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Short-horizon regulation for long-term investors

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ABSTRACT

We study the effects of imposing repeated short-horizon regulatory constraints on long-term investors. We show that Value-at-Risk and Expected Shortfall constraints, when imposed dynamically, lead to similar optimal portfolios and wealth distributions. We also show that, in utility terms, the costs of imposing these constraints can be sizeable. For a 96% funded pension plan, both an annual Value-at-Risk constraint and an annual Expected Shortfall constraint can lead to an economic cost of about 2.5–3.8% of initial wealth over a 15-year horizon.

1. Introduction

This paper investigates the economic consequences of a difference in the planning horizon of an institutional investor pursuing long-term investment strategies and a regulator enforcing prudential standards and practices on a repeated short-term basis. Such a misalignment of horizons is likely to exist in most developed financial markets and affect, for example, banks, insurance companies, and, notably, pension funds.

Consider, for example, a pension fund which typically faces long-term pension liabilities with maturities of 15 years or more. However, standard regulatory frameworks impose short-term solvency constraints. A recent example can be observed in the Netherlands where a pension regulatory regime ("Financieel Toetsings Kader", FTK) is effective as of January 2007. According to the Dutch regulation, pension funds should always keep the probability of underfunding 1 year ahead below 2.5%. Underfunding refers to the situation where the market value of a pension fund's assets falls below the market value of the pension fund's liabilities. In the Netherlands these liabilities are, for now, taken as nominally guaranteed pensions. This will likely change in the near future where pensions are no longer considered to contain guaranteed minimal payments. Other examples of such a misalignment include Basel II regulation for banks and the newly proposed Solvency II regulation for insurance companies.

The existence of such funding constraints can be understood in light of the recent experience of a simultaneous decrease in pension assets due to a poor stock market performance and an increase in pension liabilities due to low interest rates. For the UK, KPMG estimated the aggregate funding deficit of the FTSE-100 companies reaches GBP 40 billion at the end of 2008. De Nederlandsche Bank (the Dutch regulator) reports that the average Dutch pension funding ratio dropped from 144% in 2007 to 99% in the third quarter of 2010. Of all Dutch pension funds, around 68% has a funding ratio below 105%. The situation in the US is even more alarming. The funding deficit in America's corporate pension funds is estimated to be USD 350 billion (Jørgensen, 2007).

A Value-at-Risk-type (hereafter, VaR-type) constraint aims to limit the probability that the institutional investor generates a portfolio wealth loss and an Expected Shortfall-type constraint aims to control the expected loss given default. Despite the theoretical shortcomings (c.f., Artzner et al., 1999, concerning the VaR-type constraint), both types of regulatory constraints are widely adopted within the current international regulatory regimes, e.g., Basel II and Solvency II.

This paper compares the optimal portfolio wealth and the economic costs of dynamically imposed regulation when the regulatory horizon is as long as the investment horizon and when the regulatory horizon is shorter than the investment horizon. In the latter case, within the investor's investment horizon, there are a number of subsequent and non-overlapping regulatory checks and the investment horizon is divided into a few equal-length sub-periods. In general, the investor has to insure his portfolio against the bad performance of the financial market to guarantee

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that (1) the current period's regulatory constraint is satisfied and (2) there is enough wealth to fulfill next periods' regulatory constraints. To do so, the investor has to hold more risk-free assets and less risky assets, thus, his ability to profit from favorable financial market performance is limited. The economic costs are measured by the equivalent amount of wealth lost due to the regulatory constraints.

We show that, for both types of regulatory constraints, a short regulatory horizon can prevent portfolio wealth loss very effectively but at the same time also introduces a large opportunity cost by limiting the investor's ability to invest in risky assets and profit from favorable stock market performance. We also reconfirm the well-known result (see below) that, when the regulatory horizon is as long as the investment horizon, different types of regulation result in a very different optimal portfolio wealth and investment strategy. However, when the regulatory horizon is actually shorter than the investment horizon and regulation is enforced repeatedly, both types of regulatory constraints lead to very similar portfolio wealth distributions and economic cost due to the fact that both types of regulation require the institutional investor to hold enough wealth to satisfy future regulatory constraints. It is important to note that, for insurance companies, regulation aims at minimizing the risk of (partial) default while obviously the induced costs are borne by the management and/or shareholders.

The strategic asset allocation problem has been studied extensively. For example, Kim and Omberg (1996) and Wachter (2002) study the optimal portfolio allocation where the price of risk is mean-reverting. Bajeux-Besnainou et al. (2003) and Sørensen (1999) solve the optimal investment problem when interest rates are stochastic. This paper is related to the literature studying the optimal portfolio trading strategy under constraints. Grossman and Vila (1992) provide explicit solutions to optimal portfolio problems containing leverage and minimum portfolio return constraints. Basak (1995) and Grossman and Zhou (1995) focus on the impact of a specific VaR constraint, the portfolio insurance,[1] on asset price dynamics in a general equilibrium model. Van Binsbergen and Brandt (2009) assess the influence of ex ante (preventive) and ex post (punitive) risk constraints on dynamic portfolio trading strategies. Ex ante risk constraints include, among others, VaR and short sell constraints. Ex post risk constraints include the loss of the investment manager's personal compensation and reputation when the portfolio wealth turns out to be low. They found that ex ante risk constraints tend to decrease gains from dynamic investment while ex post risk constraints can be welfare improving.

Basak and Shapiro (2001) compare the impact of VaR-type and Expected Shortfall-type regulation on the institutional investors' portfolio wealth and trading strategies. Their results show that these two types of regulatory constraints lead to different portfolio wealth distributions. The VaR constraint keeps the portfolio value above or at the threshold value, e.g., the value of a pension fund's liability, when the investment environment (state of the world) is favorable but generates a sizeable loss in unfavorable states of the world. The favorable (unfavorable) states are the ones in which it is cheap (expensive) for the investor to raise his portfolio wealth to the level of the threshold value. Thus, ironically, the loss under a VaR constraint is even larger than the one without a VaR constraint in unfavorable states. The unfavorable states of the world occur with probability α. This probability is set by the regulator. The explanation is as follows. The VaR constrained investor is only concerned about the probability but neither the magnitude of the loss, nor in which (cheap or expensive) states this loss occurs. Therefore, it is optimal for him to incur losses in unfavorable states where it is most expensive to raise his portfolio wealth. An Expected Shortfall-type constraint, on the contrary, limits the expected magnitude of a loss given default, and thus, does not allow an institutional investor under regulation to incur excessive loss in all market circumstances.

In Basak and Shapiro (2001), the regulatory horizon equals the investment horizon and interest rates are deterministic. We extend the Basak and Shapiro (2001) setting by embedding a number of subsequent and non-overlapping short-term regulatory constraints in the portfolio optimization problem and allowing for a stochastic interest rate. We show that (1) more frequent regulation can prevent the investor from generating losses in unfavorable states due to the fact that there is a minimum amount of portfolio wealth required to fulfill future regulatory constraints and (2) both types of regulation result in a similar portfolio wealth distribution and economic costs if the regulatory constraint is imposed repeatedly.

Cuoco et al. (2008) consider the optimal trading strategy of institutional investors under short-horizon VaR constraints assuming that the portfolio allocation over the VaR horizon is constant and the interest rate is deterministic. We extend Cuoco et al. (2008) by allowing for optimal and time-varying portfolio allocations over the regulatory horizon, having a stochastic interest rate and analyzing the impacts of imposing Expected Shortfall-type regulatory constraints. The extensions enables us to (1) quantify the costs and benefits of both VaR-type and Expected Shortfall-type regulatory constraints given that institutional investors behave optimally, (2) study the hedge strategies of investors under both types of constraints and (3) investigate the difference of imposing these two types of constraints.

This paper is also related to the literature about dynamic trading strategies of pension funds. Sundaresan and Zapatero (1997) consider an optimal asset allocation with a power utility function in final surplus. Boulier et al. (1995) assume a constant investment opportunity set with a risky and a risk-free asset. In their paper, the pension plan sponsor aims to minimize the expected discounted value of future contributions over a given horizon. Inkmann and Blake (2011) propose a new approach to the valuation of pension obligations taking into account the asset allocation strategy and the underfunding risk of a pension fund. This paper focuses on the optimal portfolio wealth of a pension fund when the regulatory horizon is shorter than its investment horizon and evaluates the economic costs of such a regulation. Advantages of having frequent short-term VaR or Expected Shortfall constraints include, among others, smaller expected portfolio wealth losses.

The rest of this paper is organized as follows. Section 2 describes the investment environment our institutional investor operates in. Subsequently, Section 3 introduces the various regulatory constraints and studies the optimal portfolio wealth and trading strategies under a single-regulatory constraint and multiple regulatory constraints respectively. Section 4 discusses the costs of imposing 15 short-term regulatory constraints. Section 5 concludes.

2. The investment environment

We consider a stochastically complete continuous-time financial market with a finite horizon $[0,T]$. In this market, four assets are available: a zero-coupon bond maturing at time T, a cash account, a stock index, and a constant maturity zero-coupon bond fund with maturity M. The stock index (with reinvested dividends) is assumed to follow:

$$dS_t = (r_t + \Phi_S)S_t\,dt + \sigma_S S_t\,dZ_{S,t}, \tag{1}$$

where r_t denotes the short-term interest rate, Φ_S is the stock risk premium, σ_S is the instantaneous stock price volatility and $Z_{S,t}$ is a

[1] Portfolio insurance is a special case of a VaR constraint, which requires the probability that the portfolio wealth falls below a certain threshold value to be zero.

standard Brownian motion. For the short-term interest rate r_t, we impose a Vasicek process:

$$dr_t = \kappa(\bar{r} - r_t)\mathrm{d}t - \sigma_r\,\mathrm{d}Z_{r,t}, \tag{2}$$

where κ determines the mean-reversion speed of the interest rate towards the long-term average value $\bar{r}$. Furthermore, σ_r is the instantaneous volatility of the interest rate and $Z_{r,t}$ is a standard Brownian motion. The two Brownian motions $Z_{r,t}$ and $Z_{S,t}$ may be correlated and we denote their correlation coefficient by ρ_{Sr}. Vasicek (1977) derives the induced no-arbitrage price of a zero-coupon bond at time t with $T - t$ years to maturity and unit face value as

$$P_t^{T-t} = \exp(-A(T-t) - B(T-t)r_t), \tag{3}$$

where

$$A(T-t) = R_\infty[(T-t) - B(T-t)] + \frac{\sigma_r^2}{4\kappa}B(T-t)^2,$$

$$R_\infty = \bar{r} + \frac{\Phi_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2},$$

$$B(T-t) = [1 - \exp(-\kappa(T-t))]/\kappa,$$

with Φ_r a free parameter capturing the interest rate risk premium. Applying the Itô-Doeblin lemma to (3), we find for the dynamics of the zero-coupon bond price

$$\frac{\mathrm{d}P_t^{T-t}}{P_t^{T-t}} = [r_t + \Phi_r B(T-t)]\mathrm{d}t + \sigma_r B(T-t)\mathrm{d}Z_{r,t}. \tag{4}$$

In this stochastic interest rate model, if the zero coupon bond maturing at time T is the only bond available for investment, this bond has two tasks. First, it serves to achieve the optimal interest rate risk exposure for speculative purposes. Secondly, it can be used to hedge interest rate risk as it is a risk-free asset over the investment horizon. To uncouple these two functions we also consider a bond fund implementing a constant M-year to maturity (see Bajeux-Besnainou et al., 2003). The price dynamics of such a fund are given by

$$\frac{\mathrm{d}P_t^{M}}{P_t^{M}} = [r_t + \Phi_r B(M))\mathrm{d}t + \sigma_r B(M)\mathrm{d}Z_{r,t}. \tag{5}$$

All our bonds are assumed to be free of default risk.

It is well-known that, following the martingale method for optimal investment, the pricing kernel plays a crucial role in describing optimal investment strategies. For the market introduced, Merton (1992) shows that the pricing kernel can be constructed as the inverse of the growth-optimum portfolio. It is well-known that, when the interest rate follows the Vasicek process, the dynamics of ζ_t are given by

$$\frac{d\zeta_t}{\zeta_t} = -r_t\mathrm{d}t - \phi_S\mathrm{d}Z_{S,t} + \phi_r\mathrm{d}Z_{r,t}, \tag{6}$$

where

$$\phi_s = \frac{\sigma_r\Phi_S - \rho_{Sr}\Phi_r\sigma_S}{\sigma_r\sigma_S(1-\rho_{Sr}^2)},$$
$$\phi_r = \frac{\sigma_r\Phi_S\rho_{Sr} - \Phi_r\sigma_S}{\sigma_r\sigma_S(1-\rho_{Sr}^2)}.$$

3. Optimal portfolio wealth and trading strategies

We consider the problem of an institutional investor who starts with an endowment W_0 and must dynamically select a portfolio $\pi \in \Pi$ so as to maximize the expected utility $E[u(W_T)]$ of the terminal value of the trading portfolio. We assume that the institutional investor has a power utility function with constant relative risk aversion (CRRA) parameter γ and an investment horizon of T years.

The regulator imposes regulatory constraints, of either a VaR or an Expected Shortfall type, on the institutional investor. The VaR-type constraint aims to control the probability of having a portfolio wealth loss and is defined as the probability that the portfolio wealth at time $t + \tau$ falls below $\underline{W}$ should not be larger than α, where α is usually a small number in the interval [0,1]. The VaR constraint can be formulated as

$$\Pr_t(W_{t+\tau} < \underline{W}) \leqslant \alpha, \quad t \in [0, T],$$

where τ, $\tau > 0$, is the regulatory horizon, $\alpha \in [0,1]$ and the "floor" $\underline{W}$ is specified exogenously. For a pension fund, the "floor" is the value of its liability at time $t + \tau$.

An Expected Shortfall-type constraint aims to limit the magnitude of portfolio wealth loss. This paper considers two Expected Shortfall-type constraints, namely, the Expected Discounted Shortfall constraint (EDS) and the Expected Shortfall constraint (ES). The difference between an EDS constraint and an ES constraint lies in whether the expected portfolio wealth shortfall is discounted with the pricing kernel ζ_t or not. An EDS constraint can be stated as that the expected discounted portfolio wealth shortfall at time t cannot be larger than ϵ_{EDS}, that is,

$$E_t\left[\frac{1}{\zeta_t}\zeta_{t+\tau}(\underline{W} - W_{t+\tau})\,\mathbb{I}_{W_{t+\tau}\leqslant\underline{W}}\right] \leqslant \epsilon_{EDS}. \tag{7}$$

The ES constraint can be stated as that the expected portfolio wealth shortfall at time t cannot be larger than ϵ_{ES}, that is,

$$E_t[(\underline{W} - W_{t+\tau})\,\mathbb{I}_{W_{t+\tau}\leqslant\underline{W}}] \leqslant \epsilon_{ES}. \tag{8}$$

Here both ϵ_{EDS} and ϵ_{ES} are small numbers, say, 1% of the initial wealth.

In the single-constraint models, the horizon of the regulatory constraint τ equals the investment horizon. In the two-constraint models, the regulatory constraint horizon τ is half as long as the investment horizon and there are two subsequent and non-overlapping regulatory constraints in the investment horizon. In the more general multi-constraint models, say m constraints ($m > 2$), there are m non-overlapping regulatory constraints and each of these constraints has a horizon of T/m.

3.1. Single-constraint models

In this subsection, we will compare the optimal portfolio choice and portfolio wealth distribution under a single-VaR constraint, a single-EDS constraint and a single-ES constraint respectively.

3.1.1. Investment problem in the single-constraint models

The investment problem under a single-regulatory constraint is

$$\max_{W_T} \quad \mathrm{E}_0\frac{W_T^{1-\gamma}}{1-\gamma} \tag{9}$$
$$\text{s.t.} \quad \mathrm{E}_0\{\zeta_T W_T\} \leqslant \zeta_0 W_0 \tag{10}$$
$$\mathrm{Pr}_0\{W_T \leqslant \underline{W}\} \leqslant \alpha, \tag{11}$$

when a single-VaR constraint is imposed. Alternatively, when a single-EDS constraint is imposed, the regulatory constraint (11) is replaced by

$$\mathrm{E}_0\left[\frac{1}{\zeta_0}\zeta_T(\underline{W} - W_T)\,\mathbb{I}_{W_T\leqslant\underline{W}}\right] \leqslant \epsilon_{EDS}. \tag{11a}$$

Finally, when a single-ES constraint is imposed, we have

$$\mathrm{E}_0[(\underline{W} - W_T)\,\mathbb{I}_{W_T\leqslant\underline{W}}] \leqslant \epsilon_{ES}. \tag{11b}$$

Basak and Shapiro (2001) solves the optimal portfolio wealth under a single-VaR constraint, a single-EDS constraint, and a single-ES constraint when the interest rate is constant. In their paper, the optimal portfolio wealth is obtained by pointwise maximization. Therefore, their method can be applied here even though the interest rate in this model follows a Vasicek process. For the single-VaR-constraint problem (indicated by $1VaR$), the optimal wealth at time T W_T^{1VaR} is

$$W_T^{1VaR} = \begin{cases} \left(y_0^{1VaR}\zeta_T\right)^{-1/\gamma} & \text{for } \zeta_T \leqslant \underline{\zeta}_T^{1VaR}, \\ \underline{W} & \text{for } \underline{\zeta}_T^{1VaR} < \zeta_T \leqslant \bar{\zeta}_T^{1VaR}, \\ \left(y_0^{1VaR}\zeta_T\right)^{-1/\gamma} & \text{for } \zeta_T \geqslant \bar{\zeta}_T^{1VaR}, \end{cases} \tag{12}$$

where y_0^{1VaR} is the Lagrange multiplier of the budget constraint, the lower bound $\underline{\zeta}_T^{1VaR}$ is defined by $\left(y_0^{1VaR}\underline{\zeta}_T^{1VaR}\right)^{-1/\gamma} = \underline{W}$ and the upper bound $\bar{\zeta}_T^{1VaR}$ by $\Pr_0\left(\zeta_T \geqslant \bar{\zeta}_T^{1VAR}\right) = \alpha$. This means that, conditional on the information available at time 0, the probability that ζ_T will be larger than $\bar{\zeta}_T^{1VaR}$ equals the VaR level α. When the VaR constraint is actually not binding, we have $\underline{\zeta}_T^{1VaR} \geqslant \bar{\zeta}_T^{1VaR}$ and, obviously, $W_T^{1VaR} = W_T^u$, where W_T^u represents the optimal portfolio wealth without any regulatory constraints and the superindex u indicates that the optimization is not constrained other than through the budget constraint. Basak and Shapiro (2001) show that W_T^{1VaR} equals the sum of the unconstrained portfolio wealth $\left(y_0^{1VaR}\zeta_T\right)^{-1/\gamma}$ and a "corridor" option from which the investor will get $\underline{W} - \left(y_0^{1VaR}\zeta_T\right)^{-1/\gamma}$ when $\underline{\zeta}_T^{1VaR} \leqslant \zeta_T \leqslant \bar{\zeta}_T^{1VaR}$ holds and nothing otherwise.

The optimal portfolio wealth, at time T, under a single-EDS constraint, W_T^{1EDS}, is

$$W_T^{1EDS} = \begin{cases} \left(y_0^{1EDS}\zeta_T\right)^{\left(-\frac{1}{\gamma}\right)} & \text{for } \zeta_T \leqslant \underline{\zeta}_T^{1EDS}, \\ \underline{W} & \text{for } \underline{\zeta}_T^{1EDS} < \zeta_T < \bar{\zeta}_T^{1EDS}, \\ \left[\left(y_0^{1EDS} - y_1^{1EDS}\right)\zeta_T\right]^{-\frac{1}{\gamma}} & \text{for } \zeta_T \geqslant \bar{\zeta}_T^{1EDS}, \end{cases} \tag{13}$$

where the superindex 1EDS denotes a single-EDS constraint, the lower and upper bounds, $\underline{\zeta}_T^{1EDS}$ and $\bar{\zeta}_T^{1EDS}$, are defined as

$$\underline{\zeta}_T^{1EDS} \equiv \frac{\underline{W}^{-\gamma}}{y_0^{1EDS}},$$
$$\bar{\zeta}_T^{1EDS} \equiv \frac{\underline{W}^{-\gamma}}{y_0^{1EDS} - y_1^{1EDS}},$$

respectively. Here y_0^{1EDS} and y_1^{1EDS} are two Lagrange multipliers which solve the budget constraint and the EDS constraint (11a) respectively.

The optimal portfolio wealth, at time T, under a single-ES constraint is

$$W_T^{1ES} = \begin{cases} \left(y_0^{1ES}\zeta_T\right)^{\left(-\frac{1}{\gamma}\right)} & \text{for } \zeta_T \leqslant \underline{\zeta}_T^{1ES}, \\ \underline{W} & \text{for } \underline{\zeta}_T^{1ES} < \zeta_T < \bar{\zeta}_T^{1ES}, \\ \left(y_0^{1ES}\zeta_T - y_1^{1ES}\right)^{-\frac{1}{\gamma}} & \text{for } \zeta_T \geqslant \bar{\zeta}_T^{1ES}, \end{cases} \tag{14}$$

where y_0^{1ES} and y_1^{1ES} are two Lagrange multipliers which solve the budget constraint and the ES constraint (11b) respectively. The lower bound $\underline{\zeta}_T^{1ES}$ is defined as

$$\underline{\zeta}_T^{1ES} \equiv \frac{\underline{W}^{-\gamma}}{y_0^{1ES}}, \tag{15}$$

and the upper bound $\bar{\zeta}_T^{1ES}$ as

$$\bar{\zeta}_T^{1ES} \equiv \frac{\underline{W}^{-\gamma} + y_1^{1ES}}{y_0^{1ES}}. \tag{16}$$

Under both a single-EDS constraint and a single-ES constraint, the unfavorable states $\left(\zeta_T \geqslant \bar{\zeta}_T^{1EDS} \text{ or } \zeta_T \geqslant \bar{\zeta}_T^{1ES}\right)$ are classified according to, among others, the risk aversion of the investor γ and the initial wealth of the investor W_0, while in a single-VaR-constraint model, the unfavorable states $\left(\zeta_T \geqslant \bar{\zeta}_T^{1VaR}\right)$ are classified exogenously.

Let t stand for any prehorizon time. The portfolio wealth before time T is the expected discounted final portfolio wealth, i.e., $W_t^{1c} = \zeta_t^{-1}E_t\zeta_T W_T^{1c}$, where c stands for VaR, EDS, or ES respectively. Applying the Itô-Doeblin lemma to W_t^{1c}, we can obtain the optimal portfolio allocation. Let $\pi_{S_t}^{1c}$, $\pi_{P_t^M}^{1c}$, and $\pi_{P_t^{T-t}}^{1c}$ stand for the percentages of portfolio wealth, at time t, invested in the stock index, the constant-maturity bond fund and the zero-coupon bond with maturity T, respectively. The percentage of portfolio wealth invested in the cash account is the remainder, $1 - \pi_{S_t}^{1c} - \pi_{P_t^M}^{1c} - \pi_{P_t^{T-t}}^{1c}$. The optimal portfolio allocation under a single-VaR constraint is

$$\begin{bmatrix} \pi_{S_t}^{1c} \\ \pi_{P_t^M}^{1c} \\ \pi_{P_t^{T-t}}^{1c} \end{bmatrix} = \frac{1}{\gamma}X_{spec,t}^{1c}\begin{bmatrix} \frac{\phi_s}{\sigma_S} \\ -\frac{\phi_r}{\sigma_r B(M)} \\ 0 \end{bmatrix} + \left(1 - \frac{1}{\gamma}\right)X_{hedge,t}^{1c}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{17}$$

where

$$X_{spec,t}^{1c} = -\frac{dW_t^{1c}}{d\zeta_t}\frac{\zeta_t}{W_t^{1VaR}}\gamma, \tag{18}$$

$$X_{hedge,t}^{1c} = -\frac{dW_t^{1c}}{dr_t}\frac{1}{W_t^{1c}B(T-t)(1-1/\gamma)}. \tag{19}$$

In the single-VaR constraint case and the single-EDS constraint case, $X_{spec,t}^{1VaR}, X_{hedge,t}^{1VaR}, X_{spec,t}^{1EDS}$, and $X_{hedge,t}^{1EDS}$ can be derived analytically and the exact forms are provided in Appendix A. In the single-ES constraint case, since there are no analytical solutions to W_t^{1ES}, neither $dW_t^{1ES}/d\zeta_t$ nor dW_t^{1ES}/dr_t can be derived analytically. Thus, in our numerical implementation, they are approximated as follows:

$$\frac{dW_t^{1ES}}{d\zeta_t} \approx \frac{W_{t,\zeta_t+\Delta\zeta_t}^{1ES} - W_{t,\zeta_t-\Delta\zeta_t}^{1ES}}{2\times\Delta\zeta_t},$$
$$\frac{dW_t^{1ES}}{dr_t} \approx \frac{W_{t,r_t+\Delta r_t}^{1ES} - W_{t,r_t-\Delta r_t}^{1ES}}{2\times\Delta r_t},$$

where $W_{t,\zeta_t+\Delta\zeta_t}^{1ES}\left(W_{t,\zeta_t-\Delta\zeta_t}^{1ES}\right)$ refers to the portfolio wealth at time t in the single-ES-constraint model when the pricing kernel takes the value $\zeta_t + \Delta\zeta_t\,(\zeta_t - \Delta\zeta_t)$ while other values are keep unchanged and $W_{t,r_t+\Delta r_t}^{1ES}\left(W_{t,r_t-\Delta r_t}^{1ES}\right)$ refers to the portfolio wealth at time t in the single-ES-constraint model when the interest rate takes the value of $r_t + \Delta r_t \quad (r_t - \Delta r_t)$ while other parameter values are kept constant.

The optimal portfolio allocation with a single-VaR constraint, a single-EDS constraint, or a single-ES constraint, consists of a speculative fund and a hedge fund. The speculative fund consists of the stock index and the constant-maturity bond fund. The hedge fund consists only of the bond with $T - t$ years to maturity. For an investor with log utility ($\gamma = 1$), the hedge term vanishes. As the investor becomes more and more risk-averse ($\gamma \to \infty$), the speculative fund's weight tends to 0%. Note that the optimal portfolio weights without any regulatory constraints are

$$\begin{bmatrix} \pi_{S_t}^{u} \\ \pi_{P_t^M}^{u} \\ \pi_{P_t^{T-t}}^{u} \end{bmatrix} = \frac{1}{\gamma}\begin{bmatrix} \frac{1}{\sigma_S}\phi_S \\ -\frac{1}{\sigma_r B(M)}\phi_r \\ 0 \end{bmatrix} + \left(1 - \frac{1}{\gamma}\right)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{20}$$

Therefore,

$$X_{spec,t}^{1c} = \frac{\pi_{S_t}^{1c}}{\pi_{S_t}^{u}} = \frac{\pi_{P_t^M}^{1c}}{\pi_{P_t^M}^{u}}, \tag{21}$$

represents the exposure to the risky assets (i.e., the speculative fund) relative to the case without any regulatory constraints and

$$X^{1c}_{hedge,t} = \frac{\pi^{1c}_{P^{T-t}_t}}{\pi^{u}_{P^{T-t}_t}}, \tag{22}$$

represents the exposure to the riskless asset (i.e., the hedge fund) relative to the case without any regulatory constraints. When $X^{1c}_{spec,t}\left(X^{1c}_{hedge,t}\right)$ takes a value of 1, the exposures to the speculative (hedge) fund with and without a single-regulatory constraint are the same. When $X^{1c}_{spec,t}\ \left(X^{1c}_{hedge,t}\right)$ takes a value which is larger than 1, the exposure to the speculative (hedge) fund with a single-regulatory constraint is larger than the one without any regulatory constraints. When $X^{1c}_{spec,t}\left(X^{1c}_{hedge,t}\right)$ takes a value which is smaller than 1, the exposure to the speculative (hedge) fund with a single-regulatory constraint is smaller than the one without any regulatory constraints.

3.1.2. Comparison of optimal portfolio wealth and allocation under a single regulatory constraint

Fig. 1 shows the optimal wealth levels, subject to a single-regulatory-constraint at the investment horizon T, as a function of the prevailing value of the pricing kernel. These optimal wealth levels have been determined analytically using the results in Section 3.1.1. As these results closely follow Basak and Shapiro (2001), we only discuss them briefly. Observe that all optimal wealth levels are decreasing functions of the pricing kernel. This is a result of the assumed state-independent utility functions. Furthermore, note that various constraints effectively redistribute wealth across the various states of the world.

In order to make the Value-at-Risk constraint comparable to the Expected Shortfall and Expected Discounted Shortfall constraints, we put ε_{ES} equal to the induced Expected Shortfall of the optimal wealth subject to the VaR constraint, that is,

$$\varepsilon_{ES} = \mathrm{E}_0\left\{\left(\underline{W} - W^{1VaR}_T\right)\mathbb{I}_{W^{1VaR}_T \leqslant \underline{W}}\right\}. \tag{23}$$

Similarly, we put ε_{EDS} equal to the induced Expected Discounted Shortfall of W^{1VaR}_T.

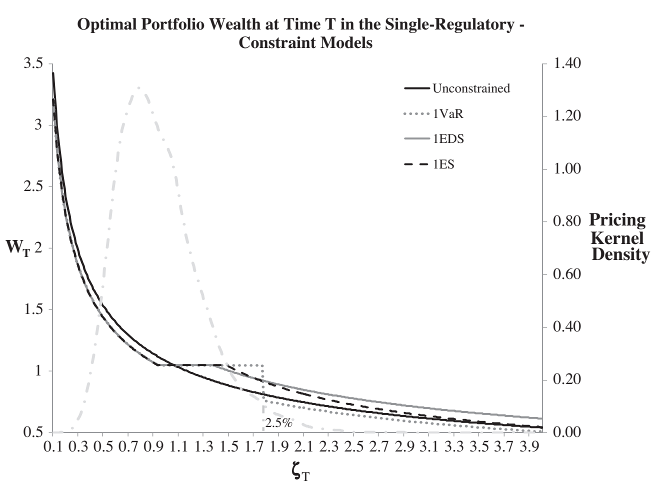

Fig. 1. This figure depicts the portfolio wealth at time T in the single-regulatory-constraint models. The parameter values are $W_0 = 1.04$, $\kappa = 0.15$, $\bar{r} = 0.05$, $\sigma_r = 0.015$, the bond Sharpe ratio $\lambda_P = 0.05$, the maturity of the bond fund $M = 10$ years, $\sigma_S = 0.25$, the stock Sharpe ratio $\lambda_S = 0.25$, $\rho_{Sr} = 0.2$, $\underline{W} = 1.05$, $r_0 = 2\%$, $\gamma = 2$, $\alpha = 0.025$. ε_{ES} is set equal to the induced Expected Shortfall of the optimal wealth subject to the VaR constraint. ε_{EDS} is set equal to the induced Expected Discounted Shortfall of W^{1VaR}_T. The vertical axis on the left (right) side measures the portfolio wealth at time T (pricing kernel density). At time 0, there is 2.5% probability that the pricing kernel value at time T, ζ_T, will be larger than about 1.78.

From Fig. 1 we see that the optimal wealth for the various constraints imposed is actually quite different. For the VaR constraint, there is no punishment on the severity of the loss once it occurs. As a result, it is optimal to accept large losses in bad states of the world, that is, when ζ_T is large. Both Expected Shortfall and Expected Discounted Shortfall constraints do weigh the size of the loss, leading to less severe losses in bad states. As EDS puts more weight on the very bad scenarios, the optimal wealth under the EDS constraint, for bad states of the world, is largest.

Fig. 2 depicts the pre-horizon exposures to the risky assets (i.e., the speculative fund) and the riskless asset (i.e., the hedge fund) relative to the ones in the unconstrained model for these three regulatory constraints when the interest rate is fixed at 4%. Let us first investigate the relative portfolio weights in a single-VaR-constraint model $X^{1VaR}_{spec,t}$ and $X^{1VaR}_{hedge,t}$. The larger the values of $X^{1VaR}_{spec,t}$ and $X^{1VaR}_{hedge,t}$ deviate from 1 the larger the difference in portfolio weights with a single-VaR constraint and without any regulatory constraints. When the market is booming, the portfolio choices with and without a VaR constraint is the same. As the market deteriorates, the relative exposure to the risky (riskless) assets first decreases (increases) and then increases (decreases). The decrease in $X^{1VaR}_{spec,t}$ is caused by the desire to make sure that the portfolio wealth at time T is at or above the wealth lower-bound ($\underline{W}$). The following increase (decrease) in $X^{1VaR}_{spec,t}$ is a moral hazard behavior or a "gambling" behavior. The "gambling" behavior occurs when the market condition at time $T/2$ is bad but it is still likely that the market might end up in "good" or "intermediate" states at time T where the VaR constraint is binding. The idea behind increasing (decreasing) in $X^{1VaR}_{spec,t}\left(X^{1VaR}_{hedge,t}\right)$ is to bring the portfolio wealth to the level of $\underline{W}$ when the market at time T turns out to be "good". However, when the market at time T turns out to be "bad", this strategy will bring a large portfolio wealth loss but this is not a concern for an investor under a single-VaR constraint since the VaR constraint does not punish the severity of a loss. On the contrary, as shown in Fig. 2, an institutional investor under the Expected Discounted Shortfall constraint does not have the incentive to "gamble", because the Expected Discounted Shortfall constraint imposes a heavy punishment on a loss. Therefore, the existence of the "gambling" behavior is the major difference between the optimal portfolio weights under a VaR constraint and

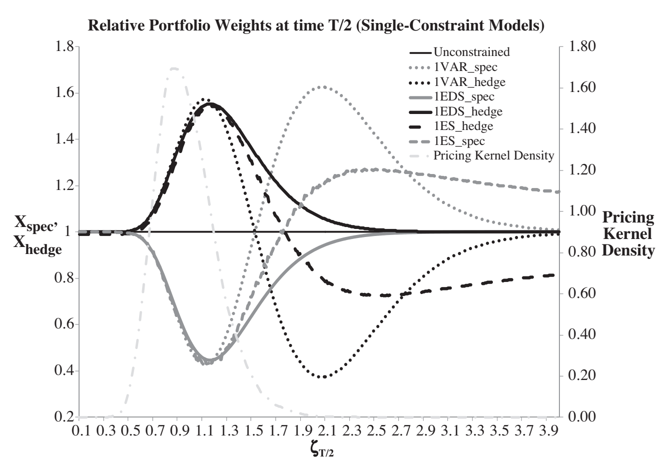

Fig. 2. This figure shows the optimal portfolio allocation at time $t = T/2$ under single-regulatory constraints. In this figure, the interest rate at time t is fixed at 4%. The parameter values are $W_0 = 1.04$, $\kappa = 0.15$, $\bar{r} = 0.05$, $\sigma_r = 0.015$, the bond Sharpe ratio $\lambda_P = 0.05$, the maturity of the bond fund $M = 10$ years, $\sigma_S = 0.25$, the stock Sharpe ratio $\lambda_S = 0.25$, $\rho_{Sr} = 0.2$, $\underline{W} = 1.05$, $r_0 = 2\%$, $\gamma = 2$, $\alpha = 0.025$. ε_{ES} is set equal to the induced Expected Shortfall of the optimal wealth subject to the VaR constraint. ε_{EDS} is set equal to the induced Expected Discounted Shortfall of W^{1VaR}_T.

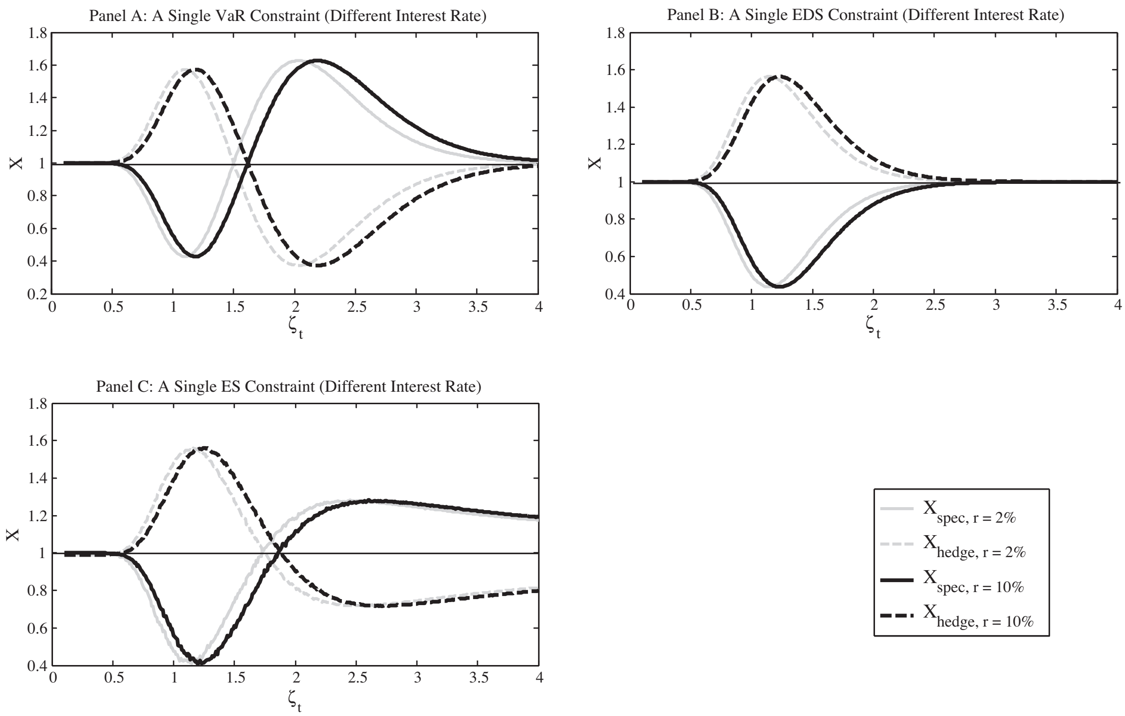

Fig. 3. This figure shows the optimal portfolio allocation at time $t = T/2$ under single-regulatory constraints with different interest rates. The parameter values are $W_0 = 1.04$, $\kappa = 0.15$, $\bar{r} = 0.05$, $\sigma_r = 0.015$, the bond Sharpe ratio $\lambda_P = 0.05$, the maturity of the bond fund $M = 10$ years, $\sigma_S = 0.25$, the stock Sharpe Ratio $\lambda_S = 0.25$, $\rho_{Sr} = 0.2$, $\underline{W} = 1.05$, $r_0 = 2\%$, $\gamma = 2$, $\alpha = 0.025$. ε_{ES} is set equal to the induced Expected Shortfall of the optimal wealth subject to the VaR constraint. ε_{EDS} is set equal to the induced Expected Discounted Shortfall of $W_T^{1\text{VaR}}$.

under an Expected Discounted Shortfall constraint. An Expected Shortfall constraint punishes a portfolio wealth loss less severe than an Expected Discounted Shortfall constraint. Consequently, the investor under an ES constraint still has an incentive to "gamble". The incentive is, naturally, much weaker than the one under a VaR-type constraint.[2]

Fig. 3 compares the portfolio weights for different levels of prevailing interest rate at time $T/2$. In general, a higher interest rate is related to a higher yield on the default-risk-free zero-coupon bond which makes the regulatory constraints easier to be fulfilled. For example, consider a single-VaR-constraint model, when the interest rate is 2% and the value of the pricing kernel is about 1.2, the investor considers to increase his allocation to the risky assets to finance his portfolio wealth at time T to the level of $\underline{W}$. However, when the interest rate is 10% and the value of the pricing kernel is 1.2, the investor will not consider to increase his allocation to the risky assets since the high yield from the riskless bond is sufficient to finance his portfolio wealth at time T to the level of $\underline{W}$.

3.2. Optimal investment under multiple regulatory constraints

3.2.1. Two-VaR-constraint model

The optimal portfolio wealth under two repeated VaR constraints, W_T^{2VaR}, is the solution to the investment problem

$$\begin{aligned} \max_{W_T} \quad & \mathrm{E}_0 \frac{W_T^{1-\gamma}}{1-\gamma}, \\ \text{s.t.} \quad & \mathrm{E}_0 W_T \zeta_T = \zeta_0 W_0, \\ & \Pr_0\left(W_{\frac{T}{2}} \leqslant \underline{W}\right) \leqslant \alpha, \\ & \Pr_{\frac{T}{2}}(W_T \leqslant \underline{W}) \leqslant \alpha. \end{aligned} \tag{24}$$

We are going to use a backward iterative solution procedure to solve (24). First, we solve the maximization problem in the second period, that is, $[T/2,T]$. This second period problem is identical to the single-constraint model. We assume that, at time $T/2$, the investor starts with wealth $W_{T/2}$. Following the same solution method as the one in the single-constraint model, we find the optimal wealth at time T, W_T^{2VaR}, and the indirect utility function at time $T/2$ $J_{T/2}^{2VaR}(W_{T/2})$. Second, we solve the maximization problem in the first period, that is, $[0,T/2]$. One of the differences between the maximization problem in the second and first period is that in the second period, the objective function is $\max E_{T/2} W_T^{1-\gamma}/(1-\gamma)$ while in the first period the objective function is the indirect utility of the problem, namely, max $E_0 J_{T/2}^{2VaR}\left(W_{T/2}\right)$. The superindex 2VaR represents two repeated VaR constraints. The value function, or the indirect utility, is defined as

$$J_{T/2}^{2VaR}\left(W_{T/2}, r_{T/2}\right) = \max_{\left\{W_T : \mathrm{E}_{T/2}\{\zeta_T W_T\} \leqslant \zeta_{T/2} W_{T/2}, \Pr_{T/2}(W_T \leqslant \underline{W}) \leqslant \alpha\right\}} \mathrm{E}_{T/2} \frac{W_T^{1-\gamma}}{1-\gamma}. \tag{25}$$

We now discuss the maximization problem for the first period in more details. The optimal trading strategy in the period from time 0 to time $T/2$ is a solution to the problem

$$\max \quad E_0 J_{\frac{T}{2}}^{2VaR}\left(W_{\frac{T}{2}}\right), \tag{26}$$

$$\text{s.t.} \quad E_0 W_{\frac{T}{2}} \zeta_{\frac{T}{2}} = \zeta_0 W_0, \tag{27}$$

$$\Pr_0\left(W_{\frac{T}{2}} \leqslant \underline{W}\right) \leqslant \alpha, \tag{28}$$

$$\Pr_0\left(W_{\frac{T}{2}} < W_{\frac{T}{2},\min}^{2VaR}\right) = 0, \tag{29}$$

where $W_{T/2,\min}^{2VaR}$ is the minimal portfolio wealth required to fulfill the next period's VaR constraint. The minimum wealth needed at time T to fulfill the VaR constraint in the second period is

[2] The relative portfolio weights under a single ES constraint, $X_{spec,T/2}^{1ES}$ and $X_{hedge,T/2}^{1ES}$, converge to 1 at a very large value of $\zeta_{T/2}$. Since the chance that such a large value of $\zeta_{T/2}$ occur is small, therefore, it is not shown in Fig. 2.

$$W_{T,\min}^{2VaR} = \begin{cases} \underline{W} & \text{if } \zeta_T < \bar{\zeta}_T^{2VaR} \\ 0 & \text{if } \zeta_T \geqslant \bar{\zeta}_T^{2VaR} \end{cases},$$

i.e., keeping the portfolio wealth at time T at the level of $\underline{W}$ in the "good" and "intermediate" states and leaves the portfolio wealth at almost 0 in the "bad" states. Therefore, the minimum wealth at time $T/2, W_{T/2,\min}^{2VaR}$, equals the present value of the minimum wealth at time T, that is,

$$W_{\frac{T}{2},\min}^{2VaR} = \frac{1}{\zeta_{\frac{T}{2}}} E_{\frac{T}{2}} \underline{W} \zeta_T \, \mathbb{I}_{\zeta_T < \bar{\zeta}_T^{2VaR}}. \tag{30}$$

If the wealth at time $T/2$ is smaller than $W_{T/2,\min}^{2VaR}$, it is not possible to fulfill the VaR constraint in the second period.

The Lagrangian for the constrained optimization problem (26) is given by

$$\mathbb{L}^{2VaR}\left(W_{\frac{T}{2}}, \zeta_{\frac{T}{2}}, r_{\frac{T}{2}}\right) = E_0\left[J_{\frac{T}{2}}^{2VaR}\left(W_{\frac{T}{2}}\right) - y_0^{2VaR}\zeta_{\frac{T}{2}} W_{\frac{T}{2}} + y_1^{2VaR}\, \mathbb{I}_{W_{\frac{T}{2}} \geqslant \underline{W}} - y_2^{2VaR}\, \mathbb{I}_{W_{\frac{T}{2}} < W_{\frac{T}{2},\min}^{2VaR}}\right] + y_0^{2VaR}\zeta_0 W_0 - y_1^{2VaR}(1-\alpha), \tag{31}$$

where y_0^{2VaR}, y_1^{2VaR} and y_2^{2VaR} are Lagrange multipliers solving the budget constraint (27), the VaR constraint (28) and the minimum wealth constraint (29) respectively, with $y_0^{2VaR} \geqslant 0$, $y_1^{2VaR} \geqslant 0$ and $y_2^{2VaR} = \infty$. Since the last two terms of (31) are constants, finding a $W_{T/2}^{2VaR}$ which maximizes the value of (31) is equivalent to finding a portfolio wealth which maximizes the value of the function within $E_0[\cdot]$ in (31).

Due to the complexity of the function $J_{T/2}^{2VaR}(W_{T/2}, r_{T/2})$, it is not possible to analytically derive the first order derivative of $J_{T/2}^{2VaR}(W_{T/2})$ with respect to the wealth $W_{T/2}$. Therefore, a numerical method using a pointwise optimization is adopted to find the optimal portfolio wealth at time $T/2$, $W_{T/2}^{2VaR}$. That is, for each pair of interest rate at time $T/2$ and the pricing kernel value at time $T/2$, the optimal portfolio wealth is the one which maximizes the function $\mathbb{L}^{2VaR}(W_{T/2}, \zeta_{T/2}, r_{T/2})$.

The numerical procedures to find the optimal portfolio wealth $W_{T/2}^{2VaR}$ are as follows. First, we simulate N scenarios of interest rates, $r_{T/2,i}$, and pricing kernel values at time $T/2$, $\zeta_{T/2,i}$, with $i = 1,2,\ldots,N$. Second, we create a vector with N' different portfolio wealth in a very broad range, $W_{T/2,j}$ with $j = 1,2,\ldots,N'$. Third, since the indirect utility (25) depends on both the interest rate at time $T/2$ and the portfolio wealth at time $T/2$, for each interest rate $r_{T/2,i}$, we evaluate the value of $J_{T/2}^{2VaR}(r_{T/2,i}, W_{T/2,j})$ for all $W'_{T/2,j}$s.[3] Fourth, for each scenario of interest rate and pricing kernel value, i.e., $r_{T/2,i}$ and $\zeta_{T/2,i}$ with $i = 1,2,\ldots,N$, we evaluate the function value $\mathbb{L}^{2VaR}(W_{T/2,j}, \zeta_{T/2,i}, r_{T/2,i})$, for all $W'_{T/2,j}$s with $j = 1,2,\ldots,N'$. Finally, for each pair of $r_{T/2,i}$ and $\zeta_{T/2,i}$, the optimal portfolio wealth is the one which maximizes the value of $\mathbb{L}^{2VaR}(\cdot)$.

At time t, $0 \leqslant t \leqslant T/2$, the optimal portfolio allocation is

$$\begin{bmatrix} \pi_{S_t}^{2VaR} \\ \pi_{P_t^M}^{2VaR} \\ \pi_{P_t^{T-t}}^{2VaR} \end{bmatrix} = \frac{1}{\gamma} X_{spec,t}^{2VaR} \begin{bmatrix} \frac{\phi_S}{\sigma_S} \\ -\frac{\phi_r}{\sigma_r B(M)} \\ 0 \end{bmatrix} + \left(1 - \frac{1}{\gamma}\right) X_{hedge,t}^{2VaR} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{32}$$

where the portfolio wealth at time t, W_t^{2VaR}, is

$$W_t^{2VaR} = \frac{1}{\zeta_t} E_t\left(\zeta_{\frac{T}{2}} W_{\frac{T}{2}}^{2VaR}\right), \tag{33}$$

$X_{spec,t}^{2VaR}$ and $X_{hedge,t}^{2VaR}$ again measure the risk exposures relative to the ones without regulatory constraints and

$$X_{spec,t}^{2VaR} = -\frac{dW_t^{2VaR}}{d\zeta_t} \frac{\zeta_t}{W_t^{VaR2}} \gamma, \tag{34}$$

$$X_{hedge,t}^{2VaR} = -\frac{dW_t^{2VaR}}{dr_t} \frac{1}{W_t^{2VaR} B(T-t)\left(1 - \frac{1}{\gamma}\right)}. \tag{35}$$

Since the two first order derivatives $dW_t^{2VaR}/d\zeta_t$ and dW_t^{2VaR}/dr_t cannot be derived analytically, $X_{spec,t}^{2VaR}$ and $X_{hedge,t}^{2VaR}$ are approximated as before.

3.2.2. Two-EDS-constraint and two-ES-constraint models

The optimization problems under two repeated EDS constraints and ES constraints can also be solved with a similar backward induction procedure, except at time $T/2$ the minimum wealth is $W_{T/2,\min}^{2EDS}$ if two repeated EDS constraints are imposed, where

$$\begin{aligned} W_{T/2,\min}^{2EDS} &= E \frac{1}{\zeta_{T/2}} \zeta_T \underline{W} \mathbb{I}_{\zeta_T \leqslant \bar{\zeta}_T^{2EDS}} + E \\ &\quad \times \frac{1}{\zeta_{T/2}} \zeta_T \left[\left(y_{0,T/2}^{2EDS} - y_{1,T/2}^{2EDS}\right)\zeta_T\right]^{-\frac{1}{\gamma}} \mathbb{I}_{\zeta_T > \bar{\zeta}_T^{2EDS}} \\ &= \underline{W} e^{\mu_{\zeta,T/2,T,r_{T/2}} + \frac{1}{2}\sigma^2_{\zeta,T/2,T}} - \epsilon_{EDS}, \end{aligned} \tag{36}$$

and $W_{T/2,\min}^{2ES}$ if two repeated ES constraints are imposed, where

$$\begin{aligned} W_{T/2,\min}^{2ES} &= E \frac{1}{\zeta_{T/2}} \zeta_T \underline{W} \mathbb{I}_{\zeta_T \leqslant \bar{\zeta}_T^{2ES}} + E \\ &\quad \times \frac{1}{\zeta_{T/2}} \zeta_T \left(y_{0,T/2}^{2ES}\zeta_T - y_{1,T/2}^{2ES}\right)^{-\frac{1}{\gamma}} \mathbb{I}_{\zeta_T > \bar{\zeta}_T^{2ES}}. \end{aligned} \tag{37}$$

In both equations above, $y_{0,T/2}^{2EDS}$ and $y_{0,T/2}^{2ES}$ solve the second period's budget constraints in the EDS model and the ES model respectively and $y_{1,T/2}^{2EDS}$ and $y_{1,T/2}^{2ES}$ solves the second period's EDS constraint and ES constraint respectively. The superindices 2EDS and 2ES stand for two repeated EDS constraints and ES constraints respectively.

The method to derive optimal portfolio choices under EDS or ES constraints is the same as the one under VaR constraints. At time t, $0 \leqslant t \leqslant T/2$, the optimal portfolio allocation is

$$\begin{bmatrix} \pi_{S_t}^{2c} \\ \pi_{P_t^M}^{2c} \\ \pi_{P_t^{T-t}}^{2c} \end{bmatrix} = \frac{1}{\gamma} X_{spec,t}^{2c} \begin{bmatrix} \frac{\phi_S}{\sigma_S} \\ -\frac{\phi_r}{\sigma_r B(M)} \\ 0 \end{bmatrix} + \left(1 - \frac{1}{\gamma}\right) X_{hedge,t}^{2c} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{38}$$

where c stands for the EDS, or ES constraint. The relative risk exposures $X_{spec,t}^{2c}$ and $X_{hedge,t}^{2c}$ are given as

$$X_{spec,t}^{2c} = -\frac{dW_t^{2c}}{d\zeta_t} \frac{\zeta_t}{W_t^{2c}} \gamma, \tag{39}$$

$$X_{hedge,t}^{2c} = -\frac{dW_t^{2c}}{dr_t} \frac{1}{W_t^{2c} B(T-t)\left(1 - \frac{1}{\gamma}\right)}. \tag{40}$$

Finally, W_t^{2c} is the portfolio wealth at time $T/2$, and again

$$W_t^{2c} = \frac{1}{\zeta_t} E_t\left(\zeta_{\frac{T}{2}} W_{\frac{T}{2}}^{2c}\right).$$

3.2.3. Comparison of optimal portfolio wealth distributions and allocations under two-repeated-constraint models

When the regulatory constraint is as long as the investment horizon (i.e., the single-constraint models), Fig. 1 shows that the three regulatory constraints lead to significantly different final

[3] To speed up the numerical process, we could first evaluate the indirect utility function value $J_{T/2}(\cdot)$ for a small sample of interest rates. For each given portfolio wealth the value function value is almost linearly increasing with interest rates. This relationship enable us to evaluate the function value $J_{T/2}(r_{T/2,i}, W_{T/2,j})$ for $i \in [1,N]$ by interpolation.

wealth levels. However, when these regulatory constraints are imposed repeatedly and when the regulatory horizon is significantly shorter than the investment horizon, the differences in portfolio wealth disappears. Fig. 4 compares the portfolio wealth at time $T/2$ under two repeated VaR constraints, EDS constraints and ES constraints when the interest rate is 4%. Observe that all three types of regulatory constraints are able to prevent large portfolio wealth losses when the financial market is unfavorable. These results stem from the necessity to hold the portfolio wealth above the minimum portfolio wealth level to fulfill the second period's regulatory constraint. Only in the "intermediate" states, the portfolio wealth under two repeated VaR constraints is different from others because of the requirement to keep the portfolio wealth at or above $\underline{W}$ in $1-\alpha$ percent of cases.

Fig. 5 compares the portfolio wealth at time $T/2$ when the interest rates are 2% and 10% respectively. We can conclude that (1) in "good" states, the optimal portfolio wealth decreases as interest rates increase, and (2) the minimum wealth at time $T/2\left(W^{2VaR}_{T/2,\min}, W^{2EDS}_{T/2,\min}, W^{2ES}_{T/2,\min}\right)$ decreases as the interest rate increases. The latter occurs as a high interest rate leads to a high zero-coupon bond yield and thus reduces the minimum amount of wealth necessary to fulfill the next period's regulatory constraint.

With respect to VaR-type constraints, the states in which the investor keeps his portfolio wealth at $\underline{W}$ no longer depends only on the pricing kernel values. For example, in Fig. 5, the investor under two repeated VaR constraints chooses to keep his portfolio wealth at $\underline{W}$ in the state where $\zeta_t = \zeta_1$ and $r_t = 2\%$ while leave the portfolio wealth at $W^{VaR}_{T/2,\min}$ in the state where $\zeta_t = \zeta_2$ and $r_t = 10\%$ even though ζ_1 is larger than $\bar{\zeta}^{2VaR}_{T/2}$ and ζ_2 is smaller than $\bar{\zeta}^{2VaR}_{T/2}$. The investor decides in which states he keeps his portfolio wealth at $\underline{W}$ not only on the value of the pricing kernel but also on the interest rates. For each state at time t, $t \in [0,T]$, the cost of raising the portfolio wealth from the unconstrained portfolio wealth W^u_t to $\underline{W}$ equals $\zeta_t(\underline{W} - W^u_t)$. At time T, as shown in Fig. 1, the unconstrained portfolio wealth W^u_T monotonically decreases as ζ_T increases. Therefore, at time T, it is always cheaper to raise the wealth level to $\underline{W}$ in states where $\zeta_T < \bar{\zeta}^{2VaR}_T$. While at time $T/2$, the unconstraint portfolio wealth depends on both the pricing kernel value and the interest rate. For any given value of $\zeta_{T/2}$, the unconstrained portfolio wealth $W^u_{T/2}$ decreases when the interest rate increases. Thus, the cost of raising the portfolio wealth to $\underline{W}$ in states with large pricing kernel value, i.e., $\zeta_{T/2} > \bar{\zeta}^{2VaR}_{T/2}$ and low interest rates might be cheaper than the one in states with small pricing kernel values, i.e., $\zeta_{T/2} \leqslant \bar{\zeta}^{2VaR}_{T/2}$, and high interest rates.

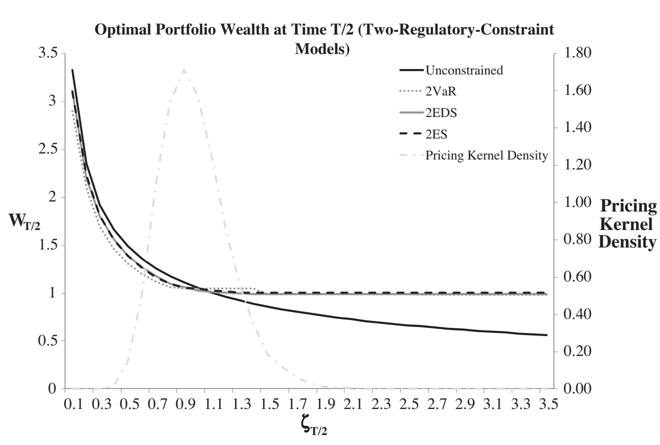

Fig. 4. This figure compares the optimal portfolio wealth in the two-constraint models at time $T/2$. The interest rate $r_{T/2}$ is 0.04. The parameter values are $W_0 = 1.04$, $\kappa = 0.15$, $\bar{r} = 0.05$, $\sigma_r = 0.015$, the bond Sharpe ratio $\lambda_P = 0.05$, the maturity of the bond fund $M = 10$ years, $\sigma_S = 0.25$, the stock Sharpe ratio $\lambda_S = 0.25$, $\rho_{Sr} = 0.2$, $\underline{W} = 1.05$, $r_0 = 2\%$, $\gamma = 2$, $\alpha = 0.025$, $\varepsilon_{ES} = 0.008$ and $\varepsilon_{EDS} = 0.017$.

Optimal portfolio allocation under two repeated regulatory constraints is significantly different from the one under a single-regulatory constraint. Figs. 6–8 depict the pre-horizon relative portfolio allocation under two repeated VaR, EDS and ES constraints respectively when the interest rate $r_{T/3}$ is 4%. The pre-horizon relative portfolio allocations under a single-VaR, a single-EDS, and a single-ES constraint are also shown as a comparison. Generally speaking, all two types of regulatory constraints force the investor to put more weight on the hedge fund as the financial market performance deteriorates. The difference between VaR-type constraints and Expected Shortfall-type constraints still lies in the existence of the incentive to gamble. At the time when the weight of the speculative fund is increasing, however, unlike the investor under a single-VaR constraint who decreases his holdings in the hedge fund, the investor under two VaR constraints *increases* his holdings in the hedge fund to compensate for the possible loss generated by the speculative fund. By doing so, the investor under two VaR constraints guarantees that his portfolio wealth is large enough to fulfill next period's VaR constraint in all circumstances.

3.2.4. Multi-constraint models

The analysis above can easily be extended to more than three constraints. For example, if there are m subsequent and non-overlapping VaR constraints within the investment horizon, we start by solving the optimal portfolio wealth in the last period and then proceed backwards by repeating the numerical procedures developed for finding the first period's optimal portfolio wealth in the two-constraint model. As is to be expected from the previous results, the optimal portfolio wealth under the two types of regulatory constraints, when implemented dynamically, are very similar.

4. Certainty equivalent loss

In this section, we consider a pension fund with 15-year investment horizon as an example to analyze the cost and the benefit of both the VaR-type and the Expected Shortfall-type prudential regulation. The regulatory horizon considered here is 1 year, meaning that in the 15-year investment horizon, there are 15 non-overlapping regulatory constraints.

To avoid agency conflicts between a pension fund's participants and the pension fund's managers, regulatory constraints are needed to protect the liabilities of participants. But in a country like the Netherlands where Defined Benefit pension benefits are linked to the pension fund's portfolio returns, the costs of regulation are, therefore, borne by all participants, with some more than others.

The economic cost is measured by the certainty equivalent loss ce relative to the unconstrained portfolio allocation problem. The certainty equivalent loss ce is defined as the equivalent amount of wealth lost due to the regulation, i.e.,

$$J^u_0(W_0 - ce) = J^{cm}_0(W_0),$$

where $J^u_0(\cdot)$ stands for the indirect utility at time 0 without any regulatory constraints, and $J^{cm}_0(W_0)$ is the indirect utility at time 0 with m regulatory constraints. The economic benefit is measured by a reduction in the Expected Shortfall at time 0. The Expected Shortfall, SF^{cm}_0, is defined as

$$SF^{cm}_0 = E_0 \max(\underline{W} - W_T, 0).$$

We assume that $\kappa = 1.5\%$, $\bar{r} = 5\%$, $\sigma_r = 1.5\%$, the bond Sharpe Ratio $\lambda_P = 5\%$, $M = 10$ years, $\sigma_S = 25\%$, $\rho_{Sr} = 20\%$, $\underline{W} = 1.05$, and the stock

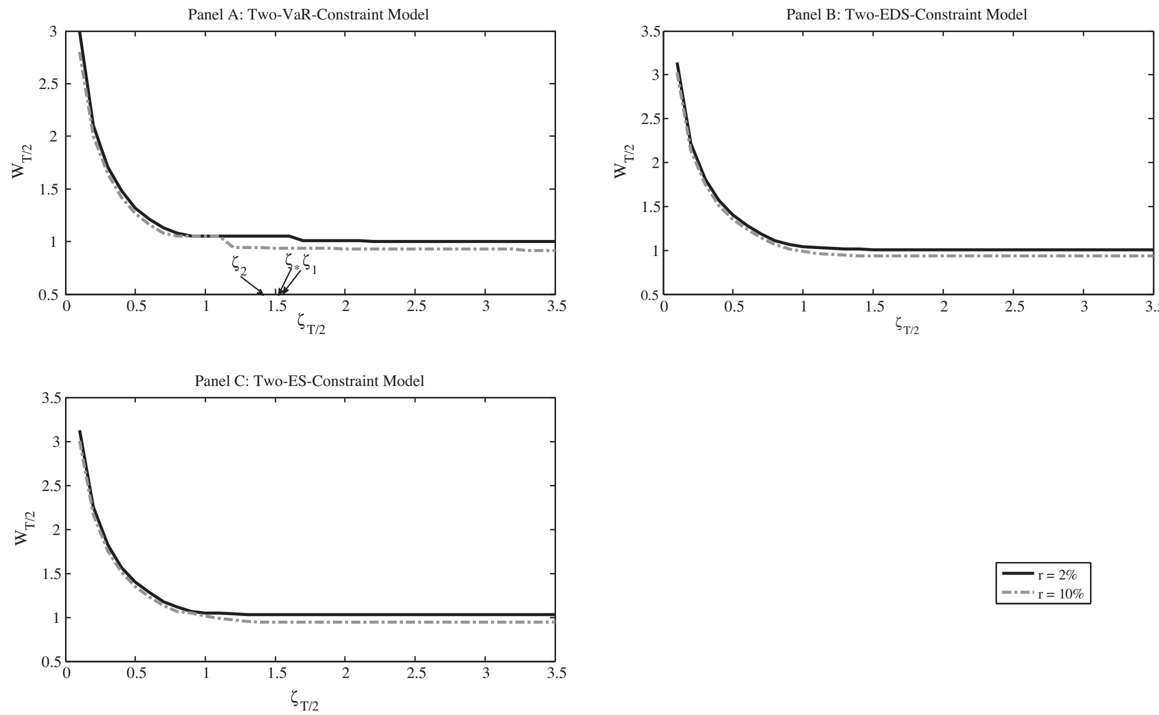

Fig. 5. This figure compares the optimal portfolio wealth in the two-constraint models at time $T/2$ with different interest rates. The parameter values are $W_0 = 1.04$, $\kappa = 0.15$, $\bar{r} = 0.05$, $\sigma_r = 0.015$, the bond Sharpe ratio $\lambda_P = 0.05$, the maturity of the bond fund $M = 10$ years, $\sigma_S = 0.25$, the stock Sharpe ratio $\lambda_S = 0.25$, $\rho_{Sr} = 0.2$, $\underline{W} = 1.05$, $r_0 = 2\%$, $\gamma = 2$, $\alpha = 0.025$, $\varepsilon_{ES} = 0.008$ and $\varepsilon_{EDS} = 0.017$. ζ_* represents $\bar{\zeta}^{2VaR}_{T/2}$.

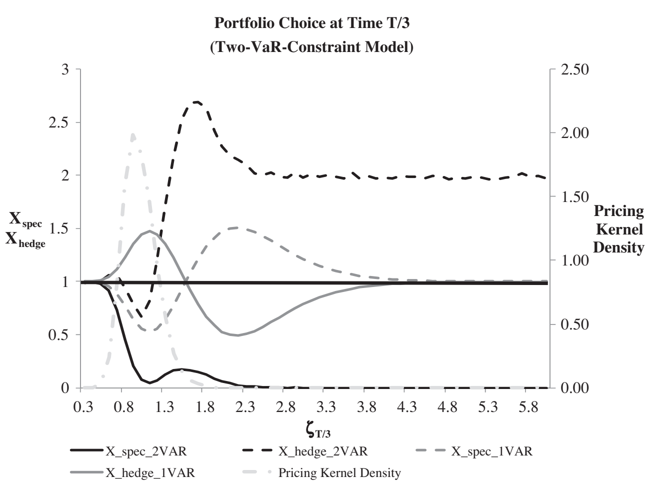

Fig. 6. This figure shows the relative portfolio allocation at time $T/3$ in two-VaR-constraint model. The parameter values are $W_0 = 1.04$, $\kappa = 0.15$, $\bar{r} = 0.05$, $\sigma_r = 0.015$, the bond Sharpe ratio $\lambda_P = 0.05$, the maturity of the bond fund $M = 10$ years, $\sigma_S = 0.25$, the stock Sharpe ratio $\lambda_S = 0.25$, $\rho_{Sr} = 0.2$, $\underline{W} = 1.05$, $r_0 = 2\%$, $\gamma = 2$, $\alpha = 0.025$, $\varepsilon_{ES} = 0.008$ and $\varepsilon_{EDS} = 0.017$.

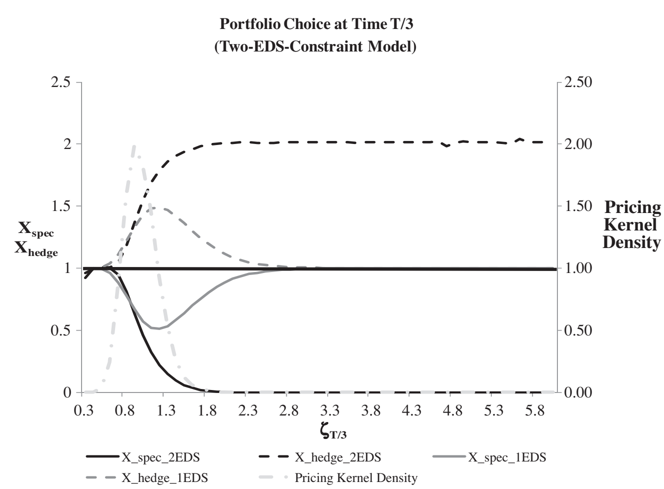

Fig. 7. This figure shows the relative portfolio allocation at time $T/3$ in two-EDS-constraint model. The parameter values are $W_0 = 1.04$, $\kappa = 0.15$, $\bar{r} = 0.05$, $\sigma_r = 0.015$, the bond Sharpe ratio $\lambda_P = 0.05$, the maturity of the bond fund $M = 10$ years, $\sigma_S = 0.25$, the stock Sharpe ratio $\lambda_S = 0.25$, $\rho_{Sr} = 0.2$, $\underline{W} = 1.05$, $r_0 = 2\%$, $\gamma = 2$, $\alpha = 0.025$, $\varepsilon_{ES} = 0.008$ and $\varepsilon_{EDS} = 0.017$.

Sharpe ratio $\lambda_s = 25\%$. These set of parameters are close to those obtained by empirical studies, for example, Chan et al. (1992). In particular, ρ_{Sr} is chosen to be positive so that the correlation between interest rate and stock price is negative which is suggested by Campbell (1987). The short-term interest rate at time 0, r_0, is 4% which is close to the average US 3-month T-Bill rate from 1985 to 2010. For a pension fund, the natural choice of the "floor" $\underline{W}$ is its liability. In this paper, we assumed that the value of the pension fund's liability is constant over time but it can be easily extended to the case when the liability value is stochastic as long as the liability value is exogenously determined.

Fig. 9 shows the certainty equivalent loss and the expected portfolio wealth shortfall of a pension fund with $\gamma = 2$ and $\alpha = 2.5\%$. ϵ_{EDS} and ϵ_{ES} are about 0.017 and 0.008 respectively.

We find that the 15 regulatory constraints can significantly reduce the portfolio wealth shortfall. It is almost guaranteed that at the end of the investment horizon the pension fund's portfolio wealth will be above $\underline{W}$. For example, when the initial portfolio

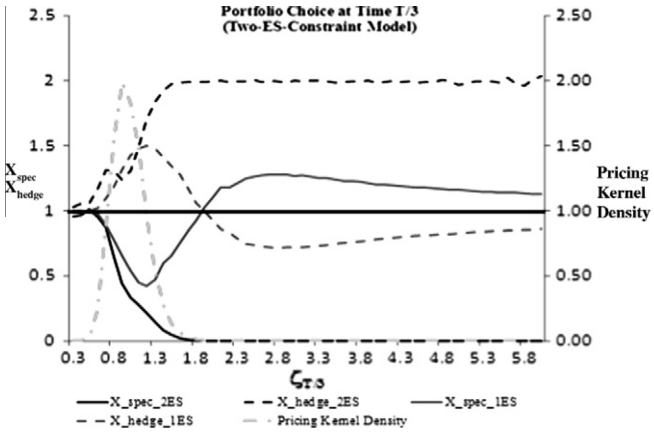

Fig. 8. This figure shows the relative portfolio allocation at time $T/3$ in two-ES-constraint model. The parameter values are $W_0 = 1.04$, $\kappa = 0.15$, $\bar{r} = 0.05$, $\sigma_r = 0.015$, the bond Sharpe ratio $\lambda_P = 0.05$, the maturity of the bond fund $M = 10$ years, $\sigma_S = 0.25$, the stock Sharpe ratio $\lambda_S = 0.25$, $\rho_{Sr} = 0.2$, $\underline{\underline{W}} = 1.05$, $r_0 = 2\%$, $\gamma = 2$, $\alpha = 0.025$, $\varepsilon_{ES} = 0.008$ and $\varepsilon_{EDS} = 0.017$.

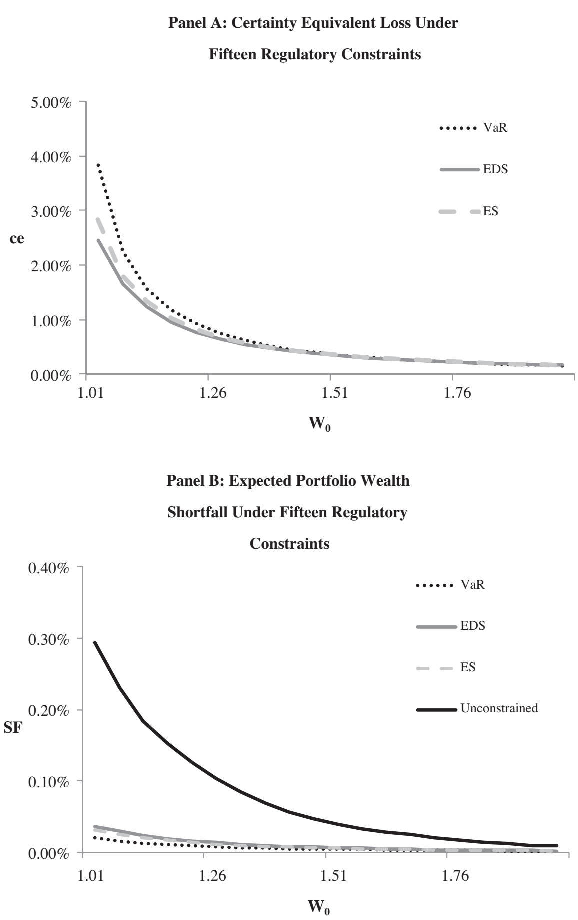

Fig. 9. This figure shows the certainty equivalent loss (Panel A) and the Expected Shortfall (Panel B) of a pension fund with a 15-year investment horizon. The parameter values are $\kappa = 0.15$, $\bar{r} = 0.05$, $\sigma_r = 0.015$, the bond Sharpe ratio $\lambda_P = 0.05$, the maturity of the bond fund $M = 10$ years, $\sigma_S = 0.25$, the stock Sharpe Ratio $\lambda_S = 0.25$, $\rho_{Sr} = 0.2$, $\underline{\underline{W}} = 1.05$, $r_0 = 4\%$, $\gamma = 2$, $\alpha = 0.025$, $\varepsilon_{ES} = 0.008$ and $\varepsilon_{EDS} = 0.017$.

wealth is about 1.01, which corresponds to a funding ratio of about 0.96, the Expected Shortfall is about 0.3% when no regulatory constraints are imposed. The Expected Shortfall decreases to almost 0 as the regulatory frequency increases. However, the certainty equivalent loss is about 3.8% when there are 15 VaR constraints and about 2.5% (2.8%) when there are 15 EDS (ES) constraints. It shows that the economic costs of imposing VaR-type constraints or Expected Shortfall-type constraints are very similar when these constraints are imposed repeatedly.

5. Conclusions

Value-at-Risk and Expected Shortfall constraints are often adopted by regulators to limit the portfolio risk of institutional investors. However, the regulatory horizon is usually much shorter than the institutional investors' investment horizon. In this paper, we compare the optimal portfolio wealth, optimal portfolio allocation and the overall economic costs when VaR and Expected Shortfall constraints are imposed repeatedly over an institutional investor's investment horizon. We find, e.g., that a constrained investor, as expected, invests more in the risk-free asset than unconstrained investors. However, unintendedly, VaR constraints may under certain market conditions also lead to gambling behavior in order to be able to meet future regulatory constraints. When adopting multiple repeated constraints, we observe that the differences in portfolio allocation and wealth under VaR-type and Expected Shortfall-type constraints disappears. The theoretical shortcomings of VaR, as not being a coherent risk measure (see Artzner et al., 1999), may thus be less severe in a dynamic setting. We also find that the costs of imposing the constraints, in terms of certainty equivalent wealth, can be sizeable.

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Appendix A. The optimal portfolio allocation in the single-constraint model

A.1. The single-VaR-constraint model

The optimal portfolio allocation under a single-VaR constraint is

$$\begin{bmatrix} \pi_{S_t}^{1VaR} \\ \pi_{P_t^M}^{1VaR} \\ \pi_{P_t^{T-t}}^{1VaR} \end{bmatrix} = \frac{1}{\gamma}\begin{bmatrix} \frac{\phi_S}{\sigma_S} X_{spec,t}^{1VaR} \\ -\frac{\phi_r}{\sigma_r B_M} X_{spec,t}^{1VaR} \\ 0 \end{bmatrix} + \left(1 - \frac{1}{\gamma}\right)\begin{bmatrix} 0 \\ 0 \\ X_{hedge,t}^{1VaR} \end{bmatrix}, \tag{41}$$

where

$$X_{spec,t}^{1VaR} = -\frac{dW_t}{d\zeta_t}\frac{\zeta_t}{W_t}\gamma,$$

$$X_{hedge,t}^{1VaR} = -\frac{1}{W_t \frac{1-e^{-\kappa(T-t)}}{\kappa}\left(1 - \frac{1}{\gamma}\right)}\frac{dW_t}{dr_t},$$

$$\begin{aligned}\frac{dW_t}{dr_t} = &-\frac{e^{\Gamma_t}}{(y_0^{1VaR}\zeta_t)^{\frac{1}{\gamma}}}N(d_1(\underline{\zeta}))\frac{1-e^{-\kappa(T-t)}}{\kappa}\left(1-\frac{1}{\gamma}\right)\\ &+\frac{e^{\Gamma_t}}{(y_0^{1VaR}\zeta_t)^{\frac{1}{\gamma}}}N'(d_1(\underline{\zeta}))\frac{(1-e^{-\kappa(T-t)})}{\kappa\sigma_{\zeta,t,T}}\\ &-\underline{W}e^{\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}}N(-d_2(\underline{\zeta}))\frac{1-e^{-\kappa(T-t)}}{\kappa}\\ &-\underline{W}e^{\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}}N'(-d_2(\underline{\zeta}))\frac{(1-e^{-\kappa(T-t)})}{\kappa\sigma_{\zeta,t,T}}\\ &+\underline{W}e^{\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}}N(-d_2(\bar{\zeta}))\frac{1-e^{-\kappa(T-t)}}{\kappa}\\ &+\underline{W}e^{\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}}N'(-d_2(\bar{\zeta}))\frac{(1-e^{-\kappa(T-t)})}{\kappa\sigma_{\zeta,t,T}}\\ &-\frac{e^{\Gamma_t}}{(y_0^{1VaR}\zeta_t)^{\frac{1}{\gamma}}}N(-d_1(\bar{\zeta}))\frac{1-e^{-\kappa(T-t)}}{\kappa}\left(1-\frac{1}{\gamma}\right)\\ &-\frac{e^{\Gamma_t}}{(y_0^{1VaR}\zeta_t)^{\frac{1}{\gamma}}}N'(-d_1(\bar{\zeta}))\frac{(1-e^{-\kappa(T-t)})}{\kappa\sigma_{\zeta,t,T}},\end{aligned} \tag{42}$$

$$\begin{aligned}\frac{dW_t}{d\zeta_t} = &\left(-\frac{1}{\gamma}\right)\frac{e^{\Gamma_t}}{(y_0^{1VaR}\zeta_t)^{\frac{1}{\gamma}}\zeta_t}N(d_1(\underline{\zeta}))\\ &+\frac{e^{\Gamma_t}}{(y_0^{1VaR}\zeta_t)^{\frac{1}{\gamma}}}N'(d_1(\underline{\zeta}))\left(\frac{-\zeta_t^{-1}}{\sigma_{\zeta,t,T}}\right)\\ &+\underline{W}\mathbf{e}^{\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}}N'(-d_2(\underline{\zeta}))\frac{\zeta_t^{-1}}{\sigma_{\zeta,t,T}}\\ &-\underline{W}\mathbf{e}^{\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}}N'(-d_2(\bar{\zeta}))\frac{\zeta_t^{-1}}{\sigma_{\zeta,t,T}}\\ &+\left(-\frac{1}{\gamma}\right)\frac{e^{\Gamma_t}}{(y_0^{1VaR}\zeta_t)^{\frac{1}{\gamma}}\zeta_t}N(-d_1(\bar{\zeta}))\\ &+\frac{e^{\Gamma_t}}{(y_0^{1VaR}\zeta_t)^{\frac{1}{\gamma}}}N'(-d_1(\bar{\zeta}))\left(\frac{\zeta_t^{-1}}{\sigma_{\zeta,t,T}}\right),\end{aligned} \tag{43}$$

$$\begin{aligned}\mu_{\zeta,t,T} = &(\bar{r}-r_t)B(T-t)-\bar{r}(T-t)\\ &-\frac{1}{2}(\phi_s^2-2\rho_{sr}\phi_s\phi_r+\phi_r^2)(T-t),\end{aligned} \tag{44}$$

$$\begin{aligned}\sigma^2_{\zeta,t,T} = &-\frac{\sigma_r^2}{2\kappa}B(T-t)^2+\left(2\frac{\Phi_r}{\kappa}-\frac{\sigma_r^2}{\kappa^2}\right)(B(T-t)-(T-t))\\ &+\phi_s^2(T-t)+\phi_r^2(T-t)-2\rho_{sr}\phi_s\phi_r(T-t),\end{aligned} \tag{45}$$

and

$$d_2(x) = \frac{\log\left(\frac{x}{\zeta_t}\right)-\left(\mu_{\zeta,t,T}+\sigma^2_{\zeta,t,T}\right)}{\sigma_{\zeta,t,T}},$$

$$d_1(x) = d_2(x)+\frac{1}{\gamma}\sigma_{\zeta,t,T},$$

$$\begin{aligned}\Gamma_t &= \mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}-\frac{1}{\gamma}\left(\mu_{\zeta,t,T}+\sigma^2_{\zeta,t,T}\right)+\frac{1}{2}\frac{1}{\gamma^2}\sigma^2_{\zeta,t,T}\\ &= \left(1-\frac{1}{\gamma}\right)\mu_{\zeta,t,T}+\left(\frac{1}{2}-\frac{1}{\gamma}+\frac{1}{2\gamma^2}\right)\sigma^2_{\zeta,t,T}.\end{aligned}$$

A.2. The single-EDS-constraint model

The optimal portfolio choices under a single-EDS constraint is

$$\begin{bmatrix}\pi^{1EDS}_{s,t}\\ \pi^{1EDS}_{P_t^M}\\ \pi^{1EDS}_{P_t^{T-t}}\end{bmatrix} = \frac{1}{\gamma}X^{1EDS}_{spec,t}\begin{bmatrix}\frac{\phi_s}{\sigma_S}\\ -\frac{\phi_r}{\sigma_r B(M)}\\ 0\end{bmatrix}+\left(1-\frac{1}{\gamma}\right)X^{1EDS}_{hedge,t}\begin{bmatrix}0\\0\\1\end{bmatrix}, \tag{46}$$

where

$$X^{1EDS}_{spec,t} = -\frac{dW_t^{1EDS}}{d\zeta_t}\frac{\zeta_t}{W_t^{1EDS}}\gamma, \tag{47}$$

$$X^{1EDS}_{hedge,t} = -\frac{1}{W_t^{1EDS}\frac{\mathbf{1-e^{-\kappa(T-t)}}}{\boldsymbol{\kappa}}\left(1-\frac{1}{\gamma}\right)}\frac{dW_t^{1EDS}}{dr_t}, \tag{48}$$

$$\begin{aligned}\frac{dW_t^{1EDS}}{dr_t} = &-\frac{e^{\Gamma_t}}{(y_0^{1EDS}\zeta_t)^{\frac{1}{\gamma}}}N(d_1(\underline{\zeta}))\frac{1-e^{-\kappa(T-t)}}{\kappa}\left(1-\frac{1}{\gamma}\right)\\ &+\frac{e^{\Gamma_t}}{(y_0^{1EDS}\zeta_t)^{\frac{1}{\gamma}}}N'(d_1(\underline{\zeta}))\frac{(1-e^{-\kappa(T-t)})}{\kappa\sigma_{\zeta,t,T}}\\ &-\underline{W}e^{\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}}N(-d_2(\underline{\zeta}))\frac{1-e^{-\kappa(T-t)}}{\kappa}\\ &-\underline{W}e^{\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}}N'(-d_2(\underline{\zeta}))\frac{(1-e^{-\kappa(T-t)})}{\kappa\sigma_{\zeta,t,T}}\\ &+\underline{W}e^{\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}}N(-d_2(\bar{\zeta}))\frac{1-e^{-\kappa(T-t)}}{\kappa}\\ &+\underline{W}e^{\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}}N'(-d_2(\bar{\zeta}))\frac{(1-e^{-\kappa(T-t)})}{\kappa\sigma_{\zeta,t,T}}\\ &-\frac{e^{\Gamma_t}}{((y_0^{1EDS}-y_1^{1EDS})\zeta_t)^{\frac{1}{\gamma}}}N(-d_1(\bar{\zeta}))\\ &\times\frac{1-e^{-\kappa(T-t)}}{\kappa}\left(1-\frac{1}{\gamma}\right)\\ &-\frac{e^{\Gamma_t}}{((y_0^{1EDS}-y_1^{1EDS})\zeta_t)^{\frac{1}{\gamma}}}N'(-d_1(\bar{\zeta}))\frac{(1-e^{-\kappa(T-t)})}{\kappa\sigma_{\zeta,t,T}},\end{aligned} \tag{49}$$

and

$$\begin{aligned}\frac{dW_t^{1EDS}}{d\zeta_t} = &\left(-\frac{1}{\gamma}\right)\frac{e^{\Gamma_t}}{(y_0^{1EDS}\zeta_t)^{\frac{1}{\gamma}}\zeta_t}N(d_1(\underline{\zeta}))\\ &+\frac{e^{\Gamma_t}}{(y_0^{1EDS}\zeta_t)^{\frac{1}{\gamma}}}N'(d_1(\underline{\zeta}))\left(\frac{-\zeta_t^{-1}}{\sigma_{\zeta,t,T}}\right)\\ &+\underline{W}\mathbf{e}^{\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}}N'(-d_2(\underline{\zeta}))\frac{\zeta_t^{-1}}{\sigma_{\zeta,t,T}}\\ &-\underline{W}\mathbf{e}^{\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}}N'(-d_2(\bar{\zeta}))\frac{\zeta_t^{-1}}{\sigma_{\zeta,t,T}}\\ &+\left(-\frac{1}{\gamma}\right)\frac{e^{\Gamma_t}}{((y_0^{1EDS}-y_1^{1EDS})\zeta_t)^{\frac{1}{\gamma}}\zeta_t}N(-d_1(\bar{\zeta}))\\ &+\frac{e^{\Gamma_t}}{((y_0^{1EDS}-y_1^{1EDS})\zeta_t)^{\frac{1}{\gamma}}}N'(-d_1(\bar{\zeta}))\left(\frac{\zeta_t^{-1}}{\sigma_{\zeta,t,T}}\right).\end{aligned} \tag{50}$$

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