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Portfolio frontiers with restrictions to tracking error volatility and value at risk

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ABSTRACT

Asset managers are often given the task of restricting their activity by keeping both the value at risk (VaR) and the tracking error volatility (TEV) under control. However, these constraints may be impossible to satisfy simultaneously because VaR is independent of the benchmark portfolio. The management of these restrictions is likely to affect portfolio performance and produces a wide variety of scenarios in the risk-return space. The aim of this paper is to analyse various interactions between portfolio frontiers when risk managers impose joint restrictions upon TEV and VaR. Specifically, we provide analytical solutions for all the intersections and we propose simple numerical methods when such solutions are not available. Finally, we introduce a new portfolio frontier.

1. Introduction

It is well known that investors assign part of their funds to asset managers with the task of beating a benchmark and the risk management usually imposes a maximum value on the tracking error volatility (TEV) in order to keep the portfolio risk close to that of a selected benchmark; in literature, the TEV is typically defined as the mean of squared deviations from the return of a benchmark portfolio (see for example, Clarke et al., 1994). Starting from the seminal contribution of Markowitz (1959) in the risk-return space (σ_P, μ_P), various contributions dedicate a lot of attention to constrained asset allocation strategies: for example, Jagannathan and Ma (2003) provide evidence explaining why constraints are useful, while others, such as Wagner (2002) or Boyle and Tian (2007), study the topic of outperforming a benchmark[1] in the presence of constraints. Other contributions, such as Alexander and Baptista (2006), show that a constrained portfolio selection negatively affects the asset manager's ability to track a benchmark because it substantially reduces the region of feasible portfolios. The most commonly used constraint on relative risk is the TEV, which is associated with the investment goal expressed in terms of the excess return over a benchmark (see Franks, 1992)[2]; furthermore, Roll (1992) shows that asset managers who aim to produce positive return performance over a benchmark whilst keeping TEV to a minimum, usually select portfolios that are not mean/variance efficient.

The literature proposes several asset allocation strategies. Roll (1992) suggests a restriction on the portfolio's beta, whereas Jorion (2003) shows that a TEV constraint produces an elliptic portfolio frontier in variance-return space.[3] The contribution of Bajeux-Besnainou et al. (2011) improves this approach by deriving analytically the efficient portfolio frontier under a TEV limit and a portfolio weights constraint dictated by the fund policy or by regulatory restrictions. Chekhlov et al. (2005) determine a mean-drawdown boundary by imposing several drawdown limits; in a related article, Alexander and Baptista (2006) analyse the impact of a maximum drawdown constraint to both the mean-variance and the mean-TEV space. Other contributions use the value at risk (VaR) as a

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[1] The field of benchmarking involves various topics, such as how to optimise costs in passive management by selecting the optimal number of assets to use (Jansen and Van Dijk, 2002) or by deciding when to rebalance (Gaivoronski et al., 2005). Whether an active manager can beat the benchmark using a specific division of labour (Lee, 2000) or other specific strategies (Browne, 1999) is another key topic in benchmarking. This subject is strictly related to the evaluation of the asset manager (see for instance, Clarke et al., 2002; Cremers and Petajisto, 2009; Grinold and Kahn, 2000; Lo, 2008).

[2] However, Rudolf et al. (1999) apply four linear models for minimising the distance between the return of a portfolio and a benchmark: specifically, they use absolute deviations instead of squared deviations.

[3] Other contributions use this methodology: for example, El-Hassan and Kofman (2003) add further constraints such as no short-selling, Palomba (2008) inserts portfolio frontiers into an econometric model for asset allocation and Riccetti (2012) generalises the model of Jorion (2003) by inserting portfolio commissions.

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measure of portfolio risk: for example, Campbell et al. (2001) introduce a model in which the maximisation of the expected portfolio return is subject to a VaR limit, while Alexander and Baptista (2008) impose a VaR constraint upon the standard asset allocation framework in (σ_P, μ_P) space. Subsequently, Alexander and Baptista (2010) present a strategy of active portfolio management in which they use a target upon *ex ante* alpha, defined as the intercept of the linear regression of the portfolio return on the benchmark return; in this case, the portfolio frontier contains all portfolios which minimise the TEV for any given *ex ante* portfolio alpha.

As a consequence of the above, different (possibly conflicting) asset allocation strategies may be equally justifiable on the basis of different definitions of risk and of the different frontiers they generate. Hence, it would be interesting to identify one or more portfolios which are able to satisfy several criteria at the same time. This paper compares different portfolio frontiers and provides a summary of their graphical and analytical properties. The field of investigation is the usual framework of unlimited short sales, quadratic utility function and normally distributed returns; these assumptions rule out skewed and leptokurtic return distributions, so that the portfolio standard deviation is the unique factor of risk. We calculate and discuss several portfolios of interest, focussing on those that lie on the intersections between the different frontiers.

The aim of this work is to analyse the situations in which managers have to keep both the VaR and TEV under control. In doing so, from the economic perspective, managers have to face two problems: first, TEV constrained portfolios could not satisfy the VaR restriction and second, TEV-VaR constrained portfolios are usually inefficient because they lie at the right of the so-called "Mean-Variance Frontier" (hereafter MVF).

The remainder of the paper proceeds as follows: Section 2 contains a summary of the principal portfolio frontiers put forward in the literature; we dedicate particular attention to the frontiers introduced by Jorion (2003) and Alexander and Baptista (2008) whose possible intersections are successively discussed in Section 3, together with numerical methods for determining common portfolios. In Section 4, we introduce a new boundary for which TEV and VaR constraints can be satisfied at the same time. Section 5 closes the analysis with a short empirical example and Section 6 concludes. Finally, we also provide an Appendix containing some useful results.

2. Review of portfolio frontiers

Before introducing the portfolio frontiers, we define some notation: assuming that the available data consist of n risky assets, an n-dimensional column vector μ contains their expected returns, while the full rank $n \times n$ matrix Ω represents the covariance matrix. In accordance with the literature, we define the following constants: $a = \imath'\Omega^{-1}\imath$, $b = \imath'\Omega^{-1}\mu$, $c = \mu'\Omega^{-1}\mu$ and $d = c - b^2/a$, where $\imath$ is an n-dimensional column vector of ones. As all these parameters depend exclusively on the data, they are independent of any allocation strategy. In this setup, some subjective inputs are also relevant because risk managers could impose some constraints upon asset managers activity: in particular, they could set a desired level of total return (μ_P) or impose restrictions upon TEV (T_0) and/or VaR (V_0). We mostly conduct the geometric analysis in the (σ_P^2, μ_P) space, representing all the graphical implications in the usual (σ_P, μ_P) space in which the axes refer to the absolute risk and total return respectively.

Our study takes two fundamental portfolio frontiers into account: the popular MVF, first introduced by Markowitz (1959), and the "Mean-TEV Frontier" (hereafter MTF) defined by Roll (1992). The well known MVF consists of all the portfolios which minimise the total portfolio variance, given a desired portfolio return; its equation is

$$\sigma_P^2 = \sigma_C^2 + \frac{1}{d}(\mu_P - \mu_C)^2, \tag{1}$$

which produces a parabola in the (σ_P^2, μ_P) space or a hyperbola in the (σ_P, μ_P) space. The expected return $\mu_C = b/a$ and the variance $\sigma_C^2 = a^{-1}$ are of the minimum variance portfolio (portfolio C), which is independent of the desired portfolio return μ_P. All the portfolios for which $\mu_P \geqslant \mu_C$ belong to the efficient subset of MVF.

The MTF, on the other hand, shifts the asset allocation strategies from the absolute risk perspective to that of the risk relative to a benchmark portfolio $B \equiv (\sigma_B^2, \mu_B)$; in this context, we assume that the manager deals with the risk component of portfolios by minimising the TEV instead of the total portfolio variance. The equation for the MTF is

$$\sigma_P^2 = \sigma_B^2 + \frac{1}{d}(\mu_P - \mu_B)^2 + 2\frac{\Delta_1}{d}(\mu_P - \mu_B), \tag{2}$$

where $\Delta_1 = \mu_B - \mu_C$. The constant Δ_1/d does not depend upon the expected portfolio return. Comparing Eqs. (1) and (2), it is evident that the MTF is a horizontal translation of the MVF in the mean-variance space; hence, it is easy to show that these curves have no intersections. These frontiers have the same analytical form with the only exception of the third addend in Eq. (2) which contributes to the above mentioned translation. The MVF is calculated "around" the minimum variance portfolio (C), while the benchmark is the reference portfolio for the MTF, but does not correspond to its minimum.[4] However, Roll (1992) claims that portfolios belonging to the MTF are generally suboptimal because they lie to the right of the MVF and are thus overly risky. The horizontal distance between the frontiers in the (σ_P^2, μ_P) space represents the efficiency loss (δ_B): for each value of the portfolio return (μ_P), we obtain this distance by subtracting Eq. (1) from Eq. (2), hence

$$\delta_B = \Delta_2 - \frac{\Delta_1^2}{d}, \tag{3}$$

where $\Delta_2 = \sigma_B^2 - \sigma_C^2$. Given the impossibility of any intersection between MVF and MTF, δ_B is positive for all μ_P by construction. However, in the special case of the benchmark lying on the mean-variance boundary, such frontiers coincide: in this context, the benchmark minimises both the portfolio variance and the TEV at the same time and the relationship $\Delta_2 = \Delta_1^2/d$ corresponds to Eq. (1) for $\mu_P = \mu_B$; in this situation, the efficiency loss is clearly zero.

2.1. The Constrained TEV Frontier (CTF)

Jorion (2003) adds to the Markowitz setup a specific TEV constraint

$$T_0 = (\omega_P - \omega_B)'\Omega(\omega_P - \omega_B)$$

where ω_P and ω_B are vectors of portfolio and benchmark weights respectively. Thus, he obtains the "Constrained TEV Frontier" (hereafter CTF), a closed and bounded frontier in the (σ_P^2, μ_P) space whose equation is

$$d(\sigma_P^2 - \sigma_B^2 - T_0)^2 + 4\Delta_2(\mu_P - \mu_B)^2 - 4\Delta_1(\sigma_P^2 - \sigma_B^2 - T_0)(\mu_P - \mu_B) - 4d\delta_B T_0 = 0, \tag{4}$$

where Δ_1, Δ_2 and δ_B are as previously defined. Jorion (2003) shows that Eq. (4) is that of an ellipse for which the horizontal axis has a positive (negative) slope when $\Delta_1 > 0$ ($\Delta_1 < 0$). The horizontal centre of the ellipse is $\sigma_B^2 + T_0$, hence an increase in T_0 produces a surface area expansion. This elliptic frontier becomes somewhat

[4] In (σ_P^2, μ_P) space, the minimum portfolio in Eq. (2) is $G \equiv \left(\sigma_B^2 - \Delta_1^2/d, \mu_C\right)$.

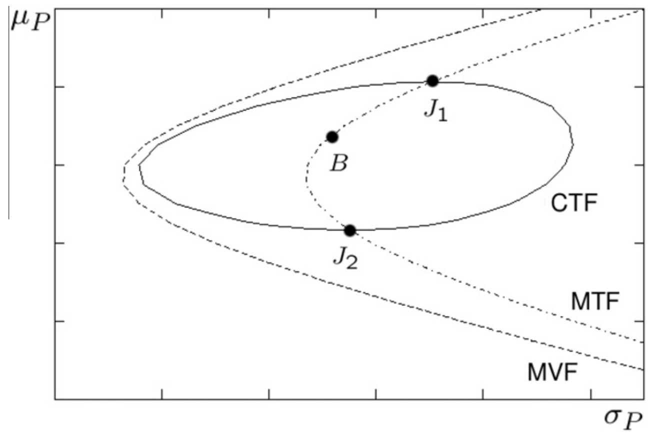

Fig. 1. Portfolio frontiers when $\Delta_1 > 0$.

distorted in the (σ_P, μ_P) space. Since we assume that asset managers generally face constrained optimisation, the CTF contains the benchmark and all the feasible portfolios for which TEV $\leqslant T_0$; Jorion (2003) and Palomba (2008) show that constraining TEV influences the ellipse eccentricity, thus intersections between the CTF and the MVF are possible. Specifically, the number of contacts depends on the equation

$$\Psi = d\,T_0 - d\Delta_2 + \Delta_1^2. \tag{5}$$

When $\Psi < 0$ the frontiers do not intersect, but when $\Psi = 0$ they have one portfolio in common. When $\Psi > 0$, the frontiers have two intersections which tend to move along the MVF as the value of T_0 increases. These contacts define two arcs on the ellipse, where the left arc coincides with the mean-variance boundary. The most interesting situation is $\Psi = 0$ for which the frontiers are tangent and the tangency TEV is

$$T_H = \delta_B = \Delta_2 - \frac{\Delta_1^2}{d}, \tag{6}$$

where $H \equiv (\sigma_C^2 + \Delta_1^2/d, \mu_B)$ is the contact point between the frontiers.[5] Jorion (2003) also shows that the CTF intersects the MTF in portfolios

$$\begin{cases} J_1 \equiv \left(\sigma_B^2 + T_0 + 2\Delta_1\sqrt{T_0/d}, \mu_B + \sqrt{dT_0}\right) \\ J_2 \equiv \left(\sigma_B^2 + T_0 - 2\Delta_1\sqrt{T_0/d}, \mu_B - \sqrt{dT_0}\right), \end{cases} \tag{7}$$

corresponding to those portfolios with the maximum and minimum expected return respectively. The economic interpretation of this result is straightforward: the minimum TEV boundary coincides with the constrained TEV frontier for those portfolios with TEV = T_0. Given that portfolios in Eq. (7) belong to the MTF, their efficiency loss is δ_B. Defining $J_1 \equiv (\sigma_1^2, \mu_1)$ and $J_2 \equiv (\sigma_2^2, \mu_2)$, where $\mu_1 > \mu_2$, all the portfolio frontiers are shown in Fig. 1. Managers could reduce the efficiency loss by maintaining a TEV = T_0: it is sufficient to select a portfolio in the left arc $\widehat{J_1J_2}$ on the CTF because the absolute risk of portfolios which lie on this arc is less than that of portfolios belonging to the arc formed by J_1 and J_2 on the MTF.

2.2. The Constrained Value at Risk Frontier (CVF)

As is widely known, the VaR is the θ-quantile of the portfolio distribution, where $0.5 < \theta < 1$; that is, the minimum loss that will be sustained with probability $1 - \theta$. Under normality, its equation is $V_0 = z_\theta\sigma_P - \mu_P$, where z_θ is the critical value taken from the standardised normal distribution. The risk managers fix the restriction VaR = V_0 which defines the intercept of the Constrained VaR Frontier (hereafter CVF):

$$\mu_P = z_\theta\sigma_P - V_0, \tag{8}$$

a linear frontier in the (σ_P, μ_P) space, where z_θ represents the slope which is always positive, while the intercept ($-V_0$) should be negative.[6] This frontier is independent of the benchmark and the space lying to its left satisfies the VaR restriction. Clearly, the CVF may intersect the MVF or not: if the straight line (8) lies to the left of the mean-variance bound, they do not intersect and no feasible portfolios exist which satisfy the VaR restriction. Conversely, if the CVF intersects the MVF, a portfolio that is efficient by construction exists (Alexander, 2009).

Using the asymptotic slope of the MVF as the critical value, Alexander and Baptista (2008) distinguish a low confidence level ($0 < z_\theta \leqslant \sqrt{d}$) from a high confidence level ($z_\theta > \sqrt{d}$) and then provide a detailed discussion about the VaR constrained frontiers for different scenarios. Focussing on the slope and the intercept in Eq. (8), this type of analysis consists of an analytical geometry problem: in this context, the objective is reaching intersections between the hyperbola MVF and a sheaf of straight lines depending upon parameters V_0 and z_θ. When this problem admits a solution, they define the "Constrained Mean-TEV Frontier" (hereafter CMTF) in the (σ_P, μ_P) space: this is the frontier which satisfies the VaR constraint and it contains all the portfolios with the smallest TEV. According to this definition, the CMTF could be

(i) an empty set if the CVF does not intersect the MVF,
(ii) a single portfolio if the CVF is tangent to the MVF,
(iii) a segment if the CVF crosses the MVF only,
(iv) an arc consecutive to two segments if the CVF crosses both the MVF and the MTF (see Alexander and Baptista, 2008, for details).

3. Intersections between the CTF and the CVF

Searching for the intersections between the CTF and the CVF is interesting in practical situations in which risk managers have to select a portfolio with restrictions on both TEV and VaR; this constrained strategy implies allocations which are generally suboptimal because they do not belong to the MVF. However, the objective is to determine a non-empty subset of the (σ_P, μ_P) space in which risk managers could set a bound on VaR in the presence of restrictions on TEV. Given that the CVF is a straight line with a positive slope whose left half-plane contains all the portfolios with VaR $\leqslant V_0$, it can have zero, one or two contacts points with the CTF, depending on T_0, V_0 and θ. Specifically, the following situations may arise:

1. if the CVF lies to the left of the CTF, then the VaR constraint is too stringent. In such a case an intersection does not exist and it is thus impossible to satisfy both constraints TEV = T_0 and VaR = V_0 simultaneously;
2. if the CVF intersects the CTF, then at least one portfolio satisfies both restrictions upon TEV and VaR. Specifically, a unique solution exists when the CVF is tangent to the CTF on the left, whereas two contacts occur and an infinite number of solutions are available when the CVF crosses the CTF;
3. if the CVF lies to the right of the CTF, the VaR constraint is non-binding.

[5] Moreover, Eq. (6) confirms that $\delta_B \geqslant 0$ for all μ_P.

[6] Formally, a negative intercept of the CVF represents a VaR (loss). Nevertheless, we focus upon the mathematical relationships among portfolio frontiers in which the condition $-V_0 < 0$ is not guaranteed. The opposite condition $-V_0 > 0$ produces a situation that we define as the "worst expected return".

The first two (relevant) scenarios are in Fig. 2 which focusses on the VaR constraint. A value V_K exists for which the curves become tangent. In particular, only when $V_0 \geqslant V_K$ at least one contact point between the frontiers exists; such intersections occur independently of the sign of Δ_1 and the positions of portfolios J_1 and J_2. Analytically, the intersections between the CTF and the CVF correspond to the solutions of a system that includes the ellipse in Eq. (4) and the parabola $\sigma_P^2 = (\mu_P + V_0)^2/z_\theta^2$ in the (σ_P^2, μ_P) space derived from Eq. (8). The resolvent of such a system is the quartic equation

$$c_0 + c_1\mu_P + c_2\mu_P^2 + c_3\mu_P^3 + c_4\mu_P^4 = 0, \tag{9}$$

where

$$\begin{cases} c_0 = f_0(\theta, V_0, T_0, \mu_B, \sigma_B^2) \\ c_1 = f_1(\theta, V_0, T_0, \mu_B, \sigma_B^2) \\ c_2 = f_2(\theta, V_0, T_0, \mu_B, \sigma_B^2) \\ c_3 = f_3(\theta, V_0, \mu_B) \\ c_4 = f_4(\theta). \end{cases}$$

Section A.3 provides the analytical expressions for the coefficients. These parameters determine the positions of the frontiers (see Fig. 2), thus:

- θ influences the magnitude of the parabola,
- V_0 determines the position of the parabola's vertex along the μ_P-axis,
- T_0 affects the eccentricity of the ellipse,
- the benchmark portfolio indicates the position of the ellipse.

In principle, Eq. (9) has four roots; however, the nature of the problem ensures that at least two of those are complex conjugates. The nature of the other two determines the existence of solutions to the portfolio problem: if the remaining two roots are themselves complex conjugates, then no solutions exist. On the other hand, if real roots exist, then there are multiple solutions unless the real roots coincide, in which case the solution is unique.

3.1. Two contacts between CTF and CVF

We calculate the solutions of Eq. (9) via the property

$$1 + c_1^*\mu_P + c_2^*\mu_P^2 + c_3^*\mu_P^3 + c_4^*\mu_P^4 = \prod_{i=1}^{4}(1 - \lambda_i\mu_P), \tag{10}$$

where $c_i^* = c_i/c_0$ and $i = 1, 2, 3, 4$; the parameters λ_is are the inverse roots of the quartic equation corresponding to the eigenvalues of the companion matrix

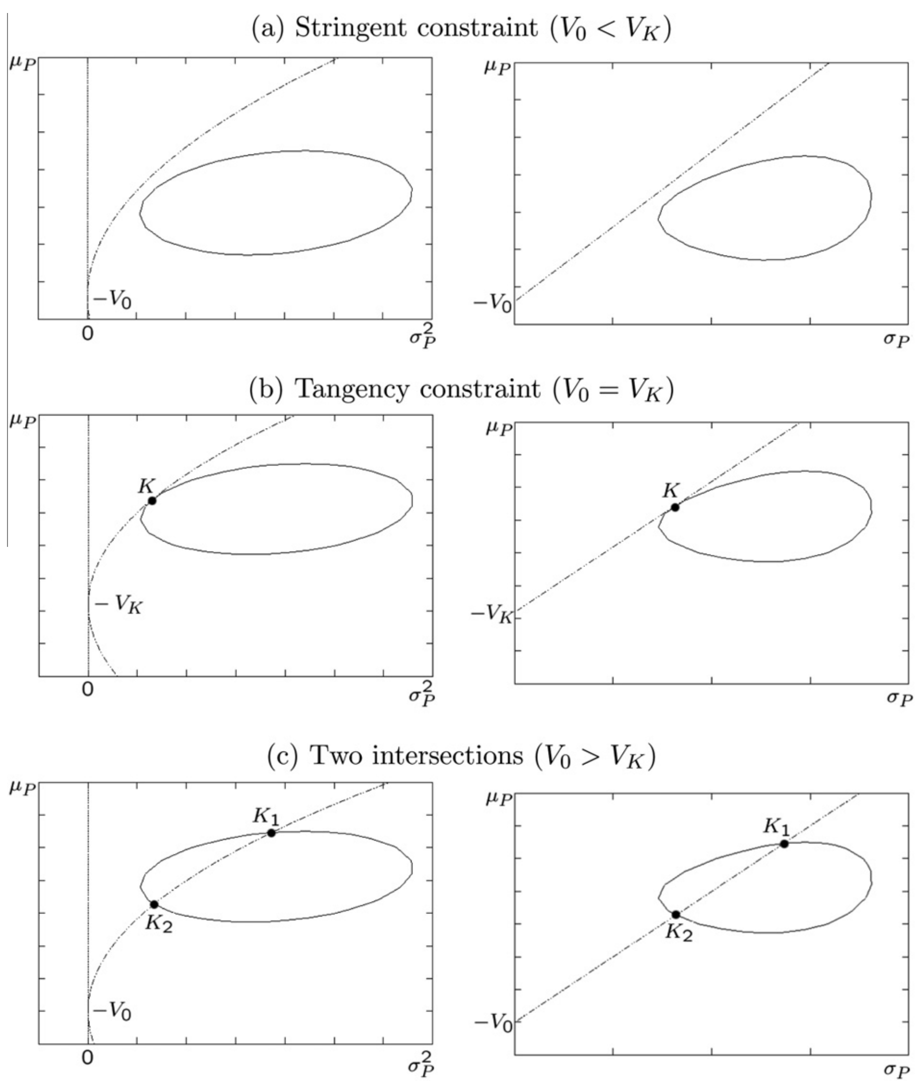

Fig. 2. Contacts between the CTF and the CVF.

$$C^* = \begin{bmatrix} -c_1^* & -c_2^* & -c_3^* & -c_4^* \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Proof. In short, the eigenvalues of the matrix C^* are the solutions of

$$\det(C^* - \lambda I_4) = 0 \Rightarrow \det\left(\begin{bmatrix} -c_1^* - \lambda & -c_2^* & -c_3^* & -c_4^* \\ 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{bmatrix}\right) = 0,$$

where I_4 is the identity matrix of order 4. After some algebra, such determinant is zero for those λs which solve the following characteristic equation

$$c_4^* + c_3^*\lambda + c_2^*\lambda^2 + c_1^*\lambda^3 + \lambda^4 = 0.$$

Eq. (10) arises after the substitution of λ with $1/\mu_P$. □

This is a suitable and computationally convenient method for obtaining numerically accurate solutions for the expected returns of contact portfolios $K_1 \equiv (\sigma_{K_1}, \mu_{K_1})$ and $K_2 \equiv (\sigma_{K_2}, \mu_{K_2})$. Analytically, these contact portfolios arise only when the polynomial (9) admits two real solutions and two complex conjugates, so it is sufficient to invert the real eigenvalues of C^* to have two solutions for μ_P. This technique remains still valid when $c_0 = 0$ for which a solution is $\mu_P = 0$: under this conditions, the quartic equation reduces to a cubic and C^* is 3×3 matrix with one real and two complex eigenvalues.

3.2. Tangency

We obtain the tangency portfolio $K \equiv (\sigma_K, \mu_K)$ by searching for the value V_K which makes the CVF tangent to the CTF for a given T_0. According to Section 3.1, this portfolio corresponds to the situation in which the companion matrix has one real eigenvalue with multiplicity two.

A computationally convenient way of determining the tangency portfolio is the grid search:

$$\begin{cases} \text{for} & \mu_i = \mu_1, \mu_1 - \varepsilon, \mu_1 - 2\varepsilon, \ldots, \mu_2 \\ \text{Min}_{\mu_i} & V_i = z_\theta S_T^{1/2}(\mu_i) - \mu_i \end{cases} \tag{11}$$

where ε is an arbitrary and numerically small increment and the function

$$S_T(\mu_P) = \sigma_B^2 + T_0 + \frac{2}{d}\left\{\Delta_1(\mu_P - \mu_B) - \sqrt{d\delta_B[dT_0 - (\mu_P - \mu_B)^2]}\right\}, \tag{12}$$

with $\mu_2 \leqslant \mu_P \leqslant \mu_1$ (see Eq. (7)), derives from Eq. (4), as the following proof documents.

Proof of equation. (12).Setting $x = \sigma_P^2 - \sigma_B^2 - T_0$, the function (4) becomes the following second order equation

$$dx^2 - 4\Delta_1(\mu_P - \mu_B)x + 4\Delta_2(\mu_P - \mu_B)^2 - 4dT_0\delta_B = 0. \tag{13}$$

Solving x, after some algebra, one can obtain

$$x = \frac{2}{d}\left\{\Delta_1(\mu_P - \mu_B) \pm \sqrt{d\delta_B[dT_0 - (\mu_P - \mu_B)^2]}\right\}.$$

Given that the CVF intersect the CTF on the left, we have to consider the smaller solution of Eq. (13), therefore the function

$$\sigma_P^2 = \sigma_B^2 + T_0 + \frac{2}{d}\left\{\Delta_1(\mu_P - \mu_B) - \sqrt{d\delta_B[dT_0 - (\mu_P - \mu_B)^2]}\right\} \tag{14}$$

defines $S_T(\mu_P)$ in Eq. (12). □

This algorithm uses Eqs. (4) and (8), is fast to compute and returns a numerical solution whose accuracy strictly depends upon the magnitude of the increments. The solution is $V_K = \min\{V_i\}$.

4. The Fixed VaR-TEV Frontier (FVTF)

In this section we derive a new portfolio frontier. Before starting the analysis, we have to make a fundamental remark about parameter $\Delta_1 = \mu_B - \mu_C$, because its sign is that of the horizontal axis of the ellipse in the (σ_P^2, μ_P) space (see Jorion, 2003). The most interesting model in practice arises under the assumption $\Delta_1 > 0$, which means that the horizontal axis of the CTF has a positive slope. When a VaR constraint is at work, it becomes relevant, but only when the CVF intersects the left arc $\widehat{J_1 J_2}$ on the CTF; in this context, the slope z_θ becomes crucial in determining the relationship between VaRs V_1 and V_2 for which the straight line passes through J_1 and J_2 respectively (see Table A.3 for a summary).

In this section we carry out the whole analysis using the triple condition $\Delta_1 > 0, z_\theta > \sqrt{d}, T_0 < T_H$, indicating that the ellipse has a positive slope, that the confidence level is high (see Alexander and Baptista, 2008) and that the CTF does not intersect the MVF. In practice, a high confidence level is the most realistic, provided that risk managers generally impose the VaR constraint with a $\theta \geqslant 0.9$. The Technical supplement[7] associated with this paper contains a detailed discussion about the scenarios with $\Delta_1 \leqslant 0$, low confidence level ($z_\theta \leqslant \sqrt{d}$) and contacts between the frontiers MVF and CTF.

4.1. The general setup

In this section we substantially revise the scenarios introduced by Alexander and Baptista (2008) by also taking the TEV restriction into account; hence, we carry out the analysis in order to identify the intersections lying between the portfolio frontiers and a parallel sheaf of lines. Fig. 3 illustrates our point from a geometric perspective.

(a) *Small bound:* the VaR constraint is $V_0 < V_M$, where V_M is the VaR at which the CVF is tangent to the MVF. The analytical solution for V_M is

$$V_M = -\mu_C + \sqrt{\sigma_C^2(z_\theta^2 - d)} \tag{15}$$

corresponding, in (σ_P^2, μ_P) space, to portfolio

$$M \equiv \left(\sigma_C^2 + d\frac{\sigma_C^2}{z_\theta^2 - d}, \mu_C + d\frac{\sigma_C}{\sqrt{z_\theta^2 - d}}\right) \tag{16}$$

which only depends upon the confidence level θ (proof in Section A.1). Under the small bound condition, the straight line CVF lies to the left of the MVF. This implies that there are no feasible portfolios which satisfy the VaR restriction which is too stringent.

(b) *Minimum bound:* the point M, at which $V_0 = V_M$, is the tangency portfolio between the MVF and the CVF and thus provides the only admissible solution. Nevertheless, this VaR restriction is not compatible with the restriction TEV $\leqslant T_0$.

(c) *Strong bound:* this situation is only available when $V_M < V_0 < V_K$, where V_K is the value of VaR at which the CVF is

[7] This Technical supplement is available at http://dx.doi.org/10.1016/j.jbankfin.2012.05.014

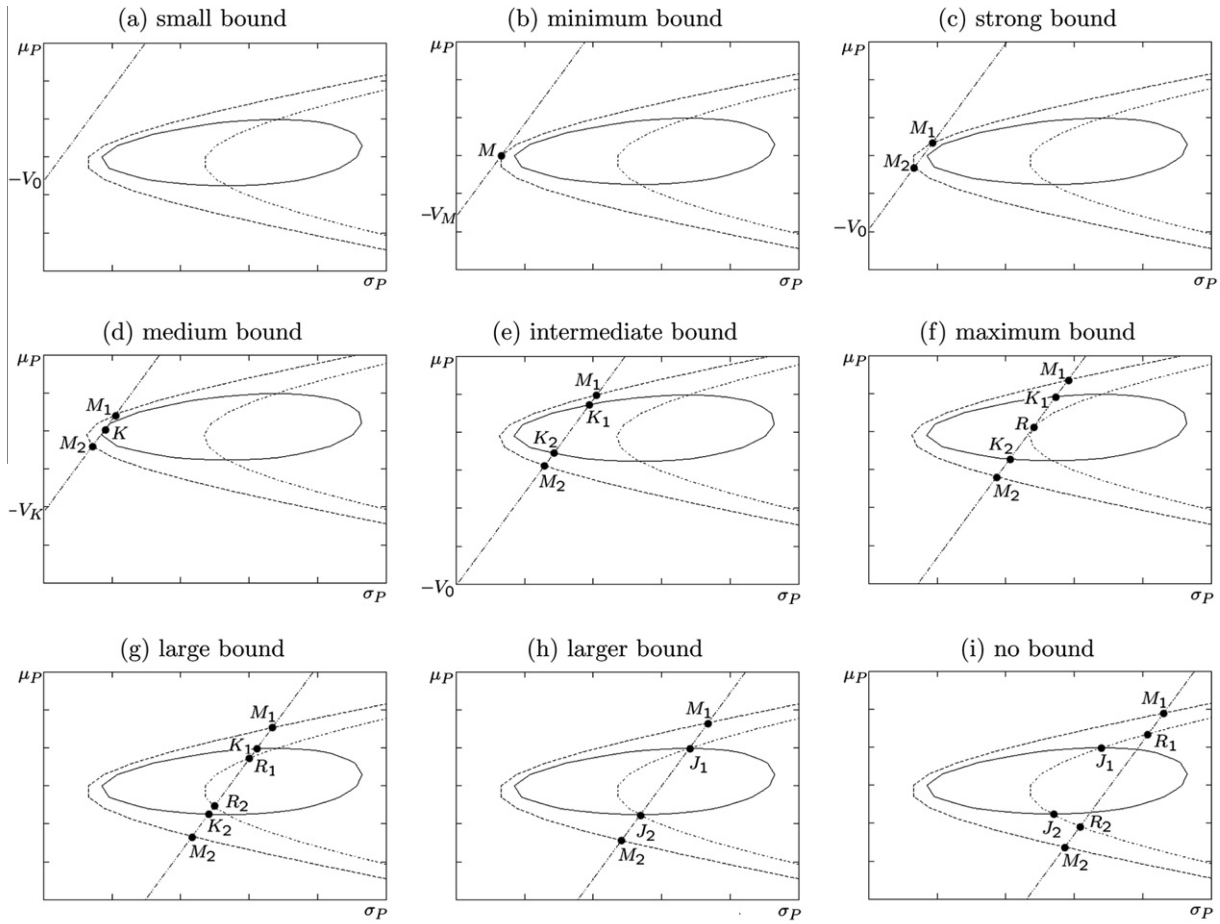

Fig. 3. $\Delta_1 > 0$, high confidence level, $T_0 < T_H$.

tangent to the CTF. When this restriction holds, the CVF only intersects the MVF in portfolios $M_1 \equiv \left(\sigma^2_{M_1}, \mu_{M_1}\right)$ and $M_2 \equiv \left(\sigma^2_{M_2}, \mu_{M_2}\right)$. Section A.1 provides the analytical expressions for the expected returns. According to Alexander and Baptista (2008), the admissible solution is the closed and bounded region between arc $\widehat{M_1M_2}$ and segment $\overline{M_1M_2}$; segment $\overline{M_1M_2}$ represents the CMTF. Nevertheless, the restriction on TEV can not be satisfied in this portion of the (σ_P, μ_P) space.

(d) *Medium bound:* in this case, $V_0 = V_K$, thus the CVF is tangent to the CTF in portfolio $K(\sigma^2_K, \mu_K)$ which allows asset managers to attain TEV = T_0. As in the previous case, all the portfolios lying to the left of the CVF satisfy the VaR constraint. Here portfolio $K \in \overline{M_1M_2}$ defines a new frontier: this is the "Fixed VaR-TEV Frontier" (FVTF), which includes all admissible portfolios which satisfy the VaR constraint and guarantee a TEV that does not exceed an *ex ante* fixed value T_0.

(e) *Intermediate bound:* when $V_K < V_0 < V_R$, the CVF crosses the MVF (portfolios M_1 and M_2) and the CTF (portfolios K_1 and K_2). The constraint on VaR is thus less stringent than in the previous case and the segment $\overline{M_1M_2}$ corresponds to the CMTF, as Alexander and Baptista (2008) observe. The value

$$V_R = -\mu_C + \sqrt{\left(\sigma^2_B - \frac{\Delta^2_1}{d}\right)(z^2_\theta - d)}, \tag{17}$$

where $V_R > V_M$, represents the VaR constraint at which the CVF is tangent to MTF (see Section A.2). Given the slope z_θ, this tangency occurs in portfolio

$$R \equiv \left(\sigma^2_C + \frac{d\sigma^2_B - \Delta^2_1}{z^2_\theta - d}, \mu_C + \sqrt{d\frac{d\sigma^2_B - \Delta^2_1}{z^2_\theta - d}}\right), \tag{18}$$

which is independent of the VaR constraint V_0. In such a situation, asset managers can satisfy both the VaR and the TEV restrictions within the closed and bounded FVTF, the region inside the CTF lying to the left of the CMTF. In Fig. 3 (e), the FVTF corresponds to the left arc $\widehat{K_1K_2}$ and the segment $\overline{K_1K_2}$, where $\mu_{K_1} > \mu_{K_2}$. For each $\mu_{K_2} < \mu_P < \mu_{K_1}$, asset managers have to make a choice: they can reduce the TEV below T_0 by augmenting the overall risk or they can maintain TEV = T_0 and consequently reduce the efficiency loss.

(f) *Maximum bound:* when $V_0 = V_R$, the FVTF is identical to the previous case with the exception of portfolio R in which the TEV is optimal by definition. Hence, the VaR restriction binds all along the segment $\overline{M_1M_2}$, while in segment $\overline{K_1K_2}$ asset managers maintain TEV $\leqslant T_0$. On the other hand, all along the arc $\widehat{K_1K_2}$, managers can reduce the overall portfolio risk by maintaining a fixed TEV.

(g) *Large bound:* in this situation $V_R < V_0 < \widehat{V}$ with $\widehat{V} = \max\{V_1, V_2\}$; thus, the CVF crosses the MVF in portfolios M_1 and M_2, the CTF in portfolios K_1 and K_2 and the MTF in portfolios R_1 and R_2. The arc $\widehat{K_1K_2}$ to the left of the CMTF ($\overline{K_1R_1}, \widehat{R_1R_2}$ and $\overline{R_2K_2}$) corresponds to the FVTF. Furthermore, the portfolios lying within the segment $\overline{R_1R_2}$ do not belong to the FVTF since they are dominated by those in $\widehat{R_1R_2}$: on the one hand, the TEV in $\overline{R_1R_2}$ is not minimised and, on the other hand, it is possible to obtain the same TEV for portfolios in $\overline{R_1R_2}$ to the left of the MTF (arc $\widehat{R_1R_2}$), thus reducing the efficiency loss.

(h) *Larger bound:* this bound occurs when $V_0 = \widehat{V}$. In this case, the FVTF corresponds to the two arcs formed by portfolios J_1 and J_2 on the frontiers CTF and MTF. Portfolios lying within the CTF to the right of the arc $\widehat{J_1J_2}$ on the hyperbola MTF are all dominated portfolios: this is the reason why the restriction on $V_0 = \widehat{V}$ is the largest that risk managers can reasonably impose.

Table A.3 shows that $V_1 < V_2$ when $\sqrt{d} < z_\theta < z_\theta^*$ and $V_2 < V_1$ when $z_\theta > z_\theta^*$, where the reference slope z_θ^* guarantees that the CVF passes through both J_1 and J_2; formally, this value depends on T_0 hence

$$z_\theta^* = \frac{d}{2\Delta_1}(\sigma_1 + \sigma_2). \tag{19}$$

For simplicity, the plot in Fig. 3 (h) illustrates the condition $z_\theta = z_\theta^*$ for which the following theorem applies.

Theorem 1. *In the (σ_P, μ_P) space, when $\Delta_1 > 0$, the straight line passing through portfolios J_1 and J_2 has a slope that is steeper than the asymptotic slope of the MVF.*

Proof. We demonstrate this theorem from two viewpoints; from the geometric perspective, the straight line passes through portfolios $J_1(\sigma_1, \mu_1)$ and $J_2(\sigma_2, \mu_2)$, with $\sigma_1 > \sigma_2$ and $\mu_1 > \mu_2$, both belonging to the MTF. Recalling that both the MVF and the MTF have the same asymptotic slope $\sqrt{d}$, the secant line passing through segment $\overline{J_1J_2}$ has a larger slope than the asymptotic slope of the frontier.

The analytical proof consists of a comparison between the equation of the asymptotic slope of the MVF (or MTF) and the slope of the line passing through portfolios J_1 and J_2, hence $\sqrt{d} < z_\theta^*$. Using Eq. (19) one can obtain

$$\mu_B < \mu_C + \sqrt{d}\left(\frac{\sigma_1 + \sigma_2}{2}\right), \tag{20}$$

where $\mu_C + \sqrt{d}\bar{\sigma}$ represents the value of the asymptote of MVF calculated in $\bar{\sigma} = (\sigma_1 + \sigma_2)/2$; for the convexity of the hyperbola MTF $\bar{\sigma} > \sigma_B$, so

$$\mu_B < \mu_C + \sqrt{d}\sigma_B < \mu_C + \sqrt{d}\bar{\sigma}.$$

This demonstrates Eq. (20). □

(i) *No bound:* when $V_0 > \widehat{V}$, the VaR constraint does not affect the FVTF which remains the same already defined in the larger bound case.

4.2. Financial implications

The principal novelty of our approach is the concept of tolerable constraints: contrary to previous literature, our proposed framework is not strictly based on any minimisation process because it combines the features of different portfolio frontiers. Hence, the resulting FVTF is not a smooth function in the (σ_P, μ_P) space, but it is a closed and bounded set inside which two indicators of risk do not exceed a pre-set maximum value. In the context of a restricted asset allocation activity this property produces three main financial implications.

First, the FVTF models the possibility of managing portfolio risk by constraining both the VaR and the TEV at the same time. Here the compatibility of the maximum TEV and VaR is crucial: if the CVF crosses that portion of the CTF lying between the arcs $\widehat{J_1J_2}$, the FVTF is a subset of the (σ_P, μ_P) space where both restrictions are satisfied. In general, along this frontier there are portfolios with VaR = V_0 and TEV < T_0 or portfolios with TEV = T_0 and VaR < V_0 (left arc $\widehat{K_1K_2}$ of the CTF). The equalities TEV = T_0 and VaR = V_0 hold only for special portfolios given by the contacts K_1, K_2, J_1, J_2 or by the tangency portfolio K in the medium bound case (see Fig. 3). When asset managers face intermediate, maximum or large bounds, they often have to make a choice between TEV or VaR, therefore a trade off emerges. When bounds are larger or non-existent altogether, the VaR constraint is always satisfied by definition because the FVTF is entirely formed by the CTF and the MTF. In this case, it is sufficient to set only a restriction on the TEV.

Second, our approach makes it possible to analyse situations in which stringent VaR or TEV constraints are incompatible to one another. In this situation, the CVF lies to the left of the CTF, the FVTF is an empty set and asset managers can only satisfy one constraint. In particular:

- when the bound is small, managers can only invest in portfolios close to the benchmark because all the portfolios lying to the left of the CVF line are outside the admissible region provided by the MVF;
- when the bound is minimum or strong, managers have to make a choice between maximum VaR and maximum TEV. On the one hand, they could satisfy the VaR constraint only via an active strategy which selects a position far from the benchmark and out of the CTF. On the other hand, keeping a small TEV value implies the impossibility of keeping the overall portfolio risk under a certain level because the benchmark itself is rather risky.

When the above situation arises, clearly it is impossible to pursue the VaR and TEV objectives at the same time. This, in turn, poses a fundamental problem to the risk manager, because either the VaR requirement is unrealistically conservative or the benchmark is an uncaracteristically volatile asset. In both cases the risk management should revise its strategies.

Third, since the TEV minimisation implies a horizontal distance δ_B between the MTF and the MVF (see Roll, 1992), the setting of a tolerable risk relative to a benchmark could lead managers to a substantial efficiency loss reduction. This is the case of active management strategies in which it is possible to invest also in those portfolios lying far from the benchmark. In this situation a high performance portfolio manager can lower the VaR and the overall portfolio risk thus obtaining a substantial reduction of the efficiency loss.

4.3. Extreme benchmarks

When the confidence level is high, our analysis shows that portfolio R provides the position at which asset managers can minimise TEV using the most stringent VaR constraint possible; Eq. (18) provides its analytical coordinates (σ_R^2, μ_R). Moreover, for each $z_\theta > \sqrt{d}$, portfolio R represents the tangency portfolio between the MTF and the CVF which is independent of V_0 and managers are only able to minimise the TEV when $V_0 \geqslant V_R$.

However, asset managers can opt for a very stringent TEV and the low eccentricity of the ellipse in the (σ_P^2, μ_P) space can generate a scenario which is more complex than those presented in the previous sections. In particular, once T_R defines the TEV of the tangency portfolio R, such portfolio could be placed outside the CTF, thus $T_0 < T_R$ and $\mu_R \notin [\mu_1, \mu_2]$. In such a situation, the maximum bound shifts from V_R to V_1 when $\Delta_1 \leqslant 0$, or to V_2 when $\Delta_1 > 0$. In both cases, these bounds are the most stringent VaR constraints that allow managers to select a portfolio on the MTF lying inside CTF.

Hence, given $z_\theta > \sqrt{d}$, we consider the benchmark portfolio B for which $T_0 < T_R$, as extreme; heuristically, an extreme benchmark is a benchmark that lies far from the point R. Moreover, when $\Delta_1 > 0$, the so-called aggressive benchmarks could belong to this category.

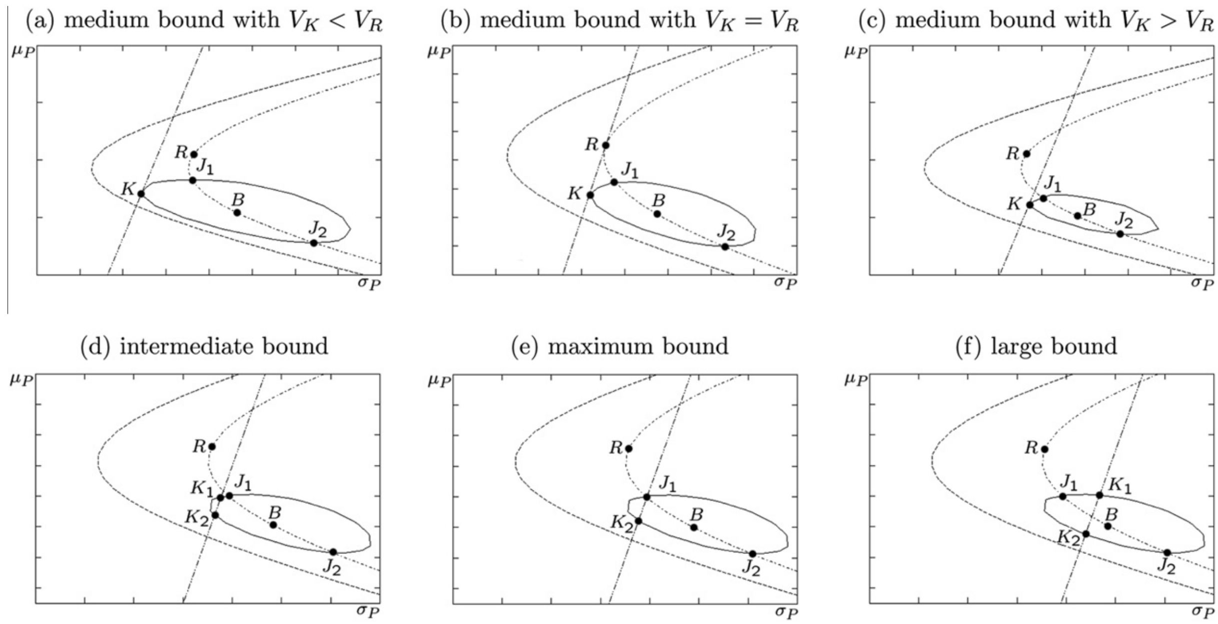

Fig. 4. Extreme benchmarks ($T_0 < T_R$).

Fig. 4 shows various situations in which portfolio R lies outside the CTF and $\Delta_1 < 0$ (when $\Delta_1 > 0$ the scenarios are the same). Specifically, using V_R as the reference VaR constraint, Fig. 4(a)–(b)–(c) shows the medium VaR bound in the presence of extreme benchmarks. In particular, Fig. 4(b)–(c) highlight that the relationship $V_K \geqslant V_R$ could hold, which implies that the straight line CVF in the medium bound scenario intersects the MTF. Fig. 4(d)–(e)–(f) illustrates the intermediate ($V_K < V_0 < V_1$), maximum ($V_0 = V_1$) and large ($V_1 < V_0 < V_2$) bounds respectively, when the benchmark is extreme.

When the confidence is low (see Technical supplement), a benchmark can not be extreme because asset managers are always able to minimise the TEV by simply setting a VaR constraint. More precisely, for each VaR bound, a straight line CVF crossing the MTF must exists, thus an infinite number of intersections is available; these contacts are not tangency portfolios and their coordinates strictly depend on V_0 (see Eq. (A.8)). As a consequence, it is not possible to select a unique reference portfolio R on the MTF.

5. An empirical example

It is important to note that our approach does not aim at solving an asset allocation problem; it only identifies points in the (σ_P^2, μ_P) space which are endowed with certain properties. While it is the risk manager's job to decide on TEV and VaR limits, it is the responsibility of the asset managers to find a set of portfolio weights which results in a portfolio with the desired characteristics. The merit of our approach is that it translates the TEV and VaR indications from the risk managers into a set of constraints that can be used by the asset managers to make a decision on the portfolio weights. In order to illustrate how our approach works in practice, now we provide a short empirical example.

The available data contain the quarterly returns (in percentages) of the 50 stocks composing the DJ Eurostoxx 50 index. The data set runs from the first quarter of 2003 to the fourth quarter of 2010 and the sample size is 32.[8] We use the Standard & Poor 500 Composite index as the benchmark portfolio (B). In line with Palomba (2008), we group all the stocks into 10 distinct asset classes, as Table A.1 shows.

We carry out the analysis by imposing an expected portfolio return $\mu_P = 5.00$ and setting the constraints $T_0 = 20.00$ and $V_0 = 15.00$. Table A.2 provides the results. We set a high confidence level of $\theta = 99\%$ and a low confidence level of $\theta = \Phi(\sqrt{d})$, while Δ_1 is positive ($\mu_B = 1.484$ and $\mu_C = 1.337$).[9] Our analysis provides the results for the following battery of portfolios:

- P is the portfolio lying on the MVF with an expected return of 5%,
- T is the portfolio lying on the MTF with an expected return of 5%,
- J is the portfolio lying on the CTF with an expected return of 5%,
- AB is the portfolio lying on the CMTF (see Alexander and Baptista, 2008) with an expected return of 5%,
- B is the benchmark,
- C is the minimum variance portfolio on the MVF,
- Q is the portfolio with the maximum Sharpe Ratio (Sharpe, 1994),
- J_1 and J_2 are the intersections between the MTF and the CTF,
- H is the portfolio on the MVF with the same return as the benchmark. It is also the contact portfolio between the MVF and the CTF when $\Psi = 0$,
- M and R are the tangency (intersection) portfolios between the CVF and the two hyperbolic frontiers when the confidence level is high (low),
- K represents the tangency portfolio between the CTF and the CVF,
- K_2 is the left intersection between the CVF and the CTF.

For each of these portfolios we evaluate the expected return, variance, risk (standard deviation), Sharpe Ratio, Alpha, TEV and Information Ratio (IR=Alpha/TEV, see for instance Lee, 2000). Moreover, we also calculate the efficiency loss and the intercept

[8] Thomson Datastream is the source of data. The prices for Alcatel and Crédit Agricole are unavailable prior to the last quarter of 2001, thus the dataset starts from the first quarter of 2002. We also restrict the sample to the fourth quarter of 2010 to avoid the negative effects of the recent crisis: in particular, the benchmark expected returns are negative when the 2011 data are considered.

[9] Table T-1 in the Technical supplement provides also an example where the DJ Eurostoxx 50 index is the benchmark portfolio and $\Delta_1 < 0$.

of the straight line CVF (the VaR restriction for a given θ). Obviously, portfolios P, T, J, B, C, Q, J_1, J_2 and H are independent of the confidence level θ. For the intersections M_1, M_2, K_1, K_2, R_1 and R_2, illustrated in Fig. 3, we provide only the coordinates within (σ_P, μ_P).

When the confidence level is high, the scenario is that of Fig. 3; portfolio J on the CTF does not appear because the return of portfolio J_1 is less than 5%. The relationship $V_M < V_K < V_R < \widehat{V}$, where $\widehat{V} = V_2$, indicates that risk managers could impose any of the different restrictions upon VaR: small ($V_0 < V_M$), minimum ($V_0 = V_M$), strong ($V_M < V_0 < V_K$), medium ($V_0 = V_K$), intermediate ($V_K < V_0 < V_R$), maximum ($V_0 = V_R$), large ($V_R < V_0 < V_2$), larger ($V_0 = V_2$) or no bound ($V_0 > V_2$). The restriction $V_0 = 15.00$ corresponds to an intermediate VaR bound for which the FVTF is given by the segment $\overline{K_1K_2}$ and the arc $\widehat{K_1K_2}$ on the CTF; the intersection portfolios are M_1, M_2, K_1 and K_2 (see Fig. 3(e)). The TEV in portfolio R is $T_R = 5.888$, thus the Standard & Poor index is not an extreme benchmark.

In the low confidence scenario, we set $V_0 = 5.00$ because the value of 15.00 is too feeble (it corresponds to the "no bound" situation) when the CVF has a small slope. Portfolios M and R are the intersections between the CVF and the hyperbolic frontiers (the MVF and the MTF respectively); K_2 represents the portfolio lying on the CVF with the maximum reduction of efficiency loss, while the VaR restrictions in J_1 and J_2 (V_1 and V_2 respectively) depend on the change in the confidence level. The coordinates of the tangency portfolio K differ from those calculated when the confidence level is high, while portfolio AB lies on the MTF ($\mu_P > \mu_M$), thus coinciding with portfolio T. However, segment $\overline{RK_2}$ and arcs $\widehat{J_1K_2}$ and $\widehat{RJ_1}$ form the FVTF.

6. Concluding remarks and further research

The key task which asset managers have to face is that of beating a benchmark. Considering also that risk management usually aim to keep risks under control, this paper attempts to formalise asset allocation strategies in the presence of constraints put upon tracking error volatility (TEV) and value at risk (VaR); all the results arise under the classical hypothesis of normally distributed expected returns which translates into an optimisation in the (σ_P^2, μ_P) space.

Since traditional portfolio optimisation based upon a relative risk measure is generally overly risky (see Roll, 1992), Alexander and Baptista (2008) try to reduce the portfolio's efficiency loss by considering the "Constrained Mean-TEV Frontier" (CMTF) which contains portfolios that satisfy a VaR constraint and minimise the TEV. Unfortunately, this frontier does not take two economic problems into consideration. First, the VaR constraint is independent of the benchmark portfolio, hence it is not related to the maximum TEV constraint. On the other hand, Jorion (2003) highlights that asset managers can choose within a closed and bounded set of feasible portfolios ("Constrained TEV Frontier", CTF) that lie around the benchmark. The resulting asset allocation strategies thus suffer from the fact that those restrictions reduce substantially the available subset of the (σ_P, μ_P) space. In such a situation, asset managers could not be able to satisfy simultaneously restrictions on VaR and TEV. Second, portfolios lying on the CMTF usually have a higher efficiency loss when compared to those lying to the left-hand side of the CTF; this depend on the definition of the CMTF (see Alexander and Baptista, 2008) which is focussed on finding the smallest TEV.

This paper introduces the concept of tolerable constraints and shows that the imposition of a maximum TEV or VaR allows managers to move away from the CMTF and select less risky portfolios, thereby reducing efficiency loss. As shown in the paper, it is possible to summarise the above two problems as follows: if maximum TEV and VaR limits are not compatible, there are no feasible portfolios and, at most, only one of the two constraints can be satisfied. Otherwise, there are portfolio allocations for which the TEV and VaR restrictions hold at the same time; in such a situation, or in the absence of a constraint on TEV, portfolios on the CMTF are generally inefficient.

The whole analysis presents various scenarios exemplifying all the possible interactions between different portfolio frontiers and provides analytical solutions for all the intersections; moreover, when TEV and VaR restrictions are not too stringent, we introduce a new portfolio boundary, the "Fixed VaR-TEV Frontier" (FVTF). When this frontier operates, an interesting trade-off between relative and absolute risk arises, consistent with Roll (1992). In other words, for any given expected return within the FVTF, managers can choose to reduce the relative risk (TEV) by augmenting the absolute risk (overall portfolio variance) or increase the relative risk by decreasing the absolute risk. This can also create a principal-agent problem between the fund investor and the asset managers because the former is typically interested in the reduction of the portfolio efficiency loss, while the latter has an incentive to maintain the TEV under a fixed threshold of tolerability.

To conclude, generalising the results to non normally distributed returns and disallowing short sales would surely represent the natural extensions of our analysis: the possibility of allowing for skewed or leptokurtotic return distributions would enable us to consider additional risk factors into the optimal portfolio selection problem. As for the impossibility of short sales, a more realistic constraint on portfolio weights arises, which is consistent with several fund policies or contracts between managers and investors; moreover, fixing a percentage for the share of certain types of assets could comply with some regulatory restrictions (see Bajeux-Besnainou et al., 2011).

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Appendix A

A.1. Intersection between the MVF and the CVF

Alexander and Baptista (2008) discuss the relationships between the MVF and the CVF, but they omit the analytic solutions for the contact portfolios M, M_1 and M_2 (see Fig. 3). This section proves the existence of such analytical solutions. The condition under which the frontiers intersect in (σ_P^2, μ_P) space is the equality between Eqs. (1) and (8)

$$\left(\frac{\mu_P + V_0}{z_\theta}\right)^2 = \sigma_C^2 + \frac{1}{d}(\mu_P - \mu_C)^2. \tag{A.1}$$

After some algebra, the resolvent becomes the quadratic expression

$$(z_\theta^2 - d)\mu_P^2 - 2(z_\theta^2\mu_C + dV_0)\mu_P + cz_\theta^2\sigma_C^2 - dV_0^2 = 0, \tag{A.2}$$

where $c = \mu'\Omega^{-1}\mu$ (see page 2); the solutions for portfolios M_1 and M_2 are

$$\mu_P = \frac{(z_\theta^2\mu_C + dV_0) \pm \sqrt{D_0}}{(z_\theta^2 - d)}. \tag{A.3}$$

If $z_\theta^2 - d > 0$ the discriminant

$$D_0 = \left(z_\theta^2\mu_C + dV_0\right)^2 - \left(z_\theta^2 - d\right)\left(cz_\theta^2\sigma_C^2 - dV_0^2\right)$$
$$= V_0^2 + 2\mu_C V_0 + \sigma_C^2\left(c - z_\theta^2\right) \tag{A.4}$$

is strictly positive when

$$V_0 \geqslant -\mu_C + \sqrt{\sigma_C^2(z_\theta^2 - d)}, \tag{A.5}$$

which represents the VaR restriction according to which the CVF intersects the efficient set.[10] However, $D_0 > 0$ holds when the confidence level is low ($z_\theta^2 - d < 0$), hence a contact point always exists.

The tangency condition $D_0 = 0$ demonstrates Eq. (15) for the portfolio M. Moreover, the result in Eq. (16) derives from the substitution of $D_0 = 0$ in Eq. (A.3).

When the tangency between the MVF and the CVF holds, some mathematical aspects have to be taken into consideration:

- if $z_\theta^2 < d$ (low confidence level), the radical is negative hence the solution (15) is not real and the tangency between the frontiers is not possible (see Figure T-2 of the Technical supplement);
- if $z_\theta^2 = d$ (low confidence level), the solution is the small bound $V_0 = -\mu_C$ (see Fig. 3): in this case, the straight line CVF is the asymptote of the MVF, hence the tangency can not exist for finite values of σ_P or μ_P. This situation corresponds to Proposition 1. (ii) in Alexander and Baptista (2008);
- if $z_\theta^2 > d$ (high confidence level), the tangency condition always holds when VaR=V_M in Eq. (15).

A.2. Intersection between the MTF and the CVF

The process of finding the intersection between the MTF and the CVF is similar to that of the previous case: the frontiers show common portfolios in (σ_P^2, μ_P) space when

$$\left(\frac{\mu_P + V_0}{z_\theta}\right)^2 = \sigma_B^2 + \frac{1}{d}(\mu_P - \mu_B)^2 + 2\frac{\Delta_1}{d}(\mu_P - \mu_B). \tag{A.6}$$

The resolvent is the following second degree equation

$$(z_\theta^2 - d)\mu_P^2 - 2(z_\theta^2\mu_C + dV_0)\mu_P + z_\theta^2(d\sigma_B^2 - \mu_B^2 + 2\mu_C\mu_B) - dV_0^2 = 0, \tag{A.7}$$

and the solutions (for portfolios R_1 and R_2) are

$$\mu_P = \frac{(z_\theta^2\mu_C + dV_0) \pm \sqrt{D_1}}{(z_\theta^2 - d)}, \tag{A.8}$$

where the discriminant is

$$D_1 = \left(z_\theta^2\mu_C + dV_0\right)^2 - \left(z_\theta^2 - d\right)\left[z_\theta^2\left(d\sigma_B^2 - \mu_B^2 + 2\mu_B\mu_C\right) - dV_0\right]$$
$$= V_0^2 + 2\mu_C V_0 - z_\theta^2\left(\sigma_B^2 - \frac{\Delta_1^2}{d}\right) + d\sigma_B^2 - \Delta_1^2 + \mu_C^2. \tag{A.9}$$

Given that the objective of the MTF is to optimise TEV, Eq. (A.9) takes a benchmark portfolio into consideration. The tangency condition $D_1 = 0$ is satisfied when the value of the constrained VaR is

$$V_R = -\mu_C + \sqrt{\left(\sigma_B^2 - \frac{\Delta_1^2}{d}\right)(z_\theta^2 - d)}. \tag{A.10}$$

As in the previous case, the analytical solution for this constraint depends on the data and the confidence level θ. The relationship $V_R > V_M$ indicates that this constraint is less stringent. In short:

$$V_R > V_M \Rightarrow z_\theta^2\left(\sigma_B^2 - \frac{\Delta_1^2}{d}\right) > \sigma_C^2(z_\theta^2 - d) \;\Rightarrow z_\theta^2 > -d\frac{\sigma_C^2}{\delta_B}.$$

This is true because $d > 0, \sigma_C^2 > 0$ and $\delta_B > 0$ by construction. Furthermore, the analysis conducted for $(z_\theta^2 - d) \lesseqgtr 0$, for the intersection MVF-CVF, is substantially confirmed. Finally, substituting $V_0 = V_R$ in Eq. (A.7), the resulting portfolio is R, as already defined in Eq. (18).

A.3. Intersection between the CTF and the CVF: the system

Starting from Eqs. (4) and (8), the intersections between the CTF and the CVF correspond to the solutions of the system

$$\begin{cases} x = \left(\dfrac{\mu_P + V_0}{z_\theta}\right)^2 - A_0 \\ dx^2 + \phi_1(\mu_P - \mu_B)^2 + \phi_2 x(\mu_P - \mu_B) + \phi_3 = 0 \end{cases} \tag{A.11}$$

where $A_0 = \sigma_B^2 + T_0, x = \sigma_P^2 - A_0, \phi_1 = 4\Delta_2, \phi_2 = -4\Delta_1$ and $\phi_3 = -4d\delta_B T_0$. The resolvent is the quartic function

$$d\left[\left(\frac{\mu_P + V_0}{z_\theta}\right)^2 - A_0\right]^2 + \phi_1(\mu_P - \mu_B)^2$$
$$+ \phi_2\left[\left(\frac{\mu_P + V_0}{z_\theta}\right)^2 - A_0\right](\mu_P - \mu_B) + \phi_3 = 0. \tag{A.12}$$

The algebra for obtaining Eq. (9) is:

$$d\left[\frac{\mu_P^4 + 4V_0\mu_P^3 + 6V_0^2\mu_P^2 + 4V_0^3\mu_P + V_0^4}{z_\theta^4} - 2A_0\frac{\mu_P^2 + 2V_0\mu_P + V_0^2}{z_\theta^2} + A_0^2\right]$$
$$+ \phi_1\mu_P^2 - 2\phi_1\mu_B\mu_P + \phi_1\mu_B^2 + \phi_2\frac{\left(\mu_P^2 + 2V_0\mu_P + V_0^2\right)(\mu_P - \mu_B)}{z_\theta^2}$$
$$- \phi_2 A_0(\mu_P - \mu_B) + \phi_3 = 0$$

then

$$\frac{d}{z_\theta^4}\left[\mu_P^4 + 4V_0\mu_P^3 + 2(3V_0^2 - z_\theta^2 A_0)\mu_P^2 + 4V_0\left(V_0^2 - z_\theta^2 A_0\right)\mu_P + \left(V_0^2 - z_\theta^2 A_0\right)^2\right]$$
$$+ \phi_1\mu_P^2 - 2\phi_1\mu_B\mu_P + \phi_1\mu_B^2$$
$$+ \frac{\phi_2}{z_\theta^2}\left(\mu_P^3 + 2V_0\mu_P^2 + V_0^2\mu_P - \mu_B\mu_P^2 - 2V_0\mu_B\mu_P - V_0^2\mu_B\right). + \phi_2 A_0\mu_P$$
$$+ \phi_2 A_0\mu_B + \phi_3 = 0.$$

Thus, the coefficients are:

$$\begin{cases} c_0 = \dfrac{d}{z_\theta^4}\left(V_0^2 - z_\theta^2 A_0\right)^2 + \phi_1\mu_B^2 - \phi_2\mu_B\left(\dfrac{V_0^2}{z_\theta^2} - A_0\right) + \phi_3 \\ c_1 = 4\dfrac{d}{z_\theta^4}V_0^3 - 4\dfrac{d}{z_\theta^2}A_0V_0 - 2\phi_1\mu_B - \phi_2 A_0 + \dfrac{\phi_2}{z_\theta^2}V_0(V_0 - 2\mu_B) \\ c_2 = 6\dfrac{d}{z_\theta^4}V_0^2 - 2\dfrac{d}{z_\theta^2}A_0 + \phi_1 + 2\dfrac{\phi_2}{z_\theta^2}V_0 - \dfrac{\phi_2}{z_\theta^2}\mu_B \\ c_3 = 4\dfrac{d}{z_\theta^4}V_0 + \dfrac{\phi_2}{z_\theta^2} \\ c_4 = \dfrac{d}{z_\theta^4}. \end{cases}$$

System (A.11) should return two distinct real solutions for the expected return μ_P when the parabola CVF crosses the ellipse in (σ_P^2, μ_P) space, a double root when the curves are tangent and no solutions when the frontiers do not have any portfolios in common. This implies that, when the intersections occur, the polynomial of the fourth degree (9) always has two complex conjugate roots; this result is very difficult to handle, so we determine the contacts points between the frontiers via the methods proposed in Section 3.

[10] We omit all the algebra for brevity. It is clear that if Eq. (A.4) admits two real solutions, only the relationship (A.5) has to be considered; in practice, the other condition $V_0 \leqslant -\mu_C + \sqrt{\sigma_C^2(z_\theta^2 - d)}$ corresponds to a small bound (see for instance, Fig. 3).

Summary

see TableA.3.

Appendix B. Supplementary material

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.jbankfin.2012.05.014.

Table A.1
Asset classes.

Asset class	Expected return (%)	Std. dev. (%)	Correlations									
			Auto	Bank	Chem.	Cons.	Ener.	Ind.	Ins.	Tel.	Util.	Oth.
Automobiles	4.068	15.620	1.000	–	–	–	–	–	–	–	–	–
Banks	0.953	16.532	0.661	1.000	-	–	–	–	–	–	–	–
Chemicals	2.993	9.858	0.606	0.708	1.000	–	–	–	–	–	–	–
Constructions	3.353	12.449	0.564	0.725	0.729	1.000	–	–	–	–	–	–
Energy	1.412	8.822	0.592	0.715	0.768	0.727	1.000	–	–	–	–	–
Industrial	3.674	9.682	0.696	0.784	0.805	0.749	0.750	1.000	–	–	–	–
Insurance	1.119	16.970	0.698	0.845	0.819	0.768	0.748	0.777	1.000	–	–	–
Telecommunications	0.597	10.199	0.440	0.661	0.620	0.722	0.727	0.650	0.672	1.000	–	–
Utilities	2.788	10.873	0.512	0.698	0.731	0.749	0.843	0.707	0.729	0.761	1.000	–
Other	0.630	13.171	0.658	0.855	0.661	0.629	0.558	0.690	0.744	0.539	0.545	1.000
DJ Eurostoxx 50	0.985	10.004	0.726	0.887	0.893	0.838	0.882	0.887	0.918	0.788	0.864	0.779
Standard & Poor 500	1.484	8.510	0.687	0.829	0.791	0.774	0.799	0.866	0.826	0.761	0.775	0.686

Notes: The expected return of asset class j is calculated via the formula $R_j = \frac{1}{n_j}\sum_{i=1}^{nj} R_i$, where n_j is the number of stocks included in class j.
ASSET CLASSES
Automobiles: Daimler, Renault, Volkswagen.
Banks: Banco Santander, BBV Argentaria, BNP. Paribas, Crédit Agricole, Deutsche Bank, Fortis, Intesa Sanpaolo, Société Générale, Unicredit.
Chemicals: Air Liquide, Basf, Bayer, Sanofi Aventis.
Constructions: Saint Gobain, Vinci.
Energy: Enel, ENI, Repsol, GDF Suez, Total.
Industrial: Arcelor Mittal, Danone, L'Oréal, LVMH, Philips, SAP, Schneider Electric, Siemens, Unilever.
Insurance: Aegon, Allianz, AXA, Generali, Ing Groep, Münchener Ruck.
Telecommunications: Alcatel, Deutsche Telekom, France Telecom, Nokia, Telecom Italia, Telefonica, Vivendi.
Utilities: E.On, Iberdrola, RWE.
Other: Carrefour (Retail), Deutsche Börse (Financial services).

Table A.2
Empirical results (benchmark portfolio: standard and poor 500 composite index).

Portfolio expected return: 5.000
TEV constraint (T_0): 20.000
Δ_1: 0.147, Δ_2: 37.176
Tangency TEV (T_H): 37.130, Ψ: -8.141
The benchmark is not extreme ($T_0 > T_R$, $V_K < V_R$)

Intermediate bound
VaR constraint (V_0): 15.000
High confidence level, θ: 99%, z_θ: 2.326, Threshold ($\sqrt{d}$): 0.689, $\widehat{V} = V_2$
Intersections in (σ_P, μ_P) space: $M_1 \equiv (7.086,4.004)$ and $M_2 \equiv (5.937,1.330)$
Intersections in (σ_P, μ_P) space: $K_1 \equiv (8.250,4.192)$ and $K_2 \equiv (6.514,0.154)$
Efficiency loss – δ_{K_1}: 15.657, δ_{K_2}: 4.239
z_θ^H: 19.204, $\theta^* \approx 1$, $z_\theta^* \approx +\infty$, V^*: 297.248

Portfolios	P	T	J	AB	B	C	Q	J_1	J_2	H	M	K	R
Exp. return	5.000	5.000	–	5.000	1.484	1.337	13.870	4.567	−1.599	1.483	2.606	2.295	3.156
Variance	63.485	100.620	–	73.911	72.423	35.247	365.760	94.330	90.515	35.293	38.641	40.345	79.345
Risk	7.968	10.031	–	8.597	8.510	5.937	19.125	9.712	9.514	5.941	6.216	6.352	8.908
Sharpe Ratio	0.628	0.498	–	0.582	0.174	0.225	0.725	0.470	−0.168	0.250	0.419	0.361	0.354
Alpha	3.516	3.516	–	3.516	–	−0.147	12.386	3.083	−3.083	–	1.123	0.811	1.673
TEV	63.147	26.017	–	112.920	–	37.175	359.940	20.000	20.000	37.130	39.783	20.000	5.888
Information ratio	0.056	0.135	–	0.031	–	−0.004	0.034	0.154	−0.154	–	0.028	0.041	0.284
Efficiency loss	–	37.130	–	10.426	37.130	–	–	37.130	37.130	–	–	3.165	37.130
VaR	13.536	18.335	–	15.000	18.314	12.475	30.621	18.028	23.732	12.337	11.854	12.481	17.566

Large bound
VaR constraint (V_0): 5.00, with $V_0 > \mu_C$
Low confidence level, θ: 0.755 (see Technical supplement)
Threshold $z_\theta = \sqrt{d}$: 0.689
V_1: 2.129, V_2: 8.1583
$\mu_P > \mu_R \Rightarrow AB = T$

Portfolios:	AB	M	K_2	K	R
Exp. return	5.000	−0.510	−0.288	3.875	0.883
Variance	100.620	42.421	46.720	59.503	72.811
Risk	10.031	6.513	6.835	7.714	8.533
Sharpe Ratio	0.498	−0.078	−0.042	0.502	0.103
Alpha	3.516	−1.993	−1.771	2.391	−0.601
TEV	26.017	45.492	20.000	20.000	0.760
Information ratio	0.135	−0.043	−0.089	0.120	−0.791
Efficiency loss	37.130	–	5.920	10.698	37.130
VaR	10.742	5.000	5.000	1.443	5.000

Table A.3
Hierarchy of VaR constraints ($T_0 \leqslant T_H$).

High confidence level			
$\Delta_1 > 0$		$\Delta_1 \leqslant 0$ (see Technical supplement)	
$\sqrt{d} < z_\theta < z_\theta^H$	$V_M < V_K < V_H < V_R < V_1 < V_2$	$z_\theta > \sqrt{d}$	$V_M < V_K < V_H < V_R < V_1 < V_2$
$z_\theta = z_\theta^H$	$V_M \leqslant V_K \leqslant V_H < V_R < V_1 < V_2$		
$z_\theta^H < z_\theta < z_\theta^*$	$V_M < V_K < V_H < V_R < V_1 < V_2$		
$z_\theta = z_\theta^*$	$V_M < V_K < V_H < V_R < V_1 = V_2$		
$z_\theta > z_\theta^*$	$V_M < V_K < V_H < V_R < V_2 < V_1$		
Extreme benchmarks			
$z_\theta > \sqrt{d}$, $T_0 < T_R$	$V_M < V_H < V_R < \widehat{V}$, where $V_K \lesseqgtr V_R$		
Low confidence level (see Technical supplement)			
	Small bound	Minimum bound	Strong bound
$0 < z_\theta < \sqrt{d}$	-	-	$V_M = V_R < V_K < V_1 < V_2$
$z_\theta = \sqrt{d}$[a]	No solutions ($V_M < -\mu_C$)	No solutions ($V_M = -\mu_C$)	$-\mu_C < V_M < V_K < V_1 < V_2$
	Medium bound	Intermediate bound	Maximum bound
$0 < z_\theta < \sqrt{d}$	$V_M = V_R = V_K < V_1 < V_2$	$V_K < V_M = V_R < V_1 < V_2$	$V_K < V_M = V_R = V_1 < V_2$
$z_\theta = \sqrt{d}$[a]	$-\mu_C < V_M = V_K < V_1 < V_2$	$-\mu_C < V_K < V_M < V_1 < V_2$	$-\mu_C < V_K < V_M = V_1 < V_2$
	Large bound	Larger bound	No bound
$0 < z_\theta < \sqrt{d}$	$V_k < V_1 < V_M = V_R < V_2$	$V_K < V_1 < V_M = V_R = V_2$	$V_K < V_1 < V_2 < V_M = V_R$
$z_\theta = \sqrt{d}$[a]	$-\mu_C < V_K < V_1 < V_M < V_2$	$-\mu_C < V_K < V_1 < V_M = V_2$	$-\mu_C < V_K < V_1 < V_2 < V_M$

[a] When $V_M > \gamma_B = \mu_C - \sqrt{d\delta_B}$, where γ_B is the intercept of the MTF asymptote, it follows that $V_M = V_R$.

These are a Technical Supplement, all the data and the routines for the analysis carried out in this paper.

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