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Fuzzy multi period portfolio selection with different rates for borrowing and lending

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ABSTRACT

Investment strategic planning is one of the most important areas of research in financial engineering. The primary concern of this research is to determine the amount of investment in different planning areas especially when the rate of borrowing is greater than that of lending. The proposed research method in this paper is a form of fuzzy linear programming which is capable of determining the amount of investment in different time cycles. In this paper return rates and borrowing/lending rate are presented as fuzzy triangular numbers instead of crisp representations. The developed model can instruct the balance between cash and margin for investors and using fuzzy set theory, their confidence level can be obtained for each produced portfolio. The method is also implemented using some numerical examples and the output results are discussed.

1. Introduction

Portfolio management plays an important role for many financial institutions. In a typical portfolio management, one is responsible to allocate funding to different assets by buying and selling them. Markowitz [1–3] is believed to be the first who introduced portfolio management where a quadratic objective function is minimized subject to some linear constraints. Markowitz has presented three nonlinear models and explained that the unique optimal solution for all three models is equal. The primary objective of Markowitz model is to build a portfolio with the highest expected return at a given level of risk.

Modern portfolio theory (MPT) that was introduced by Markowitz has led to a new paradigm in portfolio selecting for investors. However, many researches in the field have attempted to solve and develop Markowitz's seminal model. Lots of these attempts have tried to make his model more useful and practical.

In spite of comprehensive success of Markowitz' model, the portfolio selection strategy was extended for a planning horizon by many researchers using different approaches such as Samuelson [4] and Merton [5] because the single-period framework suffers from an important problem where it is practically impossible to apply a single framework work to a long term horizon. Merton [6,7] presents a mathematical model for the optimum consumption and the portfolio rules in a continuous time horizon. Merton [6,7] in his work shows how to construct and analyze optimal continuous-time allocation problems under uncertain parameters. Merton [6,7] considers the model in which the prices of the risky assets are generated by correlated geometric Brownian motions, and assumes that the portfolio can be rebalanced instantly and free of cost. His objective is to maximize the net expected utility of consumption plus the expected utility of terminal wealth. Mossin [8] also presents a multi-period optimization technique. Chryssikou [9] uses approximate dynamic programming algorithms to provide a near-optimal dynamic trading strategy for special types of utility functions when a closed form solution to the discrete-time multi period problem with quadratic transaction costs is not attainable. Hakansson [10,11] uses mean-variance and quadratic approximations in implementing dynamic investment strategies. Techniques from approximate dynamic programming have been successfully employed for efficient optimal policy computations. For example, Sadjadi et al. [12] propose a dynamic programming approach to solve efficient frontier with the consideration of transaction cost. Their approach led to a closed form solution of the mean variance portfolio selection.

Li et al. [13,14] consider a two-step method where a dynamic programming is employed to solve an auxiliary problem in the first phase and the solution to the auxiliary problem is then manipulated to obtain the optimal mean-variance portfolio policy and the corresponding efficient frontier.

Yu et al. [15] present a model for multi-period portfolio selection with maximum absolute deviation. Their model obtains optimal strategy via dynamic programming in a closed form solution. Chen

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Table 1
Some of approaches reviewed in the state of the art.

Selective scope of related topic		Reviewed literatures
Markowitz seminal model to multi period case		Samuelson (1969), Merton (1969, 1971, 1996), Mossin (1968), Zenios et al. (1998), Morey & Morey (1999), Leippold et al. (2004), Briec & Kerstens (2009)
	Dynamic programming	Chryssikou (1998), Hakansson (1971), Grauer & Hakansson (1993), Sadjadi et al.(2004), Li et al. (1998), Li & Ng (2000), Yu et al. (2010), Chen et al. (2010)
Portfolio optimization with uncertainty	Stochastic programming	Leippold et al. (2004), Wei & Ye (2007), Celikyurt & Ozekici (2007), Calafiore (2008), Costa & Araujo (2008), Rapach & Wohar (2009)
	Robust optimization	Shen & Zhang (2008), Quaranta & Zaffaroni (2008), Bertsimas & Pachamanova (2008), Chen & Tan (2009)

et al. [16] introduce a dynamic portfolio optimization which is adapted to change in stock prices based on a genetic network programming. Their proposed model uses technical indices and candlestick chart to generate portfolio investment advice.

Note that the primary assumption with these developed models is that the rates of return of the assets during consecutive periods are uncorrelated. Leippold et al. [17] introduce a geometric approach to multi period mean variance optimization of assets and liabilities. Morey and Morey [18] introduce the same idea in a multi-period or temporal setting. They propose two types of efficiency measures: The first efficiency measure attempts to contract all risk dimensions proportionally where the second one focuses on augmenting all return dimensions as much as possible in a proportional way.

Briec and Kerstens [19] develop a multi-horizon mean-variance portfolio analysis developed by Morey and Morey in the [18], in several ways. First, instead of either proportionally contracting risk dimensions or proportionally expanding return dimensions, a more general efficiency measure simultaneously attempts to reduce the risk and to expand the return over all time periods.

The multi period models have been developed in a variety of directions. Zenios et al. [20] develop a fixed income portfolio model in a multi-stage form. The uncertainty which exists on the input and the output parameters of the multi-period portfolio optimization could be investigated in different forms. In Ref. [17] a method to minimize the variance between the assets and liabilities is proposed.

Wei and Ye [21] introduce a multi period portfolio selection model constrained with bankruptcy control in a stochastic market. They use dynamic programming to solve developed model. Calafiore [22] proposed an asset allocation model which periodic optimal portfolio adjustments are determined with the objective of minimizing a cumulative risk measure over the investment horizon. In developed model, portfolio diversity constraints at each period are satisfied. Celikyurt and Ozekici [23] consider a multi period portfolio model where the market consists of a riskless asset and several risky assets. They can describe the stochastic evaluation of market by a Markov chain. Oswaldo et al. [24] propose a generalized multi period mean-variance model with market parameters such as Markov switching parameters. They can obtain some closed formulas with necessary and sufficient conditions for obtaining an optimal control policy for this Markovian generalized multi period mean-variance problem. Rapach and Wohar [25] carried out a comparative study in eight countries to investigate the inter-temporal hedging demands for stocks and the bonds for investors. They solve multi period portfolio selection problem with an infinite horizon and with asset returns which are described by a vector autoregressive process.

Shen and Zhang [26] also apply the concept of robust optimization to the portfolio selection problems. Their proposed model is formulated based on multi-stage scenario trees. They use SeDuMi to solve their robust portfolio selection problem. Quaranta and Zaffaroni [27] use robust optimization in portfolio selection problem for the minimization of the conditional value at risk of a portfolio of shares. They can obtain a linear robust copy of the bi-criteria minimization model. Chen and Tan [28] can successfully incorporate interval random chance-constrained programming to robust mean-variance portfolio selection under interval random uncertainty sets in the elements of mean vector and covariance matrix.

Bertsimas and Pachamanova [29] suggest robust optimization formulations of the multi period portfolio optimization problem that are linear and computationally efficient. Robust optimization models deal with future asset returns as uncertain coefficients in an optimization problem. Bertsimas and Pachamanova in [29] impose non negativity constraints on the investor's holdings at each time

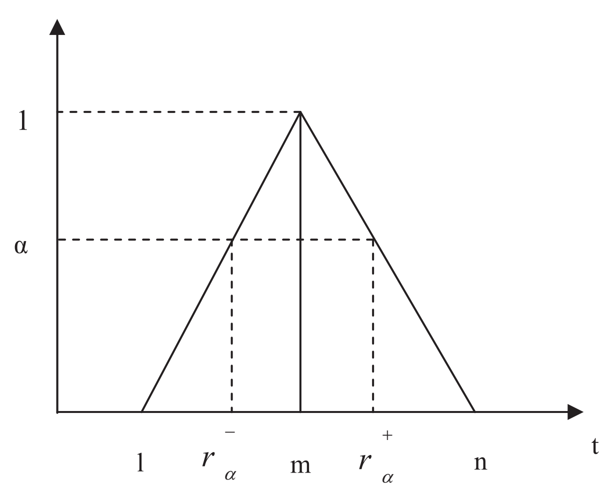

Fig. 1. The membership function of $\tilde{r}$.

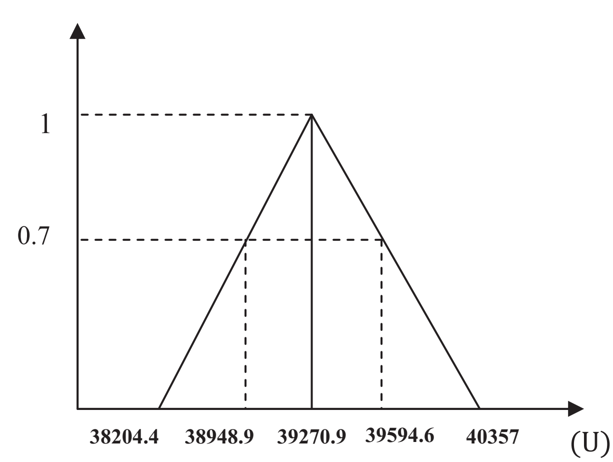

Fig. 2. The membership function of $\tilde{U}$.

interval which prevents any borrowing and short selling. Bertsimas and Pachamanova's model considers transaction cost as part of their model. However, the transaction cost does not play an important role in the optimization results since many brokerage houses are planning to remove transaction costs in order to create a motivation to absorb more investment. Table 1 demonstrates a précis of reviewed approaches in the state of the art.

The multi-period portfolio optimization proposed by Bertsimas and Pachamanova could be developed to incorporate realistic features such as borrowing and lending rates. The proposed method of this paper considers borrowing and lending rates as part of multi-period investment planning. We believe this feature makes our proposed method more realistic since most of the brokerage houses provide the opportunity to make an acquisition on different assets by borrowing the money from the brokerage. The proposed method of the paper considers the rates in fuzzy form and the results are discussed using a practical example. This paper is organized as follows. We first present the problem formulation in Section 2. The model is reformulated in fuzzy form in Section 3. The numerical results are presented in Section 4 and the conclusion remarks are given in Section 5 to summarize the contribution of the paper.

2. The proposed model formulation

The following notations and parameters are used in the problem formulation, M = the number of risky assets

N = the number of trading periods

X_t^m = the investor's dollar holdings in stock m at the beginning of period t, (which are funded with his capital); (m = 0, 1...M); (t = 0, 1...N)

X'^m_t = the investor's dollar holdings in stock m at the beginning of period t, (which are fund with borrowing); (m = 0, 1. M); (t = 0, 1...N)

r_t^m = the return of stock m over time period (t, t + 1)]; (m = 1, 2. M)

r_t^b = the riskless borrowing rate over time period (t, t + 1)]; (t = 0, 1...N)

r_t^l = the riskless lending rate over time period (t, t + 1)]; (t = 0, 1 N)

u_t^m = the amount of stock m which is sold in period t; (m = 1. M); (t = 1...N)

v_t^m = the amount of stock m which is purchased in period t; (m = 1...M); (t = 1...N)

u'^m_t = the amount of X'^m_{t-1} which is sold in period t; (m = 1...M); (t = 1...N)

v'^m_t = the amount of stock m which is purchased using credit in period t; (m = 1...M); (t = 1...N)

V = the maximum permitted amount of buying for each stock in each period

W_N = the investor's final wealth at period N

$U(X)$ = the investor utility function

In this model, there are M risky assets and one riskless asset (asset 0) with $r_t^l = r_t^b = r_t^o$. Therefore, in each period we have,

$$X_t^m = \left(1 + r_{t-1}^m\right) \quad \left(X_{t-1}^m - u_{t-1}^m + v_{t-1}^m\right), \quad t = 1 \ldots N,$$
$$m = 1 \ldots M, \tag{1}$$

$$X_t^0 = \left(1 + r_{t-1}^0\right) \left(X_{t-1}^0 + \sum_{m=1}^{M} \mu_{t-1}^m - \sum_{m=1}^{M} v_{t-1}^m\right), \quad t = 1 \ldots N. \tag{2}$$

Note that there is no borrowing or lending rates used in Eq. (1) and (2). Let r_t^b and r_t^l be the risk-less borrowing and the risk-less lending rates with $r_t^b \geq r_t^l$, respectively. Now, one may invest using the existing cash or purchase more shares using the credit with the borrowing rate. Let X_t^m and X'^m_t be the asset allocation held using the cash and the credit, respectively. Therefore, we have,

$$Max \quad U \quad \left(\sum_{m=0}^{M} X_N^m + \sum_{m=0}^{M} X'^m_N\right) \tag{P}$$

s.t.

$$X_t^m = (1 + r_{t-1}^m)(X_{t-1}^m - u_{t-1}^m + v_{t-1}^m),$$
$$t = (1 \ldots N); \quad m = (1 \ldots M), \tag{3}$$

$$X_t^0 = (1 + r_{t-1}^l) \left(X_{t-1}^0 + \sum_{m=1}^{M} u_{t-1}^m - \sum_{m=1}^{M} v_{t-1}^m\right), \quad t = (1 \ldots N) \tag{4}$$

$$X'^m_t = (1 + r_{t-1}^m) \left(X'^0_{t-1} - u'^m_{t-1} - v'^m_{t-1}\right),$$
$$t = (1 \ldots N); \quad m = (1 \ldots M), \tag{5}$$

$$X'^0_t = (1 - r_{t-1}^b) \left(X'^0_{t-1} + \sum_{m=1}^{M} u'^m_{t-1} - \sum_{m=1}^{M} v'^m_{t-1}\right), \quad t = (1 \ldots N), \tag{6}$$

$$\sum_{m=0}^{M} X_t^m \geq \beta \left(\sum_{m=o}^{M} X'^m_t\right), \quad t = (1 \ldots N), \tag{7}$$

$$v_t^m \leq V \quad t = (1 \ldots N;), \quad m = (1 \ldots M), \tag{8}$$

$\beta \in [0,1]$

Future returns are not known at time 0, realistically. Practically, the investor has to treat the portfolio optimization problem as a rolling horizon problem, i.e., he has to act upon the information available at time t, and rebalance his portfolio at time t + 1 after obtaining additional information over time period (t, t + 1]. It is assumed that at each time period, the investor takes only the first step of the optimal allocation strategy computed with the information up to that time period.

In the classical literature on portfolio optimization, the investor's utility function is assumed to be concave to reflect aversion to risk. We consider a linear objective instead:

$$U\left(\sum_{m=0}^{M} X_N^m + \sum_{m=0}^{M} X'^m_N\right) \approx \sum_{m=0}^{M} X_N^m + \sum_{m=0}^{M} X'^m_N$$

Note that any brokerage fund asks its investors to have a balance between the margin and the cash which are allocated on different risky assets and this regulation is imposed on Eq. (7) where β determines the rate of balance.

3. The proposed fuzzy multi-period investment strategy

As explained before, the proposed method of this paper is used to allocate the assets in different periods. However, the rates of borrowing and lending are not exactly known for future planning. In other words, the rates of borrowing and lending are normally adjusted based on the prime rate and the prime rate is determined by Federal Reserve of central banks based on what is happening on macro economy. Therefore, a decision maker may speculate on rate hikes in future horizon and one good way of treating this issue is to handle it through fuzzy logic [30]. One simple way to use fuzzy numbers is to use the triangular fuzzy number, $\tilde{r} = (l, m, n)$, and its membership function which is defined as follows:

$$\mu_{\tilde{r}} = \begin{cases} (t-l)/(m-l); & l \le t \le m \\ l; & t = m \\ (n-t)/(n-m); & m \le t \le n \\ 0; & t \le l \quad \text{or}\, t \ge n \end{cases} \tag{9}$$

The triangular fuzzy number is shown in Fig. 1.

A simple implementation of α-cut on membership function yields the α-level confidence of $\tilde{r}$ in terms of interval values corresponding to the triangular fuzzy number $\tilde{r} = (l, m, n)$:

$$\tilde{r}_\alpha = [r_\alpha^-, r_\alpha^+] = [(m-1)\alpha + 1, n - (n-m)\alpha];\ \forall \alpha \in [0, 1] \tag{10}$$

Hereby the lower and upper bounds of α-level confidence can be obtained easily. The fuzzy portfolio selection model (P) can be represented with fuzzy return rates as follow:

$$Max\, U = \sum_{m=0}^{M} X_N^m + \sum_{m=0}^{M} X'^m_N \tag{P1}$$

$$\text{s.t.}X_t^m = \left(1 + \tilde{r}_{t-1}^m\right)\left(X_{t-1}^m - u_{t-1}^m + v_{t-1}^m\right), \quad t = (1 \ldots N); \quad m = (1 \ldots M), X_t^0 = \left(1 + \tilde{r}_{t-1}^l\right)\left(X_{t-1}^0 + \sum_{m=1}^{M} u_{t-1}^m - \sum_{m=1}^{M} v_{t-1}^m\right), \quad t =$$

$$X'^m_t = \left(1 + \tilde{r}_{t-1}^m\right)\left(X_{t-1}^m - u'^m_{t-1} + v'^m_{t-1}\right),$$
$$t = (1 \ldots N); \quad m = (1 \ldots M),$$
$$X'^0_t = \left(1 + \tilde{r}_{t-1}^b\right)\left(X'^0_{t-1} + \sum_{m=1}^{M} u'^m_{t-1} - \sum_{m=1}^{M} v'^m_{t-1}\right),$$
$$(1 \ldots N),\ t = (1 \ldots N),$$
$$\sum_{m=0}^{M} X_t^m \ge \beta\left(\sum_{m=0}^{M} X'^m_t\right), \quad t = (1..N),$$
$$v_t^m \le V \quad t = (1 \ldots N); \quad m = (1..M),$$
$$\beta \in [0, 1]$$

The above problem can be reformulated with the α-level confidence of fuzzy numbers as in the following form:

$$(\text{P2})Max\, U = \sum_{m=0}^{M} X_N^m + \sum_{m=0}^{M} X'^m_N$$

s.t.

$$X_t^m = \left(1 + \left[\tilde{r}_{\alpha,t-1}^{-,m}, \tilde{r}_{\alpha,t-1}^{+,m}\right]\right)\left(X_{t-1}^m - u_{t-1}^m + v_{t-1}^m\right),$$
$$t = (1 \ldots N); \quad m = (1 \ldots M),$$
$$X_t^0 = \left(1 + \left[\tilde{r}_{\alpha,t-1}^{-,l}, \tilde{r}_{\alpha,t-1}^{+,l}\right]\right)\left(X_{t-1}^0 + \sum_{m=1}^{M} u_{t-1}^m - \sum_{m=1}^{M} v_{t-1}^m\right),$$
$$t = (1 \ldots N)$$
$$X'^m_t = \left(1 + \left[\tilde{r}_{\alpha,t-1}^{-,m}, \tilde{r}_{\alpha,t-1}^{+,m}\right]\right)\left(X'^m_{t-1} - u'^m_{t-1} + v'^m_{t-1}\right)$$
$$t = (1 \ldots N); \quad m = (1 \ldots M),$$
$$X'^0_t = \left(1 - \left[\tilde{r}_{\alpha,t-1}^{-,b}, \tilde{r}_{\alpha,t-1}^{+,b}\right]\right)\left(X'^0_{t-1} + \sum_{m=1}^{M} u'^m_{t-1} - \sum_{m=1}^{M} v'^m_{t-1}\right),$$
$$t = (1 \ldots N)$$
$$\sum_{m=0}^{M} X_t^m \ge \beta\left(\sum_{m=0}^{M} X'^m_t\right), \quad t = (1 \ldots N),$$
$$v_t^m \le V \quad t = (1 \ldots N); \quad m = (1 \ldots M),$$
$$\beta \in [0, 1]$$

To obtain α-level confidence of investor utility in selected portfolio the above problem can, therefore, be solved based on the lower and the upper bounds separately at different α-levels.

4. Numerical results

To illustrate the results of the proposed model an example is considered, with $M = 5$ (one risk free asset and four risky assets) and $N = 4$.

The borrowing and the lending rates and the return rates of risky assets are represented with fuzzy numbers as follow:

$$\tilde{r}_b = \left[.0\tilde{8}, .0\tilde{7}, .0\tilde{8}, .0\tilde{9}\right]$$

$$\tilde{r}_l = \left[.0\tilde{6}, .0\tilde{7}, .0\tilde{5}, .0\tilde{7}\right]$$

$$\tilde{r} = \begin{bmatrix} .0\tilde{9} & .\tilde{1}0 & .0\tilde{8} & .0\tilde{9} \\ .0\tilde{9} & .0\tilde{9} & .\tilde{1}0 & .0\tilde{8} \\ .0\tilde{8} & .0\tilde{9} & .0\tilde{9} & .\tilde{1}0 \\ .\tilde{1}0 & .0\tilde{8} & .0\tilde{9} & .0\tilde{8} \end{bmatrix}$$

$\tilde{r}_{ij}$ = return rate for risky asset i at time period j

The proposed method of this paper have been solved using this data with $\beta = 1$ (this value for β enforces rigorous situations for investor to keep balance between cash and margin) for different values of α. Moreover the initial values for investor's holding in the first period and the maximum permitted amount of buying for each stock in each period are need to be considered. Previews data can be rewritten for each $\alpha \in [0,1]$ in the form of Eq. (10) i.e. each fuzzy number which is introduced can be represented as the confidence interval for different values of α which is assumed to $n - m = m - l = 0.01$ in Eq. (10) for all fuzzy numbers (Table 2).

Table 3 demonstrates the details of the implementation of the proposed method with three confidence levels i.e., $\alpha = 0$, 0.7 and 1. As we can observe, when the α-level increases, the interval length of return rates will decrease to reflect the higher confidence. Another observation is that using fuzzy triangular numbers instead of crisp data will lead us to have a confidence level for the objective function. As an illustration Fig. 2 shows the optimal utility (objective function) for investor in this example is 39,270.9 at $\alpha = 1$. In addition, the utility will not fall out side the range of [38204.48,40357.00] at $\alpha = 0$.

With this strategy investor's holdings also have upper bound and lower bound at each period. This means that the investor will approach to interval of holding for any asset at each time period

Table 2
α-level confidence of fuzzy numbers in each period.

N	$\tilde{r}_t^1$	$\tilde{r}_t^2$	$\tilde{r}_t^3$	$\tilde{r}_t^4$	$\tilde{r}_t^l$	$\tilde{r}_t^b$
1	$[\alpha + .08, -\alpha + .1]$	$[\alpha + .08, -\alpha + .1]$	$[\alpha + .07, -\alpha + .09]$	$[\alpha + .09, -\alpha + .11]$	$[\alpha + .05, -\alpha + .07]$	$[\alpha + .07, -\alpha + .09]$
2	$[\alpha + .09, -\alpha + .11]$	$[\alpha + .08, -\alpha + .1]$	$[\alpha + .08, -\alpha + .1]$	$[\alpha + .07, -\alpha + .09]$	$[\alpha + .06, -\alpha + .08]$	$[\alpha + .06, -\alpha + .08]$
3	$[\alpha + .07, -\alpha + .09]$	$[\alpha + .09, -\alpha + .11]$	$[v + .08, -\alpha + .1]$	$[\alpha + .08, -\alpha + .1]$	$[\alpha + .04, -\alpha + .06]$	$[\alpha + .07, -\alpha + .09]$
4	$[\alpha + .08, -\alpha + .1]$	$[\alpha + .07, -\alpha + .09]$	$[\alpha + .09, -\alpha + .11]$	$[\alpha + .07, -\alpha + .09]$	$[\alpha + .06, -\alpha + .08]$	$[\alpha + .08, -\alpha + .1]$

Table 3
α-level confidence of model objective and variables for different values of α.

Variable		$\alpha = 0$	$\alpha = 0.7$	$\alpha = 1$
U		[38204.48,40357.00]	[38948.92,39594.66]	[39270.9,39270.9]
	X_1^0	[1000,1000]	[1000,1000]	[1000,1000]
	X_1^1	[2000,2000]	[2000,2000]	[2000,2000]
	X_1^2	[3000,3000]	[3000,3000]	[3000,3000]
	X_1^3	[4000,4000]	[4000,4000]	[4000,4000]
	X_1^4	[5000,5000]	[5000,5000]	[5000,5000]
$t = 1$	X'^0_1	[1000,1000]	[1000,1000]	[1000,1000]
	X'^1_1	[2000,2000]	[2000,2000]	[2000,2000]
	X'^2_1	[3000,3000]	[3000,3000]	[3000,3000]
	X'^3_1	[4000,4000]	[4000,4000]	[4000,4000]
	X'^4_1	[5000,5000]	[5000,5000]	[5000,5000]
	X_2^0	[0,0]	[0,0]	[0,0]
	X_2^1	[834.862,801.8]	[1734.7,813.24]	[818.18,818.18]
	X_2^2	[13383.5,13716.2]	[11597.6,13599.9]	[13550,13550]
	X_2^3	[0,0]	[0,0]	[0,0]
	X_2^4	[2000,2000]	[3000,2000]	[2000,2000]
	X'^0_2	[0,0]	[0,0]	[0,0]
	X'^1_2	[834.832,0]	[0,0]	[0,0]
	X'^2_2	[13383.5,14518]	[14323.2,14413.13]	[14368.18,14368.18]
	X'^3_2	[0,0]	[0,0]	[0,0]
$t = 2$	X'^4_2	[2000,2000]	[2000,2000]	[2000,2000]
	X_3^0	[0,0]	[0,0]	[0,0]
	X_3^1	[2000,2000]	[3000,2000]	[2000,2000]
	X_3^2	[15534.17,16187.84]	[13693.6,15957.69]	[15859.5,15859.5]
	X_3^3	[0,0]	[1087,0]	[0,0]
	X_3^4	[0,0]	[0,0]	[0,0]
	X'^0_3	[0,0]	[0,0]	[0,0]
	X'^1_3	[2000,1110]	[1097,1103]	[1100,1100]
	X'^2_3	[15534.17,17069.82]	[16656.3,16846.55]	[16751.3,16751.3]
	X'^3_3	[0,0]	[0,0]	[0,0]
	X'^4_3	[0,0]	[0,0]	[0,0]
	X_4^0	[0,0]	[0,0]	[0,0]
	X_4^1	[0,0]	[0,0]	[0,0]
	X_4^2	[18022.24,19078.5]	[16118.9,18704.33]	[18545.4,18545.4]
	X_4^3	[1080,1100]	[2268.5,0]	[0,0]
$t = 4$	X_4^4	[0,0]	[1087,1093]	[1090,1090]
	X'^0_4	[0,0]	[0,0]	[0,0]
	X'^1_4	[0,0]	[0,0]	[0,0]
	X'^2_4	[18022.24,20057.5]	[19369, 19684.75]	[19526.4,19526.4]
	X'^3_4	[0,0]	[105.4,0]	[0,0]
	X'^4_4	[1080,121]	[0, 112.58]	[109,109]

which adopts any value in the holding's interval by investor. This leads us to have a utility in the objective function interval with α-level confidence. Obviously, if a market stability conditions causes a higher confidence level (higher value for α), one can make his/her decisions in accurate manner among shorter interval near to the optimal.

As mentioned before, the borrowing and the lending rates are considered to be different, i.e. borrowing and short selling with different interest rates are permitted.

5. Conclusions

In this paper a fuzzy multi period portfolio selection model has been presented where the rates of borrowing are greater than the lending. We have discussed that one easy way to treat the uncertainty in the lending and the borrowing rates is to use triangular fuzzy numbers. This way, one may adjust the rates based on the changes on the prime rates imposed by regulators. The proposed model is considered under different interest rates for borrowing and lending in a practical example. The model is capable of imposing a balance between the lending and the cash purchases which makes it easy to adjust the portfolio when lender changes its regulation.

As can be seen, with numerical results, using fuzzy returns gives confidence interval of investors' utility to help them to imagine the optimistic and pessimistic situations in a portfolio hereby to make proper policies to hedge risk resources.

For future researches, this paper proposes some areas, such as adding other features of real market in constraints of model, i.e. transaction costs or considering normally distributed returns and using stochastic programming to be looked into.

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