A dynamical model of market under- and overreaction

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In this article we introduce a dynamical model of securities prices based on a particular notion of "irrational exuberance" and "market fear" generated by the under- or overreaction of market participants. The addition of a small amount of irrational behavior (behavioral noise) allows us to reproduce realistic distribution of price returns that would be absent in a perfectly rational world. Our model of asset dynamics is based on cognitive biases of market participants and their reaction to price momentum. The resulting price dynamics yields fat-tailed distributions of returns. When we apply our model to the pricing of option contracts we obtain implied volatility skews and smiles generated by the irrational behavior of market participants.

1 Introduction

In this article we introduce a dynamical model of securities prices based on a specific (and restrictive) notion of "irrational exuberance" and "market fear" generated by the under- or overreaction of market participants.[1] The under- or overreaction generates price momentum, particularly for short time horizons. The addition of a small amount of behavioral noise allows us to reproduce the basic characteristics of empirically observed distributions of price returns. These characteristics are absent in perfectly rational pricing models. Importantly, while interesting from the perspective of understanding securities price dynamics, these distributional properties and the mechanisms that generate them have significant implications for derivative pricing models.

Models of securities prices are key to pricing derivatives on securities such as options. Among the few options pricing models that have been truly successful in both theory and practice is the valuation model derived by Black and Scholes (1973). In the ideal Black–Scholes (BS) framework implied volatility would be constant across option maturities and exercise price (or moneyness). In practice, these conditions are violated more often than not. The implied volatility of

The analysis and conclusions set forth are those of the authors. Citigroup is not responsible for any statement or conclusion herein, and opinions or theories presented herein do not necessarily reflect the position of the institution. The authors would like to thank Sean C. Keenan, Yimin Shih, and the editor and anonymous referee of the Journal of Risk for their helpful comments and suggestions.

traded options exhibits persistent biases commonly referred to as skews, smirks and smiles (Bakshi, Cao and Chen, 1998; Derman, 1999); and Blynt and Uglum, 1999). Plausible explanations of this phenomenon include asymmetries caused by a combination of fat and thin tails of the distribution of stock prices and changes in state variables such as interest rates and volatility (Lipton, 2002). Furthermore, a plot of daily implied volatility with fixed time-to-maturity and exercise price looks highly irregular. These issues have become central to the debate over alternative models for the underlying security, and are the focus of this article.

Over the years, three basic approaches have been used to extend the BS model (see, for example, Babbel and Merril, 1996, and references therein). These approaches relax one or more of the simplifying assumptions made by Black and Scholes for analytical convenience, providing more realistic extensions to the original model. Each one of these approaches has its advantages and problems. The first approach makes volatility a function of the underlying security's price (local volatility models). Thus, volatility only fluctuates because price does. In practice, price and volatility are not perfectly correlated, which makes this approach questionable. The second approach introduces additional stochastic processes for state variables such as volatility and interest rates. Because this approach requires only a handful of phenomenological assumptions for the dynamics of the underlying state variables, it has been adopted by most academics and practitioners (Lipton, 2002). The third approach includes models that replace the standard Wiener random walk process for security prices with more realistic processes such as jump processes, Levy flights, fractal walks or similarly complex stochastic processes (Bouchard and Sornette, 1994; Bibby and Sorensen, 2001; Barndorff-Nielsen and Sheppard, 2001; Antonuccio and Proebsting, 2003; Dragulescu and Yakovenko, 2002; and Sornette, Malevergne and Muzy, 2003). Because of their mathematical complexity none of these radical models has been firmly established to date. The model we introduce here combines the second and third approaches using phenomenological assumptions about the reaction of market participants to price changes.

One of the fundamental tenets of all of these models is the impossibility of sustainable risk-free profits in efficient markets. This has lead researchers to postulate that prices are random walks with no memory or concept of price momentum. Although this extreme form of the market efficiency hypothesis is intuitively appealing and cannot be rejected under certain types of statistical tests, it is rarely spelled out carefully when it is used to describe the dynamics of security prices over extremely short periods of time. Unfortunately, real financial markets do not resemble the ones described by the ideal random walk models. In practice, investors and arbitrageurs are limited in their ability to restore price changes instantaneously, as explained in Fama (1997), O'Hara (1998), Thaler (1999), Shefrin (1999) and Schiffler (2000) and references therein.

Furthermore, most pricing models are based on the additional assumption that market participants are perfectly "rational" in the sense that they do not follow price trends because prices are unpredictable in nature. Even if security prices are set in the aggregate under quasi-equilibrium conditions that may eliminate sustainable arbitrage opportunities, the equilibrium may not necessarily be "rational" in this restricted sense. As a practical example, recall the recent "irrational exuberance" period of the NASDAQ, when the market value of some Internet and telecom firms was exceedingly higher than their intrinsic value. Investors kept buying these hot stocks driven by price momentum despite the lack of sustainable earnings (or any earnings at all) and the poor business models of those firms. The "tronic" boom and the "Nifty Fifty" craze of the 1960s and 1970s showed similar speculative behavior (see Malkiel, 1981).

Unfortunately, models based on standard random walks with Wiener processes fail to capture not only these basic issues but also many other crucial aspects of real price fluctuations – for example, the possibility of market crashes or extraordinary returns. More importantly, under the standard assumption of random walks with normally distributed returns, severe crashes are absent as the probability of extreme events is embarrassingly small. However, from the practical point of view the most interesting phenomena are precisely these extreme events such as bubbles and crashes – for which relatively little data are available because few of these events occur. Catastrophic events aside, the frequency of extreme stock returns is systematically larger than the extrapolation based on small and moderate events would suggest under a normal distribution of returns (Bouchaud, Cizeau, Leloux, and Potters, 1999; and Johansen and Sornette, 2002). There is, therefore, a need for better theoretical description of crashes and extreme events to supplement the shortage of data.

Because of the important role of human behavior in determining the dynamics of securities prices, we incorporate a specific form of irrational behavior into our model consistent with typical trading patterns of investors (Thaler, 1999; Schiller, 2000; Hirshleifer, 2001; and Baker and Stein, 2002). Our model includes cognitive biases of market participants based on price momentum and the level of equity prices. The resulting price dynamics yields fat-tailed distributions of returns and allows for severe price changes such as crashes and bubbles. Finally, when we apply our model to the pricing of option contracts we obtain implied volatility skews and smiles generated by the irrational behavior of market participants.

The rest of the article is organized as follows. In Section 2 we analyze the evolution of security prices introducing a dynamical approach to irrational trading that leads to fat-tailed distributions of returns. Section 3 focuses on hedging strategies. Section 4 discusses the pricing of options on the underlying security. Our results are discussed in Section 5.

2 A model of irrational exuberance and market fear

In principle, an ideal model of the dynamics of security prices must include a market-microstructure description of supply-demand dynamics and investors' preferences that lead to changes in returns. Here we simplify the problem by including a phenomenological description of the basic features of the dynamics of security prices when market participants under- or overreact to price trends, forcing the security's return to randomly deviate from its fundamental value.

At the intuitive level the economic interpretation for our particular choice of security prices and returns as state variables driven by stochastic feedback is as follows. Investors have some prior views about the company in question and some idiosyncratic responsiveness to new information. If they are too conservative when they receive news about the company, they may tend not to re-evaluate their view as much as the information warrants. This behavior gives rise to market underreaction to earnings announcements which impacts on stock returns. We describe this situation with a (dissipative) mean reversion process for the security's return. While market underreaction to news is well documented empirically it may not describe some important instances of price momentum such as bubbles, crashes and extreme events (Shleifer, 2000). During price bubbles and crashes investors tend to react to past price trends as opposed to actual news. For example, bullish investors may attach themselves to an over-optimistic view of the company during upward price trends. Similarly, bearish investors may attach themselves to an over-pessimistic view during a downward price trend. In doing so, they discount the possibility that the bullish (bearish) price trends are the result of behavioral dynamics rather than a change in the future prospects of the company. This gives rise to the market overreaction to upward and downward price moves that can create positive feedback and, therefore, potentially unstable dynamics for both stock prices and excess returns (Shliefer, 2000). We describe this situation assuming that the security's return can deviate temporarily from its fundamental value through random feedback.

We begin by describing a model for security prices. We seek to quantify several of the effects of using a model that includes a reasonable and behaviorally motivated form of price momentum. Let $\chi = \log(S/S_0)$ where S is the security price and S_0 is a reference price. The standard assumption is that the security's return follows a Wiener process (or Brownian motion) with normally distributed independent increments. That is,

$$\mathrm{d}\chi = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma \mathrm{d}Z \tag{1}$$

Here the random variable Z follows a Wiener process, μ is the security's growth rate and σ its volatility.

The model described in the following reduces to equation (1) when all market participants are rational and information is incorporated into prices instantaneously. Technically, there are two random variables we are interested in, the log-price and the excess return of the security. These variables are described with a stochastic dynamical system that has embedded in it the concept of price momentum (bullish–bearish directional change). The particular model introduced here is based on a mean-reverting Ornstein–Uhlenbeck process with multiplicative and additive noises that generate random feedback:

Log-price adjustment $$d\chi = \left(\mu + \xi - \frac{\sigma^2}{2}\right) dt + \sigma_0 dZ_0 \tag{2}$$

Excess return adjustment $$d\xi = -\xi\left(\theta dt + \varepsilon dZ_1\right) + \eta dZ_2 \tag{3}$$

Here ξ is the random excess return above (below) the expected return μ. The parameter θ is the sensitivity of market participants to price return discrepancies (which leads to mean reversion), η is the magnitude of random changes in excess return and ε is the strength of the force of price momentum, ie, the herd-like "irrational behavior". For simplicity, parameters μ, σ_0, θ, ε and η are assumed constant, and $\theta \geq 0$ for a stable mean reverting process. For analytical convenience we also assume that the random variables Z_0, Z_1 and Z_2 follow independent Wiener processes.

The effective volatility of the security σ is a function of the parameters of the model as shown in equation (8). For perfectly efficient and rational markets the effective volatility is

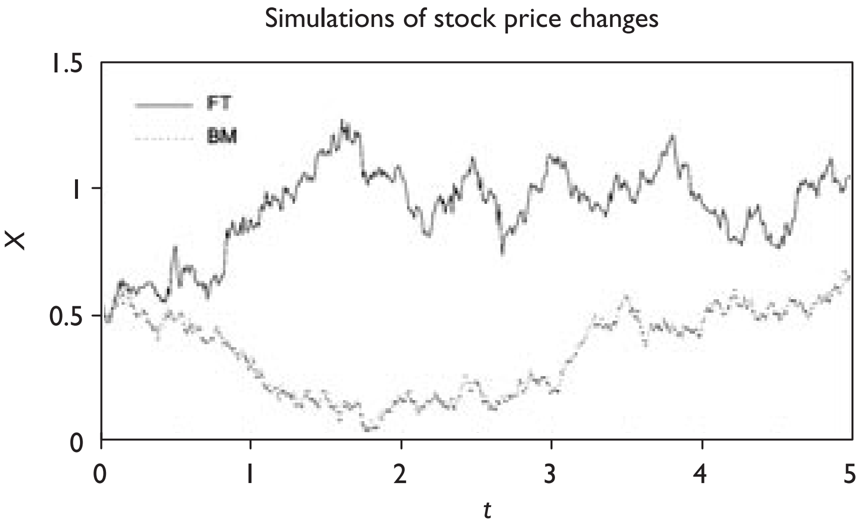
$$\sigma(\varepsilon = 0) = \sqrt{(\eta/\theta)^2 + \sigma_0^2}$$

where σ_0 is the volatility of the standard Brownian motion. The introduction of the additional parameter σ_0 allows us to model any arbitrary combination of rational and irrational dynamics. For example, for $\theta = \varepsilon = \eta = 0$ the stochastic term in equation (2) comes from Z_0 only, producing a standard Brownian motion.

The additional term $\varepsilon\xi dZ_1$ in equation (3) may seem a minor correction to readers familiar with the standard Ornstein–Uhlenbeck process, but in fact, it changes the fundamental structure of the familiar mean reverting process completely. This additional level of price randomness introduces a concept absent from the standard rational investor framework: the possibility of extreme price changes, bubbles and crashes caused by irrational behavior. Although similar effects are obtained from nonlinear and/or stochastic volatility models, note that our model of market over/underreaction is fundamentally different from the models of stochastic volatility and market feedback reported in the literature (Dragulescu and Yakovenko, 2002, and Sornette and Andersen, 2002). The model described in equations (2) and (3) is characterized by strong inertial effects caused by random positive feedback, which lead to extreme changes in price returns, as opposed to extreme price changes caused by random deviations of the stock volatility.

To illustrate this point Figure 1 shows a random path obtained from equations (2) and (3) and a Brownian motion (random walk) obtained from the standard Wiener process in equation (1). This is obtained from equations (2) and (3) in

FIGURE 1 Typical realization of equations (2) and (3) for a five-year period for daily values, μ = 0.00028, σ_0 = 0, θ = 0.3, η = 0.0032, ε = 0.3 (FT) and the standard Wiener process equation (1) with $\sigma = \eta/\theta$ (BM).



the limit $\theta = \varepsilon = \eta = 0$ with volatility σ_0. (Note that the same Brownian motion limit is obtained when the relaxation process is extremely fast. That is, for $\varepsilon = 0$, $\sigma_0 = 0$ and $\theta \to \infty$ with constant volatility $\sigma = \eta/\theta$). The fat-tailed stochastic process in equations (2) and (3) (upper curve labeled FT) and the standard Brownian motion (lower curve labeled BM) may seem similar but there are some noticeable differences caused by extreme events. More precisely, notice the sharp drop in the stock price a little after $t = 1.5$ and 2.5 years. These extreme events are similar to market crashes although there are not actual price "jumps" but price moves with very steep trends.

Although equations (2) and (3) may appear somewhat unusual, they have an insightful economic interpretation. For stable market conditions with rational participants ($\varepsilon = 0$) one might believe that markets prevent price returns from wandering too far from the expected return μ. The price adjustment process is such that if the actual return is less than the expected return μ, the adjustment may consist of some buyers rising their bids for the security in anticipation of a future price increase. In contrast, if the actual return is higher than the expected return μ, the adjustment may consist of some sellers increasing their offers of the security in anticipation of a downward price correction. Thus, for stable rational markets ($\varepsilon = 0$), the price return $\nu = \mu + \xi$ fluctuates, but it will revert to a mean return μ. The speed of price corrections is driven by the sensitivity of market participants to expected return discrepancies: $\theta > 0$. In this limit, whatever errors investors make in forecasting the future are random errors rather than the result of stubborn bias towards either optimism or pessimism.

Note that in the limit of perfectly rational market participants ($\varepsilon = 0$) and ideal market efficiency ($\theta \to \infty$), prices follow a random walk in which price changes have a completely unpredictable component. That is, given a small time interval $\theta\Delta t > 1$, equation (3) yields $\xi\Delta t \approx (\eta/\theta)\Delta Z_2$, and the stochastic price adjustments in equation (2) become completely random. This limit corresponds to the adiabatic elimination of fast relaxation processes. This is the result of rational market participants incorporating all possible information into prices and quickly eliminating any profit opportunity (see Farmer and Lo, 1999). In this limit, equations (2) and (3) play a role similar to the differential equation $\mathrm{d}S/S = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}Z$ in the BS model and its extensions. We return to this point in Section 3.

When $\varepsilon \neq 0$ (no matter how small) the dynamic is completely different. In this case, market participants tend to overreact, under-react or exhibit random "irrational behavior", leading to explosive situations such as bubbles and crashes. Let us consider the price adjustment process (2) and (3) during a small time interval Δt. Although for $\theta \geq 0$ the stabilizing mean reverting mechanism for price returns is still present, when $\theta\Delta t < \theta\Delta t + \varepsilon\Delta Z_1$ market participants overreact to price return discrepancies, and when $0 \leq \theta\Delta t + \varepsilon\Delta Z_1 < \theta\Delta t$ they under-react with respect to the "rational" case $\varepsilon = 0$. In contrast, when $\theta\Delta t + \varepsilon\Delta Z_1 < 0$ market participants exhibit irrational behavior, and the price adjustment process may be unstable during a small time interval Δt. More precisely, when $\theta\Delta t + \varepsilon\Delta Z_1 < 0$ the price adjustment process is such that if the actual return is less than the expected return μ, the adjustment may consist of some sellers actually increasing their offers of the security in anticipation of a further price decreases ("bearish" view). In contrast, if the actual return is higher than the expected return μ, the adjustment may consist of some buyers increasing their bids for the security hoping for future increases in price ("bullish" view). In simple terms, market participants struggle to find their way through the give and take between risk and return, one moment engaging in rational behavior and the next showing irrational emotional impulses. The result of this random mixture is a market that fails to perform consistently with the way perfectly rational models predict it will perform. In this case, emotions driven by cognitive difficulties destroy the self-control essential to rational decision-making. This "irrational" regime leads to situations with extreme price adjustments.

Due to the natural tendency for investors to weight losses more heavily than gains (loss aversion), one cannot expect the dynamics of irrational behavior described here with a single parameter to be valid for arbitrarily large security prices and excess returns. A more realistic model may include different values of ε for positive and negative excess returns and different security prices. When prices and returns are bigger than some psychological reference values, market participants may be asymptotically more rational ($\varepsilon \to \infty$) and may show different sensitivity parameters than the constant values required in equations (2) and (3). For simplicity, here we assume that our simplified model of irrational behavior is valid for prices and returns lower than some practical cut-off values.

In the general case $\varepsilon \neq 0$ the stochastic process followed by the log-price χ and its instantaneous excess return ξ in equations (2) and (3) can be described in terms of the probability distribution $P(\chi, \xi, t)$ in phase space of a random log-price χ_t in the interval $[\chi, \chi + d\chi]$ and random excess return ξ_t in the interval $[\xi, \xi + d\xi]$ at time t. From equations (2) and (3) the differential equation describing the evolution of the probability distribution of log-prices and excess returns is:

$$\frac{\partial P}{\partial t} + \left(\xi + \mu - \frac{\sigma^2}{2}\right)\frac{\partial P}{\partial \chi} = \frac{\partial}{\partial \xi}\left[\theta \xi P\right] + \frac{1}{2}\frac{\partial^2}{\partial \xi^2}\left[\left(\eta^2 + \varepsilon^2\xi^2\right)P\right] + \frac{\sigma_0^2}{2}\frac{\partial^2 P}{\partial \chi^2} \tag{4}$$

The analysis of the stationary distribution of returns implied from equation (4) provides a first look at the mechanics of the model and the origin of the fat-tailed distribution of returns. The stationary and homogeneous marginal distribution of the instantaneous excess return ξ is obtained by setting the time and log-price derivatives to zero:

$$\frac{\partial}{\partial \xi}\left\{\theta \xi P + \frac{1}{2}\frac{\partial}{\partial \xi}\left[\left(\eta^2 + \varepsilon^2\xi^2\right)P\right]\right\} = 0 \tag{5}$$

The solution to this equation (except for a normalization factor) is

$$P_0(\xi) = \frac{1}{\left(1 + \left(\frac{\varepsilon\xi}{\eta}\right)^2\right)^{\beta+1}}, \qquad \beta = \frac{\theta}{\varepsilon^2} \tag{6}$$

The stationary distribution of excess returns $P_0(\xi)$ is a modified t-distribution with non-integer degrees of freedom that shows heavy tails. As market participants become more rational ($\varepsilon \to 0$), the distribution is asymptotically normally distributed ($\beta \to \infty$) and security prices behave like ideal random walks with normal random increments. The role of the probability distribution in equation (6) will become clear in equations (A9) and (A10) in the appendix.

For illustration purposes the above stationary and homogeneous distribution can be compared to the average distribution of daily returns observed over an extended period of time and aggregated over all stock prices. Although, this average distribution is not equivalent to equation (6), it is a first-order approximation that shows very similar patterns for extreme returns. Figures 2 and 3 show the empirical average distributions of daily returns for the S&P500 index for the period January 1980 to February 2002, and the NASDAQ index for the period January 1985 to February 2002. Figures 2 and 3 also show the stationary distribution of instantaneous excess return in equation (6) and a normal distribution (standard Brownian motion) fitted to the central peak of the distribution. Figure 4 shows similar results for IBM stock for the period January 1980 to

FIGURE 2 Distribution of daily returns for the S&P500 index, a normal distribution and equation (6) with $\beta = 1.5$ and $\eta/\varepsilon = 0.019$.

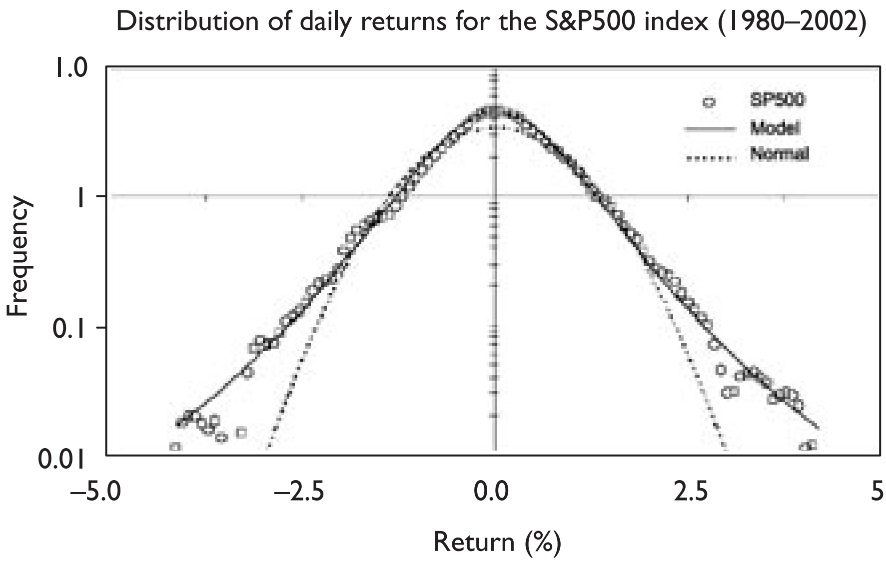
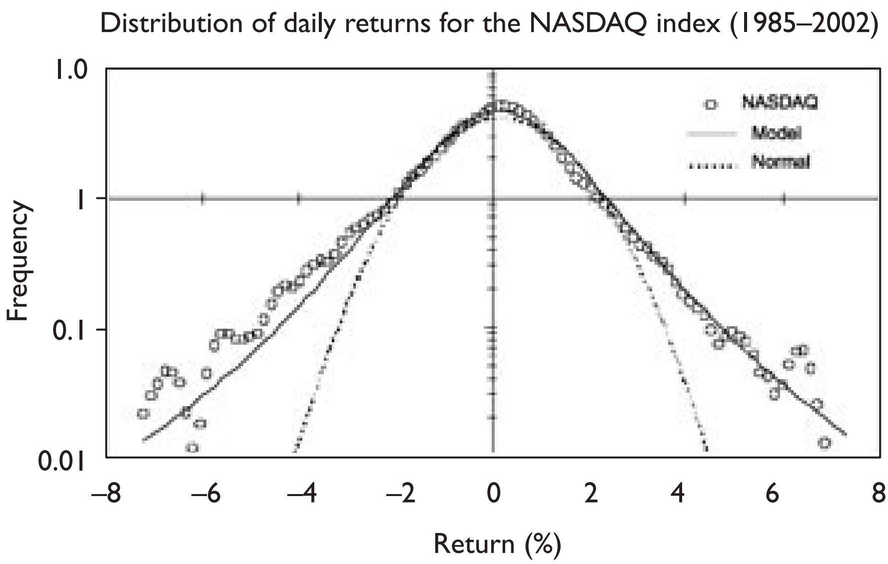


FIGURE 3 Distribution of daily returns for the NASDAQ index, a normal distribution and equation (6) with $\beta = 1.5$ and $\eta/\varepsilon = 0.024$.



February 2002. The parameter $\beta \approx 1.5$ is the same in all figures, and is in agreement with the values reported in the literature (Plerou, Gopikrishnan, Gabaix, Nunez-Amaral and Stanley, 2001). The daily-volatility parameter η/ε is consistently higher for individual securities than for well-diversified security indices. Shorter time periods show similar patterns for indices and stocks with more noise in the tails of the distributions.

Although the distribution $P_0(\xi)$ exhibits the characteristic heavy tails of the average empirical distributions, it does not represent the actual distribution of prices and returns because it is obtained by setting the log-prices and time derivatives to zero (homogeneous and stationary solution). The actual distribution is obtained from equation (4) for an arbitrary log-price χ, excess return ξ and time t. This requires several transformations and approximations that are described in the appendix. Following the derivation and notation in the appendix, the approximate solution to the marginal distribution of log-prices obtained from equation (4) is:

$$\begin{aligned} p(\chi,t) &= \int_{-\infty}^{\infty} P(\chi,\xi,t)\,\mathrm{d}\xi \\ &= \frac{1}{\sqrt{2\pi\sigma^2(t-t_0)}} \exp\left\{-\frac{\left(\chi-\chi_0-\left(\mu-\sigma^2/2\right)(t-t_0)\right)^2}{2\sigma^2(t-t_0)}\right\} \\ &\quad \times \frac{1}{A_0}\sum_{n=0}^{\infty}\frac{i^n}{n!\left(A_0\sigma\sqrt{t-t_0}\right)^n} \\ &\qquad \times \frac{\partial^n A}{\partial k^n}(0,t)H_n\left(\frac{\left(\chi-\chi_0-\left(\mu-\sigma^2/2\right)(t-t_0)\right)}{\sigma\sqrt{t-t_0}}\right) \end{aligned} \quad (7)$$

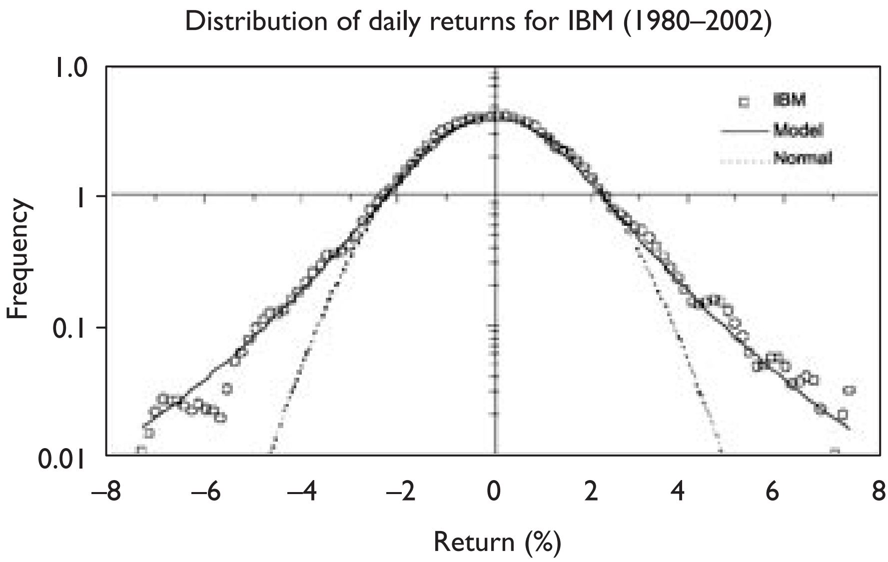
Equation (7) indicates that the bulk of the marginal distribution of log-prices resembles a normal distribution with an effective volatility given by the "irrationality" of market participants (bullish-bearish views).

$$\sigma^2 = \frac{\eta^2}{\varepsilon^4\phi(\phi+1)} + \sigma_0^2 \sim \frac{\eta^2}{\varepsilon^4\phi^2} + \sigma_0^2 \sim \frac{\eta^2}{\theta^2\left(1+\dfrac{\varepsilon^2}{2\theta}\right)^2} + \sigma_0^2 \quad (8)$$

Here $\phi = \beta + 1/2$. That is, in the limit $\varepsilon \to 0$ market participants become rational and the effective volatility approaches the value

$$\sqrt{(\eta/\theta)^2 + \sigma_0^2}$$

FIGURE 4 Distribution of daily returns for IBM stock, a normal distribution and equation (6) with $\beta = 1.5$ and $\eta/\varepsilon = 0.026$.



Also note that equations (7) and (8) indicate that the normal-like central peak of the distribution of log-prices is narrower than in the "rational" normal case as observed in practice.

One should expect the leading normal distribution term in equation (7) to be valid for a moderate range of security price changes confined to the central peak of the distribution. The tails of the distribution are driven by the higher-order corrections in equation (7), which also modify the actual return and volatility of log-prices:

$$\hat{\mu} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left[\chi(t+\Delta t) - \chi(t)\right] \approx \mu - \frac{\sigma^2}{2} \tag{9}$$

$$\begin{aligned} \hat{\sigma}^2 &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left[\left(\chi(t+\Delta t) - \chi(t)\right)^2\right] \\ &\approx \sigma^2 + \frac{1}{A_0^2 (t-t_0)} \left(\left(\frac{1}{A_0} \frac{\partial A}{\partial k} \right)^2 - \frac{1}{A_0} \frac{\partial^2 A}{\partial k^2} \right) \end{aligned} \tag{10}$$

Here we introduced $A(0, t) \approx A_0$. Equations (7)–(10), which are based on a simple model of the mechanics of market under- or overreaction, show some of the basic characteristics of the observed distribution of real stock prices. For long-term horizons the impact of the fat tail contributions diminishes, leading to a normal distribution of log-prices. This suggests that models of investors' behavior might help to explain some empirical issues that have remained unanswered for years.

3 Hedging strategies for fat-tailed distributions of returns

The exact hedging of options for the type of fat-tailed processes described here is beyond the scope of standard hedging strategies based on a tangent approximation with continuous and costless hedging. It is clear that a more meaningful concept of hedging can be generalized to minimize measures of the residual risk in the spirit of the models such as the ones described in Bouchaud and Sornette (1994), Schal (1994) and Schweizer (1995). However, given that the standard delta hedging strategy is still widely used among practitioners for hedging options, one may ask what kind of corrections can be added to preserve the spirit of the strategy while accounting for the fat-tailed nature of the distribution of returns and realistic time steps for hedging. In this section we provide a simplified first-order correction to the standard delta hedging strategy for small values of the parameter ε, and discuss why the hedging strategy may fail in practice due to imperfect and incomplete market conditions caused by the particular nature of the stochastic processes. Our first-order correction also provides the basis for the discussion in Section 4.

When the dynamics of price changes is given by equation (1) (Brownian motion), a hedge portfolio composed of options and the underlying security can be made immune to changes in the security price using a delta-hedging strategy. In principle, the same strategy could be applied to the processes described in equations (2) and (3) under the (unrealistic) assumption of instantaneous trading. However, since the stochastic processes in equations (2) and (3) do not have the same self-similar diffusion characteristics as the Brownian motion in equation (1), a change in any small time step Δt used for hedging will produce different results. The differences between the stochastic processes in equations (2) and (3) and the process in equation (1), together with the fact that expression (7) is only an approximate solution, introduce uncertainty in the valuation of options when there is frequent but not instantaneous hedging. Here we approach this problem introducing an argument similar to the one described in Sobehart and Keenan (2002, 2003) to analyze the impact of market uncertainty and curvature effects in options pricing under imperfect and incomplete market conditions.

Deriving an approximate pricing methodology for fat-tailed distributions of returns and economically meaningful hedging time steps Δt requires several steps. First we solve the stochastic equations (2) and (3). Second we construct a hedging strategy for a small but finite time interval and take the appropriate mathematical limit as $\Delta t \to 0$. Third we show the impact of the non-normal nature of the price changes. Finally, we provide a mechanism for pricing options in the limit when the discrepancy between the random walk with normally distributed increments and the stochastic process in equations (2) and (3) is small. The tricky point is that, since the problem contains a stochastic variable χ with non-normal random increments and relaxation processes, the hedging procedure is ambiguous until we specify exactly how the limit is approached and what is the relevance of the different terms being neglected as $\Delta t \to 0$.

Equations (2) and (3) represent a two-dimensional linear system of stochastic differential equations. These equations can be solved under certain technical restrictions for the homogenous and inhomogeneous solutions. From equation (3) the excess return $\xi(t)$ is

$$\xi(t) = e^{-\nu(t)}\,\xi_0 + \eta \int_{Z_2(t_0)}^{Z_2} e^{\nu(s)-\nu(t)}\,dZ_2(s) \tag{11}$$

Here ξ_0 is a constant determined by the initial conditions, and

$$\nu(t) = \left(\theta + \frac{\varepsilon^2}{2}\right)\left(t - t_0\right) + \varepsilon\left(Z_1(t) - Z_1(t_0)\right) \tag{12}$$

From equation (2) the log-price $\chi(t)$ is

$$\begin{aligned}\chi(t) = \chi_0 &+ \left(\mu - \frac{\sigma^2}{2}\right)\left(t - t_0\right) + \sigma_0\left(Z_0(t) - Z_0(t_0)\right) \\ &+ \int_{t_0}^{t} \xi_0 e^{-\nu(t')}\,dt' + \eta \int_{t_0}^{t} \int_{Z_2(t_0)}^{Z_2'} e^{\nu(s)-\nu(t')}\,dZ_2(s)\,dt'\end{aligned} \tag{13}$$

Here χ_0 is a constant determined by the initial conditions. Notice that in the limit $\varepsilon = 0$ and $\theta \to \infty$ with η/θ constant (rational and efficient markets), the general solution χ reduces to a pseudo-Brownian motion:

$$\chi \sim \left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma_0 Z_0 + \left(\frac{\eta}{\theta}\right) Z_2$$

with effective volatility

$$\sigma \approx \sqrt{\sigma_0^2 + \left(\eta/\theta\right)^2}$$

Also note that when $\varepsilon \neq 0$ the excess return ξ can be driven by the random amplification of the process Z_2 caused by the stochastic process Z_1. This leads to fat-tailed distributions of stock returns. In the following we introduce an asymptotic analysis in the limit $\varepsilon \ll \theta^{1/2}$ and $\theta \gg 1$.

Using equations (11)–(13) the change in the security price during a small time interval Δt is:

$$\begin{aligned}\Delta S_t = S_{t+\Delta t} - S_t = S_t\left(e^{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma_0 \Delta Z_0 + \Delta Y} - 1\right) \\ = S_t\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma_0 \Delta Z_0 + \Delta Y \right. \\ \left. + \tfrac{1}{2}\left(\sigma_0 \Delta Z_0 + \Delta Y\right)^2 + O\left(\Delta t^{3/2}\right)\right)\end{aligned} \tag{14}$$

Here $O(\Delta t^{3/2})$ denotes higher-order terms in powers of Δt and $\Delta Y = Y(t + \Delta t) - Y(t)$, where $Y(t)$ is defined by the last two terms in equation (13)

$$Y(t) = \int_{t_0}^{t} \xi_0 e^{-\nu(t')} \mathrm{d}t' + \eta \int_{t_0}^{t} \int_{Z_2(t_0)}^{Z_2(t')} e^{\nu(s) - \nu(t')} \mathrm{d}Z_2(s)\, \mathrm{d}t' \tag{15}$$

Notice that the increment ΔY in equation (15) does not have the same self-similar diffusion characteristics as the process ΔZ_0. The increment ΔY exhibits the characteristics of a diffusion process only for time scales greater than the relaxation time of return adjustments $1/\theta$. Figure 5 shows the effective volatility $E^Y\lfloor \Delta Y^2/\Delta t \rfloor$ as a function of $\theta \Delta t$ for different values of the parameter ε. The values were calculated averaging 10,000 Monte Carlo simulations of the relevant terms in equations (2) and (3). The effective volatility of ΔY remains relatively constant for time steps $\theta \Delta t \gg 1$. For low values of $\theta \Delta t$ the inertial effects become evident.

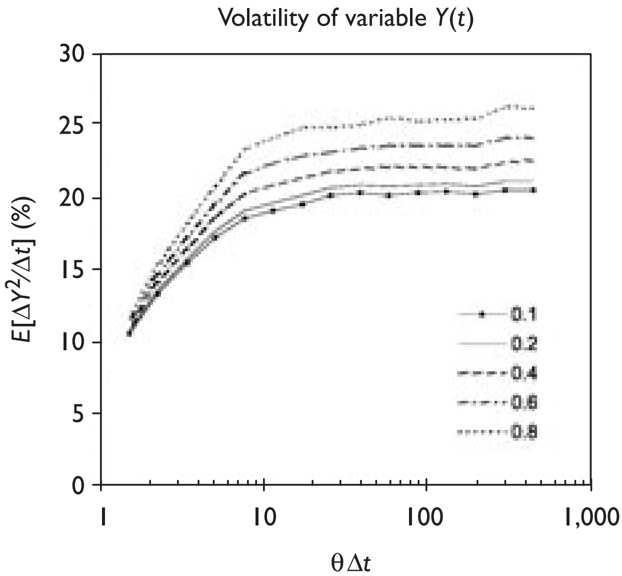
From a practical point of view, hedging for intervals shorter than $1/\theta$ is not very meaningful because $1/\theta$ is the characteristic trading time for eliminating price disturbances. In the following we consider the limit $\Delta t \to 0$ as the "economically meaningful" limit where the time interval Δt is short enough for any practical hedging purpose but it is long enough for ΔY to exhibit diffusion characteristics ($\theta \Delta t > 1$). We denote this particular limit: $\Delta t \to 0^+$. This mathematical construct allows us to calculate a meaningful "effective" volatility for ΔY preserving the standard derivation of options pricing equations using hedging strategies. A more formal approach may require a derivation based on multiple time-scales perturbation analysis.

Following the standard derivation of the BS model, assume that at time t we construct an ideal hedge portfolio Φ composed of a long position in the option $C(S, t)$ and a short position $\partial C/\partial S$ in the underlying security S.

$$\Phi(S,t) = C - \partial_S C\, S \tag{16}$$

Let us fix $t > 0$ and consider the capital gain of the portfolio at time $t + \Delta t$. A Taylor expansion of equation (16) yields a formal description of the portfolio

FIGURE 5 Effective volatility $E^Y\lfloor \Delta Y^2/\Delta t \rfloor$ as a function of $\theta\Delta t$ for different values of the ratio $\varepsilon/\theta^{1/2}$ and $\eta/\theta = 0.2$.



variation

$$\Delta\Phi_t \equiv \Phi\left(S_{t+\Delta t}, t+\Delta t\right) - \Phi(S_t, t) = \Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5+\Omega_6 \quad (17)$$

where

$$\Omega_1 \equiv S_t \frac{\partial \Phi}{\partial S}(S_t, t)\left(\sigma_0 \Delta Z_0 + \Delta Y\right)$$

$$\Omega_2 \equiv \mu S_t \frac{\partial \Phi}{\partial S}(S_t, t)\Delta t$$

$$\Omega_3 \equiv \left(\frac{\partial \Phi}{\partial t}(S_t, t) + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 \Phi}{\partial S^2}\left(S_t, t\right)\right)\Delta t$$

$$\Omega_4 \equiv \frac{1}{2} S_t \frac{\partial}{\partial S}\left(S_t \frac{\partial \Phi}{\partial S}\left(S_t, t\right)\right)\left(\left(\sigma_0 \Delta Z_0 + \Delta Y\right)^2 - \sigma^2 \Delta t\right)$$

$$\Omega_5 \equiv \frac{\sigma^2}{2} S_t^2 \left(\frac{\partial^2 \Phi}{\partial S^2}\left(\varsigma_t, t\right) - \frac{\partial^2 \Phi}{\partial S^2}\left(S_t, t\right)\right)\Delta t$$

$$\Omega_6 \equiv O\left(\Delta t^{3/2}\right) \quad (18)$$

Here $\zeta_t = S_t + \phi_t \Delta S_t$ for some appropriate ϕ_t satisfying $0 \le \phi_t \le 1$. The appropriate limit of terms Ω_1, Ω_2 and Ω_3 in equations (17) and (18) can be easily recognized as the leading contributions to Ito's lemma. Notice that $|\Omega_4| \sim O(\Delta t)$ for each random realization of ΔZ_0 and ΔY but $E^{Y,Z}[\Omega_4] \sim O(\Delta t^2, \varpi \Delta t)$ and $E^{Y,Z}[\Omega_4^2] \sim O(\Delta t^2)$. Here $E^{Y,Z}[\Delta Z_0 \Delta Y] = 0$, $\varpi(\varepsilon, \theta) = E^Y[\Delta Y^2/\Delta t] - (\sigma^2 - \sigma_0^2)$ and $E^X(f)$ is the expectation of $f(X)$ with respect to the random variable X at time t. If σ represents the correct effective volatility for the processes Z_0 and Y, one should expect $\varpi \sim O(\Delta t)$. In the limit $\Delta t \to 0$, term Ω_5 vanishes because S_t is continuous. Term Ω_6 includes higher-order contributions in Δt that vanish as $\Delta t \to 0$.

Because the hedge (17) removes the leading stochastic term in $\Delta t^{1/2}$, we can take the limit of equations (17) and (18) for the rate of change in the quasi-continuously hedged portfolio

$$\hat{L}\Phi_t \equiv \lim_{\Delta t \to 0^+} E^{Y,Z}\left[\frac{\Delta \Phi_t}{\Delta t}\right] = \frac{\partial \Phi_t}{\partial t} + \frac{1}{2}\left(\sigma^2 + \lim_{\Delta t \to 0^+} \varpi\right) S_t^2 \frac{\partial^2 \Phi_t}{\partial S^2} \tag{19}$$

Here $\hat{L}$ is the operator defined by the time and price derivatives. In the BS framework the return on the ideal hedge portfolio is claimed to be certain and equal to the riskless rate (that is, equation (19) reduces to $\hat{L}\Phi_t = r\Phi_t$). However, from equations (17) and (18) the variance of the changes in the hedge portfolio is:

$$\lim_{\Delta t \to 0^+} E^{Y,Z}\left[\left(\frac{\Delta P_t}{\Delta t} - \hat{L}\Phi_t\right)^2\right] =$$

$$\lim_{\Delta t \to 0^+} E^{\varphi,\psi}\left[\left(\frac{\sigma^2}{2} S_t \frac{\partial}{\partial S}\left(S_t \frac{\partial \Phi_t}{\partial S}\right)\right.\right.$$

$$\left.\left. \times \left(\frac{1}{\sigma^2}\left((\sigma_0\varphi + \psi)^2 - \lim_{\Delta t \to 0^+} \varpi\right) - 1\right)\right)^2\right] \geq 0 \tag{20}$$

Here $\varphi^2 = \Delta Z_0^2/\Delta t$, where $\varphi \sim N(0,1)$ is a normal random variable, and $\psi^2 = \Delta Y^2/\Delta t$, where $Y(t)$ is described in equation (15). Note from equation (20) that even if $\partial\Phi/\partial S \approx 0$ as a result of the hedging strategy, the portfolio is still exposed to additional risk if $\partial^2\Phi/\partial S^2 \neq 0$ (curvature or gamma risk). Equations (19) and (20) indicate that the hedge portfolio is not riskless due to the non-normal nature of the price increments and curvature effects that cannot be diversified or hedged away.

In the limit $\varepsilon \ll \theta^{1/2}$ and $\theta \gg 1$ we have

$$\Delta Y \to \sqrt{\sigma^2 - \sigma_0^2}\,\Delta Z_2$$

and, therefore, the discrepancy between the actual fat-tailed price return and a normally distributed return can be assumed to be a random hedging error term. In this limit we calculate a first-order correction to the standard Brownian motion neglecting the higher-order terms that describe heavy tails and other effects. Imposing that equation (19) must yield a risk-adjusted return on the portfolio ($\hat{L}\Phi_t = \rho\,\Phi_t$) to compensate for non-normal effects that cannot be hedged away, equations (17)–(19) immediately yield the alternative first-order options pricing equation

$$\frac{\partial C}{\partial t} + \rho S_t \frac{\partial C}{\partial S} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 C}{\partial S^2} - \rho C \approx 0 \qquad (21)$$

This leading-order approximate equation is equivalent to the Black–Scholes equation for instantaneous hedging but with an effective volatility and a risk-adjusted return that, in general, will depend on the risk preferences of the options market participants:

$$\rho - r = f\left(S_t, C_t, t, \lim_{\Delta t \to 0^+} E\left\lfloor\left(\Omega_4 / \Phi_t \Delta t\right)^2\right\rfloor\right)$$

This might appear to be a drawback of our calculations but the lack of perfect hedging caused by non-continuous trading, non-normal price changes and curvature clearly reflects reality. Market participants are well aware of the risk of imperfect hedging, and will require additional compensation for any residual risk exposure resulting from the non-normal distribution of returns and other effects. It is clear from equations (15)–(20) that the hedging strategy leading to equation (21) may fail in practice due to imperfect and incomplete market conditions caused by the particular nature of the stochastic processes.

4 The impact of irrational behavior in options pricing

In this section we show how the under- or overreaction to changes in returns can contribute to the observed volatility skew and smile effects recognizing, of course, that there may be other important factors that can also contribute to this phenomenon such as stochastic changes in volatility, interest rates or other state variables. Because the literature on this area is extensive, it is difficult to provide both a good technical description of our model and a fair and meaningful comparison with other approaches in this introductory article. For a comparison of different approaches we refer readers to the recent work in Potters, Cont and Bouchaud (1998), Bouchaud, Iori and Sornette (1996), Lipton (2002) and references therein. The extension of our model to many state variables is conceptually straightforward. To illustrate our ideas, in the following we focus only on one state variable: the price of the underlying security. This makes the algebra simpler and our discussion easier to follow.

Here we analyze the impact of irrational behavior in options pricing in the limit of nearly rational markets ($\varepsilon \ll \theta^{1/2}$ and $\theta \gg 1$) using the arguments explained in Section 3. More precisely, when the discrepancies between the process in equation (1) and the processes in equations (2) and (3) are small and portfolio hedging occurs frequently but not continuously ($\Delta t \to 0^+$), the discounting rate of the option may include a risk-adjusted gain required by investors as a compensation for any additional risk introduced by higher-order terms that cannot be hedged away, as shown in the first-order pricing equation (21). In the following, we assume that the distribution of prices required for pricing options is given by equation (7) where we replace the riskless rate r and price drift μ with a risk-adjusted return ρ. The derivation of the correct distribution for realistic hedging and option replication strategies remains an open problem.

To further clarify our point, let us focus on the value of a European call option at time t with exercise price K and maturity T. From equations (7)–(21) and the assumption of nearly rational markets we obtain

$$C(S, t; \rho, \sigma) = \mathrm{e}^{-\rho(T-t)} E\left(S - K\right)^+$$
$$= C_{BS} + \frac{1}{A_0} \sum_{n=1}^{\infty} \frac{i^n}{n! \left(A_0 \sigma \sqrt{T-t}\right)^n} \frac{\partial^n A}{\partial k^n} B_n \qquad (22)$$

Here

$$B_n(S, t; \rho, \sigma_\varepsilon) = S \Gamma_n\left(d_1, \sigma\sqrt{T-t}\right) - K \mathrm{e}^{-\rho(T-t)} \Gamma_n(d_2, 0) \qquad (23)$$

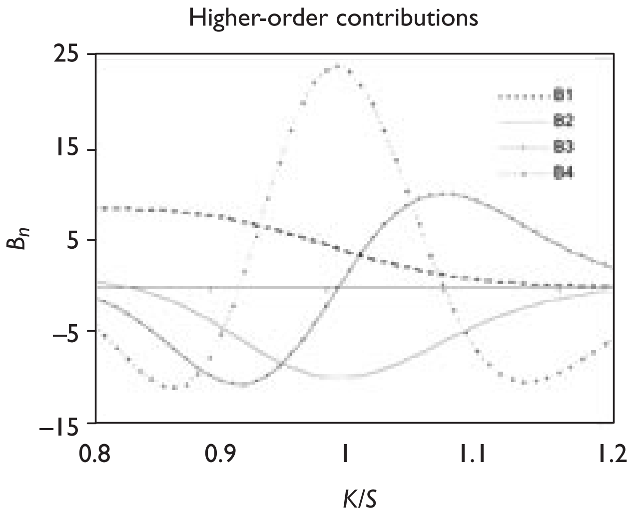
$$\Gamma_n(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} H_n(z + y)\, \mathrm{e}^{-z^2/2} \mathrm{d}z \qquad (24)$$

$$d_1 = \frac{\log(S/K) + \left(\rho + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t} \qquad (25)$$

Notice that the first term of the Taylor series ($n = 0$) is the standard BS formula $C_{BS}(S, t; \rho; \sigma)$ with risk premium ρ and volatility σ. Figure 6 shows the shape of the additional contributions to the option value for the first few terms.

If at each point in time the option value $C(S, t; \rho; \sigma)$ is fitted with the BS formula for a call option $C_{BS}(S, t; r; \sigma_t)$ containing an implied volatility σ_I and the risk-free rate r, then the irrational behavior of market participants will cause the implied parameter σ_I to differ from the observed security's volatility σ. To illustrate, consider the ideal case where the discrepancy between the implied volatility and the security's volatility is small, and where the BS formula is a good estimate of the option prices observed in the market – ie, $C_{BS}(S, t; r; \sigma_I) \approx C(S, t; \rho; \sigma_t)$. Then, a first-order Taylor series expansion in terms of the implied volatility and risk premium yields

FIGURE 6 B_n for $n \leq 4$ as a function of the option's moneyness for maturity $T - t$ = 30 days. The security's price is S = 100, the security's volatility is σ = 30%, the risk-free rate is r = 5%, the required risk premium is $\rho - r$ = 0.5%.



$$C(S, t; r, \sigma) + \frac{\partial C}{\partial r}(\rho - r) \approx C_{BS}(S, t; r, \sigma) + \frac{\partial C_{BS}}{\partial \sigma}(\sigma_I - \sigma) \tag{26}$$

Equation (26) has to be solved for the implied volatility σ_I in terms of the stock price S and effective volatility σ. This first-order approximation yields

$$\sigma_I \approx \sigma + \sqrt{\frac{2\pi}{T-t}} \frac{K}{S} e^{d_1^2/2} \left[C - C_{BS} + \frac{\partial C}{\partial r}(\rho - r) \right] \tag{27}$$

The implied volatility σ_I given in equation (27) is a function of both the option moneyness and time-to-maturity due to two related contributions. The first contribution $C - C_{BS}$ represents the actual impact of irrational behavior on the distribution of option prices. The second contribution in $\rho - r$ shows the impact of the additional risk premium required for holding an option that cannot be perfectly replicated using an ideal BS hedging strategy due to the fat-tailed nature of the distribution of returns. In practice, there could be different levels of risk premia for different exercise prices K and maturity T.

Figures 7 and 8 sketch the first-order volatility correction (equation (27)) introduced by the irrational behavior of market participants and the required risk premium $\rho - r$, across moneyness for different times to expiration. To illustrate the curves include only corrections for $n = 1$, 2 and 4. Other corrections are

FIGURE 7 Implied volatility as a function of the option's moneyness for different times to expiration $T - t$ = 30, 60 and 90 days. The security's price is S = 100, the security's volatility is σ = 30%, the risk-free rate is r = 5%, the required risk premium is $\rho - r$ = 0.5%. To illustrate the curves include only corrections for n = 1, 2 and 4.

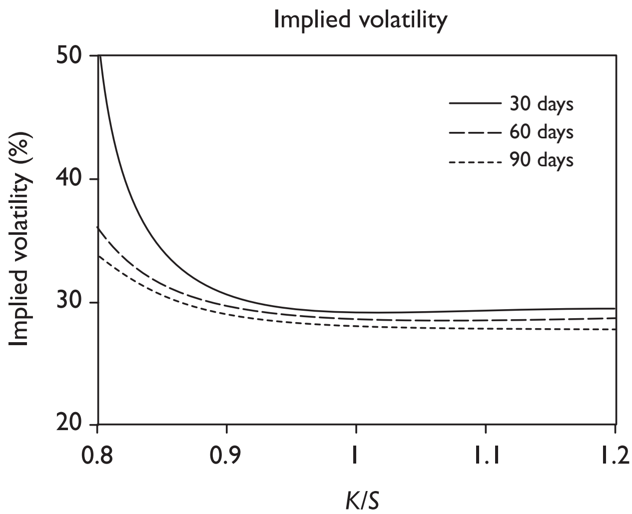
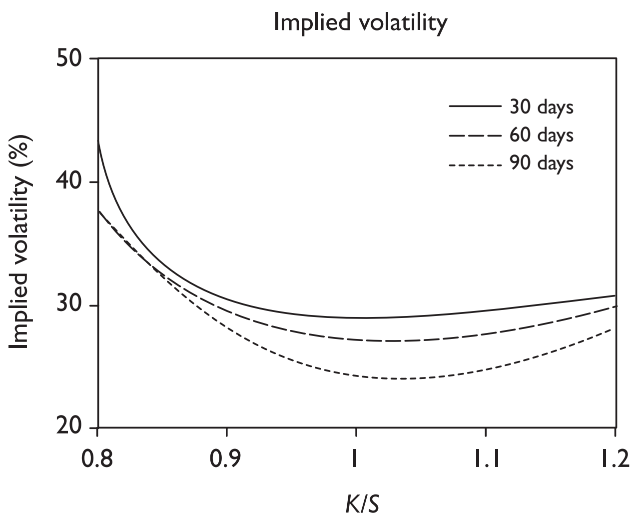


FIGURE 8 Implied volatility as a function of the option's moneyness for different times to expiration $T - t$ = 30, 60 and 90 days The security's price is S = 100, the security's volatility is σ = 30%, the risk-free rate is r = 5%, the required risk premium is $\rho - r$ = 0.5%. To illustrate the curves include only corrections for n = 2 and 4.



assumed negligible small. Note that different shapes of skews and smiles can be obtained by changing the relative weights of the contributions B_n in equation (23), which depend on the distribution of excess returns as indicated in equations (A10)–(A12) in the Appendix. Finally, notice the similarity between the results from equation (27) and the models for the skew and smile effects described in Natemberg (1994), Malz (1997), Derman (1999), Alexander (2001), and Derman and Zou (2001).

5 Conclusions

In this article we introduced a model of irrational exuberance and fear in equity markets based on the under- and overreaction of market participants to price trends. In our model market participants struggle to find their way through the give and take between risk and return, one moment engaging in rational behavior and the next yielding to irrational emotional impulses. We show that a simple dynamical model incorporating this type of behavior leads to fat-tailed distributions of returns and other characteristics that are remarkably consistent with observed market dynamics. We also analyzed the asymptotic impact of irrational behavior in options pricing in the limit of nearly rational markets and frequent trading. Note however that the derivation of the correct pricing equation remains an open problem. The results discussed here are particularly important because the same basic approach can also be applied to the development of a pricing theory for credit derivatives and corporate liabilities.

Appendix

The joint distribution of log-prices and returns is obtained from equation (4) for an arbitrary log-price χ, excess return ξ and time t. This requires several transformations and approximations.

First we normalize equation (4) using the scale transformations $x = \varepsilon^3(\chi - \chi_0)/\eta$, $y = \varepsilon\xi/\eta$, and $\tau = \varepsilon^3(t - t_0)$, where χ_0 and t_0 are initial reference values.

$$\frac{\partial P}{\partial \tau} + (y + \alpha)\frac{\partial P}{\partial x} = \beta\frac{\partial}{\partial y}[yP] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\left(1 + y^2\right)P\right] + \frac{\gamma}{2}\frac{\partial^2 P}{\partial x^2} \qquad \text{(A1)}$$

Here $\alpha = \varepsilon(\mu - \sigma_\varepsilon^2/2)/\eta$ and θ is defined in equation (6), and $\gamma = (\varepsilon^2\sigma_0/\eta)^2$.

Rather than solving equation (A1) using the rigorous Laplace transformation approach, we introduce the Fourier representation

$$P(x, y, \tau) = \frac{1}{4\pi^2}\iint G(y, k, \omega)\; \mathrm{e}^{ikx + i\omega\tau}\, \mathrm{d}k\, \mathrm{d}\omega \qquad \text{(A2)}$$

This allows us to easily analyze the basic dynamics of the solutions. Of course, this comes at the expense of a detailed description of initial conditions and transient effects. Using equation (A2), equation (A1) reduces to

$$\frac{1}{2}(1+y^2)\frac{\partial^2 G}{\partial y^2}+(2+\beta)y\frac{\partial G}{\partial y}+\left[\beta+1-i\omega-\frac{\gamma}{2}k^2-ik(y+\alpha)\right]G=0 \qquad \text{(A3)}$$

Introducing the additional transformations $y = \sinh(u)$ and $G = F/(1 + y^2)^{(\phi+1)/2}$, where $\phi = \beta + 1/2$, equation (A3) yields

$$\frac{\mathrm{d}^2 F}{\mathrm{d}u^2}+\left[\phi-2i(\omega+k\alpha)-\gamma k^2-2ik\ \sinh(u)-\phi(\phi+1)\ \tanh^2(u)\right]F=0 \qquad \text{(A4)}$$

Equation (A4) must be solved with the additional boundary conditions $F(+\infty) = F(-\infty) = 0$ to obtain sensible solutions for extreme excess returns. In principle, the numerical solution of equations (A2)–(A4) provides the distribution of prices at different points in time. However, we can obtain an insightful view on the solutions in the limit where the random excess return is small, that is $u \ll 1$. In this limit equation (A4) yields

$$\frac{\mathrm{d}^2 F}{\mathrm{d}u^2}+\left[\phi-2i(\omega+k\alpha)-\gamma k^2-2iku-\phi(\phi+1)u^2\right]F=0 \qquad \text{(A5)}$$

Notice that the first-order approximation $\phi(\phi + 1)u^2$ in equation (A5) overestimates the effective potential $\phi(\phi + 1)\tanh^2(u)$ in equation (A4) for large values of u, while the term $2iku$ in equation (A5) underestimates the term $2ik\sinh(u)$ in equation (A4). Both approximations affect the nature of the solutions for extreme values and may introduce spurious effects. A more rigorous description may require an analysis based on matched asymptotic expansions for different values of u.

A final transformation of variables in equation (A5) yields

$$\frac{\mathrm{d}^2 F}{\mathrm{d}z^2}+\left[\lambda-z^2\right]F=0, \qquad F(+\infty)=F(-\infty)=0 \qquad \text{(A6)}$$

Here $z(\xi, k) = \sqrt{c}(u + b/c^2)$ and $\lambda = (b^2/c^2 - d)/c$ where $b = ik$, $c = \sqrt{\phi(\phi+1)}$ and $d = 2i(\varpi + k\alpha) + \gamma k^2 - \theta$. The Sturm–Liouville problem (A6) can be solved analytically. Its general solution is

$$F(z)=\sum_{n=0}^{\infty}a_n H_n(z)\ \mathrm{e}^{-z^2/2}, \qquad \lambda_n=2n+1 \qquad \text{(A7)}$$

Here $H_n(z)$ is the Hermite polynomial of degree "n". The coefficients a_n are determined by additional conditions imposed on the initial distribution of prices and excess returns. Because $z = \sqrt{c}(\operatorname{arcsinh}(\varepsilon\xi/\eta) + ik/c^2)$, equation (A7) indicates that the probability distribution will exhibit fat tails as a function of the excess return ξ. This also affects the distribution of log-prices as we show below.

Also notice that in order to satisfy equation (A6), the eigenvalue λ_n and wave number k impose a constraint on the possible values for the frequencies required in equation (A7).

$$i\omega(k,n) = \frac{1}{2}\left(\phi - \sqrt{\phi(\phi+1)}\,\lambda_n - 2i\alpha k - \delta k^2\right), \qquad \delta = \gamma + \frac{1}{\phi(\phi+1)} \tag{A8}$$

This condition reduces the double integral in ω and k in equation (A6) to a single integral in k.

Finally, the distribution of the log-price χ and instantaneous excess return ξ at time t results:

$$P(\chi,\xi,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{n=0}^{\infty} g_n(\xi,k)\exp\left\{ik\frac{\varepsilon^3}{\eta}(\chi-\chi_0) + i\omega(k,n)\varepsilon^2(t-t_0)\right\}\mathrm{d}k \tag{A9}$$

Here

$$g_n(\xi,k) = \frac{a_n(k)H_n\big(z(\xi,k)\big)\,\mathrm{e}^{-z^2(\xi,k)/2}}{\left(1+\left(\dfrac{\varepsilon\xi}{\eta}\right)^2\right)^{(\phi+1)/2}} \tag{A10}$$

Equations (A9) and (A10) provide the general description of the distribution of prices and returns. In order to understand the nature of this distribution, let us analyze the asymptotic behavior of the marginal distribution of log-prices

$$\begin{aligned} p(\chi,t) &= \int_{-\infty}^{+\infty} P(\chi,\xi,t)\,\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(k,t)\exp\left\{ik\frac{\varepsilon^3}{\eta}(\chi-\chi_0) - \frac{\varepsilon^2}{2}\left(\delta k^2 + 2i\alpha k\right)(t-t_0)\right\}\mathrm{d}k \end{aligned} \tag{A11}$$

Here

$$A(k,t) = \sum_{n=0}^{\infty}\int_{-\infty}^{+\infty} g_n(\xi,k)\,\mathrm{d}\xi\,\exp\left\{-\frac{\varepsilon^2}{2}\left(\sqrt{\phi(\phi+1)}\lambda_n - \phi\right)(t-t_0)\right\} \tag{A12}$$

Note the time decay of the different contributions to the coefficient $A(k,t)$. From equation (A12) the characteristic time-scale for exponential decay for $n \geq 1$ and $\phi \gg 1$ is approximately $\tau_n \sim 1/(\varepsilon^2 n\phi)$. The slow time decay for $n = 0$ is caused by the linear approximations leading to equation (A5) and, therefore, can be neglected.

Assuming that the security price is known with certainty at time t_0 but the excess return has initial density function $f(\xi)$, the initial condition on the distribution of prices and excess return is $P(\chi, \xi, t_0) = f(\xi)\delta(\chi - \chi_0)$, where $\delta(x)$ is the Dirac delta-function. This condition immediately provides a normalization condition for the constants a_n and, therefore, for $A(k, t_0)$. Since we are interested in the marginal distribution of prices, the initial condition reduces to $p(\chi, t_0) = \delta(\chi - \chi_0)$. This condition yields $A(k, t_0) = A_0 = \varepsilon^3/\eta$. That is, the Fourier coefficients are independent of k at time t_0. Of course, this condition changes for $t > t_0$ as indicated in equation (A12). Expanding $A(k, t)$ in a Taylor series for small wave number k (diffusion approximation), equation (A11) can be recognized as the asymptotic Fourier representation of a normal distribution with higher-order corrections

$$
\begin{aligned}
p(\chi, t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[A(0,t) + k \frac{\partial A}{\partial k}(0, t) + \ldots \right] \\
&\quad \times \exp\left\{ ik \frac{\varepsilon^3}{\eta}(\chi - \chi_0) - \frac{\varepsilon^2}{2}\left(\delta k^2 + 2ik\alpha\right)(t - t_0) \right\} \mathrm{d}k \\
&= \frac{1}{\sqrt{2\pi\sigma^2 (t - t_0)}} \exp\left\{ -\frac{\left(\chi - \chi_0 - \left(\mu - \sigma^2/2\right)(t - t_0)\right)^2}{2\sigma^2 (t - t_0)} \right\} \\
&\quad \times \frac{1}{A_0} \sum_{n=0}^{\infty} \frac{i^n}{n! \left(A_0 \sigma \sqrt{t - t_0}\right)^n} \frac{\partial^n A}{\partial k^n}(0,t) \\
&\quad \times H_n\left(\frac{\left(\chi - \chi_0 - \left(\mu - \sigma^2/2\right)(t - t_0)\right)}{\sigma\sqrt{t - t_0}} \right)
\end{aligned}
\tag{A13}
$$

1. The main results of the model were recently reported in Sobehart and Farengo (2002).

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