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# Optimal Stopping, Free Boundary, and American Option in a Jump-Diffusion Model

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**Abstract.** This paper considers the American put option valuation in a jump-diffusion model and relates this optimal-stopping problem to a parabolic integro-differential free-boundary problem, with special attention to the behavior of the optimal-stopping boundary. We study the regularity of the American option value and obtain in particular a decomposition of the American put option price as the sum of its counterpart European price and the early exercise premium. Compared with the Black–Scholes (BS) [5] model, this premium has an additional term due to the presence of jumps. We prove the continuity of the free boundary and also give one estimate near maturity, generalizing a recent result of Barles *et al.* [3] for the BS model. Finally, we study the effect of the market price of jump risk and the intensity of jumps on the American put option price and its critical stock price.

**Key Words.** Optimal stopping, Jump-diffusion model, American option, Free-boundary problem.

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## Introduction

Jump-diffusion processes were incorporated by Merton [24] into the theory of option valuation in order to introduce a discontinuous sample path of the underlying stock's return dynamics, by contrast with the classical lognormal diffusion model of Black and Scholes [5]. These models allow us to account for large price changes due to sudden exogenous events on information. They are particularly adapted in the context of foreign exchange rates and may explain some systematic empirical biases with respect to the BS model (see [16] and [2]).

Contrarily to the BS model, jump-diffusion models induce incompleteness of the market in the Harrison–Pliska [12] sense. Merton [24] has developed pricing formulas for European options assuming that jump risk is unpriced. Generalizations of Merton's result can be found, for example, in [1], [27], or [28]. The aim of this paper is to study the problem of pricing the American option in a jump-diffusion model.

The earliest works on this problem are due to McKean [22], and further to Van Moerbeke [31] who transformed the American option pricing analysis into a free-boundary problem, within the framework of diffusion models. In addition to the free-boundary method, the formulation of the optimal-stopping problem by variational inequalities, as developed by Bensoussan and Lions [4], and applied to American options in diffusion models by Jaillet *et al.* [15], provide numerical computations for the pricing of American options. This approach was recently applied by Zhang [32] in the context of Merton's jump-diffusion model. However, variational inequalities lead to a somewhat less-explicit characterization of the American option value.

This paper adopts the free-boundary approach with careful attention to the behavior of the optimal-stopping boundary, by using analytical methods as well as probabilistic results.

In Section 1 we describe the financial market in the presence of jump uncertainty. Since the market is incomplete, there is an infinity of admissible prices for contingent claims (see [11]) associated to an infinity of equivalent martingale measures, that we characterize by identifying the market price of diffusion and jump risk.

In Section 2 we relate the American option pricing valuation to an optimal-stopping problem and we state some basic properties of the American option value.

In Section 3 we establish that the American put option price and its free boundary (also called the critical stock price in financial language) are a solution pair of a parabolic integrodifferential free-boundary problem arising from the optimal-stopping problem. In particular, we check the continuous differentiability of the option price with respect to the stock price, a result known as the smooth-fit condition. Generalizing results of Jacka [13] and Myneni [26], we obtain a decomposition of the American put option value as the sum of its corresponding European put price and the early exercise premium. It appears that, compared with the BS model, this premium has an extra complex term due to the fact that the stock price can jump from the exercise region to the continuation region without crossing the exercise boundary. A uniqueness result for the free-boundary problem is also provided. The main difficulty, with regard to the diffusion model, comes from the nonlocal integral term. We give a sufficient condition which ensures that the value function of the optimal-stopping problem is the unique solution of the free-boundary problem.

Section 4 is concerned with the behavior of the optimal-stopping boundary. Under a similar condition to that for the uniqueness result of Section 3, we prove continuity with respect to time of the free boundary. We also provide an estimate of the difference between the critical stock price in the BS model and the one in a jump-diffusion model. In particular, it gives an estimate of the behavior of the critical stock price near maturity within a jump-diffusion model, extending therefore a recent result derived by Barles *et al.* [3] (see also [19]) in the framework of the BS model.

The presence of jump uncertainty in the stock price dynamics introduce two essential parameters in option valuation, compared with the BS model: the market price of jump risk and the intensity of jumps. As a final contribution, we prove in Section 5 that the American option price (resp. the critical stock price) is nondecreasing (resp. nonincreasing) with respect to each of these two parameters.

## 1. The Framework

We consider a financial market where two assets $(S^0, X)$ are traded continuously up to some fixed time horizon $T$. The underlying uncertainty is generated by a probability space $(\Omega, \mathcal{F}, P)$ with $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$, a filtration satisfying the usual conditions. On this probability space are defined a standard Brownian motion $W$ and a homogeneous Poisson random measure $\upsilon(dt, dy)$ on $[0, T] \times \mathbb{R}$ identified to a marked point process $(N_t, (Y_n)_{n\in\mathbb{N}})$. We assume that $\mathbb{F}$ is equal to the information structure generated by $W$ and $\upsilon$. The intensity measure $q(dt, dy)$ of $\upsilon$ is of the form

$$q(dt, dy) = \lambda m(dy)\, dt,$$

where the constant $\lambda > 0$ is the intensity of jumps of the Poisson process $N_t = \upsilon([0, t] \times \mathbb{R})$ and $m(dy)$ is the probability measure on $\mathbb{R}$ of the independent identically distributed random variables $Y_n$, also independent of $N_t$. We say that $(\lambda, m(dy))$ are the $(P, \mathcal{F}_t)$ local characteristics of the marked point process $\upsilon$. The random measure $\tilde{\upsilon}$, defined by

$$\tilde{\upsilon}(dt, dy) = \upsilon(dt, dy) - q(dt, dy)$$

is called the $P$-compensated jump martingale of $\upsilon$. We refer the reader to [14] for a formal definition of the random measure and its characteristic.

The first asset $S^0$ is a bond whose price evolves according to the differential equation

$$\frac{dS_t^0}{S_t^0} = r\, dt,$$

where $r$ is the constant positive interest rate. The price of the risky asset is described by the stochastic equation

$$\frac{dX_t}{X_{t^-}} = \mu\, dt + \sigma\, dW_t + \int_{\mathbb{R}} \gamma(y)\tilde{\upsilon}(dt, dy). \tag{1.1}$$

The coefficients $\mu, \sigma$ are constants. $(\gamma(Y_n)_{n\in\mathbb{N}})$ are the square integrable random jump relative sizes of the stock price $X$. We assume that $1 + \gamma > 0$ in order for the price to be real valued. As is well known, the basic market $(S^0, X)$ is incomplete: mathematically formalized by Harrison and Pliska [12], it essentially means that under the absence of arbitrage opportunities, there are many equivalent martingale measures, i.e., probability measures equivalent to $P$, under which the discounted risky asset price process $X/S^0$ is a martingale. We recall from [7] or [28] the characterization of the equivalent martingale measures by their Radon–Nykodym density with respect to $P$:

$$\frac{dQ^p}{dP} = \mathcal{E}\left(-\int_0^T \theta_t\, dW_t + \int_0^T \int_{\mathbb{R}} (p_t(y) - 1)\tilde{\upsilon}(dt, dy)\right),$$

where $\mathcal{E}(\cdot)$ is the exponential semimartingale of Doléans–Dade, $\theta$ and $p$ are two predictable processes such that

$$\mu - r = \theta_t \sigma + \lambda \int_{\mathbb{R}} \gamma(y)(1 - p_t(y))m(dy) \tag{1.2}$$

together with the conditions

$$p > 0 \quad \text{and} \quad E\left(\frac{dQ^p}{dP}\right) = 1.$$

$\theta$ is interpreted as the market price of diffusion risk and $p$ as the market price of jump risk. In this paper we only consider equivalent martingale measures such that the market price of jump risk is independent of $t \in [0, T]$ and $\omega \in \Omega$: $p_t(y) = p(y)$, and $p \in L^2(m)$. Therefore, by Girsanov's theorem, $\upsilon$ is still a homogeneous Poisson random measure under $Q^p$ with local characteristics:

$$\lambda_p = \lambda \int_{\mathbb{R}} p(y)m(dy) \quad \text{and} \quad m^p(dy) = \frac{p(y)m(dy)}{\int_{\mathbb{R}} p(y)m(dy)} \tag{1.3}$$

and $W_t^p = W_t + \int_0^t \theta_s\, ds$ is a $Q^p$-Brownian motion.

In our incomplete market framework, the no arbitrage theory does not induce a unique price for a contingent claim written on the underlying asset $X$. Indeed, according to Harrison and Kreps [11], each equivalent martingale measure $Q^p$ defines an admissible price of the contingent claim. For a European option $g(X_T)$ expiring at $T$, an admissible time $t$ price is given by

$$f_t^p = E^{Q^p}[e^{-r(T-t)} g(X_T) | \mathcal{F}_t],$$

while for an American option with payoff $(g(X_t))_{0 \le t \le T}$ until time expiry $T$, it is given by

$$F_t^p = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} E^{Q^p}[e^{-r(\tau - t)} g(X_\tau) | \mathcal{F}_t],$$

where $\mathcal{T}_{t,T}$ denotes the set of all stopping times between $t$ and $T$.

We denote by $(X_s^t(x))_{s \ge t}$ a right continuous with left limits (RCLL) version of the flow of the stochastic differential equation (1.1). Therefore $(s, t, x) \to X_s^t(x)$ is RCLL for almost all $\omega \in \Omega$, $X_t^t(x) = x$ and $X^t(x)$ satisfies (1.1) on $[t, T]$. When $t = 0$, we simply denote $X_s(x) = X_s^0(x)$. Therefore, by the Doléans–Dade formula and from (1.2), we have, almost surely under $Q^p$,

$$X_s^t(x) = x \cdot \exp\left[-\lambda_p k_p(s-t) + \int_t^s \int_{\mathbb{R}} \ln(1+\gamma(y))\upsilon(du, dy)\right] \cdot \exp[\sigma(W_s^p - W_t^p) + (r - \tfrac{1}{2}\sigma^2)(s-t)], \tag{1.4}$$

where $k_p = \int_{\mathbb{R}} \gamma(y) m^p(dy)$ is the expectation under $Q^p$ of the jump relative size (note that $\gamma \in L^1(m^p)$ by the Hölder inequality since $\gamma, p \in L^1(m)$). In particular, $X$ is a time homogeneous Markov process under $Q^p$. According to the relation between Snell's envelope and réduite (see, e.g., [8]), the $Q^p$-admissible American option price process is a function only of the current price of the underlying stock and of the time to expiry of the option. It is given by $F_t^p = F^p(T-t, X_t)$, where

$$F^p(t,x) = \sup_{\tau \in \mathcal{T}_{0,t}} E^{Q^p}[e^{-r\tau} g(X_\tau(x))]. \tag{1.5}$$

In the following we study the function $F^p$ defined by this optimal-stopping problem (1.5). For simplicity, we assume that the stock pays no dividends. It is known that the American call option on a stock without dividends is equivalent to its European counterpart (see [23]). We consider therefore in the rest of the paper the American put option

$$g(x) = (K-x)^+,$$

where $K > 0$ is the exercise price.

## 2. The Optimal-Stopping Problem

It is well known that the function $F^p$ defined in (1.5) is not smooth. Jaillet *et al.* [15] in the diffusion case, and Zhang [32] in the jump-diffusion case, have studied the American option value by the method of variational inequalities, based on the work of Bensoussan and Lions [4]. Viscosity solutions, introduced by Lions [20] for diffusion processes and generalized to jump-diffusion processes in [29], is also a powerful means for characterizing the value function of a stochastic control problem: the function $F^p$, defined in (1.5), is the unique $BUC([0,T] \times \mathbb{R}_+)$ (set of bounded uniformly continuous functions) viscosity solution of

$$\min(-\mathcal{L}^p v; v - g) = 0, \qquad \forall (t,x) \in (0,T] \times \mathbb{R}_+, \tag{2.1}$$

$$v(0,x) = g(x), \qquad \forall x \in \mathbb{R}_+, \tag{2.2}$$

where $\mathcal{L}^p$ is the parabolic integrodifferential operator:

$$\mathcal{L}^p v = -\frac{\partial v}{\partial t} - rv + rx\frac{\partial v}{\partial x} + \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + \lambda \int_{\mathbb{R}} \left[ v(t, x(1+\gamma(y))) - v(t,x) - \gamma(y) x \frac{\partial v}{\partial x}(t,x)\right] p(y) m(dy). \tag{2.3}$$

Moreover, there exists $C > 0$ such that, for all $t_1, t_2 \in [0, T]$, $x_1, x_2 \in \mathbb{R}_+$,

$$|F^p(t_1, x_1) - F^p(t_2, x_2)| \leq C[|t_1 - t_2|^{1/2} + |x_1 - x_2|]. \tag{2.4}$$

Another more explicit analysis of the American option pricing problem is to transform the optimal-stopping problem into a free-boundary problem, as developed by McKean [22] and Van Moerbeke [31] in the diffusion case. We extend here such an approach to the case of jump-diffusion processes. From general results for optimal stopping of Markov processes (see, e.g., [21] or [30]), $F^p(t, x) \geq g(x)$, and an optimal stopping time for problem (1.5) is

$$\tau^*(t, x) = \inf\{0 \leq s \leq t,\ F^p(t - s, X_s(x)) = g(X_s(x))\}. \tag{2.5}$$

Moreover, the process $\{e^{-rs}F^p(t - s, X_s(x)),\ 0 \leq s \leq \tau^*(t, x)\}$ is a $Q^p$-martingale. The solution of the American option pricing is then implicitly determined by (1.5) and (2.5). We want to characterize $F^p$ more precisely by analytical methods. First, as for the BS model, the following classical properties of the American put option price can be stated.

**Proposition 2.1.** *The American put option function $F^p$ satisfies*:

(i) $F^p(t, \cdot)$ *is nonincreasing and convex on* $\mathbb{R}_+$ *for every* $t \in [0, T]$.
(ii) $F^p(\cdot, x)$ *is nondecreasing on* $[0, T]$ *for every* $x \in \mathbb{R}_+$.
(iii) $F^p(t, x) > 0$, *for every* $(t, x) \in (0, T] \times \mathbb{R}_+$.

*Proof.* Given that the American put option reward $g(x)$ is convex and nonincreasing with respect to $x$, property (i) is easily derived from the pathwise solution given in (1.4). Property (ii) follows from the fact that if $\tau \in \mathcal{T}_{0,t}$, then $\tau \in \mathcal{T}_{0,s}$ for any $s \geq t$. For $x < K$, $F^p(t, x) \geq g(x) > 0$. For $x \geq K$ and $t > 0$, $F^p(t, x) \geq e^{-rt}(K/2)Q^p\{X_t(x) \leq K/2\}$. Now, from (1.4),

$$\left\{X_t(x) \leq \frac{K}{2}\right\} = \left\{W_t^q \leq \frac{1}{\sigma}\left[\ln\left(\frac{K}{2x}\right) - (r - \lambda_p k_p - \tfrac{1}{2}\sigma^2)t - \int_0^t \int_Y \ln(1 + \gamma(y))\upsilon(ds, dy)\right]\right\}.$$

Since $W^p$ and $\upsilon$ are independent under $Q^p$, it yields

$$\begin{aligned} Q^p\left\{X_t(x) \leq \frac{K}{2}\right\} &= E^{Q^p}\left\{\Phi\left(\frac{1}{\sigma\sqrt{t}}\left[\ln\left(\frac{K}{2x}\right) - (r - \lambda_p k_p - \tfrac{1}{2}\sigma^2)t - \int_0^t \int_Y \ln(1 + \gamma(y))\upsilon(ds, dy)\right]\right)\right\} \\ &> 0, \end{aligned}$$

where $\Phi$ is the standard normal distribution function. Property (iii) is then proved. □

Noting that $F^p(t,0) = K$ since $r \geq 0$, it is clear from the preceding proposition that, for each time to expiry $t > 0$, there exists a critical stock price $b^p(t)$, below which the American put option should be exercised early:

$$\text{if} \quad 0 \leq x \leq b^p(t), \quad \text{then} \quad F^p(t,x) = g(x),$$

$$\text{if} \quad x > b^p(t) \quad \text{then} \quad F^p(t,x) > g(x).$$

The domain $(0, T] \times \mathbb{R}_+$ of the American put option price $F^p$ is therefore divided by the optimal-stopping boundary $\{(t, b^p(t)),\ t \in (0,T]\}$ into:

- The continuation region:

$$\begin{aligned} \mathcal{C}^p &:= \{(t,x) \in (0,T] \times \mathbb{R}_+,\ F^p(t,x) > g(x)\} \\ &= \{(t,x) \in (0,T] \times \mathbb{R}_+,\ x > b^p(t)\}. \end{aligned}$$

- Its complement, the exercise (or stopping) region:

$$\begin{aligned} \mathcal{S}^p &:= \{(t,x) \in (0,T] \times \mathbb{R}_+,\ F^p(t,x) = g(x)\} \\ &= \{(t,x) \in (0,T] \times \mathbb{R}_+,\ x \leq b^p(t)\}. \end{aligned}$$

By continuity of $F^p$, the region $\mathcal{C}^p$ (resp. $\mathcal{S}^p$) is open (resp. closed). From Proposition 2.1, we deduce the following property of the free boundary $b^p$.

**Proposition 2.2.** *The boundary $b^p$ is nonincreasing in the time to expiry $t$ and is bounded above by $K$.*

*Proof.* Since $F^p(\cdot, x)$ is a nondecreasing function of the time to expiry $t$, therefore $b^p(\cdot)$ must be a nonincreasing function of $t$. Moreover, since $F^p(t,x) > 0 = g(x)$, for $t > 0$ and $x \geq K$, it implies that $b^p$ is bounded above by $K$: $b^p(t) < K$. □

## 3. Free-Boundary Formulation

We now turn to the free-boundary formulation of the optimal-stopping problem described in the preceding section. We first extend, to the jump-diffusion case, the smoothness result of the American option value in the continuation region.

**Proposition 3.1.** *The American put option price $F^p$ is smooth in the continuation region and satisfies, in $\mathcal{C}^p$,*

$$\mathcal{L}^p F^p(t,x) = 0. \tag{3.1}$$

*Proof.* This follows from the martingale property of

$$\{e^{-rs} F^p(t-s, X_s(x)),\ 0 \leq s \leq \tau^*(t,x)\}$$

with $\tau^*(t,x) = \inf\{0 \leq s \leq t,\ (t-s, X_s(x)) \notin \mathcal{C}^p\}$, from Itô's formula, and the smoothness results for the parabolic integrodifferential operator $\mathcal{L}^p$ (see [29]). □

This last proposition allows us to substantiate a convexity result of the American option value.

**Corollary 3.1.** *The American option price $F^p$ is strictly convex with respect to $x$ in the continuation region $\mathcal{C}^p$.*

*Proof.* Recall that $F^p$ is convex in $x$ in the whole domain $[0, T] \times \mathbb{R}_+$. Introducing the change of variable $z = \ln x$ and defining $\hat{F}(t, z) = F^p(t, e^z)$, we immediately see that $\hat{F}$ is convex in $z$. Moreover, since $F^p_{xx} = -(1/x)F^p_x + (1/x^2)\hat{F}_{zz}$ and $F^p$ is nonincreasing in $x$, it suffices to prove the strict convexity of $\hat{F}$ with respect to $z$ in $\hat{\mathcal{C}} := \{(t, z) \in (0, T] \times \mathbb{R};\ z > \ln b^p(t)\}$. From Proposition 3.1, we deduce that $\hat{F}$ satisfies $\hat{\mathcal{L}}\hat{F}(t, z) = 0$ in $\hat{\mathcal{C}}$ where $\hat{\mathcal{L}}$ is the maximum principle operator:

$$\hat{\mathcal{L}}v = -\frac{\partial v}{\partial t} - rv + \tfrac{1}{2}\sigma^2\frac{\partial^2 v}{\partial z^2} + (r - \lambda_p k_p - \tfrac{1}{2}\sigma^2)\frac{\partial v}{\partial z}$$
$$+\lambda\int_{\mathbb{R}}[v(t, z + \ln(1 + \gamma(y))) - v(t, z)]p(y)m(dy). \tag{3.2}$$

Since $\hat{\mathcal{C}}$ is an open set of $[0, T] \times \mathbb{R}$, for any point $(t_0, z_0) \in \hat{\mathcal{C}}$, there exists an open set $\mathcal{O} \subset \hat{\mathcal{C}}$, containing $(t_0, z_0)$. Differentiating twice with respect to $z$, we obtain that $\hat{F}_{zz}$ is a solution of the Dirichlet problem: $\hat{\mathcal{L}}v = 0$ in $\mathcal{O}$, $v = \hat{F}_{zz}$ in $\mathcal{O}^c$. If $\hat{F}_{zz}(t_0, z_0) = 0$, then, by the Feynman–Kac representation theorem and since $\hat{F}_{zz} \geq 0$, it follows that $\hat{F}_{zz} = 0$ in $\mathcal{O}^c$ for any open subset $\mathcal{O} \subset \hat{\mathcal{C}}$ containing $(t_0, z_0)$. Hence, $\hat{F}_{zz} = 0$ in $\hat{\mathcal{C}}$, which is obviously false from the preceding properties of the American put price. □

The following boundary conditions are associated with the partial differential equation (3.1).

**Lemma 3.1.** *The American put option function $F^p$ satisfies*

$$\lim_{x \downarrow b^p(t)} F^p(t, x) = K - b^p(t), \qquad t \in (0, T], \tag{3.3}$$

$$F^p(0, x) = (K - x)^+, \qquad x \in \mathbb{R}_+. \tag{3.4}$$

*Proof.* Relation (3.3) is true thanks to the continuity of the value function $F^p$ and since $b^p(t) < K$, while (3.4) states that the American put is European at expiration. □

To close the free-boundary problem and in particular to determine the boundary $b^p$, an additional condition known in the theory of optimal stopping as the principle of smooth fit is required, i.e., the continuous differentiability in $x$ of the value function through the optimal-stopping boundary.

**Proposition 3.2.** *The American put option price $F^p$ is continuously differentiable with respect to $x$, in $(0, T] \times \mathbb{R}_+$, in particular across the optimal-stopping boundary:*

$$\lim_{(s,x)\to(t,b^p(t))} \frac{\partial F^p}{\partial x}(s, x) = -1, \qquad \forall t \in (0, T]. \tag{3.5}$$

*Moreover, we have*

$$-1 \le \frac{\partial F^p}{\partial x}(t,x) \le \frac{\partial f^p}{\partial x}(t,x), \qquad \forall (t,x) \in (0,T] \times \mathbb{R}_+, \tag{3.6}$$

*where $f^p$ is the European put option value:* $f^p(t,x) = E^{Q^p}[e^{-rt}(K - X_t(x))^+]$.

*Proof.* The smooth-fit condition (3.5) was derived by Zhang [32] within a jump-diffusion model in the context of variational inequalities. For completeness, we give a short proof. From relation (2.4), we deduce that $F_x^p$ is uniformly bounded in $[0,T] \times \mathbb{R}_+$ and $F_t^p$ is locally bounded in $(0,T] \times \mathbb{R}_+$. Moreover, from (2.1), $F^p$ satisfies in the viscosity sense (or distribution sense) $\mathcal{L}^p F^p \le 0$, i.e.,

$$\begin{aligned}\tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 F^p}{\partial x^2} &\le r F^p + \frac{\partial F^p}{\partial t} - rx\frac{\partial F^p}{\partial x} \\ &\quad -\lambda \int_{\mathbb{R}} \left( F^p(t, x(1+\gamma(y))) - F^p(t,x) - \gamma(y) x \frac{\partial F^p}{\partial x}(t,x) \right) \\ &\quad \times p(y) m(dy).\end{aligned}$$

Since $F^p$ is Lipschitz in $x$, uniformly in $t$, the integral term of the right-hand side of this last inequality is bounded above by $C|x|(1 + F_x^p) \int_{\mathbb{R}} |\gamma(y)| p(y) m(dy)$. Since $\gamma \in L^1(m^p)$, we deduce therefore from the local boundedness of $F_t^p$ and $F_x^p$ in $(0,T] \times \mathbb{R}_+$ that $F_{xx}^p$ is locally bounded in $(0,T] \times \mathbb{R}_+^*$. According to Lemma 3.1 in Chapter 2 of [18], it yields that $F_x^p$ is continuous in $(0,T] \times \mathbb{R}_+^*$ and in fact in $(0,T] \times \mathbb{R}_+$ from the convexity of $F^p$ with respect to $x$ in $[0,T] \times \mathbb{R}_+$.

The first inequality of (3.6) is directly obtained by observing that $F_x^p(t,x) = -1$ for $x \le b^p(t)$ and because $x \mapsto F_x^p(t,\cdot)$ is nondecreasing. To prove the second inequality, it is convenient to consider again the change of variable $z = \ln x$ and to define $\hat{F}(t,z) = F^p(t,e^z)$, $\hat{f}(t,z) = f^p(t,e^z)$, and $\hat{h}(t,z) = \hat{F}(t,z) - \hat{f}(t,z)$. Recalling that $\hat{F}$ and $\hat{f}$ are both solutions of $\hat{\mathcal{L}}v = 0$ for $z > \ln b^p(t)$, we deduce that $\hat{h}$ satisfies

$$\hat{\mathcal{L}}\hat{h} = 0, \qquad z > \ln b^p(t),$$

$$\hat{h} = K - e^z - \hat{f}, \qquad z \le \ln b^p(t),$$

$$\hat{h}(0,z) = 0, \qquad z \in \mathbb{R},$$

where $\hat{\mathcal{L}}$ is the maximum principle operator defined in (3.2). Denoting by $f^{\mathrm{BS}}$ the European put option in the BS model and observing that $f^p \ge f^{\mathrm{BS}}$ (see [28]) and $f^p(t,0) = f^{\mathrm{BS}}(t,0) = Ke^{-rt}$, we have that $f_x^p(t,0) \ge f^{\mathrm{BS}}(t,0) = -1$ and by convexity of $f^p(t,\cdot)$ that $f_x^p(t,x) \ge -1$ or equivalently that $\hat{f}_z(t,z) \ge -e^z$. We deduce, by differentiating the above equations system once with respect to $z$ and since $F^p$ is differentiable on the boundary $b^p$, that $\hat{h}_z$ satisfies

$$\hat{\mathcal{L}}\hat{h}_z = 0, \qquad z > \ln b^p(t),$$

$$\hat{h}_z \le 0, \qquad z \le \ln b^p(t),$$

$$\hat{h}_z(0,z) = 0, \qquad z \in \mathbb{R}.$$

It follows by the maximum principle that $\hat{h}_z(t,z) \le 0$, for all $t \in (0,T]$, $z \in \mathbb{R}$, which ends the proof. □

Extending the Riesz decomposition or the early exercise premium representation obtained for the BS model (see, e.g., [6], [13], or [17]), we now derive a decomposition of the American put option within a jump-diffusion model.

**Theorem 3.1.** *The value function $F^p$ of the American put option has the representation*

$$F^p(t,x) = f^p(t,x) + e^p(t,x),$$

*where $e^p$ is the early exercise premium: $e^p = e_1^p - e_2^p$, with*

$$e_1^p(t,x) = rK \int_0^t e^{-rs} Q^p[X_s(x) \le b^p(t-s)]\, ds,$$

$$\begin{aligned} e_2^p(t,x) = \lambda E^{Q^p}\Bigg[\int_0^t \int_{A^p_{t,s,x}} & e^{-rs}\chi(X_s(x) \le b^p(t-s)) \\ & \times \{F^p(t-s, X_s(x)[1+\gamma(y)]) \\ & \quad - (K - X_s(x)[1+\gamma(y)])\} p(y) m(dy)\Bigg], \end{aligned}$$

*$\chi$ is the characteristic function and $A^p_{t,s,x} = \{y \in \mathbb{R},\ X_s(x)(1+\gamma(y)) > b^p(t-s)\}$.*

*Proof.* By Propositions 3.1 and 3.2, the function $F^p$ is $C^1$, piecewise $C^2$ in $x$, and piecewise $C^1$ in $t$. According to the generalized Itô lemma for convex functions (see Chapter 6.II of [25]), we have

$$\begin{aligned} & e^{-rt} F^p(0, X_t(x)) \\ & \quad = F^p(t,x) + \int_0^t e^{-rs} \mathcal{L}^p F^p(t-s, X_s(x))\, ds \\ & \qquad + \int_0^t e^{-rs} F_x^p(t-s, X_s(x)) \sigma X_s(x)\, dW_s^p \\ & \qquad + \int_0^t \int_{\mathbb{R}} e^{-rs}[F^p(t-s, X_s(x)(1+\gamma(y))) - F^p(t-s, X_s(x))] \\ & \qquad \times [\upsilon(ds,dy) - \lambda_p m^p(dy)\, ds]. \end{aligned}$$

From the preceding propositions, $|F_x^p(t,x)| \le 1$ for all $(t,x) \in (0,T] \times \mathbb{R}_+$. Moreover, $\gamma(Y_n)$ is integrable for the measure $m^p$ and $E^{Q^p}[X_s(x)]^2 \le$ const for all $s \in [0,T]$. Therefore, the two stochastic integrals of the last relation are $Q^p$-martingales and, by taking expectation (under $Q^p$) and since $F^p(0,x) = g(x)$, we have

$$F^p(t,x) = f^p(t,x) - E^{Q^p}\left[\int_0^t e^{-rs} \mathcal{L}^p F^p(t-s, X_s(x))\, ds\right].$$

Now, an easy computation yields: if $x > b^p(t)$, then

$$\mathcal{L}^p F^p(t,x) = 0,$$

and if $x \leq b^p(t)$, then

$$\mathcal{L}^p F^p(t,x) = -rK + \lambda \int_{x(1+\gamma(y))>b^p(t)} \{F^p(t, x[1+\gamma(y)]) - (K - x[1+\gamma(y)])\} p(y) m(dy),$$

which gives the asserted result. □

**Remarks.** 1. In a jump-diffusion model the early exercise premium is the difference of two terms. The first term, $e_1^p$, analogous to the BS model's one, equals the present value of interest earned on the $K$ units in bonds while the stock price is below the critical stock price. In the case of Merton's model [24], $p = 1$ (jump risk unpriced) and $\ln(1+\gamma(Y_i))$ normally distributed with variance $\delta^2$, we can derive an explicit form of $e_1^1$:

$$e_1^1(t,x) = rK \sum_{n=0}^{+\infty} \int_0^t \exp[-(r+\lambda)s] \frac{(\lambda s)^n}{n!} \times \Phi\left( \frac{\ln((b^1(t-s))/x) - \ln(1+k) - (r - \lambda k - \frac{1}{2}\sigma_n^2)s}{\sigma_n \sqrt{s}} \right) ds,$$

where $k = E[\gamma(Y_n)]$ is the expectation jump relative size, $\sigma_n^2 = \sigma^2 + n\delta^2/s$, and $\Phi$ is the standard normal distribution function. The second nonnegative term, $e_2^p$, due to the nonlocal integral part of $\mathcal{L}^p$, is explained by the fact that the stock price can jump from below the critical stock price to the continuation region, without crossing the exercise boundary. Note that $e_2^p$ is not so explicit as $e_1^p$, and depends on $F^p$.

2. From the boundary condition (3.4), the optimal-stopping boundary $b^p$ can be viewed (at least implicitly) as the solution of the integral equation:

$$f^p(t, b^p(t)) + e^p(t, b^p(t)) = K - b^p(t), \qquad \forall t \in (0, T].$$

Propositions 3.1 and 3.2 and Lemma 3.1 show that the American put option value $F^p$ is a solution of a parabolic integrodifferential free-boundary problem. Extending Van Moerbeke's approach to a jump-diffusion model, we now prove that this free-boundary problem has essentially a unique solution, $F^p$. The main difficulty comes from the nonlocal integral term of the operator $\mathcal{L}^p$. Indeed, $v = (K-x)^+$ in $\mathcal{S}^p$ does not imply that $\mathcal{L}^p v = -rK$ in $\mathcal{S}^p$, while it was true for the BS model, but as seen in the proof of Theorem 3.1:

$$\mathcal{L}^p v(t,x) = -rK + \lambda \int_{x(1+\gamma(y))>b^p(t)} \{v(t, x(1+\gamma(y))) - [K - x(1+\gamma(y))]\} p(y) m(dy)$$

if $x \le b^p(t)$. In particular, we do not know *a priori* if a solution of the free-boundary problem is $Q^p$ $r$-excessive. We give therefore the following condition:

$$(C_p) \quad r_p := r - \lambda \int_{\gamma(y)\geq 0} \gamma(y)p(y)m(dy) \geq 0.$$

This assumption is obviously satisfied in a diffusion model. It means that the riskless interest rate corrected by the jump risk, $r_p = r - \lambda_p E^{m^p}[\gamma(Y_1)]$, is nonnegative.

**Theorem 3.2.** *Assume condition $(C_p)$ holds. Then $(F^p, b^p)$ is the unique solution pair $(v, b)$ with $v$: $[0,T]\times\mathbb{R}_+ \to \mathbb{R}$, $v(t,\cdot)$ nonincreasing and convex, and $b$: $(0,T] \to \mathbb{R}$, $0 \le b(t) < K$, of the free-boundary problem*:

$$\mathcal{L}^p v = 0, \qquad x > b(t), \tag{3.7}$$

$$\lim_{x\downarrow b(t)} v(t,x) = K - b(t), \qquad t \in (0,T], \tag{3.8}$$

$$\lim_{x\downarrow b(t)} v_x(t,x) = -1, \qquad t \in (0,T], \tag{3.9}$$

$$v(0,x) = (K-x)^+, \qquad x \in \mathbb{R}_+, \tag{3.10}$$

$$v > (K-x)^+ \quad \textit{if} \quad x > b(t), \qquad \textit{and} \qquad v = (K-x)^+ \quad \textit{if} \quad x \le b(t). \tag{3.11}$$

*Proof.* The fact that $(F^p, b^p)$ is the solution of this free-boundary problem follows from the preceding propositions. Conversely, we consider a pair $(v, b)$ as in the text of the theorem. Therefore, $v$ is $\mathcal{C}^1$ in $x$, piecewise $\mathcal{C}^2$ in $x$, and piecewise $\mathcal{C}^1$ in $t$. By Itô's formula, we have

$$\begin{aligned} e^{-rt}&v(t-s, X_s(x)) \\ &= v(t,x) + \int_0^s e^{-ru}\mathcal{L}^p v(t-u, X_u(x))\,du \\ &+ \int_0^s e^{-ru} v_x(t-u, X_u(x))\sigma X_u(x)\,dW_u^p \\ &+ \int_0^s \int_{\mathbb{R}} e^{-ru}[v(t-u, X_u(x)(1+\gamma(y))) - v(t-u, X_u(x))] \\ &\times [\nu(du,dy) - \lambda_p m^p(dy)\,du]. \end{aligned} \tag{3.12}$$

Since $v(t,\cdot)$ is nonincreasing and convex, and $v \ge (K-x)^+$, it implies that $v_x$ is bounded (by 1) on $\mathbb{R}_+$, so that the two stochastic integrals are $Q^p$-martingales. Moreover, if $x > b(t)$ then $\mathcal{L}^p v = 0$, and if $x \le b(t)$, then

$$\begin{aligned} \mathcal{L}^p v(t,x) &= -rK + \lambda \int_{x(1+\gamma(y))>b(t)} \{v(t, x(1+\gamma(y))) \\ &\qquad\qquad - [K - x(1+\gamma(y))]\}p(y)m(dy) \\ &\le -rK + \lambda \int_{x(1+\gamma(y))>b(t)} \{v(t,b(t)) - [K - x(1+\gamma(y))]\}p(y)m(dy) \end{aligned}$$

$$
\begin{aligned}
&\leq -rK + \lambda x \int_{x(1+\gamma(y))>b(t)} \gamma(y)p(y)m(dy) \\
&\leq -rK + \lambda x \int_{\gamma(y)\geq 0} \gamma(y)p(y)m(dy) \\
&\leq -K\left[r - \lambda \int_{\gamma(y)\geq 0} \gamma(y)p(y)m(dy)\right],
\end{aligned}
$$

where the first inequality is true because $v(t, \cdot)$ is nonincreasing, the second since $v(t, b(t)) = K - b(t) \leq K - x$, the third because $\{x(1+\gamma(y)) \geq b(t)\} \subset \{\gamma(y) \geq 0\}$, when $x \leq b(t)$, and the fourth since $x \leq b(t) \leq K$. Therefore, from condition $(C_p)$, it yields $\mathcal{L}^p v \leq 0$ and finally we deduce that $\{e^{-rs}v(t-s, X_s(x)),\ 0 \leq s \leq t\}$ is a $Q^p$-surmartingale. This means also that $v$ is $Q^p$ $r$-excessive in terms of potential theory.

We show then that $v = F^p$. From the $Q^p$ $r$-excessivity property of $v$, we have, for all $\tau \in \mathcal{T}_{0,t}$,

$$
\begin{aligned}
v(t,x) &\geq E^{Q^p}[e^{-r\tau}v(t-\tau, X_\tau(x))] \\
&\geq E^{Q^p}[g(X_\tau(x))],
\end{aligned}
$$

where the second ineqality is derived from $v \geq g$. This implies therefore that $v \geq F^p$, by definition (1.5) of $F^p$. If $x \leq b(t)$, then $v(t,x) = g(x) \leq F^p(t,x)$. If $x > b(t)$, then $v(t,x) > g(x)$. We then define the stopping time:

$$
\tau = \inf\{0 \leq s \leq t,\ v(t-s, X_s(x)) = g(X_s(x))\}.
$$

The preceding set is not empty since $v(0,x) = g(x)$ from (3.10) and so $\tau \in \mathcal{T}_{0,t}$. It follows from (3.7) that $\mathcal{L}^p v(t-u, X_u(x)) = 0$ for $0 \leq u \leq \tau$ and then from (3.12) that the process $\{e^{-rs}v(t-s, X_s(x)),\ 0 \leq s \leq \tau\}$ is a $Q^p$ martingale:

$$
\begin{aligned}
v(t,x) &= E^{Q^p}[e^{-r\tau}v(t-\tau, X_\tau(x))] \\
&= E^{Q^p}[e^{-r\tau}g(X_\tau(x))].
\end{aligned}
$$

This implies that $v \leq F^p$, which ends the proof. □

## 4. Behavior of the Free Boundary

The shape and smoothness of the free boundary for parabolic variational inequalities have been studied by various authors (see, e.g., [9], [31], and [13]). To our knowledge, results on the free boundary for parabolic integrodifferential operators have not appeared so far in the literature (see, however, [10] for a study of the free boundary for elliptic variational inequalities with nonlocal operators).

Adapting arguments of Jacka [13], we prove a continuity property of the free boundary $b^p(t)$ within a jump-diffusion model. As for the uniqueness result (Theorem 3.2), we need the following condition, slightly stronger than $(C_p)$:

$$
(C'_p) \quad r_p := r - \lambda \int_{\gamma(y)\geq 0} \gamma(y)p(y)m(dy) \ > \ 0.
$$

**Theorem 4.1.** *Under condition* $(C_p')$, *the free boundary* $b^p$ *is continuous in* $(0, T]$.

Before proving Theorem 4.1, we have the following lemma:

**Lemma 4.1.** *Assume* $(C_p')$ *holds. Let* $t_0 \in (0, T]$. *Then there exists* $\varepsilon > 0$ *such that, for all* $t \in [t_0, T]$, $x \in (b^p(t), K]$,

$$\tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 F^p}{\partial x^2}(t, x) \ \geq\ \varepsilon.$$

*Proof.* According to Proposition 3.1 and since $t \mapsto F^p(\cdot, x)$ is nondecreasing, we have, for all $x > b^p(t)$,

$$\begin{aligned}\tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 F^p}{\partial x^2} &\geq rF^p - rx\frac{\partial F^p}{\partial x} - \lambda \\ &\quad \times \int_{\mathbb{R}} \left[ F^p(t, x(1+\gamma(y))) - F^p(t, x) - \gamma(y) x \frac{\partial F^p}{\partial x}(t, x) \right] \\ &\quad \times p(y) m(dy).\end{aligned}$$

From the continuity of $F^p$ and $F_x^p$, and by the dominated convergence theorem, we have (recall that $F^p(t, b^p(t)) = K - b^p(t)$ and $F_x^p(t, b^p(t)) = -1$)

$$\begin{aligned}&\liminf_{x \downarrow b^p(t)} \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 F^p}{\partial x^2} \\ &\quad \geq r(K - b^p(t)) + rb^p(t) - \lambda \\ &\qquad \times \int_{\mathbb{R}} [F^p(t, b^p(t)(1+\gamma(y))) - (K - b^p(t)(1+\gamma(y)))] p(y) m(dy) \\ &\quad = rK - \lambda \int_{\gamma(y) \geq 0} [F^p(t, b^p(t)(1+\gamma(y))) - (K - b^p(t)(1+\gamma(y)))] \\ &\qquad \times p(y) m(dy) \\ &\quad \geq rK - \lambda \int_{\gamma(y) \geq 0} [F^p(t, b^p(t)) - (K - b^p(t)(1+\gamma(y)))] p(y) m(dy) \\ &\quad = rK - \lambda b^p(t) \int_{\gamma(y) \geq 0} \gamma(y) p(y) m(dy) \\ &\quad \geq K \left( r - \lambda \int_{\gamma(y) \geq 0} \gamma(y) p(y) m(dy) \right)\end{aligned}$$

since $F^p(t, \cdot)$ is nonincreasing and $b^p(t) \leq K$. Therefore under condition $(C_p')$, there exists a neighborhood $V$ of the optimal-stopping boundary, $V \subset \mathcal{C}^p$, such that $\frac{1}{2}\sigma^2 x^2 F_{xx}^p \geq K r_p / 2 > 0$ in $V$. We conclude by noting, thanks to Proposition 3.1 and Corollary 3.1, that the positive continuous function $(t, x) \mapsto \frac{1}{2}\sigma^2 x^2 F_{xx}^p(t, x)$ attains a positive minimum in the compact set $\mathrm{cl}([t_0, T] \times [0, K] \cap \mathcal{C}^p \backslash V)$. □

*Proof of Theorem* 4.1. Since $S^p$ is closed, if $t_n \nearrow t$, then $(t, b^p(t^-) = \lim b^p(t_n)) \in S^p$, hence $b^p(t^-) \leq b^p(t)$. The left-continuity of $b^p$ is thus obtained from the nonincreasing nature of $b^p$.

Let $t \in (0, T)$ and $0 < t_0 \leq t$. For $\eta > 0$: $b^p(t) + \eta < K$, and $(t_n)$ a sequence in $(t, T]$: $t_n \searrow t$, as $n \to +\infty$, we have, since $F^p$ and $g$ agree on $b^p$ up to the first derivative (see (3.3) and (3.5)),

$$\begin{aligned} F^p(t_n, b^p(t_n) + \eta) &- g(t_n, b^p(t_n) + \eta) \\ &= \int_{b^p(t_n)}^{b^p(t_n)+\eta} \int_{b^p(t_n)}^{y} \left( \frac{\partial^2 F^p}{\partial x^2} - \frac{\partial^2 g}{\partial x^2} \right) (t_n, u) \, du \, dy \\ &\geq \frac{\eta^2}{\sigma^2 K} \varepsilon > 0, \end{aligned}$$

where the inequality follows from Lemma 4.1 and since $\partial^2 g / \partial x^2$ vanishes in $[0, T] \times [0, K]$. Sending $n \to +\infty$, it comes from the continuity of $F^p$ and $g$ that $F^p(t, b^p(t^+) + \eta) > g(t, b^p(t^+) + \eta)$ and hence that $(t, b^p(t^+) + \eta) \in C^p$, for all $\eta > 0$. It implies that $b^p(t^+) \geq b^p(t)$ and then $b^p(t^+) = b^p(t)$ by the nonincreasing property of $b^p$. The right-continuity and finally the continuity of $b^p$ in $t$ is then proved. □

Our interest is now in the behavior of the critical stock price near maturity, i.e., as $t \to 0$. We denote by $f^{\mathrm{BS}}$, $F^{\mathrm{BS}}$, and $b^{\mathrm{BS}}$, the European, American option put price, and its critical stock price in the framework of the BS model. We have therefore the following estimate.

**Proposition 4.1.** *Under condition* $(C'_p)$, *there exists a positive constant* $C > 0$, *such that, for all* $t \in (0, T]$,

$$0 \leq b^{\mathrm{BS}}(t) - b^p(t) \leq C\sqrt{t}.$$

Before proving Proposition 4.1, we state the following lemma:

**Lemma 4.2.**

$$\lim_{t \downarrow 0^+} \frac{\partial F^p}{\partial x}(t, b^{\mathrm{BS}}(t)) = -1.$$

*Proof.* According to relation (3.6), it suffices to prove that $\lim_{t \downarrow 0^+} f_x^p(t, b^{\mathrm{BS}}(t)) = -1$. From the explicit expression of the delta hedge ratio $f_x^p$ (see [28]), we easily obtain, by recalling that $f_x^p \leq 0$ and $f_x^p(t, \cdot)$ is nondecreasing,

$$-1 \leq f_x^p(t, x) \leq e^{-\lambda_p (k_p + 1) t} f_x^{\mathrm{BS}}(t, x).$$

Now from the closed-form expression of the delta BS (see [5]),

$$f_x^{\mathrm{BS}}(t, x) = -\left[ 1 - \Phi\left( \frac{\ln(x/K)}{\sigma\sqrt{t}} + \left( \frac{r}{\sigma} + \frac{\sigma}{2} \right) \sqrt{t} \right) \right]$$

we deduce that $\lim_{t\downarrow 0^+} f_x^{\mathrm{BS}}(t,x) = -1$, for $x < K$. Moreover, thanks to the estimate[1] of the BS critical stock price near expiration (as $t \to 0^+$) derived in [3] and [19],

$$\ln \frac{b^{\mathrm{BS}}(t)}{K} \sim \frac{b^{\mathrm{BS}}(t) - K}{K} \sim -\sigma\sqrt{t|\ln t|},$$

we conclude also that $\lim_{t\downarrow 0^+} f_x^{\mathrm{BS}}(t, b^{\mathrm{BS}}(t)) = -1$, which states Lemma 4.2. □

*Proof of Proposition* 4.1. The first inequality, $0 \le b^{\mathrm{BS}}(t) - b^p(t)$, is always true (without condition $(C_p')$) and follows from $F^p(t,x) \ge F^{\mathrm{BS}}(t,x)$, which is a particular assertion of Proposition 5.1. Before proving the other inequality, recall that $F^p(t,\cdot)$ is $C^2$ for $x > b^p(t)$, and $C^1$ in $\mathbb{R}_+$. When $b^{\mathrm{BS}}(t) > b^p(t)$ (if $b^{\mathrm{BS}}(t) = b^p(t)$, there is nothing to prove), we then obtain, by Taylor's formula,

$$\begin{aligned} F^p(t, b^{\mathrm{BS}}(t)) &= F^p(t, b^p(t)) + (b^{\mathrm{BS}}(t) - b^p(t))\frac{\partial F^p}{\partial x}(t, b^p(t)) \\ &\quad + \tfrac{1}{2}(b^{\mathrm{BS}}(t) - b^p(t))^2 \frac{\partial^2 F^p}{\partial x^2}(t, \zeta(t)) \end{aligned}$$

with $b^p(t) < \zeta(t) < b^{\mathrm{BS}}(t)$. Hence, using $F^p(t, b^p(t)) = K - b^p(t)$ and $F_x^p(t, b^p(t)) = -1$, we have

$$\begin{aligned} F^p(t, b^{\mathrm{BS}}(t)) &= K - b^{\mathrm{BS}}(t) + \tfrac{1}{2}(b^{\mathrm{BS}}(t) - b^p(t))^2 \frac{\partial^2 F^p}{\partial x^2}(t, \zeta(t)) \\ &= F^{\mathrm{BS}}(t, b^{\mathrm{BS}}(t)) + \tfrac{1}{2}(b^{\mathrm{BS}}(t) - b^p(t))^2 \frac{\partial^2 F^p}{\partial x^2}(t, \zeta(t)). \end{aligned} \tag{4.1}$$

From the decomposition of the American put option price (see Theorem 3.1) we have

$$F^p(t,x) - F^{\mathrm{BS}}(t,x) \le f^p(t,x) - f^{\mathrm{BS}}(t,x) + rKt.$$

Now, from the explicit expression of $f^p$ (see [28]), we have

$$\begin{aligned} f^p(t,x) &= \sum_{n=0}^{\infty} e^{-\lambda_p t}\frac{(\lambda_p t)^n}{n!} E^{Q^p}\left[ f^{\mathrm{BS}}\left(t, xe^{-\lambda_p k_p t}\prod_{i=1}^{n}(1+\gamma(Y_i))\right)\right] \\ &\le e^{-\lambda_p t} f^{\mathrm{BS}}(t, xe^{-\lambda_p k_p t}) + K(1 - e^{-\lambda_p t}) \\ &\le f^{\mathrm{BS}}(t,x) + x(e^{-\lambda_p k_p t} - 1)\frac{\partial f^{\mathrm{BS}}}{\partial x}(t,\xi) + K(1 - e^{-\lambda_p t}) \\ &\le f^{\mathrm{BS}}(t,x) + \lambda_p(k_p x + K)t, \end{aligned}$$

where the first inequality follows from $f^{\mathrm{BS}}(t,x) \le K$, the second from Taylor's formula and the third from $|f_x^{\mathrm{BS}}| \le 1$ and since $1 - e^{-ct} \le ct$ for $c \ge 0$. Therefore, we have, for all $x \le K$,

$$0 \le F^p(t,x) - F^{\mathrm{BS}}(t,x) \le K(r + \lambda_p(k_p + 1))t.$$

[1] Given two functions $f$ and $g$ defined in $(0,T]$, we write that $f(t) \sim g(t)$ as $t \to 0^+$ if $\lim_{t\to 0^+}(f(t)/g(t)) = 1$.

Going back to (4.1), we then obtain

$$\tfrac{1}{2}(b^{\mathrm{BS}}(t) - b^p(t))^2 \frac{\partial^2 F^p}{\partial x^2}(t, \zeta(t)) \leq K(r + \lambda_p(k_p + 1))t. \tag{4.2}$$

According to Proposition 3.1, we have, since $\zeta(t) > b^p(t)$, $F^p \geq 0$ and $F_t^p \geq 0$,

$$\begin{aligned} &\tfrac{1}{2}\sigma^2\zeta(t)^2 \frac{\partial^2 F^p}{\partial x^2}(t, \zeta(t)) \\ &\geq -r\zeta(t)\frac{\partial F^p}{\partial x}(t, \zeta(t)) - \lambda \\ &\quad \times \int_{\mathbb{R}} \left[ F^p(t, \zeta(t)(1 + \gamma(y))) - F^p(t, \zeta(t)) - \gamma(y)\zeta(t)\frac{\partial F^p}{\partial x}(t, \zeta(t)) \right] \\ &\quad \times p(y)m(dy). \end{aligned} \tag{4.3}$$

Since $\zeta(t) < b^{\mathrm{BS}}(t)$ and $x \mapsto F_x^p(t, \cdot)$ is nondecreasing, we have $-1 \leq F_x^p(t, \zeta(t)) \leq F_x^p(t, b^{\mathrm{BS}}(t))$ and therefore, from Lemma 4.2, $\lim_{t\downarrow 0^+} F_x^p(t, \zeta(t)) = -1$. By the dominated convergence theorem and by sending $t \to 0$ in the last inequality (4.3), we obtain

$$\begin{aligned} &\liminf_{t\downarrow 0^+} \tfrac{1}{2}\sigma^2\zeta(t)^2 \frac{\partial^2 F^p}{\partial x^2}(t, \zeta(t)) \\ &\geq r\zeta(0^+) - \lambda \int_{\mathbb{R}} [(K - \zeta(0^+)(1 + \gamma(y)))^+ - (K - \zeta(0^+)(1 + \gamma(y)))] \\ &\quad \times p(y)m(dy) \\ &= r\zeta(0^+) - \lambda \int_{\zeta(0^+)(1+\gamma(y))\geq K} [\zeta(0^+)(1 + \gamma(y)) - K]p(y)m(dy) \\ &\geq \zeta(0^+)\left[ r - \lambda \int_{\gamma(y)\geq 0} \gamma(y)p(y)m(dy) \right], \end{aligned}$$

since $\zeta(0^+) \leq K$ and $\zeta(0^+)(1 + \gamma(y)) - K \geq 0$ in the set $\{\zeta(0^+)(1 + \gamma(y)) \geq K\} \subset \{\gamma(y) \geq 0\}$. Note that $\zeta(0^+) > 0$ since otherwise $b^p(0^+) = 0$ and $b^p(t) = 0$ for all $t \in (0, T]$ by the nonincreasing nature of $b^p$, which is obviously false. Therefore, under condition $(C_p')$, there exists a positive constant $C > 0$ such that $F_{xx}^p(t, \zeta(t)) \geq C$ for $t$ close to 0, which combined with (4.2) suffices to prove Proposition 4.1. □

Thanks to the estimate of Proposition 4.1, we can provide within a jump-diffusion model the same estimate of the critical stock price near maturity, obtained recently by Barles *et al.* [3] (see also [19]) for the BS model.

**Theorem 4.2.** *Under condition* $(C_p')$, *we have* $\lim_{t\downarrow 0^+} b^p(t) = K$ *and*

$$\frac{K - b^p(t)}{K} \sim \sigma\sqrt{t|\ln t|}$$

*as t approaches* 0.

## 5. Impact of Jump Intensity and Jump Risk on the American Option Price and Its Critical Stock Price

Compared with a diffusion model, the presence of jump uncertainty in stock price behavior introduces new parameters for the option valuation, essentially the intensity of jumps $\lambda$ and the market price of jump risk $p(\cdot)$. In this section we study the influence of these two factors on the critical stock price and on the American put option price. We denote by $F_\lambda^p$ (resp. $b_\lambda^p$) the American put option value (resp. the critical stock price) corresponding to a market price of jump risk $p$ and to an intensity of jumps $\lambda$. The proofs of the following proposition are essentially based on comparison principles for parabolic integrodifferential operators.

**Proposition 5.1.** *The American put option value and its free-boundary satisfy:*

— *For a fixed market price of jump risk $p$, $F_\lambda^p$ (resp. $b_\lambda^p$) is a nondecreasing (resp. nonincreasing) function of the intensity of jumps $\lambda$: if $\lambda_1 \leq \lambda_2$, then*

$$F_{\lambda_1}^p(t,x) \leq F_{\lambda_2}^p(t,x), \qquad \forall (t,x) \in [0,T] \times \mathbb{R}_+,$$

$$b_{\lambda_2}^p(t) \leq b_{\lambda_1}^p(t), \qquad \forall t \in (0,T].$$

— *For a fixed intensity of jumps $\lambda$, $F_\lambda^p$ (resp. $b_\lambda^p$) is a nondecreasing (resp. nonincreasing) function of the market price of jump risk $p$: if $p_1(\cdot) \leq p_2(\cdot)$, then*

$$F_\lambda^{p_1}(t,x) \leq F_\lambda^{p_2}(t,x), \qquad \forall (t,x) \in [0,T] \times \mathbb{R}_+,$$

$$b_\lambda^{p_2}(t) \leq b_\lambda^{p_1}(t), \qquad \forall t \in (0,T].$$

*Proof.* For a fixed market price of jump risk $p$, we denote, for notational simplicity, by $F_i$ (resp. $b_i$) the American option price (resp. the critical stock price) when the intensity of jumps (under $P$) is $\lambda_i$, and also $F = F_2 - F_1$.

$$\text{If} \quad x \leq b_1(t), \quad \text{then} \quad F(t,x) = F_2(t,x) - g(x) \geq 0. \tag{5.1}$$

If $x > b_1(t)$, then $F_1(t,x) > g(x)$ and, from Proposition 3.1, $F_1$ is smooth and satisfies

$$\mathcal{L}_1 F_1(t,x) = 0, \qquad \forall x > b_1(t), \tag{5.2}$$

where $\mathcal{L}_i$ is the operator $\mathcal{L}^p$ defined in (2.3) with $\lambda$ replaced by $\lambda_i$. We do not know *a priori* if $b_2 \geq b_1$, and so if $F_2$ is smooth for $x > b_1(t)$. However, $F_2$ satisfies, in the viscosity sense,

$$\mathcal{L}_2 F_2(t,x) \leq 0, \qquad \forall (t,x) \in (0,T] \times \mathbb{R}_+. \tag{5.3}$$

Relations (5.2) and (5.3) imply that $\mathcal{L}_2 F_2(t,x) - \mathcal{L}_1 F_1(t,x) \leq 0$ for every $x > b_1(t)$, which can also be written as

$$\mathcal{L}_2 F \leq -(\lambda_2 - \lambda_1) \times \int_{\mathbb{R}} \left[ F_1(t, x(1+\gamma(y))) - F_1(t,x) - \gamma(y) x \frac{\partial F_1}{\partial x}(t,x) \right] p(y) m(dy)$$

for every $x > b_1(t)$. From convexity of $F_1(t, \cdot)$, the right-hand side of this inequality is nonpositive whenever $\lambda_1 \leq \lambda_2$. Moreover, since $F(0, x) = 0$, we deduce from (5.1) by the maximum principle that $F \geq 0$ for all $(t, x) \in [0, T] \times \mathbb{R}_+$, which also implies that $b_2(t) \leq b_1(t)$.

The second assertion of the proposition is proved by the same means and is omitted here. □

**Remarks.** 1. The main point is the convexity of $F^p(t, \cdot)$. Therefore, Proposition 5.1 remains valid for any American option with payoff, a convex function $g$, and generalizes statements proved for European options by [28].

2. The first assertion of the last proposition is consistent with the following economic intuition. As the intensity of jumps increases, the market is more risky and American put option holders are more stringent in their exercise decision. This makes the optimal exercise boundary lower. We find in particular the intuitive result that an American option on a stock with a jump component is more valuable than its counterpart without jump uncertainty.

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