left\{0, \max_{1 \leqslant k \leqslant n} W_k\right\} \quad \text{with } U_0 = 0. \tag{2.6}$$

Define another Markov chain as

$$V_0 = 0, \quad V_n = Y_n \max\{0, X_n + V_{n-1}\}, \quad n = 1, 2, \ldots. \tag{2.7}$$

The following result shows that the relation

$$\psi(x, n) = \mathbb{P}(V_n > x) \tag{2.8}$$

holds for each $n = 1, 2, \ldots$ under the assumptions P_1, P_2 and P_3.

Theorem 2.1. *Let the assumptions P_1, P_2 and P_3 hold simultaneously. Then for each $n = 0, 1, \ldots$, the two r.v.'s U_n and V_n, which are, respectively, given by (2.6) and (2.7), have the same distribution, denoted by*

$$U_n \overset{d}{=} V_n. \tag{2.9}$$

Proof. Result (2.9) is trivial for the case when $n = 0$. Now we aim at (2.9) for each $n = 1, 2, \ldots$. Let $n \geqslant 1$ be fixed. In view of the assumptions P_1, P_2 and P_3, we replace X_i and Y_j in U_n, respectively, by X_{n+1-i} and Y_{n+1-j} in deriving the following relations:

$$U_n = \max\left\{0, \max_{1 \leqslant k \leqslant n} \sum_{i=1}^{k} X_i \prod_{j=1}^{i} Y_j\right\}$$

$$\overset{d}{=} \max\left\{0, \max_{1 \leqslant k \leqslant n} \sum_{i=1}^{k} X_{n+1-i} \prod_{j=1}^{i} Y_{n+1-j}\right\}$$

$$= \max\left\{0, \max_{1 \leqslant k \leqslant n} \sum_{i^*=n+1-k}^{n} X_{i^*} \prod_{j^*=i^*}^{n} Y_{j^*}\right\}$$

$$= \max\left\{0, \max_{1 \leqslant k^* \leqslant n} \sum_{i^*=k^*}^{n} X_{i^*} \prod_{j^*=i^*}^{n} Y_{j^*}\right\}. \tag{2.10}$$

If we write the right-hand side of (2.10) as $\tilde{V}_n$, then it satisfies the recurrence equation that

$$\tilde{V}_n = Y_n \max\{0, X_n + \tilde{V}_{n-1}\}, \quad n = 1, 2, \ldots,$$

which is just the same as (2.7). So we immediately conclude that $\tilde{V}_n = V_n$ for each $n = 1, 2, \ldots$. Finally, it follows from (2.10) that (2.9) holds for each $n = 1, 2, \ldots$. This ends the proof of Theorem 2.1. □

Remark 2.2. Consider the risk model (2.2). For this case the ruin probability is $\psi(x, n) = \mathbb{P}(U'_n > x)$ with

$$U'_n = \max \left\{ 0, \ \max_{1 \leqslant k \leqslant n} \sum_{i=1}^{k} X_i \prod_{j=1}^{i-1} Y_j \right\}.$$

Going along the same line as Theorem 2.1 a similar result can be established as $U'_n \stackrel{d}{=} V'_n$, $n = 1, 2, \ldots$, where V'_n is defined by a Markov chain as

$$V'_0 = 0, \quad V'_n = \max\{0, \ X_n + Y_n V'_{n-1}\}, \quad n = 1, 2, \ldots.$$

Theorem 2.1 generalizes Lindley chain for one server queueing system $G/G/1/\infty$ (see Lindley, 1952) to the stochastic risk model (2.1) under stochastic interest force. It gives convenient statistical simulation algorithms for the ruin probability calculation. Especially, relation (2.8) allows us to build asymptotic formulae for the finite time ruin probability $\psi(x, n)$ step by step.

3. Tails of sum and product of independent random variables

3.1. Tails of convolution

In the sequel, for any r.v. X distributed by F and any real number γ we denote its moment generating function by $\hat{F}(\gamma) = \mathbb{E}\exp\{\gamma X\}$. We say that X or its d.f. F is heavy tailed on right hand if $\hat{F}(\gamma) = \infty$ for any $\gamma > 0$.

A d.f. F concentrated on $(0, \infty)$ is said to belong to the class $\mathscr{S}(\gamma)$, $\gamma \geqslant 0$, if and only if

(1) $\lim_{x \to \infty} \overline{F^{*2}}(x)/\bar{F}(x) = 2c < \infty$;
(2) $\lim_{x \to \infty} \bar{F}(x - t)/\bar{F}(x) = e^{\gamma t}$ for all t real.

The d.f. F belongs to $\mathscr{L}(\gamma)$, $\gamma \geqslant 0$, if and only if F satisfies the condition (2).

The two classes above were introduced independently by Chistyakov (1964) and Chover et al. (1973a, b). Applications to the classical ruin theory can be found in Veraverbeke (1977), Embrechts and Veraverbeke (1982), among others. We remark that γ is the right abscissa of convergence of $\hat{F}(\cdot)$, and that the convergence in item (2) is automatically uniform on t in any finite interval. When $\gamma = 0$, $\mathscr{S} = \mathscr{S}(0)$ is just the subexponential class and $\mathscr{L} = \mathscr{L}(0)$ is the class of all long-tailed d.f.'s, which are two of the most important classes of heavy-tailed distributions. It has been proved that for any d.f. $F \in \mathscr{S}(\gamma)$, $\gamma \geqslant 0$, the constant $c = \hat{F}(\gamma) < \infty$; see Cline (1987), Rogozin (2000) and the references therein. More generally, a d.f. F concentrated on $(-\infty, \infty)$ is also said to belong to the class $\mathscr{S}(\gamma)$ or $\mathscr{L}(\gamma)$ if its right-hand distribution $\tilde{F}(x) = F(x)\mathbb{I}_{(x>0)}$ does. By Lemma 3.2 below one sees that, if a d.f. F concentrated on $(-\infty, \infty)$

belongs to the class $\mathscr{S}(\gamma)$, the relation

$$\lim_{x \to \infty} \frac{\overline{F^{*2}}(x)}{\bar{F}(x)} = 2\hat{F}(\gamma) < \infty$$

still holds. Because of the monotonicity of the function $\bar{F}$, one also easily checks that a d.f. F concentrated on $(-\infty, \infty)$ belongs to the class $\mathscr{L}$ if and only if

$$\lim_{x \to \infty} \frac{\bar{F}(x+1)}{\bar{F}(x)} = 1. \tag{3.1}$$

For two d.f.'s F_1 and F_2 satisfying $\overline{F_1}(x) \sim c\overline{F_2}(x)$ for some constant $c \in (0, \infty)$, we know that $F_1 \in \mathscr{S}(\gamma)$ if and only if $F_2 \in \mathscr{S}(\gamma)$; for related discussions see Klüppelberg (1988), Theorem 2.1(a) and a sentence before that theorem.

One important result of Rogozin and Sgibnev (1999) says:

Lemma 3.1. *For some d.f. $F \in \mathscr{S}(\gamma)$, $\gamma \geqslant 0$, and two other d.f.'s F_1 and F_2 concentrated on $(-\infty, \infty)$ such that $k_i = \lim_{x \to \infty} \overline{F_i}(x)/\bar{F}(x)$ exists finite, $i = 1, 2$, it holds that*

$$\lim_{x \to \infty} \frac{\overline{F_1 * F_2}(x)}{\bar{F}(x)} = k_1 \hat{F}_2(\gamma) + k_2 \hat{F}_1(\gamma). \tag{3.2}$$

We remark that, in the original work by Rogozin and Sgibnev (1999), the d.f.'s F_1 and F_2 above are two general measures on $(-\infty, \infty)$, not necessarily standard probabilistic ones. So (3.2) is still valid for the defective case where $0 < F_i(-\infty) < 1$, $i = 1$ and/or 2. In fact, this can also be directly proved by some trivial adjustments. Result (3.2) in this general understanding will be applied in Theorem 6.1 below.

The merit of the following result is that it does not require the existence of the limit of $\overline{F_2}(x)/\overline{F_1}(x)$. This result can be found in Cline (1986, Corollary 1), but for the case where the d.f.'s F_1 and F_2 are concentrated on $[0, \infty)$; under some additional restriction it was first obtained by Embrechts and Goldie (1980).

Lemma 3.2. *Let F_1 and F_2 be two d.f.'s concentrated on $(-\infty, \infty)$. If $F_1 \in \mathscr{S}(\gamma)$, $F_2 \in \mathscr{L}(\gamma)$ for $\gamma \geqslant 0$, and $\overline{F_2}(x) = O(\overline{F_1}(x))$, then $F_1 * F_2 \in \mathscr{S}(\gamma)$ and*

$$\overline{F_1 * F_2}(x) \sim \hat{F}_2(\gamma)\overline{F_1}(x) + \hat{F}_1(\gamma)\overline{F_2}(x). \tag{3.3}$$

Proof. Let X_1 and X_2 be two independent r.v.'s distributed by F_1 and F_2, respectively. According to whether or not the events $(X_1 > 0)$ and $(X_2 > 0)$ happen we divide the tail of $F_1 * F_2$ into three parts as

$$\overline{F_1 * F_2}(x) = \sum_{k=1}^{3} \mathbb{P}(X_1 + X_2 > x, \Omega_k) = I_1 + I_2 + I_3, \tag{3.4}$$

where $\Omega_1 = (X_1 > 0, X_2 > 0)$, $\Omega_2 = (X_1 \leqslant 0, X_2 > 0)$, and $\Omega_3 = (X_1 > 0, X_2 \leqslant 0)$. Applying Corollary 1 of Cline (1986), we immediately obtain that

$$I_1 = \mathbb{P}(\Omega_1)\mathbb{P}(X_1 + X_2 > x | \Omega_1)$$

$$\sim \mathbb{P}(\Omega_1) \left(\mathbb{P}(X_1 > x | X_1 > 0) \int_0^\infty e^{\gamma t} \mathbb{P}(X_2 \in dt | X_2 > 0) \right.$$

$$\left. + \mathbb{P}(X_2 > x | X_2 > 0) \int_0^\infty e^{\gamma t} \mathbb{P}(X_1 \in dt | X_1 > 0) \right)$$

$$= \overline{F_1}(x) \int_0^\infty e^{\gamma t} F_2(dt) + \overline{F_2}(x) \int_0^\infty e^{\gamma t} F_1(dt). \tag{3.5}$$

Since $F_2 \in \mathscr{L}(\gamma)$, the dominated convergence theorem gives that

$$I_2 = \overline{F_2}(x) \int_{-\infty}^0 \frac{\overline{F_2}(x-t)}{\overline{F_2}(x)} F_1(dt) \sim \overline{F_2}(x) \int_{-\infty}^0 e^{\gamma t} F_1(dt). \tag{3.6}$$

Similarly, it holds that

$$I_3 \sim \overline{F_1}(x) \int_{-\infty}^0 e^{\gamma t} F_2(dt). \tag{3.7}$$

Substituting (3.5)–(3.7) into (3.4) leads to the announced result (3.3).

In order to verify that $F_1 * F_2 \in \mathscr{S}(\gamma)$, we recall Theorem 2.1(a) of Klüppelberg (1988) and a sentence before that theorem, where it is indicated that, if F and F_1 are two elements of the class $\mathscr{L}(\gamma)$ and satisfy the weak tail-equivalence, i.e. $\bar{F}(x) \asymp \overline{F_1}(x)$, then $F \in \mathscr{S}(\gamma) \Leftrightarrow F_1 \in \mathscr{S}(\gamma)$. In the original form of this result the d.f.'s involved are concentrated on $[0, \infty)$, but the generalization to $(-\infty, \infty)$ is trivial since we define a d.f. belonging to the class $\mathscr{S}(\gamma)$ by its right-hand distribution. By virtue of (3.3), the weak tail-equivalence of $F_1 * F_2$ and F_1 is straightforward. Hence, $F_1 * F_2 \in \mathscr{S}(\gamma)$ follows from the condition $F_1 \in \mathscr{S}(\gamma)$. This ends the proof of Lemma 3.2. □

3.2. Heavy-tailed distributions

Like many recent researchers in the fields of applied probability and risk theory, we restrict our interest to the case of heavy-tailed risks. As mentioned above, among the most important classes of heavy-tailed distributions are the classes $\mathscr{S}$ and $\mathscr{L}$. There is another class of heavy-tailed distributions, the class $\mathscr{D}$ of d.f.'s with dominatedly varying tails, which is closely related to the classes $\mathscr{S}$ and $\mathscr{L}$. A d.f. F concentrated on $(-\infty, \infty)$ belongs to the class $\mathscr{D}$ if and only if its tail $\bar{F}$ is of dominated variation in the sense that the relation

$$\limsup_{x \to \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} < \infty \tag{3.8}$$

holds for any $0 < y < 1$ (or equivalently for some $0 < y < 1$). It is well known that

$$\mathscr{D} \cap \mathscr{L} \subset \mathscr{S} \subset \mathscr{L};$$

see Embrechts et al. (1997, Chapters 1.3, 1.4 and A3) and the references therein. We comment that the intersection $\mathscr{D} \cap \mathscr{L}$ is rich enough to contain many useful heavy-tailed distributions in modelling risk variables. A famous subclass of the intersection $\mathscr{D} \cap \mathscr{L}$ is $\mathscr{R}$, which is the class of d.f.'s with regularly varying tails. By definition, a d.f. F concentrated on $(-\infty, \infty)$ belongs to the class $\mathscr{R}$ if and only if there is some $\alpha \geqslant 0$ such that the relation

$$\lim_{x \to \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} = y^{-\alpha}$$

holds for any $y > 0$. In this case we denote $F \in \mathscr{R}_{-\alpha}$. Another useful subclass of the intersection $\mathscr{D} \cap \mathscr{L}$ is the so-called extended regularly varying (ERV) class, which is slightly larger than the class $\mathscr{R}$. By definition, a d.f. F concentrated on $(-\infty, \infty)$ belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$ for some $0 \leqslant \alpha \leqslant \beta < \infty$ if and only if the relation

$$y^{-\beta} \leqslant \liminf_{x \to \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} \leqslant \limsup_{x \to \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} \leqslant y^{-\alpha} \tag{3.9}$$

holds for any $y > 1$. For more details about the classes of heavy-tailed distributions and their applications to insurance and finance, the readers are referred to Bingham et al. (1987) and Embrechts et al. (1997).

In this paper we are particularly interested in the class $\mathscr{D}$, which, as mentioned above, is a very large heavy-tailed subclass. Obviously, if $F \in \mathscr{D}$ then $\bar{F}(cx) \asymp \bar{F}(x)$ for any $c > 0$. By definition, we easily see that the class $\mathscr{D}$ is closed under weak tail equivalence, i.e. that, for two d.f.'s with tails weakly equivalent to each other, one belongs to the class $\mathscr{D}$ if and only if the other does. The following lemma is due to Tang and Yan (2002), which provides us with a certain taste of robustness of the class $\mathscr{D}$ under convolution. Similar discussions on the class $\mathscr{L}$ will be given in Section 3.3.

Lemma 3.3. *Let $F_1 \in \mathscr{D}$. If F_2 is a d.f. such that $\overline{F_2}(x) = \mathrm{O}(\overline{F_1}(x))$, then $F_1 * F_2 \in \mathscr{D}$ and*

$$\overline{F_1 * F_2}(x) \asymp \overline{F_1}(x). \tag{3.10}$$

3.3. Some indices of heavy-tailed distributions

Now we consider some indices of a general random variable. For any r.v. X with a d.f. F concentrated on $(-\infty, \infty)$ we define its moment index by

$$\mathbb{I}_F = \mathbb{I}(X) = \sup\{v : \mathbb{E}X_+^v < \infty\}. \tag{3.11}$$

This index indeed describes a characteristic of the right tail of the r.v. X. We refer to Daley (2001) for some interesting discussions on the moment index $\mathbb{I}_F$. Trivially, for any proper distribution F we have $0 \leqslant \mathbb{I}_F \leqslant \infty$. However, if $F \in \mathscr{D}$ then $\mathbb{I}_F < \infty$; see, for example, Appendix A3 of Seneta (1976). By the definition in (3.11), it holds for any $v < \mathbb{I}_F$ that

$$\bar{F}(x) = \mathrm{o}(x^{-v}). \tag{3.12}$$

An inverse relation of (3.12) will be built in (3.17) below. We shall show in Section 4 that the index $\mathbb{I}_F$ is a very convenient tool in analyzing the tail behavior of the risk process that we are handling. Some related discussions to the following lemma can be found in Daley (2001).

Lemma 3.4. *Let X and Y be two independent r.v.'s, where Y is nonnegative. Then we have*

(1) $\mathbb{I}(X + Y) = \min\{\mathbb{I}(X), \mathbb{I}(Y)\}$;
(2) $\mathbb{I}(XY) = \min\{\mathbb{I}(X), \mathbb{I}(Y)\}$.

Proof. (1) We choose some real numbers $a < b$ such that $\mathbb{P}(a < X \leqslant b) > 0$. Then for any $r \geqslant 0$, it is easy to check that

$$2\mathbb{E}(X + Y)_+^r \geqslant \mathbb{E}X_+^r + \mathbb{E}(a + Y)_+^r \mathbb{P}(a < X \leqslant b).$$

It follows that $\mathbb{I}(X + Y) \geqslant \min\{\mathbb{I}(X), \mathbb{I}(Y)\}$. Conversely,

$$\mathbb{E}(X + Y)_+^r \leqslant \mathbb{E}(X_+ + Y)^r \leqslant \max\{1, 2^{r-1}\}\{\mathbb{E}X_+^r + \mathbb{E}Y^r\},$$

from which we conclude that $\mathbb{I}(X + Y) \leqslant \min\{\mathbb{I}(X), \mathbb{I}(Y)\}$. This proves (1).
  (2) The proof of this part is trivial since for any $r \geqslant 0$ it holds that

$$\mathbb{E}(XY)_+^r = \mathbb{E}X_+^r \mathbb{E}Y^r.$$

This ends the proof of Lemma 3.4. $\square$

  We further recall two other significant indices, which are crucial for our purpose. Let X be an r.v. concentrated on $(-\infty, \infty)$ with a d.f. F. For any $y > 0$ we set

$$\bar{F}_*(y) = \liminf_{x \to \infty} \frac{\bar{F}(xy)}{\bar{F}(x)}, \quad \bar{F}^*(y) = \limsup_{x \to \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} \tag{3.13}$$

and then define

$$\mathbb{J}_F^+ = \mathbb{J}^+(X) = \inf\left\{-\frac{\log \bar{F}_*(y)}{\log y} : y > 1\right\} = -\lim_{y \to \infty} \frac{\log \bar{F}_*(y)}{\log y}, \tag{3.14}$$

$$\mathbb{J}_F^- = \mathbb{J}^-(X) = \sup\left\{-\frac{\log \bar{F}^*(y)}{\log y} : y > 1\right\} = -\lim_{y \to \infty} \frac{\log \bar{F}^*(y)}{\log y}. \tag{3.15}$$

In the terminology of Bingham et al. (1987), here the quantities $\mathbb{J}_F^+$ and $\mathbb{J}_F^-$ are the upper and lower Matuszewska indices of the nonnegative and nondecreasing function $f(x) = (\bar{F}(x))^{-1}$, $x \geqslant 0$. Without any confusion we simply call the $\mathbb{J}_F^\pm$ as the upper/lower Matuszewska index of the d.f. F. The latter equalities in (3.15) and (3.14) are due to Theorem 2.1.5 in Bingham et al. (1987). For more details of the Matuszewska indices, see Bingham et al. (1987, Chapter 2.1), Cline and Samorodnitsky (1994). Let F_1 and F_2 be two d.f.'s satisfying $\overline{F_1}(x) \asymp \overline{F_2}(x)$. It is not difficult to see

that $\mathbb{I}_{F_1} = \mathbb{I}_{F_2}$, $\mathbb{J}_{F_1}^{\pm} = \mathbb{J}_{F_2}^{\pm}$. That is, all the three indices introduced above are invariant under weak tail equivalence.

We shall need the following lemma in the sequel:

Lemma 3.5. *For a d.f. $F \in \mathcal{D}$ with its moment index $\mathbb{I}_F$ and Matuszewska indices $\mathbb{J}_F^{\pm}$, we have*

$$\bar{F}(x) = o(x^{-v}) \quad \text{for any } v < \mathbb{J}_F^-; \tag{3.16}$$

$$x^{-v} = o(\bar{F}(x)) \quad \text{for any } v > \mathbb{J}_F^+; \tag{3.17}$$

$$0 \leqslant \mathbb{J}_F^- \leqslant \mathbb{I}_F \leqslant \mathbb{J}_F^+ < \infty. \tag{3.18}$$

Proof. From Proposition 2.2.1 in Bingham et al. (1987) we know that, for any $p_1 < \mathbb{J}_F^-$ and $p_2 > \mathbb{J}_F^+$, there are positive constants C_i and D_i, $i = 1, 2$, such that the inequality

$$\frac{\bar{F}(y)}{\bar{F}(x)} \geqslant C_1(x/y)^{p_1} \tag{3.19}$$

holds for all $x \geqslant y \geqslant D_1$, and that the inequality

$$\frac{\bar{F}(y)}{\bar{F}(x)} \leqslant C_2(x/y)^{p_2} \tag{3.20}$$

holds for all $x \geqslant y \geqslant D_2$. Hence, fixing the variable y in (3.19) and (3.20) leads to the results (3.16) and (3.17), respectively.

We next prove (3.18). The assertion $\mathbb{J}_F^- \geqslant 0$ is trivial since the function $\bar{F}(x)$ is non-increasing in x. From inequality (2.1.9) in Theorem 2.1.8 in Bingham et al. (1987) we easily see that $F \in \mathcal{D}$ if and only if $\mathbb{J}_F^+ < \infty$. Clearly, result (3.16) implies $\mathbb{E}X_+^{p_1} < \infty$ for any $p_1 < \mathbb{J}_F^-$, whereas result (3.17) implies $\mathbb{E}X_+^{p_2} = \infty$ for any $p_2 > \mathbb{J}_F^+$. Hence, the inequalities $\mathbb{J}_F^- \leqslant \mathbb{I}_F \leqslant \mathbb{J}_F^+$ hold immediately. This ends the proof of Lemma 3.5. $\quad\square$

Based on Lemma 3.5, one easily proves the results below:

Lemma 3.6. *Let X be an r.v. with a d.f. F.*

(1) *If $F \in \mathrm{ERV}(-\alpha, -\beta)$ for some $0 \leqslant \alpha \leqslant \beta < \infty$ then $\alpha \leqslant \mathbb{J}_F^- \leqslant \mathbb{I}_F \leqslant \mathbb{J}_F^+ \leqslant \beta$;*
(2) *If $F \in \mathcal{R}_{-\alpha}$ for some $\alpha \geqslant 0$ then $\mathbb{J}_F^- = \mathbb{I}_F = \mathbb{J}_F^+ = \alpha$.*

The proof of the following lemma is also immediate if we apply Lemma 3.5.

Lemma 3.7. *If $F \in \mathcal{D}$ with its upper Matuszewska index $\mathbb{J}_F^+ \geqslant 0$ and $\mathbb{E}Y^p < \infty$ for some $p > \mathbb{J}_F^+$, then $\mathbb{P}(Y > x^{1-\varepsilon}) = o(\bar{F}(x))$ holds for any $0 < \varepsilon < 1 - \mathbb{J}_F^+/p$.*

Proof. Since $p(1 - \varepsilon) > \mathbb{J}_F^+$, from (3.17) we know $x^{p(1-\varepsilon)}\bar{F}(x) \to \infty$. Hence,

$$\frac{\mathbb{P}(Y > x^{1-\varepsilon})}{\bar{F}(x)} \leqslant \frac{\mathbb{E}Y^p}{x^{p(1-\varepsilon)}\bar{F}(x)} \to 0.$$

This ends the proof of Lemma 3.7. $\quad\square$

3.4. Product of independent random variables

Let X and Y be two independent r.v.'s, where X is concentrated on $(-\infty, \infty)$ with a d.f. F, but Y is strictly positive with a d.f. G. We write

$$Z = XY \tag{3.21}$$

and denote by H the d.f. of Z. In this subsection we study the tail behavior of the product Z. The following first result can be found in Cline and Samorodnitsky (1994, Theorem 3.3(iv)):

Lemma 3.8. *Consider the independent product model* (3.21). *If $F \in \mathcal{D}$ with its upper Matuszewska index $\mathbb{J}_F^+ \geqslant 0$ and $\mathbb{E}Y^p < \infty$ for some $p > \mathbb{J}_F^+$, then*

$$0 < \mathbb{E}(\bar{F}_*(Y^{-1})) \leqslant \liminf_{x \to \infty} \frac{\bar{H}(x)}{\bar{F}(x)} \leqslant \limsup_{x \to \infty} \frac{\bar{H}(x)}{\bar{F}(x)} \leqslant \mathbb{E}(\bar{F}^*(Y^{-1})) < \infty, \tag{3.22}$$

where $\bar{F}_*$ and $\bar{F}^*$ are defined by (3.13).

Recall the closure property of the class $\mathcal{D}$ and the invariance of the Matuszewska indices under weak tail equivalence. From (3.22) we immediately obtain that:

Lemma 3.9. *Under the assumptions of Lemma* 3.8, $H \in \mathcal{D}$ *and* $\mathbb{J}_H^{\pm} = \mathbb{J}_F^{\pm}$.

We establish below a similar result as Lemma 3.3 but for the class $\mathcal{L}$:

Lemma 3.10. *Consider the product model* (3.21). *If $F \in \mathcal{L}$ and*

$$\bar{G}(x) = o(\bar{F}(cx)) \tag{3.23}$$

for some $0 < c < \infty$, then $H \in \mathcal{L}$.

Proof. We formulate the proof into two parts according to whether or not the r.v. Y is bounded.

1. As a special case of (3.23) we first assume that Y is bounded, i.e. there exists some $M > 0$ such that $G(M) = 1$. Clearly, for any $\varepsilon > 0$, there exists some $0 < a < 1$ sufficiently small such that $\bar{G}(a) > 0$ and $G(0, a]/\bar{G}(a) \leqslant \varepsilon$. By $F \in \mathcal{L}$ we have, for $x > 0$,

$$
\begin{aligned}
\bar{H}(x+1) &\geqslant \int_a^M \bar{F}\left(\frac{x}{t} + \frac{1}{a}\right) G(\mathrm{d}t) \\
&\sim \left(\int_0^M - \int_0^a\right) \bar{F}\left(\frac{x}{t}\right) G(\mathrm{d}t) \\
&\geqslant \bar{H}(x)\left(1 - \frac{\int_0^a \bar{F}(x/t)G(\mathrm{d}t)}{\int_a^M \bar{F}(x/t)G(\mathrm{d}t)}\right) \\
&\geqslant \bar{H}(x)\left(1 - \frac{\bar{F}(x/a)G(0,a]}{\bar{F}(x/a)\bar{G}(a)}\right) \\
&\geqslant (1 - \varepsilon)\bar{H}(x).
\end{aligned}
$$

This, together with the arbitrariness of $\varepsilon > 0$ and the monotonicity of $\bar{H}$, gives

$$\bar{H}(x+1) \sim \bar{H}(x),$$

which is just the definition of $H \in \mathscr{L}$ (recall (3.1)).

2. Now we consider the remaining case that the r.v. Y is unbounded, i.e. $\bar{G}(x) > 0$ for any real value of x. According to Cline and Samorodnitsky (1994, Theorem 2.2(iii)), it suffices to verify that, for any $b > 0$,

$$\bar{G}(x) = \mathrm{o}(\bar{H}(bx)).$$

In fact, this can be proved as follows:

$$\limsup_{x \to \infty} \frac{\bar{G}(x)}{\bar{H}(bx)} \leqslant \limsup_{x \to \infty} \frac{\bar{G}(x)}{\int_{b/c}^{\infty} \bar{F}(bx/t) G(\mathrm{d}t)} \leqslant \frac{1}{\bar{G}(b/c)} \limsup_{x \to \infty} \frac{\bar{G}(x)}{\bar{F}(cx)} = 0.$$

This ends the proof of Lemma 3.10. □

4. A rough look via the moment index

Let U be an r.v. with its moment index $\mathbb{I}(U)$ defined by (3.11). When U is heavy tailed on its right hand, it is believable that the index $\mathbb{I}(U)$ can act as a critical quantity in characterizing the heaviness of its right tail. Roughly speaking, smaller value of $\mathbb{I}(U)$ usually suggests that U is heavier on its right-hand side.

As we qualitatively mentioned in the introduction, many results in the literature confirm that the ruin probability is mainly determined by whichever of insurance risk and financial risk is heavier. Recall (2.5), which says that the ruin probability $\psi(x, n)$ is just the tail probability of the maximum U_n. If we are satisfied with a rough description of the heaviness of the right tail of a risk variable by its moment index, the following result gives the folklore a very explicit explanation.

Theorem 4.1. *Suppose that the assumptions P_1, P_2 and P_3 hold simultaneously. Then we have that*

$$\mathbb{I}(U_n) = \min\{\mathbb{I}(X), \mathbb{I}(Y)\}, \qquad n = 1, 2, \ldots . \tag{4.1}$$

Proof. From Theorem 2.1, and applying Lemma 3.4 again and again, we obtain, for $n = 1, 2, \ldots,$

$$\begin{aligned}
\mathbb{I}(U_n) &= \mathbb{I}(Y_n \max\{0, \; X_n + V_{n-1}\}) \\
&= \min\{\mathbb{I}(Y_n), \mathbb{I}(X_n + V_{n-1})\} \\
&= \min\{\mathbb{I}(Y_n), \mathbb{I}(X_n), \mathbb{I}(V_{n-1})\} \\
&= \cdots \\
&= \min\{\mathbb{I}(Y_n), \mathbb{I}(X_n), \mathbb{I}(Y_{n-1}), \mathbb{I}(X_{n-1}), \cdots, \mathbb{I}(Y_1), \mathbb{I}(X_1)\} \\
&= \min\{\mathbb{I}(X), \mathbb{I}(Y)\}.
\end{aligned}$$

Hence result (4.1) holds. This ends the proof of Theorem 4.1. □

Remark 4.1. Consider the risk model (2.2). Going along the same line as Theorem 4.1 the same result can be established as

$$\mathbb{I}(U_n') = \min\{\mathbb{I}(X), \mathbb{I}(Y)\}, \quad n = 1, 2, \dots.$$

Hence, all the results in Theorem 4.2 below hold true for model (2.2).

Applying Theorem 4.1 to estimating the finite time ruin probability, we obtain:

Theorem 4.2. *Suppose that the assumptions P_1, P_2 and P_3 hold simultaneously.*

(1) *If $m_0 = \min\{\mathbb{I}(X), \mathbb{I}(Y)\} < \infty$, then, for any $n = 1, 2, \dots$ and any $\varepsilon > 0$, it holds that*

$$\psi(x, n) = \mathrm{o}(x^{-m_0+\varepsilon}).$$

(2) *If the insurance risk X has a d.f. $F \in \mathscr{D}$ with moment index $\mathbb{I}_F$ and upper Matuszewska index $\mathbb{J}_F^+$ (hence finite) and the financial risk Y satisfies $\mathbb{E}Y^{\mathbb{I}_F} < \infty$, then, for any $n = 1, 2, \dots$ and any $\varepsilon > 0$,*

$$\psi(x, n) = \mathrm{o}(x^{-\mathbb{I}_F+\varepsilon}), \quad x^{-\mathbb{J}_F^+-\varepsilon} = \mathrm{o}(\psi(x, n)). \tag{4.2}$$

Analogously, if the financial risk Y has a d.f. $G \in \mathscr{D}$ with moment index $\mathbb{I}_G$ and upper Matuszewska index $\mathbb{J}_G^+$ (hence finite) and the insurance risk X satisfies $\mathbb{E}X^{\mathbb{I}_G} < \infty$, then, for any $n = 1, 2, \dots$ and any $\varepsilon > 0$, the results in (4.2) hold with $\mathbb{I}_G$ and $\mathbb{J}_G^+$ replacing $\mathbb{I}_F$ and $\mathbb{J}_F^+$.

(3) *If $F \in \mathrm{ERV}(-\alpha, -\beta)$ $(G \in \mathrm{ERV}(-\alpha, -\beta))$ for some $0 \leqslant \alpha \leqslant \beta < \infty$ and $\mathbb{E}Y^\alpha < \infty$ $(\mathbb{E}X^\alpha < \infty)$, then, for any $n = 1, 2, \dots$ and any $\varepsilon > 0$,*

$$\psi(x, n) = \mathrm{o}(x^{-\alpha+\varepsilon}), \quad x^{-\beta-\varepsilon} = \mathrm{o}(\psi(x, n)).$$

(4) *If $F \in \mathscr{R}_{-\alpha}$ $(G \in \mathscr{R}_{-\alpha})$ for some $0 \leqslant \alpha < \infty$ and $\mathbb{E}Y^\alpha < \infty$ $(\mathbb{E}X^\alpha < \infty)$, then, for any $n = 1, 2, \dots$ and any $\varepsilon > 0$,*

$$\psi(x, n) = \mathrm{o}(x^{-\alpha+\varepsilon}), \quad x^{-\alpha-\varepsilon} = \mathrm{o}(\psi(x, n)).$$

Proof. Clearly, items (3) and (4) are the natural consequences of item (2), and the first relation in (4.2) is the natural consequence of item (1). What's more, recalling (3.12), item (1) is the natural consequence of Theorem 4.1. Now we aim to prove the second relation in (4.2) under the assumption that $F \in \mathscr{D}$. Let $a > 0$ be arbitrarily fixed such that $\bar{G}(a) > 0$. Since $\psi(x, n)$ is nonincreasing in n, from (2.8) we have for each $n = 1, 2, \dots$ that

$$\psi(x, n) \geqslant \psi(x, 1) = \mathbb{P}(XY > x) \geqslant \bar{F}(x/a)\bar{G}(a) \asymp \bar{F}(x),$$

which, together with (3.17), implies the second relation in (4.2). This ends the proof of Theorem 4.2. $\quad\square$

Theorem 4.2 shows that the finite time ruin probability $\psi(x, n)$ decreases approximately at a power rate as $x \to \infty$ provided that the insurance risk X or the financial risk

Y has a dominatedly varying tail. More accurate results will be built in the following two sections.

5. Approximation (1): insurance risk dominates financial risk

As before, we write by F and G the d.f.'s of the r.v.'s X and Y.

Theorem 5.1. *Suppose that the assumptions P_1, P_2 and P_3 hold simultaneously. If*

(1) $F \in \mathscr{L} \cap \mathscr{D}$ *with its upper Matuszewska index $\mathbb{J}_F^+$*
(2) $\mathbb{E}Y^p < \infty$ *for some $p > \mathbb{J}_F^+$,*

then it holds for each $n = 1, 2, \ldots$ that

$$\psi(x, n) \sim \sum_{k=1}^{n} \mathbb{P}\left(X \prod_{i=1}^{k} Y_i > x \right). \tag{5.1}$$

Proof. We prove this theorem by the mathematical induction device. By (2.8) we have, for $x > 0$,

$$\psi(x, 1) = \mathbb{P}(V_1 > x) = \mathbb{P}(X_1 Y_1 > x). \tag{5.2}$$

This shows that the asymptotic result (5.1) holds for $n = 1$. Applying Lemmas 3.7, 3.8, 3.9 and 3.10, we also know from (5.2) that

1° $\mathbb{P}(V_1 > x) \asymp \bar{F}(x)$;
2° $\mathbb{J}^+(V_1) = \mathbb{J}_F^+$;
3° the d.f. of V_1 belongs to the intersection $\mathscr{L} \cap \mathscr{D}$.

Now we assume by induction that (5.1) holds for $n = m$, $m \geqslant 1$, and that the three items above are satisfied by V_m. Based on these items, it follows from Lemma 3.2 with $\gamma = 0$ that

$$\mathbb{P}(X_{m+1} + V_m > x) \sim \mathbb{P}(X_{m+1} > x) + \mathbb{P}(V_m > x). \tag{5.3}$$

From (5.3), Lemma 3.2 with $\gamma = 0$ and Lemma 3.3, we know that the d.f. of the sum $X_{m+1} + V_m$ belongs to the intersection $\mathscr{L} \cap \mathscr{D}$ and that $\mathbb{P}(X_{m+1} + V_m > x) \asymp \bar{F}(x)$. Trivially, from (2.8) we have

$$\psi(x, m+1) = \mathbb{P}(V_{m+1} > x) = \mathbb{P}((X_{m+1} + V_m)Y_{m+1} > x). \tag{5.4}$$

So, just by copying the proof in the first step with $X_{m+1} + V_m$ replacing X_1 and Y_{m+1} replacing Y_1 in (5.2), we prove that V_{m+1} satisfies the items 1°, 2° and 3° above. We continue to verify that $\psi(x, m+1)$ satisfies (5.1). From Lemma 3.7, we see that $\bar{G}(x^{1-\varepsilon}) = o(\bar{F}(x))$ holds for any $0 < \varepsilon < 1 - J_F^+/p$. For the arbitrarily fixed ε, we derive

from (5.4), (5.3) and (5.1) with $n = m$ that

$$
\psi(x, m+1) = \left(\int_0^{x^{1-\varepsilon}} + \int_{x^{1-\varepsilon}}^{\infty} \right) \mathbb{P}(X_{m+1} + V_m > x/t) G(\mathrm{d}t)
$$

$$
\sim \int_0^{x^{1-\varepsilon}} (\mathbb{P}(X_{m+1} > x/t) + \mathbb{P}(V_m > x/t)) G(\mathrm{d}t)
$$

$$
+ \int_{x^{1-\varepsilon}}^{\infty} \mathbb{P}(X_{m+1} + V_m > x/t) G(\mathrm{d}t)
$$

$$
\sim \left(\int_0^{\infty} - \int_{x^{1-\varepsilon}}^{\infty} \right) \left(\mathbb{P}(X_{m+1} > x/t) + \sum_{k=1}^{m} \mathbb{P}\left(X \prod_{i=1}^{k} Y_i > x/t \right) \right) G(\mathrm{d}t)
$$

$$
+ \int_{x^{1-\varepsilon}}^{\infty} \mathbb{P}(X_{m+1} + V_m > x/t) G(\mathrm{d}t)
$$

$$
= \sum_{k=1}^{m+1} \mathbb{P}\left(X \prod_{i=1}^{k} Y_i > x \right) + \Delta_{m+1}, \tag{5.5}
$$

where Δ_{m+1} denotes the remaining term and is

$$
\Delta_{m+1} = \int_{x^{1-\varepsilon}}^{\infty} \left(\mathbb{P}(X_{m+1} + V_m > x/t) \right.
$$

$$
\left. - \mathbb{P}(X_{m+1} > x/t) - \sum_{k=1}^{m} \mathbb{P}\left(X \prod_{i=1}^{k} Y_i > x/t \right) \right) G(\mathrm{d}t).
$$

Clearly, by Lemma 3.7 and the fact that $\bar{F}(x) \asymp \mathbb{P}(XY > x)$, the remaining term Δ_{m+1} can be estimated by

$$
|\Delta_{m+1}| \leqslant (m+2)\bar{G}(x^{1-\varepsilon}) = \mathrm{o}(\bar{F}(x)) = \mathrm{o}(\mathbb{P}(XY_1 > x)).
$$

Substituting this into (5.5) yields that (5.1) holds for $n = m + 1$.

By the mathematical induction device we conclude that (5.1) holds for each $n = 1, 2, \ldots$. This ends the proof of Theorem 5.1. □

Remark 5.1. Consider the risk model (2.2). Going along the same line as Theorem 5.1 a similar result can be proved as

$$
\psi(x, n) \sim \sum_{k=1}^{n} \mathbb{P}\left(X \prod_{i=1}^{k-1} Y_i > x \right).
$$

Hence, all the results in Theorems 5.2 and 5.3 below can be established for model (2.2) accordingly.

Recall Lemma 3.8. Relation (5.1) shows that $\psi(x,n) \asymp \bar{F}(x)$ for each $n = 1, 2, \ldots$. More precisely, we have:

Theorem 5.2. *Let the conditions of Theorem 5.1 remain valid.*

(1) *It holds for each $n = 1, 2, \ldots$ that*

$$\bar{F}(x) \sum_{k=1}^{n} \mathbb{E}\left(\bar{F}_*\left(\prod_{i=1}^{k} Y_i^{-1}\right)\right) \lesssim \psi(x,n) \lesssim \bar{F}(x) \sum_{k=1}^{n} \mathbb{E}\left(\bar{F}^*\left(\prod_{i=1}^{k} Y_i^{-1}\right)\right);$$

(2) *If $F \in \mathrm{ERV}(-\alpha, -\beta)$ for some $0 \leqslant \alpha \leqslant \beta < \infty$, then for each $n = 1, 2, \ldots$,*

$$\bar{F}(x) \sum_{k=1}^{n} \mathbb{E}\min\left\{\prod_{i=1}^{k} Y_i^{\alpha}, \prod_{i=1}^{k} Y_i^{\beta}\right\}$$

$$\lesssim \psi(x,n) \lesssim \bar{F}(x) \sum_{k=1}^{n} \mathbb{E}\max\left\{\prod_{i=1}^{k} Y_i^{\alpha}, \prod_{i=1}^{k} Y_i^{\beta}\right\};$$

(3) *If $F \in \mathscr{R}_{-\alpha}$ for some $\alpha \geqslant 0$, then for each $n = 1, 2, \ldots$,*

$$\psi(x,n) \sim B_n \bar{F}(x), \tag{5.6}$$

where the coefficients are given by $B_n = \mathbb{E}Y^{\alpha} + (\mathbb{E}Y^{\alpha})^2 + \cdots + (\mathbb{E}Y^{\alpha})^n$.

Proof. Clearly, (3) is a natural consequence of (2), and (2) is a natural consequence of (1). What's more, (1) is also a natural consequence of Theorem 5.1 and Lemma 3.8. This ends the proof of Theorem 5.2. $\square$

We remark that result (5.6) can independently be proved from (2.8) if we apply Proposition 3 in Breiman (1965).

From (2.4) and the two theorems above we immediately obtain

Theorem 5.3. *Consider the probability that the ruin occurs exactly at year n, say $\phi(x,n)$, $n = 1, 2, \ldots$. Let the conditions of Theorem 5.1 remain valid.*

(1) *It holds for each $n = 1, 2, \ldots$ that*

$$\phi(x,n) \sim \mathbb{P}\left(X \prod_{i=1}^{n} Y_i > x\right), \tag{5.7}$$

hence that

$$\bar{F}(x)\mathbb{E}\left(\bar{F}_*\left(\prod_{i=1}^{n} Y_i^{-1}\right)\right) \lesssim \phi(x,n) \lesssim \bar{F}(x)\mathbb{E}\left(\bar{F}^*\left(\prod_{i=1}^{n} Y_i^{-1}\right)\right);$$

(2) *If $F \in \mathrm{ERV}(-\alpha, -\beta)$ for some $0 \leqslant \alpha \leqslant \beta < \infty$, then for each $n = 1, 2, \ldots$,*

$$\bar{F}(x)\mathbb{E}\min\left\{\prod_{i=1}^{n} Y_i^{\alpha}, \prod_{i=1}^{n} Y_i^{\beta}\right\} \lesssim \phi(x,n) \lesssim \bar{F}(x)\mathbb{E}\max\left\{\prod_{i=1}^{n} Y_i^{\alpha}, \prod_{i=1}^{n} Y_i^{\beta}\right\};$$

(3) *If $F \in \mathscr{R}_{-\alpha}$ for some $\alpha \geqslant 0$, then for each $n = 1, 2, \ldots,$*

$$\phi(x, n) \sim \bar{F}(x)(\mathbb{E}Y^{\alpha})^n.$$

Proof. Analogously to above, it suffices to prove (1). Actually, by Lemma 3.8 and Theorem 5.1, one easily verifies that

$$\liminf_{x \to \infty} \frac{\psi(x, n)}{\psi(x, n - 1)} > 1.$$

So (5.7) is proved by substituting (5.1) into (2.4). This ends the proof of Theorem 5.3. □

6. Approximation (2): financial risk dominates insurance risk

In this section, we consider some results for the inverse case where the tail of the financial risk Y is heavier than that of the insurance risk X. The following theorem assumes that the d.f. of $\log Y$ belongs to the class $\mathscr{S}(\alpha)$ for some $\alpha > 0$. From the discussions in Cline (1986, Section 2) it is easy to construct some parallel sufficient and/or necessary conditions for the d.f. G of the r.v. Y to meet this requirement.

Theorem 6.1. *Suppose that the assumptions P_1, P_2 and P_3 hold simultaneously. If the d.f. of the r.v. $\log Y$ belongs to the class $\mathscr{S}(\alpha)$ for some $\alpha > 0$ and $\lim_{x \to \infty} \bar{F}(x)/ \bar{G}(x) = \theta \in [0, \infty)$ exists, then it holds for each $n = 1, 2, \ldots$ that*

$$\psi(x, n) \sim C_n \bar{G}(x), \tag{6.1}$$

where the coefficients C_n satisfy the recurrence equation

$$C_0 = 0, \quad C_n = \mathbb{E}(X_n + V_{n-1})_+^{\alpha} + (\theta + C_{n-1})\mathbb{E}Y^{\alpha}, \quad n = 1, 2, \ldots.$$

Proof. We prove Theorem 6.1 by the mathematical induction device. Because the d.f. of $\log Y$ belongs to the class $\mathscr{S}(\alpha)$, we conclude from the definitions of the classes $\mathscr{S}(\alpha)$ and $\mathscr{R}_{-\alpha}$ that the d.f. G of the r.v. Y belongs to the class $\mathscr{R}_{-\alpha}$ and that $\mathbb{E}Y^{\alpha} < \infty$. By (2.8) we have, for $x > 1$,

$$\psi(x, 1) = \mathbb{P}(V_1 > x) = \mathbb{P}(\log X_+ + \log Y > \log x),$$

where we have used a convention that $\log 0 = -\infty$. Since

$$\lim_{x \to \infty} \frac{\mathbb{P}(\log X_+ > \log x)}{\mathbb{P}(\log Y > \log x)} = \lim_{x \to \infty} \frac{\bar{F}(x)}{\bar{G}(x)} = \theta \in [0, \infty)$$

by Lemma 3.1,

$$\psi(x, 1) \sim (\mathbb{E}\exp\{\alpha \log X_+\} + \theta\mathbb{E}\exp\{\alpha \log Y\})\mathbb{P}(\log Y > \log x)$$

$$= (\mathbb{E}X_+^{\alpha} + \theta\mathbb{E}Y^{\alpha})\bar{G}(x). \tag{6.2}$$

This shows that the asymptotic result (6.1) holds for $n = 1$.

Now we assume by induction that relation (6.1) holds for $n = m$. By the closure property of the class $\mathscr{S}(\alpha)$ under tail equivalence we conclude that the d.f. of $\log V_m$ belongs to the class $\mathscr{S}(\alpha)$, hence that the d.f. of V_m belongs to the class $\mathscr{R}_{-\alpha}$ and $\mathbb{E} V_m^\alpha < \infty$. By (3.2) with $\gamma = 0$ and (6.1) with $n = m$, we obtain

$$\mathbb{P}(X_{m+1} + V_m > x) \sim (\theta + C_m)\bar{G}(x). \tag{6.3}$$

From (2.8) and (3.2) once again, we derive

$$\begin{aligned}
\psi(x, m+1) &= \mathbb{P}(V_{m+1} > x) \\
&= \mathbb{P}(\log Y_{m+1} + \log(X_{m+1} + V_m)_+ > \log x) \\
&\sim \mathbb{E} \exp\{\alpha \log (X_{m+1} + V_m)_+\} \mathbb{P}(\log Y_{m+1} > \log x) \\
&\quad + \mathbb{E} \exp\{\alpha \log Y_{m+1}\} \mathbb{P}(\log (X_{m+1} + V_m)_+ > \log x) \\
&= \mathbb{E}(X_{m+1} + V_m)_+^\alpha \bar{G}(x) + \mathbb{E} Y^\alpha \mathbb{P}(X_{m+1} + V_m > x). \tag{6.4}
\end{aligned}$$

Substituting (6.3) into (6.4) yields that (6.1) holds for $n = m + 1$.

By the mathematical induction device we conclude that (6.1) holds for each $n = 1, 2, \ldots$. This ends the proof of Theorem 6.1. □

Clearly, the coefficients C_n, $n = 1, 2, \ldots$, in (6.1) can be rewritten as

$$C_0 = 0, \quad C_n = \mathbb{E} \left(\frac{V_n}{Y_n} \right)^\alpha + (\theta + C_{n-1})\mathbb{E} Y^\alpha, \quad n = 1, 2, \ldots.$$

Therefore, in the special case where $\theta = 0$, (6.1) holds with the coefficients C_n given by

$$C_0 = 0, \quad C_n = \sum_{k=1}^{n} \mathbb{E} \left(\frac{V_k}{Y_k} \right)^\alpha (\mathbb{E} Y^\alpha)^{n-k}, \quad n = 1, 2, \ldots.$$

Theorem 6.1 gives a taste that the ruin probability is mainly determined by the financial risk for the present case where the tail of the financial risk is heavier than that of the insurance risk. The disadvantage is that the coefficients C_n, $n = 1, 2, \ldots$, are quite involved. The following is a more concrete example for the present situation.

Theorem 6.2. *Suppose that assumptions P_1, P_2 and P_3 hold simultaneously, and that the inflation coefficient $B = Y^{-1}$ has a density function given by*

$$b(t) = \begin{cases} ct^\alpha, & 0 < t < t_0, \\ 0 & otherwise, \end{cases} \tag{6.5}$$

where $\alpha \geqslant 0$, $t_0 > 0$ and $c = (\alpha + 1)t_0^{-\alpha-1}$. If $\mathbb{E} X_+^{\alpha+1} < \infty$, then it holds for each $n = 1, 2, \ldots$ that

$$\psi(x, n) \sim \frac{c^n \mathbb{E} X_+^{\alpha+1}}{(\alpha + 1)(n - 1)!} \frac{\ln^{n-1} x}{x^{\alpha+1}}. \tag{6.6}$$

Proof. Analogously to the proof of Theorem 6.1, for $x > 0$, by (2.8) we derive

$$\psi(x,1) = \mathbb{P}(V_1 > x) = \int_0^{t_0} \bar{F}(xt)b(t)\,\mathrm{d}t \sim \frac{c\mathbb{E}X_+^{\alpha+1}}{\alpha+1}\frac{1}{x^{\alpha+1}}.$$

This proves that (6.6) holds for $n = 1$.

Now we assume by induction that (6.6) holds for $n = m$. By (2.8) and (6.5) we have

$$\psi(x,m+1) = \mathbb{P}(V_{m+1} > x)$$

$$= \int_0^{t_0} \mathbb{P}(X_{m+1} + V_m > xt)b(t)\,\mathrm{d}t$$

$$= \frac{c}{x^{\alpha+1}}\int_0^{t_0 x} \mathbb{P}(X_{m+1} + V_m > v)v^{\alpha}\,\mathrm{d}v. \tag{6.7}$$

Since the condition $\mathbb{E}X_+^{\alpha+1} < \infty$ implies $\mathbb{P}(X > x) = \mathrm{o}(x^{-(\alpha+1)})$, it holds that $\mathbb{P}(X > x) = \mathrm{o}(\mathbb{P}(V_m > x))$, hence that

$$\mathbb{P}(X_{m+1} + V_m > x) \sim \mathbb{P}(V_m > x) \sim \frac{c^m \mathbb{E}X_+^{\alpha+1}}{(\alpha+1)(m-1)!}\frac{\ln^{m-1}x}{x^{\alpha+1}}, \tag{6.8}$$

where we have used result (3.3) with $\gamma = 0$; see also Embrechts et al. (1979, Proposition 1). Applying (6.8) we derive

$$\lim_{x\to\infty}\frac{\int_0^{t_0 x}\mathbb{P}(X_{m+1} + V_m > v)v^{\alpha}\,\mathrm{d}v}{\ln^m x} = \lim_{x\to\infty}\frac{t_0\mathbb{P}(X_{m+1} + V_m > t_0 x)(t_0 x)^{\alpha}}{mx^{-1}\ln^{m-1}x}$$

$$= \frac{c^m\mathbb{E}X_+^{\alpha+1}}{(\alpha+1)m!}.$$

Substituting this into (6.7) yields that (6.6) holds for $n = m + 1$.

By the mathematical induction device we conclude that (6.6) holds for each $n = 1, 2, \ldots$. This ends the proof of Theorem 6.2. $\quad\square$

We remark that condition (6.5) on the r.v. B is equivalent to a direct condition on the d.f. G of the r.v. Y that

$$\bar{G}(x) = \begin{cases} (t_0 x)^{-\alpha-1}, & x > t_0^{-1}, \\ 1 & \text{otherwise.} \end{cases} \tag{6.9}$$

Hence, the asymptotic relation (6.6) reads

$$\psi(x,n) \sim D_n\bar{G}(x)\ln^{n-1}x \tag{6.10}$$

with

$$D_n = \frac{c^{n-1}}{(n-1)!}\mathbb{E}X_+^{\alpha+1}, \quad n = 1, 2, \ldots.$$

Clearly, (6.9), together with $\mathbb{E}X_+^{\alpha+1} < \infty$, indicates that

$$G \in \mathscr{R} \quad \text{and} \quad \bar{F}(x) = \mathrm{o}(\bar{G}(x)). \tag{6.11}$$

It is worthwhile mentioning that, although the assumptions on the d.f.'s of the two risks X and Y in Theorems 6.1 and 6.2 can be unified into the one as (6.11), the asymptotic results for $\psi(x, n)$, given, respectively, by (6.1) with $\theta = 0$ and (6.10), differ from each other by a significant factor $\ln^{n-1} x$, $n = 1, 2, \ldots$.

7. Numerical examples

This section is devoted to the numerical analysis on the finite time ruin probability $\psi(x, n)$ for some cases. Recall model (2.1) introduced in Section 2, where the generic r.v.'s of the insurance and financial risks are X and Y, which are distributed by F and G, respectively. In this section we assume that $X = Z - c$, where the r.v. Z, with a tail probability $\mathbb{P}(Z > x) = x^{-\alpha}$ for $x > 1$ and $\alpha > 1$, is interpreted as the generic size of the total claim amount within 1 year, and $c > \mathbb{E}Z = \alpha/(\alpha - 1)$ is the total constant incoming premium within 1 year. Hence,

$$\bar{F}(x) = (x + c)^{-\alpha} \quad \text{for } x > 1 - c \text{ and } \alpha > 1.$$

We also assume that the d.f. G satisfies

$$\bar{G}(x) = (l/x)^{\beta} \quad \text{for } x > l \text{ and } \beta > 1.$$

1. First we consider the case $\beta > \alpha$. Theorem 5.2(3), implies that the asymptotic relation

$$\psi(x, 1) \sim \mathbb{E}Y^{\alpha}\mathbb{P}(X > x) \tag{7.1}$$

holds. Now we analyze the accuracy of this asymptotic relation. For this purpose we denote the ratio of the two sides of (7.1) by

$$A(x) = \frac{\psi(x, 1)}{\mathbb{E}Y^{\alpha}\bar{F}(x)}.$$

Recalling (2.8), one easily proves that, for arbitrarily fixed $M > 0$,

$$\mathbb{P}(ZY > x + cM) - \mathbb{P}(Y > M) \leqslant \psi(x, 1) \leqslant \mathbb{P}(ZY > x).$$

Hence, we obtain two-sided bounds for the ratio $A(x)$ as

$$A^-(x) \leqslant A(x) \leqslant A^+(x), \quad x > l \tag{7.2}$$

with

$$A^+(x) = \left(\frac{x + c}{x}\right)^{\alpha} \left(1 - \frac{\alpha}{\beta}\left(\frac{l}{x}\right)^{\beta - \alpha}\right)$$

and

$$A^-(x) = \left(\frac{x + c}{x + cM}\right)^{\alpha} \left(1 - \frac{\alpha}{\beta}\left(\frac{l}{x + cM}\right)^{\beta - \alpha}\right) - \frac{(\beta - \alpha)l^{\beta - \alpha}(x + c)^{\alpha}}{\beta M^{\beta}}.$$

We can also substitute $M = x/\ln x$ to the above to obtain a more explicit form of $A^-(x)$. The bounds in (7.2) indicate that the accuracy of (7.1) depends on the difference

Table 1
$c = 2$, $\alpha = 2$, $l = 0.9$

β	$A(100)$	$A(1000)$	$A(10000)$
5	0.986976	0.998611	0.999860
10	0.998905	0.999886	0.999988

between the two parameters α and β. Direct computation by "Maple" package gives the numerical results in Table 1.

2. Next we turn to the inverse case $\beta < \alpha$. Again from (2.8) it holds that

$$\psi(x,1) = l^{\beta}\beta x^{-\beta}(J(\alpha,\beta) - J_x(\alpha,\beta)), \tag{7.3}$$

where

$$J(\alpha,\beta) = \int_0^{\infty} t^{-\beta-1}\left(c + \frac{1}{t}\right)^{-\alpha} dt \quad \text{and} \quad J_x(\alpha,\beta) = \int_0^{l/x} t^{-\beta-1}\left(c + \frac{1}{t}\right)^{-\alpha} dt.$$

It follows that

$$\psi(x,1) \sim l^{\beta}\beta x^{-\beta}J(\alpha,\beta). \tag{7.4}$$

As carried out in the previous case, we denote the ratio of the two sides of (7.4) by

$$B(x) = \frac{\psi(x,1)}{l^{\beta}\beta x^{-\beta}J(\alpha,\beta)}.$$

Clearly,

$$J(\alpha,\beta) \geqslant \int_0^{1/c} t^{\alpha-\beta-1}\frac{1}{2^{\alpha}} dt = \frac{c^{\beta-\alpha}}{2^{\alpha}(\alpha-\beta)}$$

and

$$J_x(\alpha,\beta) \leqslant \int_0^{l/x} t^{\alpha-\beta-1} dt = \frac{l^{\alpha-\beta}}{x^{\alpha-\beta}(\alpha-\beta)}.$$

Substituting these inequalities into (7.3) yields that the relation

$$B^-(x) \leqslant B(x) \leqslant B^+(x) = 1, \quad x > l \tag{7.5}$$

holds with

$$B^-(x) = \max\left\{1 - 2^{\alpha}\left(\frac{lc}{x}\right)^{\alpha-\beta}, 0\right\}.$$

The two-sided inequality (7.5) gives the accuracy of the asymptotic relation (7.4), indicating that the accuracy depends on the difference between the two parameters α and β. Several numerical results for the lower bound $B^-(x)$ are given in Table 2.

3. Finally we consider the more flexible case for $n = 1, 2, \ldots$ and $\beta > \alpha$. In this case Theorem 5.2(3), implies that the asymptotic relation

$$\psi(x,n) \sim \bar{F}(x)\sum_{k=1}^{n}(\mathbb{E}Y^{\alpha})^k \tag{7.6}$$

Table 2
$c = 2$, $\beta = 9$, $l = 0.9$

α	$B^-(100)$	$B^-(1000)$	$B^-(10000)$
10	0	0	0.81568
11	0.33648	0.9933648	0.999933648

Table 3
$n = 5$

β	$\Psi(\beta, n)$	$\hat{\Psi}(\beta, n)$	$R(\beta, n)$
3	5.7615×10^{-3}	1.367555×10^{-2}	0.421299
4	2.3001×10^{-3}	2.551048×10^{-3}	0.901629
5	1.36185×10^{-3}	1.29165×10^{-3}	1.05434218

Table 4
$n = 10$

β	$\Psi(\beta, n)$	$\hat{\Psi}(\beta, n)$	$R(\beta, n)$
5	6.27925×10^{-3}	7.083498×10^{-3}	0.886461
6	3.4815×10^{-3}	3.2648×10^{-3}	1.06635
7	2.316×10^{-3}	2.04707×10^{-3}	1.131373

Table 5
$n = 20$

β	$\Psi(\beta, n)$	$\hat{\Psi}(\beta, n)$	$R(\beta, n)$
6	2.01435×10^{-2}	2.615386×10^{-2}	0.7701192
7	9.53×10^{-3}	9.245921×10^{-3}	1.03072
8	5.4345×10^{-3}	4.75037×10^{-3}	1.1440

holds. We specify the other parameters as $c = 2$, $l = 0.9$, $\alpha = 2$, $x = 100$, and, in order to give prominence to the parameters β and n, we rewrite the two sides of (7.6) as $\Psi(\beta, n)$ and $\hat{\Psi}(\beta, n)$, respectively. We examine the accuracy of (7.6) by varying the values of the parameters β and n. To this end, we further designate the ratio of the two sides of (7.6) by

$$R(\beta, n) = \frac{\Psi(\beta, n)}{\hat{\Psi}(\beta, n)}.$$

In Tables 3–6 the values of $\Psi(\beta, n)$ are obtained by the Monte-Carlo simulations.

Table 6
$n = 40$

β	$\Psi(\beta, n)$	$\hat{\Psi}(\beta, n)$	$R(\beta, n)$
8	2.52975×10^{-2}	2.689168×10^{-2}	0.94071084
9	1.1807×10^{-2}	9.83869×10^{-3}	1.2
10	6.5385×10^{-3}	5.01087×10^{-3}	1.304861

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References

Baltrünas, A., 1999. The probability of ruin in finite time. Liet. Mat. Rink. 39 (3), 304–309 (translation in Lithuanian Math. J. 39(3), 240–244).

Bingham, N.H., Goldie, C.M., Teugels, J.L., 1987. Regular Variation. Cambridge University Press, Cambridge.

Breiman, L., 1965. On some limit theorems similar to the arc-sin law. Theory Probab. Appl. 10, 323–331.

Cai, J., 2002. Ruin probabilities with dependent rates of interest. J. Appl. Probab. 39 (2), 312–323.

Chistyakov, V.P., 1964. A theorem on sums of independent positive random variables and its applications to branching random processes. Theory Probab. Appl. 9, 640–648.

Chover, J., Ney, P., Wainger, S., 1973a. Functions of probability measures. J. Analyse Math. 26, 255–302.

Chover, J., Ney, P., Wainger, S., 1973b. Degeneracy properties of subcritical branching processes. Ann. Probab. 1, 663–673.

Cline, D.B.H., 1986. Convolution tails, product tails and domains of attraction. Probab. Theory Related Fields 72 (4), 529–557.

Cline, D.B.H., 1987. Convolutions of distributions with exponential and subexponential tails. J. Austral. Math. Soc. Ser. A 43, 347–365.

Cline, D.B.H., Samorodnitsky, G., 1994. Subexponentiality of the product of independent random variables. Stochastic Process. Appl. 49 (1), 75–98.

Daley, D.J., 2001. The moment index of minima. J. Appl. Probab. 38A, 33–36.

Embrechts, P., Goldie, C.M., 1980. On closure and factorization properties of subexponential and related distributions. J. Austral. Math. Soc. Ser. A 29 (2), 243–256.

Embrechts, P., Veraverbeke, N., 1982. Estimates for the probability of ruin with special emphasis on the possibility of large claims. Insurance Math. Econom. 1 (1), 55–72.

Embrechts, P., Goldie, C.M., Veraverbeke, N., 1979. Subexponentiality and infinite divisibility. Z. Wahrsch. verw. Geb. 49, 335–347.

Embrechts, P., Klüppelberg, C., Mikosch, T., 1997. Modelling Extremal Events for Insurance and Finance. Springer, Berlin.

Goldie, C.M., 1991. Implicit renewal theory and tails of solutions of random equations. Ann. Appl. Probab. 1 (1), 126–166.

Harrison, J.M., 1977. Ruin problems with compounding assets. Stochastic Process. Appl. 5 (1), 67–79.

Kalashnikov, V., Norberg, R., 2002. Power tailed ruin probabilities in the presence of risky investments. Stochastic Process. Appl. 98 (2), 211–228.

Klüppelberg, C., 1988. Subexponential distributions and integrated tails. J. Appl. Probab. 25 (1), 132–141.

Lindley, D.V., 1952. The theory of queues with a single server. Proc. Camb. Philos. Soc. 48, 277–289.

Malinovskii, V.K., 2000. Probabilities of ruin when the safety loading tends to zero. Adv. Appl. Probab. 32 (3), 885–923.

Mikosch, T., Nagaev, A.V., 1998. Large deviations of heavy-tailed sums with applications in insurance. Extremes 1 (1), 81–110.

Norberg, R., 1999. Ruin problems with assets and liabilities of diffusion type. Stochastic Process. Appl. 81 (2), 255–269.

Nyrhinen, H., 1999. On the ruin probabilities in a general economic environment. Stochastic Process. Appl. 83 (2), 319–330.

Nyrhinen, H., 2001. Finite and infinite time ruin probabilities in a stochastic economic environment. Stochastic Process. Appl. 92 (2), 265–285.

Paulsen, J., 1998. Ruin theory with compounding assets—a survey. Insurance Math. Econom. 22 (1), 3–16.

Rogozin, B.A., 2000. On the constant in the definition of subexponential distributions. Theory Probab. Appl. 44 (2), 409–412.

Rogozin, B.A., Sgibnev, M.S., 1999. Banach algebras of measures on the line with given asymptotics of distributions at infinity. Siberian Math. J. 40, 565–576.

Seneta, E., 1976. Regularly Varying Functions. Lecture Notes in Mathematics, Vol. 508. Springer, New York.

Tang, Q.H., Tsitsiashvili, G.Sh., 2003. Finite and infinite time ruin probabilities in the presence of stochastic investment return with emphasis on rapid and dominated variations. Ann. Appl. Probab., revised.

Tang, Q.H., Yan, J.A., 2002. A sharp inequality for the tail probabilities of sums of i.i.d. r.v.'s with dominatedly varying tails. Science in China (Series A) 45 (8), 1006–1011.

Veraverbeke, N., 1977. Asymptotic behaviour of Wiener–Hopf factors of a random walk. Stochastic Process. Appl. 5, 27–37.