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THE UNIVALENCE AXIOM FOR INVERSE DIAGRAMS

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Abstract. We prove that Voevodsky's univalence axiom for the internal type theory of a suitable category is preserved by passage to diagrams over inverse categories, using the Reedy model structure. The basic observation which makes this work is that Reedy fibrant inverse diagrams correspond to contexts of a certain sort in type theory. Applying our result to Voevodsky's univalent model in simplicial sets, we obtain new models of univalence in a number of $(\infty, 1)$-toposes, answering a question raised at the Oberwolfach workshop on homotopical type theory.

1. Introduction

Recently it has become apparent that *intensional type theory* admits semantics in *homotopy theory*. The first such model was constructed in [HS98]; more recent and general references include [War08, AW09, vdBG12, Voea]. The basic idea of such models is that intensional *identity types* are interpreted by *path spaces*.

Since there can be nontrivial paths even from a point to itself, these models make a virtue out of the failure of "uniqueness of identity proofs" in intensional type theory. In effect, they argue that intensional type theory is naturally a theory of "homotopy types", and many of its traditionally uncomfortable attributes come from trying to force it to be a theory only of sets. From the homotopical perspective, sets should be identified only with "discrete" or "0-truncated" types. This raises the possibility of using intensional type theory as a "natively homotopical" foundation for mathematics.

One of the innovations of homotopical type theory, due to Voevodsky, is the identification of the correct identity types for universes. It is natural to consider two types "equal", as terms belonging to a universe Type, if there is an isomorphism between them. However, this is hard to square with uniqueness of identity proofs, since two types can be isomorphic in more than one way, and if the equality between them doesn't remember which isomorphism it came from, how can we meaningfully substitute along that equality? But homotopically, taking isomorphisms (or, more precisely, equivalences) to form the identity type of the universe makes perfect sense and is the right thing to do; the resulting rule is called the *univalence axiom*.

The univalence axiom has proven quite valuable in the formalization of mathematics and homotopy theory in intensional type theory. Moreover, it is quite similar to the Lurie-Rezk notion of *object classifier* in $(\infty, 1)$-topos theory (see [Lur09, §6.1.6]). Thus, one may hope to model dependent type theory with the univalence axiom in any $(\infty, 1)$-topos. In addition to providing a source of examples and counterexamples for the study of univalence itself, this would imply that DTT

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with univalence could serve as an *internal logic* for higher toposes, generalizing the well-known internal logic of ordinary toposes. Since the existence of object classifiers more or less characterizes $(\infty,1)$-toposes, just as the existence of subobject classifiers characterizes ordinary toposes, there would then be a very close match between type theory and category theory.

The difficulty with this is that the rules of type theory are stricter than those of $(\infty,1)$-categories, so we must look for strict models of $(\infty,1)$-toposes which contain strict models of their object classifiers. The first set-theoretic model of univalence was constructed by Voevodsky [Voea], in the category of simplicial sets (a strict model for the $(\infty,1)$-topos $\infty\mathrm{Gpd}$, which plays the same role for $(\infty,1)$-toposes that Set does for ordinary toposes). The construction uses many technical details of simplicial sets and it is unclear how to generalize it to any other context. In fact, until now, no other truly different set-theoretic models of univalence have been known, and the question was raised at the Oberwolfach mini-workshop [AGMLV11] of whether any such models exist.

In this paper, we answer this question affirmatively. Specifically, we show how to lift any model of univalence in an appropriate category $\mathscr{C}$ to a new model in the functor category $\mathscr{C}^I$, where I is any *inverse category*. An inverse category is one containing no infinite composable strings

$$\to\to\to\to\ \cdots$$

of nonidentity morphisms. For instance, a finite category is inverse just when it is skeletal and has no nonidentity endomorphisms. This property enables us to construct diagrams and morphisms of diagrams by well-founded induction; we exploit this to construct a universe object which models the univalence axiom.

The homotopy theory we use in $\mathscr{C}^I$ is familiar to homotopy theorists—it is the *Reedy model structure*, which exists for diagrams on any inverse category. (It exists more generally than this, but I do not know how to generalize the construction presented here beyond the case of inverse categories.) In particular, if $\mathscr{C}$ is the category sSet of simplicial sets, then the Reedy model structure on $\mathscr{C}^I$ is a strict model for the presheaf $(\infty,1)$-topos $\infty\mathrm{Gpd}^I$. Thus, this is a first step towards the goal of modeling the univalence axiom in all $(\infty,1)$-toposes.

Moreover, since the construction assumes nothing about $\mathscr{C}$ other than that it models type theory with the univalence axiom in a canonical way, it can also be interpreted as a "stability" result for categories that model univalence. Probably it can even be performed internally inside of type theory. This has implications for a hypothetical definition of "elementary $(\infty,1)$-topos".

Organization. We begin in §2 by recalling how to model dependent type theory in a category $\mathscr{C}$ with suitable structure. In particular, we explain in some detail a technique (due to Voevodsky) for dealing with "coherence" for substitution, using a universe object. Then in §3 we recall some of the basic definitions of homotopical type theory, leading up to the statement of the univalence axiom. For each such definition, we characterize its meaning in the categorical semantics of §2.

The heart of the paper is in §§4–6, although inverse categories in general do not appear until §7. Sections 4–6 treat in detail the first nontrivial example of an inverse category: the arrow category $\mathbb{2} = (1 \to 0)$. Assuming a category $\mathscr{C}$ with the structure of §2, in §4–5 we build the same structure on $\mathscr{C}^{\mathbb{2}}$, and then in §6 we show that the universes in $\mathscr{C}^{\mathbb{2}}$ inherit univalence from those in $\mathscr{C}$.

Finally, in §7 we introduce general inverse categories. It turns out that once the arguments of §§4–6 are understood, very little work is required to generalize to the case of arbitrary inverse categories. The work of §§4–6 is almost exactly the same as the induction step in the corresponding proof for a general inverse category. Thus, in §7 we merely sketch the necessary modifications.

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2. Categorical models of type theory

Universes in categories play two roles in the modeling of type theory. On the one hand, they serve as models for internal universes *in* the type theory; this is their role in the context of univalence. On the other hand, Voevodsky [Voea] has observed that a universe in a category can also be used to deal quite handily with the traditional problems of strictness and coherence involved in the categorical interpretation of dependent type theory. The universe used for this purpose does not exist in the resulting type theory at all, remaining "external" to the model and in fact *defining* it. Other universes which embed into the "external" universe then serve as internal universes in the type theory. Although these two purposes are different, in general the same universe objects can serve either purpose.

In this section, we recall how we obtain a model of type theory from a category, using an "external" universe. Thus, let $\mathscr{C}$ be a category with the following structure.

(1) A terminal object 1.
(2) A subcategory $\mathcal{F} \subset \mathscr{C}$ containing all the objects and all the isomorphisms.
   - A morphism in $\mathcal{F}$ is called a **fibration**.
   - An object A is called **fibrant** if $A \to 1$ is a fibration.
   - A morphism i is called an **acyclic cofibration** if it has the left lifting property with respect to all fibrations. This means that if p is a fibration and $pf = gi$, then there is an h with $f = hi$ and $g = ph$.
(3) All pullbacks of fibrations between fibrant objects exist and are fibrations.
(4) For every fibration $g\colon A \to B$ between fibrant objects, the pullback functor $g^*\colon \mathscr{C}/B \to \mathscr{C}/A$ has a partial right adjoint Π_g, defined at all fibrations over A, and whose values are fibrations over B. This implies that g^* preserves acyclic cofibrations.
(5) If $f\colon A \to B$ is a fibration between fibrant objects, its diagonal $A \to A \times_B A$ factors as $A \to P_B A \to A \times_B A$, where $P_B A \to A \times A$ is a fibration and $A \to P_B A$ is an acyclic cofibration which is preserved by pullback along *all* maps into B.

Remark 2.1. In type theory the terms *display map* and *dependent projection* are usually used instead of *fibration*. Under this translation, conditions (1), (2), (3), and (4) make $\mathscr{C}$ into a *display map category* (see e.g. [Jac99, §10.4]) with the well-known additional structure required for interpreting a unit type, strong dependent sums, and dependent products.

Condition (5) is the analogous structure required for identity types. It is similar to the notion of *stable path objects* from [War08, AW09], but weaker in that we don't require a functorial global *choice* of such path objects. We can get away with this because we use universes for coherence, as explained below.

We have two main classes of examples in mind; here is the first.

Example 2.2. Let $\mathscr{C}$ be the category of contexts in a dependent type theory with a unit type, dependent sums, dependent products, and intensional identity types. We require the unit type, sums, and products to satisfy the definitional η-rule.

The fibrations are the closure under isomorphisms of the "dependent projections" from any context to an initial segment thereof. Note that by η for unit types and dependent sums, any context is isomorphic to one consisting of a single type. The η rule for dependent products is required to make the operations Π_g into actual adjoints (rather than weak adjoints). Finally, condition (5) is implemented by "identity contexts" as in [GG08].

The second class of examples comes from homotopy theory, so we digress briefly to relate the above structure with notions of homotopy theory. All the abstract homotopy theory we require can be found in [Hov99, Chapters 1 and 5], in [Hir03, Chapters 7, 8, and 15], or in [MP12, Chapters 14–16].

Definition 2.3. A **weak factorization system** $(\mathcal{L}, \mathcal{R})$ on a category consists of two classes of maps $\mathcal{L}$ and $\mathcal{R}$ such that

- $\mathcal{L}$ is precisely the class of maps having the left lifting property with respect to $\mathcal{R}$, and dually.
- Every morphism factors as $p \circ i$ for some $i \in \mathcal{L}$ and $p \in \mathcal{R}$.

Conditions (4) and (5) above imply that the subcategory of fibrant objects in $\mathscr{C}$ admits a weak factorization system, where $\mathcal{L}$ is the class of acyclic cofibrations and $\mathcal{R}$ is the class of retracts of fibrations. The proof is exactly that of [GG08, 4.2.1], translated into category theory. This produces factorizations $f = p \circ i$ where p is a fibration and i an acyclic cofibration; the characterization of $\mathcal{L}$ and $\mathcal{R}$ then follows by the "retract argument" of model category theory.

In particular, if all objects are fibrant and the fibrations are closed under retracts, then $\mathscr{C}$ has a weak factorization system where $\mathcal{L}$ is the acyclic cofibrations and $\mathcal{R}$ is the fibrations. Conversely, if such a weak factorization system exists, then:

- Fibrations are automatically preserved by pullback, so (3) need only assert that such pullbacks exist.
- If Π_g is defined at fibrations, then it takes fibrations as values if and only if g^* preserves acyclic cofibrations.
- The factorization in (5) almost follows: all that is missing is the additional pullback-stability of $A \to P_B A$.

Most examples from homotopy theory have the following additional structure.

Definition 2.4. A **model structure** on a complete and cocomplete category consists of three classes of maps $\mathcal{C}$ (cofibrations), $\mathcal{F}$ (fibrations), and $\mathcal{W}$ (weak equivalences) such that

- $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorization systems.
- If two of f, g, and gf are in $\mathcal{W}$, so is the third.

In a model category, the maps in $\mathcal{C} \cap \mathcal{W}$ are called *acyclic cofibrations*, and similarly the maps in $\mathcal{F} \cap \mathcal{W}$ are *acyclic fibrations* (some authors say *trivial* instead of *acyclic*). We will mostly work only with one weak factorization system, as above. But since that weak factorization system behaves like $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ in a model category, we use the names "acyclic cofibration" and "fibration".

Now we can define our second main class of examples.

Definition 2.5. A **type-theoretic model category** is a model category $\mathscr{C}$ with the following additional properties.

- $\mathscr{C}$ is right proper, i.e. weak equivalences are stable under pullback along fibrations. This is automatic if all objects are fibrant.
- Cofibrations are stable under pullback. This is automatic if the cofibrations are the monomorphisms.
- Pullback g^* along a fibration between fibrant objects has a right adjoint Π_g. This is automatic if $\mathscr{C}$ is locally cartesian closed.

Since a model category has finite limits and a weak factorization system consisting of acyclic cofibrations and fibrations, conditions (1)–(3) hold. By right properness and pullback-stability of cofibrations, g^* preserves acyclic cofibrations for any fibration g; hence condition (4) also holds. Finally, condition (5) follows for *any* factorization $A \to P_B A \to A \times_B A$ into an acyclic cofibration followed by a fibration, since cofibrations are assumed pullback-stable and weak equivalences between fibrations are always pullback-stable.

Remark 2.6. In a type-theoretic model category, any fibration g between fibrant objects yields a Quillen adjunction $g^* \dashv \Pi_g$.

Examples 2.7. Here are our basic examples of type-theoretic model categories.

- Any locally cartesian closed category, equipped with the trivial model structure in which the weak equivalences are the isomorphisms and every morphism is a cofibration and a fibration. Of course, this sort of category will only interpret *extensional* type theory.
- The category of groupoids, with its canonical model structure in which the weak equivalences are the equivalences of categories, the fibrations are the functors with isomorphism-lifting ("isofibrations"), and the cofibrations are the injective-on-objects functors. All objects are fibrant, cofibrations are clearly stable under pullback, and isofibrations are exponentiable (although the category of groupoids is not locally cartesian closed). This was the first non-extensional set-theoretic model of type theory [HS98].
- The category sSet of simplicial sets, with its traditional (Quillen) model structure. This is right proper (the fibrant objects are the Kan complexes), locally cartesian closed, and the cofibrations are the monomorphisms.

We would like to say that any category $\mathscr{C}$ satisfying (1)–(5) models dependent type theory, with contexts interpreted as fibrant objects, and dependent types $\Gamma \vdash A\colon \mathsf{Type}$ as fibrations $A \to \Gamma$. The obvious way to interpret *substitution* in such a model is by pullback. However, in type theory, substitution is strictly associative and preserves all structure strictly, but this is not generally the case for pullbacks in a category. There are several ways to resolve this (see e.g. [Hof94, War08, AW09, vdBG12]), but perhaps the cleanest is the following (due to Voevodsky [Voea]).

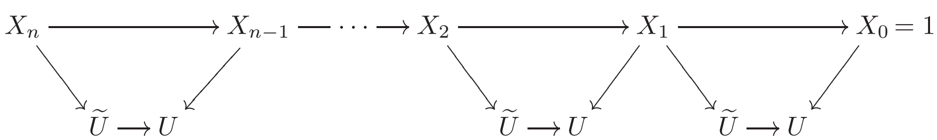
Let $p\colon \widetilde{U} \to U$ be a specified fibration between fibrant objects in $\mathscr{C}$, which we refer to as the (external) *universe*. We then interpret a dependent type $\Gamma \vdash A\colon \mathsf{Type}$ as a morphism $\Gamma \to U$. We call such a morphism a *named type*; the *object named by* a named type is the pullback of $\widetilde{U}$ along it (which is a fibration).

We say a fibration $Y \to X$ is *small* if it admits some name, i.e. it fits into some pullback square

$$\begin{array}{ccc} Y & \longrightarrow & \widetilde{U} \\ \downarrow & & \downarrow \\ X & \longrightarrow & U \end{array}$$

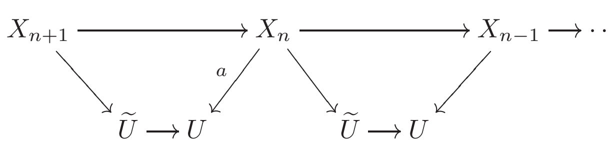
In general, a small fibration may have more than one name. Of course, a fibrant object X is *small* if $X \to 1$ is small. Note that p is small, but U usually is not.

The point is that substitution for named types can now be implemented by simple composition of maps with codomain U, and as such is strictly associative. More precisely, we interpret a *context* in type theory by an equivalence class of diagrams



in which every trapezoid is a pullback. We consider two such diagrams equivalent if they are isomorphic in the obvious sense. Note that such an isomorphism is always unique, by the universal property of pullbacks.

A *type* in such a context is interpreted by a name $X_n \to U$. Since isomorphisms of context diagrams are unique when they exist, the set of types in a given context is independent, up to canonical bijection, of the chosen representative. The *extension* of a context by a type $a\colon X_n \to U$ in that context is the equivalence class of extensions by a pullback square:



A *term* belonging to such a type-in-context is a lifting of its name to $\widetilde{U}$, or equivalently a section $s\colon X_n \to X_{n+1}$ of $X_{n+1} \to X_n$. If furthermore $b\colon X_{n+1} \to U$ is a type in the extended context, then the composite $X_n \xrightarrow{s} X_{n+1} \xrightarrow{b} U$ is the type in the original context obtained by substituting s into b. Again, note that this is independent of the chosen pullback square or representative of the original context.

We can also resolve issues of strictness in the modeling of *any* type-forming operation with a categorical operation, in the following way. Any such construction takes some input (types and terms) and produces some output, both of which correspond in the model to some fibrations in $\mathscr{C}$ and maps between them. If the corresponding categorical operation in $\mathscr{C}$ preserves smallness and is stable under pullback (in the usual category-theoretic sense, i.e. up to isomorphism), then we can perform it *once* in the "universal" case over U and then implement the type-theoretic operation in the model of named types by composition.

For instance, suppose that small fibrations (between fibrant objects) are closed under composition. Then we can interpret dependent sums as follows. Define the *universal dependent named type* to be the local exponential

$$U^{(1)} = (U \times U \to U)^{(\widetilde{U} \to U)}.$$

Then maps from any object X to $U^{(1)}$ are in bijection with pairs (a, b) where $a\colon X \to U$ is a named type over X and $b\colon a^*\widetilde{U} \to U$ is a named type over the type named by a. In particular, the identity map of $U^{(1)}$ names a pair of composable small fibrations $B \to A \to U^{(1)}$. Their composite, being small by assumption, is named by some map $U^{(1)} \to U$, and we can define the operation of dependent sum to act on named types by composition with this map.

Analogously, suppose that when f and g are small fibrations between fibrant objects, then so is $\Pi_g(f)$. In this case, the map

$$\Pi_{A \to U^{(1)}}(B \to A) \longrightarrow U^{(1)}$$

is a small fibration, and hence named by some map $U^{(1)} \to U$. Composing with this map then implements dependent product on named types.

For identity types, suppose that the path fibration $P_A X \to X \times_A X$ of any small fibration $X \to A$ is small. Then the path-object of $\widetilde{U} \to U$ in $\mathscr{C}/U$ is a small fibration $P_U \widetilde{U} \to \widetilde{U} \times_U \widetilde{U}$, which has some name $\widetilde{U} \times_U \widetilde{U} \to U$. Composing with this map implements identity types for the model of named objects, which are strictly preserved by substitution.

In the case of identity types, there is an additional concern: the *eliminator* must also be preserved by substitution. In [War08] this is called *coherence*. (This issue does not arise for dependent sums and products, since in that case all the categorical structure is defined uniquely, whereas the eliminator for identity types is defined by a non-unique lifting property.) However, we can also solve this problem using the universe. Let $U^{(\square)}$ denote the pullback

$$\begin{array}{ccc} U^{(\square)} & \longrightarrow & (U \times \widetilde{U} \to U)^{(\widetilde{U} \to U)} \\ \downarrow & & \downarrow \\ (U \times U \to U)^{(P_U \widetilde{U} \to U)} & \longrightarrow & (U \times U \to U)^{(\widetilde{U} \to U)} \end{array}$$

Then for any object X, to give a map $X \to U^{(\square)}$ is equivalent to giving a named type $a\colon X \to U$ together with a commutative square

$$\begin{array}{ccc} a^*\widetilde{U} & \longrightarrow & \widetilde{U} \\ \downarrow & & \downarrow{\scriptstyle p} \\ P_X(a^*\widetilde{U}) & \longrightarrow & U \end{array}$$

This is exactly the input for the eliminator for path-types applied to the type a in context X. In particular, we have universal such data with $X = U^{(\square)}$. If we choose a solution to this universal lifting problem:

$$\begin{array}{ccc} a_{\Box}^{*}\widetilde{U} & \longrightarrow & \widetilde{U} \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ P_{U^{(\Box)}}(a_{\Box}^{*}\widetilde{U}) & \longrightarrow & U \end{array}$$

(where $a_{\Box} : U^{(\Box)} \to U$ is the evident projection) then we can use this to define the eliminator J for all path types in a coherent way by composition. Thus, our type theory models the full structure of identity types.

Definition 2.8. By **universe structure** on $p: \widetilde{U} \to U$, we will mean a choice of particular morphisms $U^{(1)} \to U$, $U^{(1)} \to U$, $\widetilde{U} \times_U \widetilde{U} \to U$, and $P_{U^{(\Box)}}(a_{\Box}^{*}\widetilde{U}) \to \widetilde{U}$ implementing dependent sums, dependent products, and identity types as above.

Remark 2.9. It is possible to make U into an internal category in $\mathscr{C}$, and the universe structure into internal operations on this category, reflecting the type-theoretic structure of $\mathscr{C}$ itself. This is analogous to how the subobject classifier in a topos automatically becomes an internal complete Heyting algebra, reflecting the logical operations on subobjects in the topos.

Remark 2.10. There are, of course, many other type constructors one might ask for in addition to dependent sums, dependent products, and identity types. The same techniques can be used to model all of them categorically. For instance, if $\mathscr{C}$ is extensive [CLW93] and copairing preserves small fibrations, then we can interpret disjoint union types, and if $\mathscr{C}$ has a small fibrant natural numbers object we can interpret the type $\mathbb{N}$. We will not require any such type constructors in this paper (aside from cartesian products, which are just the special case of dependent sums when the dependency is trivial), so we omit the details.

Finally, we consider the interpretation of universes *in* type theory (in contrast to the use of a universe "outside" the model of type theory to make the structure coherent). Suppose then that U and U' are two universes in the above sense, that every U-small fibration is also U'-small, and moreover that U itself is a small *object* relative to U'. From the first assumption, we can choose a pullback square

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\tilde{i}} & \widetilde{U}' \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ U & \xrightarrow[i]{} & U' \end{array} \tag{2.11}$$

and from the second we can choose some name $u: 1 \to U'$ for U. In the model of type theory constructed from U'-names, as above, we can then interpret a universe type Type by the name u.

If we want to *identify* terms of type Type with certain types (rather than using an "à la Tarski" coercion from the former to the latter), then we need an additional assumption that the map $U \to U'$ is monic, and moreover strictly respects the universe structure of U and U'. For example, the square

$$\begin{array}{ccc} U^{(1)} & \xrightarrow{i^{(1)}} & (U')^{(1)} \\ {\scriptstyle \Sigma_U}\downarrow & & \downarrow{\scriptstyle \Sigma_{U'}} \\ U & \xrightarrow[i]{} & U' \end{array} \tag{2.12}$$

must commute, where $\Sigma_U \colon U^{(1)} \to U$ and $\Sigma_{U'} \colon (U')^{(1)} \to U'$ are the maps implementing dependent sums for U and U'. The top map $i^{(1)} \colon U^{(1)} \to (U')^{(1)}$ is most easily described representably: given a pair $(X \xrightarrow{a} U,\, a^*\widetilde{U} \xrightarrow{b} U)$ corresponding to a morphism $X \to U^{(1)}$, the pullback square (2.11) tells us that $a^*\widetilde{U} \cong (ia)^*\widetilde{U}'$, so the pair

$$(X \xrightarrow{a} U \xrightarrow{i} U',\, (ia)^*\widetilde{U}' \cong a^*\widetilde{U} \xrightarrow{b} U \xrightarrow{i} U')$$

corresponds to a morphism $X \to (U')^{(1)}$. If (2.12) commutes, as well as the analogous squares for dependent products and identity types, we say that $i \colon U \hookrightarrow U'$ is an **embedding of universes**.

Remark 2.13. Suppose that $i \colon U \hookrightarrow U'$ is monic, and also that it *adds no new names* in the sense that any U'-name of a U-small type factors through U. Then any morphism implementing a type-forming operation for U' must preserve U-smallness, and hence induce a unique corresponding such morphism for U which commutes with $U \hookrightarrow U'$. Thus, in this case we can always choose the universe structure of U and U' so as to make $U \hookrightarrow U'$ an embedding of universes.

The same principle applies to arbitrarily many universes: we need one more universe in the category than we want to have in the resulting type theory, and all inclusions of universes must be universe embeddings as above.

Moreover, if in $\mathscr{C}$ we have a countably infinite sequence of universe embeddings

$$U_1 \hookrightarrow U_2 \hookrightarrow U_3 \hookrightarrow \dots,$$

then it is possible to model type theory with an infinite and exhaustive sequence of universes (that is, every type belongs to some universe Type_n) without requiring an extra containing universe U_ω to exist in $\mathscr{C}$. We do this by interpreting contexts as equivalence classes of diagrams

$$\begin{array}{ccccccccc} X_n & & \longrightarrow & & X_{n-1} \longrightarrow \cdots \longrightarrow X_1 & & \longrightarrow & & X_0 = 1 \\ & \searrow & & \swarrow & & \searrow & & \swarrow & \\ & \widetilde{U}_{k_n} & \longrightarrow & U_{k_n} & & \widetilde{U}_{k_1} & \longrightarrow & U_{k_1} & \end{array}$$

for some natural numbers k_i, $1 \le i \le n$. Now the equivalence relation allows not only isomorphisms of the X_is, but composition with the universe embeddings $U_k \hookrightarrow U_{k'}$ for $k \le k'$. Since the squares (2.11) are all pullbacks, the notions of types and terms in context, and the operation of context extension, are invariant under this equivalence relation. And since the universe embeddings commute with all the structure maps, the implementation of type-forming operations is also independent of the choice of the k_is. In particular, this allows us to interpret type theories with universe polymorphism, such as the predicative Calculus of Constructions.

Remark 2.14. Without a U_ω, however, we cannot apply Remark 2.13 to obtain an infinite sequence of universe embeddings. In some cases, we can ensure in some other way that the inclusions are embeddings. However, this is rarely a problem in practice, since any *particular* construction requires only finitely many universes.

The remaining problem is to *find* universes whose small fibrations are closed under the type-theoretic operations. Of course, in a category of contexts, a universe object arises from any universe type in the original type theory, but it is more difficult to find examples in categories arising from homotopy theory.

In the extensional case of a locally cartesian closed $\mathscr{C}$ with the trivial model structure, there is the following "tautological" approach. Choose a split fibration equivalent to the self-indexing of $\mathscr{C}$, and let $U \in [\mathscr{C}^{op}, \mathbf{Set}]$ be its presheaf of objects. Let $\widetilde{U}$ be the corresponding presheaf of *sections* — that is, an element of $\widetilde{U}(X)$ is an element of $U(X)$ together with a section of the corresponding map $A \to X$. Then a map is U-small just when its pullback along any map out of a representable presheaf is representable. Such maps are closed under composition and dependent products (since $\mathscr{C}$ is locally cartesian closed), so U is a universe for the trivial model structure on $[\mathscr{C}^{op}, \mathbf{Set}]$. Moreover, all contexts consist only of representables; thus the resulting model of (extensional) type theory lives entirely in $\mathscr{C}$. This essentially coincides with the classical approach to modeling extensional type theory in locally cartesian closed categories, see e.g. [Hof94].

As an even simpler example, we can take $U = \Omega$ to be the subobject classifier in an elementary topos, with $\widetilde{U} = 1$ the universal subobject. Then the small fibrations are exactly the monomorphisms, and the resulting "propositional" model of type theory lives entirely in the subterminal objects.

More interestingly, in the case of groupoids, we can take U to be the groupoid of groupoids of cardinality $< \kappa$, for some cardinal number κ. We let $\widetilde{U}$ be the corresponding groupoid of pointed groupoids: its objects are pairs (X, x_0) where X is a κ-small groupoid and x_0 an object of X, and its morphisms $(X, x_0) \to (Y, y_0)$ are pairs $(f \colon X \xrightarrow{\sim} Y,\ \phi\colon f(x_0) \cong y_0)$. Pullback of $\widetilde{U}$ along a map $A \to U$ implements the classical "Grothendieck construction"; thus a fibration over A is U-small exactly when its fibers are κ-small groupoids and it admits a splitting. If κ is inaccessible, such split fibrations are closed under all category-theoretic operations, and if $\lambda < \kappa$ is also inaccessible, we have a universe embedding $U_\lambda \hookrightarrow U_\kappa$. We thereby obtain the groupoid model of [HS98] with internal universes. We can also restrict to the sub-universe of *discrete* groupoids (called $\mathrm{Gpd}_\triangle(V_\kappa)$ in [HS98]).

Finally, and most importantly, Voevodsky [Voea] has shown that in simplicial sets, there is a *universal Kan fibration* $p\colon \widetilde{U} \to U$ such that U is a Kan complex, and every Kan fibration with fibers of cardinality $< \kappa$ (for some chosen cardinal κ) is U-small. See [KLV12] for a detailed exposition. As before, if κ is inaccessible, such fibrations are closed under category-theoretic operations, and if $\lambda < \kappa$ is also inaccessible, we have a universe embedding $U_\lambda \hookrightarrow U_\kappa$ (either from Remark 2.13 or by choosing the structure carefully). There are also sub-universes which classify n-truncated Kan fibrations (those whose homotopy groups above n are trivial).

Thus, we obtain a model of intensional type theory, with universes, in which the fibrations are the Kan fibrations. Perhaps the most important fact about this model is that its internal universes satisfy the *univalence axiom*. We will recall this axiom in the next section.

3. Homotopy type theory in models

We recall some definitions, also due to Voevodsky [Voeb], for doing homotopy theory inside of type theory. We denote the identity type of a type A by

$$x\colon A, y\colon A \vdash (x = y)\colon \mathsf{Type}.$$

The eliminator of identity types implies, in particular, that we have an operation of *transport*, which we denote as follows:

$$A\colon \mathsf{Type}, B\colon A \to \mathsf{Type}, x\colon A, y\colon A, p\colon (x = y), b\colon B(x) \vdash p_* b\colon B(y).$$

For any type A, we have the type

$$\mathrm{isContr}(A) \coloneqq \sum_{x\colon A} \prod_{y\colon A} (x = y)$$

which expresses the assertion that A is contractible. Similarly, we define

$$\mathrm{isProp}(A) \coloneqq \prod_{x\colon A} \prod_{y\colon A} (x = y)$$

which expresses the assertion that A is "(-1)-truncated", i.e. contractible if it is inhabited. There are many other definitions of isProp, all equivalent under function extensionality (see below). We say A *is a proposition* if $\mathrm{isProp}(A)$ is inhabited.

Now given A, B and $f\colon A \to B$, we define

$$\mathrm{isEquiv}(f) \coloneqq \prod_{b\colon B} \mathrm{isContr}\left(\sum_{a\colon A} (f(a) = b)\right)$$

which expresses the assertion that f is an equivalence in the sense that its homotopy fibers are contractible. Finally, for A and B we define

$$\mathrm{Equiv}(A, B) \coloneqq \sum_{f\colon A \to B} \mathrm{isEquiv}(f).$$

representing the "space of equivalences" from A to B.

Now let $\mathscr{C}$ be a category with the structure considered in §2, so that it interprets type theory. The path objects in $\mathscr{C}$ give rise to a notion of (right) *homotopy* and thus a notion of *homotopy equivalence*. It is easy to show that every acyclic cofibration between fibrant objects is a homotopy equivalence, and that homotopy equivalences satisfy the 2-out-of-3 property.

We will call a morphism between fibrant objects an *acyclic fibration* if it is both a fibration and a homotopy equivalence. (In a type-theoretic model category where all objects are cofibrant, every weak equivalence between fibrant objects is a homotopy equivalence; so in that case this terminology agrees with the established one.) We now explain the meaning of the above type-theoretic definitions in $\mathscr{C}$.

Firstly, by the defining adjunction for dependent products, $\mathrm{isProp}(A)$ has a global element precisely when the path fibration $PA \to A \times A$ of A has a section, which is to say that the two projections $A \times A \rightrightarrows A$ are homotopic. This is equivalent to saying that *any* two morphisms $X \rightrightarrows A$ are homotopic.

Similarly, to give a global element of $\mathrm{isContr}(A)$ is to give a global element $a\colon 1 \to A$ together with a homotopy relating the composite $A \to 1 \to A$ to the identity, which is equivalent to saying that $A \to 1$ is a homotopy equivalence. Since A is assumed fibrant, this is equivalent to $A \to 1$ being an acyclic fibration. By slicing — which corresponds to working in a nonempty context in type theory —

we can then conclude that for any fibration $A \to B$ between fibrant objects, the fibration represented by the dependent type

$$b\colon B \vdash \mathrm{isContr}(A(b))\colon \mathsf{Type}$$

has a section precisely when $A \to B$ is an acyclic fibration. Moreover, this is also equivalent, by adjunction, to $\prod_{b\colon B} \mathrm{isContr}(A)$ having a global element.

Remark 3.1. In particular, this implies that *acyclic fibrations are stable under pullback*. Thus, if we define a "weak equivalence" to be a homotopy equivalence, the subcategory of fibrant objects in $\mathscr{C}$ (which is where all the type theory lives) becomes a *category of fibrant objects* in the sense of [Bro74].

Finally, we observe that any $f\colon A \to B$ (not necessarily a fibration) factors as

$$A \longrightarrow Pf \longrightarrow B$$

where $Pf \to B$ is the *mapping path fibration*, representing the dependent type

$$b\colon B \vdash \sum_{a\colon A} (f(a) = b)\colon \mathsf{Type}.$$

Moreover, $A \to Pf$ is a homotopy equivalence, so by 2-out-of-3, f is a homotopy equivalence just when $Pf \to B$ is an acyclic fibration. But by definition of isEquiv and the above remarks, this is precisely to say that $\mathrm{isEquiv}(f)$ has a global element.

In conclusion, we can say that for A, B fibrant and $f\colon A \to B$:

- $\mathrm{isProp}(A)$ has a global element $\iff$ any $f, g\colon X \rightrightarrows A$ are homotopic.
- $\mathrm{isContr}(A)$ has a global element $\iff$ $A \to 1$ is an acyclic fibration.
- $\mathrm{isEquiv}(f)$ has a global element $\iff$ f is a homotopy equivalence.

From now on, we will say simply *equivalence* rather than "homotopy equivalence".

Still following Voevodsky, we say that *function extensionality* holds if we have a term in context of the following type:

$$A\colon \mathsf{Type},\ B\colon A \to \mathsf{Type}\ \vdash\ \mathrm{funext}\colon \textstyle\prod_a \mathrm{isContr}(B(a)) \to \mathrm{isContr}(\prod_a B(a))$$

It is shown in [Voeb] (see also [HTT]) that this implies a seemingly stronger form of function extensionality, which states that the canonically defined term

$$A\colon \mathsf{Type},\ B\colon A \to \mathsf{Type},\ f\colon \textstyle\prod_a B(a),\ g\colon \prod_a B(a)$$
$$\vdash \mathrm{happly}\colon (f = g) \to \textstyle\prod_a (f(a) = g(a))$$

is an equivalence. This implies all other forms of (propositional) function extensionality, such as those considered in [Gar09] (see [Lum11]). In particular, function extensionality implies that $\mathrm{isProp}(A)$, $\mathrm{isContr}(A)$, and $\mathrm{isEquiv}(f)$ are propositions.

In terms of the category $\mathscr{C}$, function extensionality holds just when for any fibrations $P \xrightarrow{f} X \xrightarrow{g} A$ between fibrant objects, there is a map

$$\Pi_g \mathrm{isContr}_X(P) \to \mathrm{isContr}_A(\Pi_g P).$$

By Yoneda and the definition of Π_g, this means that for any $h\colon B \to A$, if there exists a map from h^*X to $\mathrm{isContr}_X(P)$ over X, then there exists a map from B to $\mathrm{isContr}_A(\Pi_g P)$ over A. By the above characterization of isContr, slicing, and preservation of all structure by pullback, this means that if the pullback $h^*P \to h^*X$ is an acyclic fibration, then so is $h^*(\Pi_g P) \to B$. In particular, this means that whenever $f\colon P \to X$ is an acyclic fibration, then so is $\Pi_g(f)$. However, this special case implies the general one, by the Beck-Chevalley condition for dependent products. Thus

- Function extensionality holds $\iff$ dependent products along fibrations between fibrant objects preserve acyclicity of fibrations.

Remark 3.2. If the acyclic fibrations are (the restriction to fibrant objects of) the right class in a weak factorization system, then this condition is equivalent to requiring pullback along fibrations between fibrant objects to preserve the corresponding left class. Thus, it holds in any type-theoretic model category.

Additionally, function extensionality implies that the following type is equivalent to isEquiv(f) (see [HTT]):

$$\mathrm{ishIso}(f) :\equiv \left(\sum_{s\colon B\to A}\prod_{b\colon B}(f(s(b))=b)\right) \times \left(\sum_{r\colon B\to A}\prod_{a\colon A}(r(f(a))=a)\right) \tag{3.3}$$

As we will see, this type is often much simpler to work with, due to the facts that it only involves "level-1" path types (no paths between paths), and that its two halves appear very symmetric. It was first suggested in this context by André Joyal at the Oberwolfach workshop [AGMLV11].

Finally, we consider Voevodsky's univalence axiom. This axiom depends on having an internal universe in the type theory, and is stated relative to a particular such universe; we denote the chosen universe by Type. Since identity maps are equivalences, we have a canonical term

$$A\colon \mathsf{Type} \vdash \mathrm{idequiv}_A\colon \mathrm{Equiv}(A,A)$$

By induction over paths, this gives rise to a canonically defined term

$$A\colon \mathsf{Type}, B\colon \mathsf{Type} \vdash \mathrm{pathToEquiv}_{A,B}\colon (A=B)\to \mathrm{Equiv}(A,B).$$

Of course, $(A = B)$ denotes the identity type of the universe Type. We say the *univalence axiom holds* for the universe Type, or that Type *is univalent*, if the type

$$\prod_{A,B}\mathrm{isEquiv}(\mathrm{pathToEquiv}_{A,B})$$

is globally inhabited.

In categorical terms, this states that the canonically defined map $PU \to E$ over $U\times U$ is an equivalence, where $E\to U\times U$ is the fibration representing the dependent type

$$A\colon \mathsf{Type}, B\colon \mathsf{Type} \vdash \mathrm{Equiv}(A,B)\colon \mathsf{Type}.$$

Since this map $PU \to E$ is defined by the lifting property of PU (i.e. path induction), by the 2-out-of-3 property this is equivalent to saying that the map $U \to E$, which sends a type A to its identity equivalence, is itself an equivalence.

In extensional type theory, a universe can probably only be univalent if all its types are subterminal. (For instance, the subobject classifier in a topos is univalent.) But in intensional type theory, we can have interesting univalent universes. For instance, the groupoid of small sets is univalent; thus the groupoid model of [HS98] can include one univalent universe. However, since this universe classifies only discrete fibrations, but is not itself discrete, this model cannot contain more than one nested univalent universe. (The groupoid of small groupoids is not univalent.)

On the other hand, Voevodsky [Voea] has shown that any universal Kan fibration in simplicial sets is univalent (see [Moe12, KLV12] for alternative proofs). These universes can be nested arbitrarily, since their small fibrations are restricted only by size and not truncation. Thus, we have:

Theorem 3.4 (Voevodsky)**.** *The model category* sSet *supports a model of intensional type theory with dependent sums and products, identity types, and with one fewer univalent universe than there are inaccessible cardinals.*

Since the homotopy theory of simplicial sets is a model for the $(\infty, 1)$-topos ∞Gpd, we can say informally that the above model lives in that $(\infty, 1)$-topos.

We can obtain a few other models of type theory easily from this theorem. For instance, since the universe of n-truncated Kan fibrations is $(n+1)$-truncated, we can also obtain a model with countably many universes in which truncation level increases with universe level—and there are other similar modifications. (The universe of 0-truncated Kan fibrations is of course closely related to the groupoid of sets.) Finally, we can pull back any univalent universe to the slice category over any fibrant object. However, it seems that until now, no other set-theoretic models of univalence have been known.

Remark 3.5. Voevodsky has also shown that the univalence axiom implies function extensionality. Specifically, if there are two nested univalent universes, then function extensionality holds for all types belonging to the smaller universe. In what follows, we will need to apply function extensionality even for Type-valued functions (that is, dependent types). This can be deduced from a third nested univalent universe—or from the observation above that any type-theoretic model category satisfies function extensionality.

4. The Sierpinski $(\infty, 1)$-topos

Before considering inverse diagrams in general, we treat in detail one particular case, which contains essentially all the ideas. Thus, let $\mathscr{C}$ have all the structure considered in §2, and let $\mathscr{C}^2$ denote the category of arrows $(\alpha \colon A_1 \to A_0)$ of $\mathscr{C}$. We will construct a model of type theory in $\mathscr{C}^2$ from the one in $\mathscr{C}$.

Definition 4.1. A morphism

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha} & A_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ B_1 & \xrightarrow[\beta]{} & B_0 \end{array}$$

in $\mathscr{C}^2$ is a **Reedy fibration** if

(1) f_0 is a fibration, and
(2) The induced map $A_1 \to A_0 \times_{B_0} B_1$ is a fibration.

On the other hand, f is a **Reedy acyclic cofibration** if f_0 and f_1 are acyclic cofibrations in $\mathscr{C}$.

Remark 4.2. An object $(\alpha \colon A_1 \to A_0)$ of $\mathscr{C}^2$ is Reedy fibrant iff A_0 is fibrant and α is a fibration. Thus, in the type theory of $\mathscr{C}$, the Reedy fibrant objects of $\mathscr{C}^2$ can be regarded as *2-type contexts* of the form

$$a_0 \colon A_0,\ a_1 \colon A_0(a_0).$$

This point of view will be crucial in what follows.

It is easy (and standard, see [Hov99, Hir03]) to prove that the Reedy acyclic cofibrations and Reedy fibrations form a weak factorization system on $\mathscr{C}^{\mathbf{2}}$. Since fibrations are closed under pullback and composition, if f is a Reedy fibration, then f_1 is also a fibration — thus Reedy fibrations are in particular levelwise fibrations. Since limits are also levelwise in $\mathscr{C}^{\mathbf{2}}$, it follows that all pullbacks of Reedy fibrations between Reedy fibrant objects exist, and such pullback preserves Reedy acyclic cofibrations. Therefore, to construct a model of type theory in $\mathscr{C}^{\mathbf{2}}$, it remains only to construct dependent sums and products, path objects, and one or more universes.

Remark 4.3. If $\mathscr{C}$ is a model category, then the Reedy fibrations are the fibrations in a model structure on $\mathscr{C}^{\mathbf{2}}$ whose cofibrations and weak equivalences are both defined levelwise. If $\mathscr{C}$ is simplicial sets, then the Reedy model structure on $\mathrm{sSet}^{\mathbf{2}}$ presents the $(\infty, 1)$-category $\infty\mathrm{Gpd}^{\mathbf{2}}$.

Moreover, if $\mathscr{C}$ is right proper, locally cartesian closed, and has pullback-stable cofibrations, then $\mathscr{C}^{\mathbf{2}}$ inherits all of these properties; thus all it lacks is a universe. However, the detailed constructions of dependent sums and products and path objects we present below are still necessary for our proof of univalence in §6.

Thus, assume that a universe $p\colon \widetilde{U} \to U$ is given in $\mathscr{C}$, defining a notion of *small fibration* in $\mathscr{C}$ which is closed under composition, dependent product, and path objects in the senses described in §2. We define a morphism $q\colon \widetilde{V} \to V$ in $\mathscr{C}^{\mathbf{2}}$ as follows. Set $V_0 = U$, $\widetilde{V}_0 = \widetilde{U}$, and $q_0 = p$. Let $V_1 = U^{(1)} = (U \times U \to U)^{(\widetilde{U}\to U)}$, with $V_1 \to V_0$ being the projection $U^{(1)} \to U$; since this is a fibration, V is Reedy fibrant. Finally, by definition V_1 comes with an evaluation map $V_1 \times_U \widetilde{U} \to U \times U$ over U, which is to say an arbitrary map $V_1 \times_U \widetilde{U} \to U$; define $\widetilde{V}_1 \to V_1 \times_{V_0} \widetilde{V}_0$ to be the fibration named by this map. Then by construction, q is a Reedy fibration.

In the type theory of $\mathscr{C}$, V_0 is the universe Type, while the fibration $V_1 \to V_0$ represents the dependent type

$$A\colon \mathsf{Type} \vdash (A \to \mathsf{Type})\colon \mathsf{Type}_1.$$

The fibration $\widetilde{V}_0 \to V_0$ is of course the universal dependent type $A\colon \mathsf{Type} \vdash A\colon \mathsf{Type}$ in $\mathscr{C}$, while $\widetilde{V}_1 \to V_1 \times_{V_0} \widetilde{V}_0$ represents the dependent type

$$A_0\colon \mathsf{Type},\ A_1\colon A_0 \to \mathsf{Type},\ a_0\colon A_0 \vdash A_1(a_0)\colon \mathsf{Type}.$$

Definition 4.4. A map $f\colon A \to B$ in $\mathscr{C}^{\mathbf{2}}$ is called a **Reedy small-fibration** if both f_0 and the induced map $A_1 \to A_0 \times_{B_0} B_1$ are small fibrations in $\mathscr{C}$.

Proposition 4.5. *A map $f\colon A \to B$ is a Reedy small-fibration if and only if it is small with respect to V, i.e. it is a pullback of q along some map $B \to V$.*

Proof. By construction, q is a Reedy small-fibration, and this property is evidently preserved under pullback. Conversely, suppose $f\colon A \to B$ is a Reedy small-fibration. Since f_0 is a small fibration, it is named by some map $a_0\colon B_0 \to U = V_0$. Then the composite $B_1 \xrightarrow{\beta} B_0 \xrightarrow{a_0} U$ names the pullback $A_0 \times_{B_0} B_1$. Since $A_1 \to A_0 \times_{B_0} B_1$ is a small fibration, it has a name which supplies a lifting, say a_1, of $a_0\beta$ to $U^{(1)} = V_1$. Then $a\colon B \to V$ is a name for f with respect to V. □

Thus, for V to be a universe in $\mathscr{C}^{\mathbf{2}}$, it suffices to check that the Reedy small-fibrations are closed under all desired type-theoretic operations.

Proposition 4.6. *If small fibrations in $\mathscr{C}$ are closed under composition, then so are Reedy small-fibrations in $\mathscr{C}^{\mathbf{2}}$.*

Proof. Suppose given small fibrations $A \xrightarrow{f} B \xrightarrow{g} C$. Then $(gf)_0 = g_0 f_0$ is small by assumption. Moreover, the induced map $A_1 \to A_0 \times_{C_0} C_1$ is the composite

$$A_1 \longrightarrow A_0 \times_{B_0} B_1 \longrightarrow A_0 \times_{B_0} (B_0 \times_{C_0} C_1) \xrightarrow{\cong} A_0 \times_{C_0} C_1$$

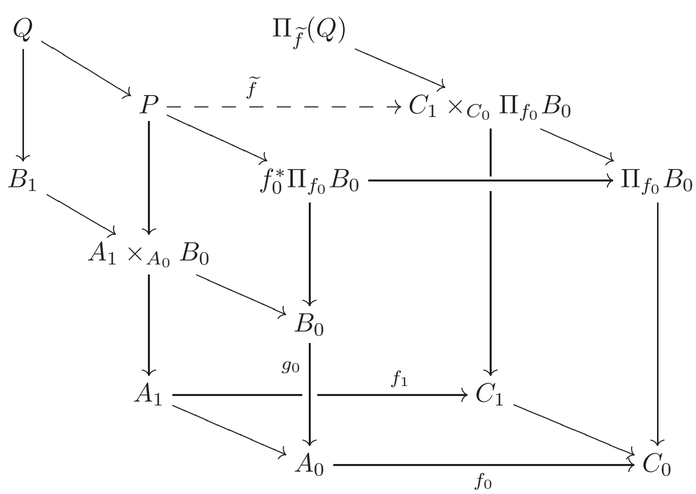
where the first map is small since f is small, and the second is small since it is a pullback of $B_1 \to B_0 \times_{C_0} C_1$, which is small since g is small. □

Remark 4.7. If small fibrations in $\mathscr{C}$ are closed under composition, then a Reedy small-fibration $f \colon A \to B$ has the property that both f_0 and f_1 are small fibrations. Conversely, if the small fibrations in $\mathscr{C}$ are "left-cancellable" (i.e. if g and f are fibrations and g and $g \circ f$ are small, then f is also small), then a Reedy fibration with this property is automatically a Reedy small-fibration. Left-cancellability holds whenever smallness is characterized by a downward-closed cardinality condition on the fibers, as is the case for the univalent universe in simplicial sets.

Proposition 4.8. *If small fibrations in $\mathscr{C}$ are closed under dependent products, then so are Reedy small-fibrations in $\mathscr{C}^{\mathbf{2}}$.*

Proof. Let $f \colon A \to C$ and $g \colon B \to A$ be Reedy small-fibrations, and consider the following diagram.

(4.9)



Here the objects P and Q are defined so as to make the squares

$$\begin{array}{ccc} P & \longrightarrow & f_0^*\Pi_{f_0} B_0 \\ \downarrow & & \downarrow \\ A_1 \times_{A_0} B_0 & \longrightarrow & B_0 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ B_1 & \longrightarrow & A_1 \times_{A_0} B_0 \end{array}$$

(which appear in the above diagram) pullback squares. By the pasting law for pullbacks, the left-hand face of the cube shown is a pullback, and since the front and right-hand faces are also pullbacks by definition, so is the back face:

$$\begin{array}{ccc} P & \xrightarrow{\widetilde{f}} & C_1 \times_{C_0} \Pi_{f_0} B_0 \\ \downarrow & & \downarrow \\ A_1 & \xrightarrow[f_1]{} & C_1. \end{array}$$

The map $\widetilde{f}$, of course, is induced by the universal property of $C_1 \times_{C_0} \Pi_{f_0} B_0$.

Now, since f_0 and g_0 are small fibrations, so is $\Pi_{f_0} B_0 \to C_0$. And since f_1 is a composite of two small fibrations

$$A_1 \to C_1 \times_{C_0} A_0 \to C_1,$$

(using the assumption that f is a Reedy small-fibration) so is its pullback $\widetilde{f}$. By assumption, this implies that $\Pi_{\widetilde{f}}$ preserves small fibrations.

However, since g is a Reedy small-fibration, the map $B_1 \to A_1 \times_{A_0} B_0$ is a small fibration, and hence so is its pullback $Q \to P$; thus the map $\Pi_{\widetilde{f}}(Q) \to C_1 \times_{C_0} \Pi_{f_0} B_0$ is also a small fibration. So if we define $(\Pi_f B)_0 = \Pi_{f_0} B_0$ and $(\Pi_f B)_1 = \Pi_{\widetilde{f}}(Q)$, we have a Reedy small-fibration $\Pi_f B \to C$. It is straightforward to verify that this is actually the dependent product of $B \to A$ along f in $\mathscr{C}^{\mathbf{2}}$. $\square$

In the type theory of $\mathscr{C}$, the construction of Proposition 4.8 can be described as follows. We are given dependent types

$$\begin{aligned} &\vdash C_0 \colon \mathsf{Type}_1 \\ c_0 \colon C_0 &\vdash C_1(c_0) \colon \mathsf{Type}_1 \\ c_0 \colon C_0 &\vdash A_0(c_0) \colon \mathsf{Type} \\ c_0 \colon C_0,\, c_1 \colon C_1(c_0),\, a_0 \colon A_0(c_0) &\vdash A_1(c_0, c_1, a_0) \colon \mathsf{Type} \\ c_0 \colon C_0,\, a_0 \colon A_0(c_0) &\vdash B_0(c_0, a_0) \colon \mathsf{Type} \\ \dots, a_1 \colon A_1(c_0, c_1, a_0),\, b_0 \colon B_0(c_0, a_0) &\vdash B_1(c_0, c_1, a_0, a_1, b_0) \colon \mathsf{Type} \end{aligned}$$

and we define

$$c_0 \colon C_0 \vdash (\Pi_f B)_0(c_0) \coloneqq \textstyle\prod_{a_0 \colon A_0(c_0)} B_0(c_0, a_0)$$

and

$$\begin{aligned} &c_0 \colon C_0,\, c_1 \colon C_1(c_0),\, f_0 \colon \textstyle\prod_{a_0 \colon A_0(c_0)} B_0(c_0, a_0) \\ &\qquad \vdash (\Pi_f B)_1(c_0, c_1, f_0) \coloneqq \textstyle\prod_{a_0 \colon A_0(c_0)} \prod_{a_1 \colon A_0(c_0, c_1, a_0)} B_1(c_0, c_1, a_0, a_1, f_0(a_0)) \end{aligned}$$

Since small types are closed under all type-theoretic operations, this is clearly small if all the relevant inputs are. (Note that C_0 and C_1 need not be small.)

Proposition 4.10. *If $\mathscr{C}$ has small stable path objects in the sense of §2, then so does $\mathscr{C}^{\mathbf{2}}$.*

Proof. Suppose $A \to B$ is a Reedy small-fibration. Let $P_{B_0} A_0 \to A_0 \times_{B_0} A_0$ and $P_{A_0} A_1 \to A_1 \times_{A_0} A_1$ be small stable path objects in $\mathscr{C}$. Define $(P_B A)_0 = P_{B_0} A_0$; then $(P_B A)_0 \to (A \times_B A)_0$ is a small fibration. For $P_B A \to A \times_B A$ to be a Reedy small-fibration, we need a small fibration

$$(P_B A)_1 \to (A_1 \times_{B_1} A_1) \times_{(A_0 \times_{B_0} A_0)} P_{B_0} A_0.$$

We will obtain this as the pullback of $P_{A_0}A_1 \to A_1 \times_{A_0} A_1$ along a map

$$(A_1 \times_{B_1} A_1) \times_{(A_0 \times_{B_0} A_0)} P_{B_0}A_0 \longrightarrow A_1 \times_{A_0} A_1.$$

Such a map is, of course, determined by two maps

$$(A_1 \times_{B_1} A_1) \times_{(A_0 \times_{B_0} A_0)} P_{B_0}A_0 \rightrightarrows A_1 \tag{4.11}$$

which agree in A_0. We take one of these maps to be simply the projection onto the second factor A_1 appearing in the domain. We cannot take the other to be projection onto the first factor, however, since these two projections do not agree in A_0. Instead, we consider the following square:

$$\begin{array}{ccccc} (A_1 \times_{B_1} A_1) \times_{(A_0 \times_{B_0} A_0)} A_0 & \xrightarrow{\cong} & A_1 \times_{A_0} A_1 & \xrightarrow{\pi_1} & A_1 \\ \downarrow & & & & \downarrow \\ (A_1 \times_{B_1} A_1) \times_{(A_0 \times_{B_0} A_0)} P_{B_0}A_0 & & \xrightarrow[\pi_2]{} & & A_0 \end{array} \tag{4.12}$$

Here π_1 denotes the projection onto the *first* factor of $A_1 \times_{A_0} A_1$, while π_2 denotes projection onto the *second* factor of A_0 appearing in its domain. The reader will easily verify that this square nevertheless commutes. Since the right-hand map is a fibration, and the left-hand map is an acyclic cofibration (being the pullback of the acyclic cofibration $A_0 \to P_{B_0}A_0$ along a fibration of fibrant objects), there is a lift, and indeed a specified coherent one (using the assumed universe U). We take this coherent lift as the second map in (4.11).

This completes the definition of a Reedy small-fibration $PA \to A \times A$. Now we need the diagonal to factor through it by an acyclic cofibration. Consider first the following diagram

$$\begin{array}{ccccc} P_{A_0}A_1 & \longrightarrow & (P_{B_1}A)_1 & \longrightarrow & P_{A_0}A_1 \\ \downarrow & & \downarrow & & \downarrow \\ A_1 \times_{A_0} A_1 & \longrightarrow & (A_1 \times_{B_1} A_1) \times_{(A_0 \times_{B_0} A_0)} P_{B_0}A_0 & \longrightarrow & A_1 \times_{A_0} A_1 \\ \downarrow & & \downarrow & & \\ A_0 & \longrightarrow & P_{B_0}A_0 & & \end{array} \tag{4.13}$$

The upper-right square is a pullback by definition, and the lower-left square is a pullback by inspection. The composite across the middle is the identity morphism of $A_1 \times_{A_0} A_1$, and thus the outer top rectangle is also a pullback. Hence, by the pasting law for pullback squares, the upper-left square is also a pullback. However, all the vertical maps are fibrations of fibrant objects, and the lower map $A_0 \to P_{B_0}A_0$ is an acyclic cofibration; hence its pullback $P_{A_0}A_1 \to (P_{B_0}A)_1$ is also. Composing this with the defining acyclic cofibration $A_1 \to P_{A_0}A_1$ gives our desired factorization. Finally, coherency of the lift in (4.12), and stability of path objects in $\mathscr{C}$, imply stability for these path objects in $\mathscr{C}^{\mathbf{2}}$. □

In terms of the type theory of $\mathscr{C}$, the path objects in $\mathscr{C}^{\mathbf{2}}$ are defined as follows:

$$a_0 \colon A_0,\, a_0' \colon A_0 \vdash\; (PA)_0(a_0, a_0') :\equiv (a_0 = a_0')$$

$$\ldots,\, a_1 \colon A_1(a_0),\, a_1' \colon A_1(a_0'), p \colon a_0 = a_0' \vdash (PA)_1(a_0, a_0', a_1, a_1', p) :\equiv (p_* a_1 = a_1')$$

where, as in §3, p_* denotes transport in the fibration $A_1 \to A_0$ along the path p.

In conclusion, we have proven the following theorem.

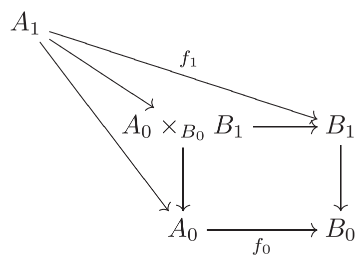
Theorem 4.14. *If $\mathscr{C}$ models dependent type theory with dependent sums and products and identity types using a universe U as described in §2, then so does $\mathscr{C}^{\mathbf{2}}$ using the universe V.* □

We end this section with two further important observations about the type theory of $\mathscr{C}^{\mathbf{2}}$.

Proposition 4.15. *Under the hypotheses of Proposition 4.10, the homotopy equivalences in $\mathscr{C}^{\mathbf{2}}$ are the levelwise homotopy equivalences in $\mathscr{C}$.*

By the observations in §2, this is immediate when $\mathscr{C}$ is a type-theoretic model category, since in that case $\mathscr{C}^{\mathbf{2}}$ has its own model structure with levelwise weak equivalences. Thus, we will only sketch the general case.

Sketch of proof. Since fibrations and acyclic cofibrations in $\mathscr{C}^{\mathbf{2}}$ are in particular levelwise, so are homotopy equivalences. For the converse, it suffices to show that any Reedy fibration which is a levelwise equivalence is a homotopy equivalence. For such an $f\colon A \to B$, in the diagram



the map $A_0 \times_{B_0} B_1 \to B_1$ is the pullback of an acyclic fibration along a fibration of fibrant objects, hence itself an acyclic fibration. Since f_1 is also an equivalence, the map $A_1 \to A_0 \times_{B_0} B_1$ is also an acyclic fibration.

Since acyclic fibrations are deformation retractions, we can find a section g_0 of f_0 with a homotopy $g_0 f_0 \sim \mathrm{id}$. By pullback, we obtain a section h of $A_0 \times_{B_0} B_1 \to B_1$ lying over f_0, with a corresponding homotopy that lies over the homotopy $g_0 f_0 \sim \mathrm{id}$ in a suitable sense. Similarly, we have a section k of $A_1 \to A_0 \times_{B_0} B_1$ with a similar homotopy, all lying over A_0. Defining $g_1 := kh$, and composing the homotopies, then gives a deformation section of f in $\mathscr{C}^{\mathbf{2}}$. □

Proposition 4.16. *If $\mathscr{C}$ satisfies function extensionality, then so does $\mathscr{C}^{\mathbf{2}}$.*

Proof. Let g be a Reedy acyclic fibration in $\mathscr{C}^{\mathbf{2}}$. By Proposition 4.15, this amounts to saying that g is a Reedy fibration and g_0 and g_1 are acyclic fibrations in $\mathscr{C}$, or equivalently (by the 2-out-of-3 property and pullback-stability of acyclic fibrations) that g_0 and the induced map $B_1 \to A_1 \times_{A_0} B_0$ are acyclic fibrations.

Now the construction of Π_g in (4.9) works regardless of smallness hypotheses. Thus, using again the pullback-stability of acyclic fibrations and the assumption on $\mathscr{C}$, we see that $\Pi_{f_0} B_0 \to C_0$ and $\Pi_{\widetilde{f}}(Q) \to C_1 \times_{C_0} \Pi_{f_0} B_0$ are acyclic fibrations. Hence $\Pi_f B \to C$ is an acyclic fibration in $\mathscr{C}^{\mathbf{2}}$, as desired. □

5. Universes in the Sierpinski $(\infty,1)$-topos

We now consider how *internal* universes in the type theory of $\mathscr{C}$ lift to $\mathscr{C}^2$. By §2, it suffices to show that any universe embedding in $\mathscr{C}$ lifts to $\mathscr{C}^2$.

Remark 5.1. Suppose $i\colon U \hookrightarrow U'$ is a monomorphism of universes in $\mathscr{C}$ such that U is U'-small, every U-small fibration is U'-small, and i adds no new names (in the sense of Remark 2.13); thus i can be made into a universe embedding. Let V and V' be the corresponding universes in $\mathscr{C}^2$; then it is easy to see that V is V'-small, every Reedy V-small fibration is Reedy V'-small, and we have a monomorphism $j\colon V \hookrightarrow V'$ which adds no new names. Hence $j\colon V \hookrightarrow V'$ can be made into a universe embedding as well.

In the rest of this section, we show that the same is true for *any* universe embedding in $\mathscr{C}$, whether or not it adds new names. In particular, this shows that a countably infinite sequence of universe embeddings can also be lifted to $\mathscr{C}^2$. A reader who is uninterested in this generalization can freely skip this section.

To show that universe embeddings are preserved, we have to be a little careful about the universe structure on the universe V defined in §4. Propositions 4.6, 4.8, and 4.10 only tell us that we can make *some* choice of universe structure on V, but in fact, any choice of universe structure on U *canonically* induces universe structure on V. To obtain this structure, we simply phrase the constructions of small dependent sums, products, and path-objects from Propositions 4.6, 4.8, and 4.10 in type theory, then translate them into morphisms in $\mathscr{C}$ using the given universe structure of U.

For instance, consider dependent sums. The Reedy fibration $V^{(1)} \to V$ can be written in terms of $\mathscr{C}$ as

$$\begin{array}{ccc} U^{(1\times 1)} & \longrightarrow & U^{(1)} \\ \downarrow & & \downarrow \\ U^{(1)} & \longrightarrow & U \end{array}$$

where $U^{(1\times 1)}$ has the universal property that maps $X \to U^{(1\times 1)}$ correspond naturally to quadruples

$$\Big(X \xrightarrow{a} U,\ a^*\widetilde{U} \xrightarrow{b} U,\ a^*\widetilde{U} \xrightarrow{c} U,\ b^*\widetilde{U} \times_{a^*\widetilde{U}} c^*\widetilde{U} \xrightarrow{d} U\Big). \tag{5.2}$$

If we denote by $\Sigma_U\colon U^{(1)} \to U$ the specified morphism implementing dependent sums for U in $\mathscr{C}$, then we can define a morphism $\Sigma_V\colon V^{(1)} \to V$:

$$\begin{array}{ccc} U^{(1\times 1)} & \longrightarrow & U^{(1)} \\ {\scriptstyle (\Sigma_V)_1}\downarrow & & \downarrow{\scriptstyle (\Sigma_V)_0=\Sigma_U} \\ U^{(1)} & \longrightarrow & U. \end{array}$$

Here $(\Sigma_V)_1$ is defined representably as follows. Given a quadruple (5.2), we have a named type $\Sigma_U(a,c)\colon X \to U$ such that $(\Sigma_U(a,c))^*\widetilde{U} \cong c^*\widetilde{U}$. Now the composite $c^*\widetilde{U} \to a^*\widetilde{U} \xrightarrow{b} U$ is a name for $b^*\widetilde{U} \times_{a^*\widetilde{U}} c^*\widetilde{U}$, so this composite together with d gives us a map $c^*\widetilde{U} \to U^{(1)}$. Composing with $\Sigma_U\colon U^{(1)} \to U$, we obtain a map $(\Sigma_U(a,c))^*\widetilde{U} \to U$. Together with $\Sigma_U(a,c)\colon X \to U$, this gives us a map $X \to U^{(1)}$, as desired. It is easy to check that this map implements dependent sums for $\mathscr{C}^2$.

The corresponding definitions for dependent products and identity types are similar. Using this canonical structure, we can deal with universe embeddings.

Proposition 5.3. *If $i\colon U \hookrightarrow U'$ is a universe embedding in $\mathscr{C}$, then there is an induced universe embedding $j\colon V \hookrightarrow V'$ in $\mathscr{C}^2$.*

Proof. We define $j_0\colon V_0 \to (V')_0$ to be $i\colon U \to U'$, and $j_1\colon V_1 \to (V')_1$ to be the map $i^{(1)}\colon U^{(1)} \to (U')^{(1)}$ defined after (2.12). To start with, we need a pullback square

$$\begin{array}{ccc} \widetilde{V} & \longrightarrow & \widetilde{V}' \\ \downarrow & & \downarrow \\ V & \longrightarrow & V' \end{array}$$

in $\mathscr{C}^2$, which will be a cube

$$\begin{tikzcd}[row sep=small] (\mathrm{ev}_U)^*\widetilde{U} \ar[rr] \ar[dd] \ar[dr] & & (\mathrm{ev}_{U'})^*\widetilde{U}' \ar[dd] \ar[dr] & \\ & \widetilde{U} \ar[rr] \ar[dd] & & \widetilde{U}' \ar[dd] \\ U^{(1)} \ar[rr] \ar[dr] & & (U')^{(1)} \ar[dr] & \\ & U \ar[rr, "i"'] & & U'. \end{tikzcd} \tag{5.4}$$

Here $\widetilde{V}_1 = (\mathrm{ev}_U)^*\widetilde{U}$ has the universal property that maps $X \to (\mathrm{ev}_U)^*\widetilde{U}$ correspond naturally to triples

$$\Big(X \xrightarrow{a} U,\ a^*\widetilde{U} \xrightarrow{b} U,\ X \xrightarrow{s} b^*\widetilde{U}\Big)$$

where s is a section of $b^*\widetilde{U} \to a^*\widetilde{U} \to X$. Of course, $(\widetilde{V}')_1 = (\mathrm{ev}_{U'})^*\widetilde{U}'$ is analogous, and the map $\widetilde{V}_1 \to (\widetilde{V}')_1$ is given by composing the components a and b with i.

Now the front face of (5.4) is a pullback since i is a universe embedding in $\mathscr{C}$, so it remains to show that the back face is also. However, the back vertical maps simply forget the sections s, so the back face being a pullback simply says that a map $X \to (\widetilde{V}')_1$ corresponding to a triple

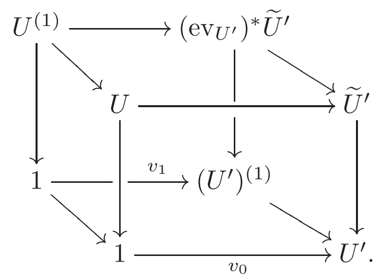
$$\Big(X \xrightarrow{a} U',\ a^*\widetilde{U}' \xrightarrow{b} U',\ X \xrightarrow{s} b^*\widetilde{U}'\Big)$$

factors through $\widetilde{V}_1$ just when a and b factor through U — which is clear.

Next, we need a pullback square

$$\begin{array}{ccc} V & \longrightarrow & \widetilde{V}' \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[v]{} & V' \end{array}$$

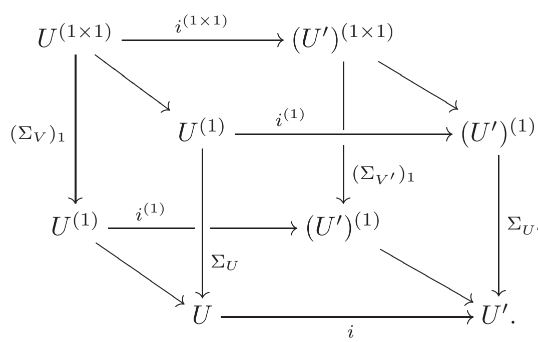
in $\mathscr{C}^2$, which will be a cube



in $\mathscr{C}$. Of course, with $v_0 \coloneqq u$, the front face of this cube is given. We define $v_1 \colon 1 \to (U')^{(1)}$ to name the dependent U'-named type $U^{(1)} \to U$, where U is named by u and $U^{(1)} \to U$ is named by $i \colon U \to U'$. It is then easy to see that the back face is also a pullback.

Now I claim that if we give V and V' their canonical universe structures induced from those of U and U', as above, then $j \colon V \hookrightarrow V'$ is a universe embedding. Consider, for instance, the case of dependent sums; we want the following cube to commute:

(5.5)



The front face commutes since i is a universe embedding, so consider the back face. A map $X \to U^{(1\times 1)}$ corresponds to a quadruple

$$\Big(X \xrightarrow{a} U,\ a^*\widetilde{U} \xrightarrow{b} U,\ a^*\widetilde{U} \xrightarrow{c} U,\ b^*\widetilde{U} \times_{a^*\widetilde{U}} c^*\widetilde{U} \xrightarrow{d} U\Big).$$

as in (5.2). The map $i^{(1\times 1)}$ acts by composing all the named types a, b, c, d with $i \colon U \hookrightarrow U'$ (which, of course, doesn't change the types that they name, up to canonical isomorphism). Since we defined $(\Sigma_V)_1$ with two applications of Σ_U applied to these named types, and i commutes with Σ_U and $\Sigma_{U'}$, it follows that the back square in (5.5) commutes as desired. The cases of dependent products and identity types are similar. □

Thus, however many internal universes there are in the type theory of $\mathscr{C}$, we can find the same number in the type theory of $\mathscr{C}^2$.

6. Univalence in the Sierpinski $(\infty, 1)$-topos

We continue with the notations of the last two sections; our goal is now to prove the following theorem.

Theorem 6.1. *Suppose that U is a universe in $\mathscr{C}$, closed under all the type-forming operations, which satisfies the univalence axiom. Then the corresponding universe V in $\mathscr{C}^2$ also satisfies the univalence axiom.*

To be precise, in order to interpret U as an internal universe in the type theory of $\mathscr{C}$, we must assume at least one further universe U' with a universe embedding $U \hookrightarrow U'$. However, as we saw in §3, we can say what univalence means in terms of $\mathscr{C}$ without explicit reference to U'.

Proof. Let $E \to V \times V$ be the universal space of equivalences in $\mathscr{C}^2$; we must show that the section $V \to E$, which assigns to each type its identity equivalence, is itself an equivalence. Since all the structure at level 0 is exactly as in $\mathscr{C}$, the univalence of U directly implies that $V_0 \to E_0$ is an equivalence; thus it remains to consider $V_1 \to E_1$.

Now since the last step in the construction of Equiv is a dependent sum, we have a pair of Reedy fibrations

$$\begin{array}{ccc} E_1 & \longrightarrow & E_0 \\ \downarrow & & \downarrow \\ F_1 & \longrightarrow & F_0 \\ \downarrow & & \downarrow \\ V_1 \times V_1 & \longrightarrow & V_0 \times V_0 \end{array}$$

in which $F \to V \times V$ represents the dependent type

$$A\colon \mathsf{Type}, B\colon \mathsf{Type} \vdash (A \to B)\colon \mathsf{Type}$$

in the internal type theory of $\mathscr{C}^2$. By construction, this means that $F_0 \to V_0 \times V_0$ represents

$$A_0\colon \mathsf{Type},\ B_0\colon \mathsf{Type} \vdash (A_0 \to B_0)\colon \mathsf{Type}$$

in $\mathscr{C}$, whereas $F_1 \to (V_1 \times V_1) \times_{V_0 \times V_0} F_0$ represents

$$A_0\colon \mathsf{Type},\ A_1\colon A_0 \to \mathsf{Type},\ B_0\colon \mathsf{Type},\ B_1\colon B_0 \to \mathsf{Type},\ f_0\colon A_0 \to B_0 \\ \vdash\ \textstyle\prod_{a_0\colon A_0}(A_1(a_0) \to B_1(f_0(a_0)))$$

Our goal is to describe E_1 in terms of the internal type theory of $\mathscr{C}$, so that we can apply univalence there. By definition, $E \to F$ represents the dependent type

$$A\colon \mathsf{Type},\ B\colon \mathsf{Type},\ f\colon A \to B\ \vdash\ \mathrm{isEquiv}(f)\colon \mathsf{Type}$$

constructed in the internal type theory of $\mathscr{C}^2$. However, because $\mathscr{C}^2$ satisfies function extensionality by Proposition 4.16, we are free to consider instead the dependent type (see (3.3))

$$A\colon \mathsf{Type},\ B\colon \mathsf{Type},\ f\colon A \to B\ \vdash\ \mathrm{ishIso}(f)\colon \mathsf{Type}.$$

We now evaluate this in terms of $\mathscr{C}$, considering separately the two factors

$$A\colon \mathsf{Type},\ B\colon \mathsf{Type},\ f\colon A \to B \vdash \textstyle\sum_{s\colon B \to A}\prod_{b\colon B}(f(s(b)) = b)\colon \mathsf{Type} \tag{6.2}$$

$$A\colon \mathsf{Type},\ B\colon \mathsf{Type},\ f\colon A \to B \vdash \textstyle\sum_{r\colon B \to A}\prod_{a\colon A}(r(f(a)) = a)\colon \mathsf{Type} \tag{6.3}$$

which are of course closely analogous.

$$\begin{array}{ccc}
\Big(\ldots, p_1 \colon (p_0)_*\big(f_1(s_0(b_0), s_1(b_0, b_1))\big) = b_1\Big) & \longrightarrow & (\ldots, p_0 \colon f_0(s_0(b_0)) = b_0) \\
\downarrow & & \downarrow \\
(\ldots, b_1 \colon B_0(b_1)) & \longrightarrow & (\ldots, b_0 \colon B_0) \\
\downarrow & & \downarrow \\
\Big(\ldots, s_1 \colon \prod_{b_0 \colon B_0} B_1(b_0) \to A_1(s_0(b_0))\Big) & \longrightarrow & (\ldots, s_0 \colon B_0 \to A_0) \\
\downarrow & & \downarrow \\
\Big(\ldots, f_1 \colon \prod_{a_0 \colon A_0} A_1(a_0) \to B_1(f_0(a_0))\Big) & \longrightarrow & (\ldots, f_0 \colon A_0 \to B_0) \\
\downarrow & & \downarrow \\
(\ldots, B_1 \colon B_0 \to \mathsf{Type}) & \longrightarrow & (\ldots, B_0 \colon \mathsf{Type}) \\
\downarrow & & \downarrow \\
(\ldots, A_1 \colon A_0 \to \mathsf{Type}) & \longrightarrow & (A_0 \colon \mathsf{Type})
\end{array}$$

Figure 1. Path spaces for the universal section

Firstly, by definition of path-spaces and pullback in $\mathscr{C}^{\mathbf{2}}$, the dependent type

$$A \colon \mathsf{Type},\ B \colon \mathsf{Type},\ f \colon A \to B,\ s \colon B \to A,\ b \colon B\ \vdash\ (f(s(b)) = b) \colon \mathsf{Type}$$

is represented by the tower of Reedy fibrations shown in Figure 1. In this diagram, each morphism is a fibration and each square is a Reedy fibration. The ellipses in each context stand for all the variables appearing in contexts below and to the right of it.

Now, applying dependent product to the top two morphisms, and using the construction from Proposition 4.8, we find that the dependent type

$$A \colon \mathsf{Type},\ B \colon \mathsf{Type},\ f \colon A \to B,\ s \colon B \to A\ \vdash\ \prod_{b \colon B} (f(s(b)) = b) \colon \mathsf{Type}$$

is represented by the tower in Figure 2. (For brevity, we have omitted the types of some variables.) Therefore, (6.2) is simply obtained by composing the top squares in Figure 2. And of course, (6.3) is directly analogous.

Now, recall that we are interested in the map $V \to E$, and specifically its 1-component $V_1 \to E_1$. This map factors through the pullback $V_0 \times_{E_0} E_1$. Moreover, since $V_0 \times_{E_0} E_1 \to E_1$ is a pullback of the equivalence $V_0 \to E_0$ along the fibration $E_1 \to E_0$ of fibrant objects, it is also an equivalence. Thus, by 2-out-of-3, $V_1 \to E_1$ is an equivalence if and only if $V_1 \to V_0 \times_{E_0} E_1$ is so.

$$
\begin{array}{ccc}
\Big(\ldots, q_1 \colon \prod_{b_0,b_1}\big((q_0(b_0))_*\big(f_1(s_0(b_0), s_1(b_0,b_1))\big) = b_1\big)\Big) & \longrightarrow & (\ldots, q_0 \colon \prod_{b_0}(f_0(s_0(b_0)) = b_0)) \\
\downarrow & & \downarrow \\
\Big(\ldots, s_1 \colon \prod_{b_0} B_1(b_0) \to A_1(s_0(b_0))\Big) & \longrightarrow & (\ldots, s_0 \colon B_0 \to A_0) \\
\downarrow & & \downarrow \\
\Big(\ldots, f_1 \colon \prod_{a_0} A_1(a_0) \to B_1(f_0(a_0))\Big) & \longrightarrow & (\ldots, f_0 \colon A_0 \to B_0) \\
\downarrow & & \downarrow \\
(\ldots, B_1 \colon B_0 \to \mathsf{Type}) & \longrightarrow & (\ldots, B_0 \colon \mathsf{Type}) \\
\downarrow & & \downarrow \\
(\ldots, A_1 \colon A_0 \to \mathsf{Type}) & \longrightarrow & (A_0 \colon \mathsf{Type})
\end{array}
$$

Figure 2. Section homotopies for the universal section

In terms of the variables appearing in Figure 2, the map $V_0 \to E_0$ acting on $A_0 \colon \mathsf{Type}$ is defined by

$$
\begin{aligned}
B_0 &\coloneqq A_0 \\
f_0 &\coloneqq \mathrm{id}_{A_0} \\
s_0 &\coloneqq \mathrm{id}_{A_0} \\
q_0 &\coloneqq \lambda_{b_0 \colon A_0}.\ \mathrm{idpath}_{b_0}
\end{aligned}
$$

and similarly for the corresponding data for r. Therefore, upon pullback along this map, the types of the data in E_1 become

$$
\begin{aligned}
f_1 &\colon \textstyle\prod_{a_0} A_1(a_0) \to B_1(a_0) \\
s_1 &\colon \textstyle\prod_{a_0} B_1(a_0) \to A_1(a_0) \\
q_1 &\colon \textstyle\prod_{a_0,a_1}\big(f_1(a_0, s_1(a_0,a_1)) = a_1\big)
\end{aligned}
$$

and similarly for r. (We have used the fact that transporting along the identity path is the identity.) Hence, the fibration $V_0 \times_{E_0} E_1 \to V_0 \times_{F_0} F_1$ is represented by the dependent type

$$
A_0, A_1, B_1, f_1 \vdash \sum_{s_1} \prod_{a_0,a_1} \big(f_1(a_0, s_1(a_0,a_1)) = a_1\big) \times \sum_{r_1} \prod_{a_0,a_1} \big(r_1(a_0, f_1(a_0,a_1)) = a_1\big)
$$

(all variables have the same types as above). However, in the presence of function extensionality, it is not hard to show that this type is naturally equivalent to

$$
\begin{aligned}
A_0, A_1, B_1, f_1 \vdash \prod_{a_0} \Bigg(& \sum_{s_1 \colon B_1(a_0) \to A_1(a_0)} \prod_{a_1} \big(f_1(a_0, s_1(a_1)) = a_1\big) \times \\
& \qquad \sum_{r_1 \colon B_1(a_0) \to A_1(a_0)} \prod_{a_1} \big(r_1(f_1(a_0,a_1)) = a_1\big) \Bigg)
\end{aligned}
$$

i.e. to

$$A_0, A_1, B_1, f_1 \vdash \prod_{a_0} \mathrm{ishIso}(f_1(a_0))$$

Therefore (using function extensionality again), the fibration $V_0 \times_{E_0} E_1 \to V_1 \times_{V_0} V_1$ may be represented by

$$A_0, A_1, B_1 \vdash \prod_{a_0} \mathrm{Equiv}(A_1(a_0), B_1(a_0))$$

Now we have a commutative square

$$\begin{array}{ccc} V_1 & \longrightarrow & V_0 \times_{E_0} E_1 \\ \downarrow & & \downarrow \\ P_{V_0} V_1 & \longrightarrow & V_1 \times_{V_0} V_1 \end{array}$$

in $\mathscr{C}/V_0$, in which the left-hand map is an acyclic cofibration and the right-hand map is a fibration. Therefore, we have an induced map $P_{V_0}V_1 \to V_0 \times_{E_0} E_1$ of fibrations over $V_1 \times_{V_0} V_1$, which it suffices to show to be an equivalence. This map is represented by a section of the dependent type

$$A_0, A_1, B_1 \ \vdash \ (A_1 = B_1) \to \prod_{a_0} \mathrm{Equiv}(A_1(a_0), B_1(a_0)) \colon \mathsf{Type}.$$

obtained from the eliminator for the path type $(A_1 = B_1)$. But this map factors, up to homotopy, as a composite

$$(A_1 = B_1) \to \prod_{a_0} (A_1(a_0) = B_1(a_0)) \to \prod_{a_0} \mathrm{Equiv}(A_1(a_0), B_1(a_0))$$

in which the first map is an equivalence by function extensionality, and the second by function extensionality and by univalence in $\mathscr{C}$. Thus, our desired map is internally a fiberwise equivalence over $V_1 \times_{V_0} V_1$, hence also an equivalence on total spaces externally. Hence V is univalent. □

Corollary 6.4. *The Reedy model category* $\mathrm{sSet}^{\mathbf{2}}$ *supports a model of intensional type theory with dependent sums and products, identity types, and with one fewer univalent universe than there are inaccessible cardinals.*

As before, since the homotopy theory of $\mathrm{sSet}^{\mathbf{2}}$ models the "Sierpinski $(\infty, 1)$-topos" $\infty\mathrm{Gpd}^{\mathbf{2}}$, we can say informally that we have a model of type theory in this $(\infty, 1)$-topos.

7. Inverse categories in general

As we have observed, what makes §§4–6 work is that a Reedy fibrant object $A_1 \to A_0$ of $\mathscr{C}^{\mathbf{2}}$ can be represented by a context in type theory:

$$a_0 \colon A_0,\ a_1 \colon A_1(a_0).$$

A corresponding fact is true for Reedy fibrant diagrams on some other categories. For instance, spans of fibrations $A_1 \to A_0 \leftarrow A_2$ correspond to contexts of the form

$$a_0 \colon A_0,\ a_1 \colon A_1(a_0),\ a_2 \colon A_2(a_0).$$

whereas cospans $A_0 \leftarrow A_2 \rightarrow A_1$ such that $A_2 \rightarrow A_0 \times A_1$ is a fibration correspond to contexts of the form

$$a_0 \colon A_0,\ a_1 \colon A_1,\ a_2 \colon A_2(a_0, a_1).$$

(This correspondence between diagrams and contexts has also been used elsewhere, e.g. [Mak95].) In this section we sketch an extension of §§4–6 to such cases (we will need a few extra assumptions in places).

Definition 7.1. An **inverse category** is a category I such that the relation "x receives a nonidentity arrow from y" on its objects is well-founded.

When I is an inverse category, we write $\prec$ for the above well-founded relation. The point of the definition is that we can construct diagrams on I and maps between them by well-founded induction, as follows.

For an object $x \in I$, we write $x /\!\!/ I$ for the full subcategory of the co-slice category x/I which excludes only the identity id_x.

Assumption 7.2. *$\mathscr{C}$ admits limits over all the categories $x /\!\!/ I$.*

If A is a diagram in $\mathscr{C}$ defined (at least) on the full subcategory $\{\, y \mid y \prec x \,\} \subset I$, then we can restrict it to $x /\!\!/ I$; we define the **matching object** $M_x A$ to be the limit

$$M_x A \coloneqq \lim_{x /\!\!/ I} A$$

To give an extension of A to x, then, is precisely to give an object A_x with a map $A_x \rightarrow M_x A$. Similarly, given a natural transformation $f \colon A \rightarrow B$ of diagrams on $\{\, y \mid y \prec x \,\}$, if A and B have extensions to x, then to give an extension of f to x is precisely to give a map

$$A_x \rightarrow M_x A \times_{M_x B} B_x.$$

Note that if x has no $\prec$-predecessors, then $x /\!\!/ I$ is empty and $M_x A$ is terminal.

Assumption 7.3. *$\mathscr{C}$ has the structure considered in §2.*

Definition 7.4. A **Reedy fibration** in $\mathscr{C}^I$ is a map $f \colon A \rightarrow B$ between I-diagrams such that each map

$$A_x \rightarrow M_x A \times_{M_x B} B_x.$$

is a fibration in $\mathscr{C}$. A **Reedy acyclic cofibration** in $\mathscr{C}^I$ is a levelwise acyclic cofibration.

In particular, A is Reedy fibrant iff each map $A_x \rightarrow M_x A$ is a fibration. If I is finite, then Reedy fibrant I-diagrams can be regarded as contexts of a certain form in the type theory of $\mathscr{C}$. In the general case, we can regard them as a certain type of "infinite context".

It is easy (and standard) to show that the Reedy acyclic cofibrations and the Reedy fibrations form a weak factorization system on $\mathscr{C}^I$. In particular, this implies:

Lemma 7.5. *The limit functor* $\lim \colon \mathscr{C}^I \rightarrow \mathscr{C}$ *takes Reedy fibrations to fibrations.*

Proof. It suffices to observe that its left adjoint, the constant diagram functor, takes acyclic cofibrations to Reedy (i.e. levelwise) acyclic cofibrations. □

If $\mathscr{C}$ is a model category, then there is a whole Reedy model structure, with the cofibrations and weak equivalences levelwise. (See, for instance, [Hov99, Ch. 5].) This implies:

Lemma 7.6. *If $\mathscr{C}$ is a type-theoretic model category, then* $\lim\colon \mathscr{C}^I \to \mathscr{C}$ *takes levelwise equivalences between Reedy fibrant diagrams to equivalences, respectively.*

Proof. The limit is a right Quillen functor, hence preserves weak equivalences between (Reedy) fibrant objects. □

Note that this can be proven by induction for arbitrary $\mathscr{C}$ if I is finite.

All our examples of type-theoretic model categories also have the property that *limits preserve cofibrations*, and hence that $\lim\colon \mathscr{C}^I \to \mathscr{C}$ also takes levelwise acyclic cofibrations between Reedy fibrant diagrams to acyclic cofibrations.

Assumption 7.7. *Each functor* $\lim\colon \mathscr{C}^{x/\!\!/I} \to \mathscr{C}$ *takes levelwise weak equivalences and levelwise acyclic cofibrations between Reedy fibrant diagrams to weak equivalences and acyclic cofibrations, respectively.*

Now suppose given a universe $\widetilde{U} \to U$ in $\mathscr{C}$; we define a Reedy fibration $\widetilde{V} \to V$ of Reedy fibrant objects in $\mathscr{C}^I$ as follows. For $x \in I$, by induction suppose $\widetilde{V} \to V$ is defined on $\{\, y \mid y \prec x \,\}$. Taking limits, we have a fibration $M_x\widetilde{V} \to M_xV$. Define

$$V_x := (M_xV \times U \to M_xV)^{(M_x\widetilde{V} \to M_xV)}$$

equipped with the evident fibration $V_x \to M_xV$. By definition, we have an evaluation map $V_x \times_{M_xV} M_x\widetilde{V} \to M_xV \times U$ over M_xV, hence a plain morphism $V_x \times_{M_xV} M_x\widetilde{V} \to U$. Let $\widetilde{V}_x \to V_x \times_{M_xV} M_x\widetilde{V}$ be the fibration named by this map. Then by construction, V is Reedy fibrant and $\widetilde{V} \to V$ is a Reedy fibration.

Assumption 7.8. *Small fibrations in $\mathscr{C}$ are closed under limits over all the categories $x /\!\!/ I$.*

This assumption can be proven by induction if each $x /\!\!/ I$ is finite. On the other hand, if small fibrations are defined by a cardinality condition on the fibers, as for the univalent universes in groupoids and simplicial sets, then it holds as long as the cardinality class in question is closed under limits of the size of each $x /\!\!/ I$.

All the proofs in §§4–6 now go through almost exactly as before. The constructions there of 1-level data from 0-level data can be easily modified into the induction steps of constructions by well-founded induction over I, making x-level data from the corresponding matching data. Fibrations such as $A_1 \to A_0$ are replaced by $A_x \to M_xA$, pullbacks such as $B_1 \times_{B_0} A_0$ are replaced by $B_x \times_{M_xB} M_xA$, and so on. Assumption 7.8 ensures that the 0-level fibrations in §§4–6 which were small by definition remain small when replaced by matching data.

Possible infiniteness of I is not a problem, since the corresponding "infinite contexts" are always represented by single objects of $\mathscr{C}$ and hence can be incarnated by single types in its internal type theory. However, we make a few remarks about places where a little thought is required.

- In the proof of Proposition 4.8, the dependent product $\Pi_{f_0}B_0$ must be replaced, not by $\Pi_{M_xf}(M_xB)$, but by $M_x(\Pi_fB)$. However, this does not affect the proof in any other way.
- Similarly, in the proof of Proposition 4.10, the path object $P_{B_0}A_0$ must be replaced, not by $P_{M_xB}(M_xA)$, but by $M_x(P_BA)$. We need the acyclic-cofibration part of Assumption 7.7 in order to conclude that the induced map $M_xA \to M_x(P_BA)$, forming the lowest horizontal morphism in the analogue of (4.13), is an acyclic cofibration.

- By the equivalence part of Assumption 7.7, limits over the categories $x \sslash I$ also take Reedy fibrations which are levelwise equivalences ("Reedy acyclic fibrations") to acyclic fibrations. By 2-out-of-3, this implies that a Reedy fibration is a levelwise equivalence if and only if each $A_x \to M_x A \times_{M_x B} B_x$ is an acyclic fibration. These facts are used in the proof of Proposition 4.15 to conclude that $M_x f \colon M_x A \to M_x B$ (hence also its pullback) is an acyclic fibration, and in the proof of Proposition 4.16 to conclude that $M_x(\Pi_f B) \to M_x C$ is an acyclic fibration.
- The equivalence part of Assumption 7.7 is also used in the proof of Theorem 6.1 to conclude by induction that $M_x V \to M_x E$ is an equivalence.

Since Voevodsky's univalent model in simplicial sets is a type-theoretic model category in which the cofibrations are the monomorphisms, it satisfies all the assumptions of this section. Thus we conclude:

Theorem 7.9. *For any inverse category I, the Reedy model category* sSet^I *supports a model of intensional type theory with dependent sums and products, identity types, and with one fewer univalent universe than there are inaccessible cardinals.* □

As before, we may say that this model lives in the $(\infty, 1)$-topos $\infty\mathrm{Gpd}^I$.

Remark 7.10. The Reedy model structure on $\mathscr{C}^I$ exists more generally than when I is an inverse category: we only need I to be a *Reedy category* or some generalization thereof (see e.g. [Ree, BM11, Cis06]). In general, however, Reedy cofibrations are not levelwise (though the weak equivalences are), and so far I have been unable to generalize the above methods to Reedy categories that are not inverse.

On the other hand, for suitable $\mathscr{C}$ (including simplicial sets) and *any* I, the category $\mathscr{C}^I$ has an *injective model structure* in which the weak equivalences and cofibrations are levelwise. (It just so happens that when I is inverse — or, more generally, *elegant* [BR11] — the Reedy and injective model structures coincide.) In general, however, the injective fibrations seem to admit no simple description, so the methods of this paper probably do not apply in that generality.

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