Optimal Portfolio Execution Strategies and Sensitivity to Price Impact Parameters *

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Abstract

When liquidating a portfolio of large blocks of risky assets, an institutional investor wants to minimize the cost as well as the risk of execution. An optimal execution strategy minimizes a weighted combination of the expected value and the variance of the execution cost, where the weight is given by a nonnegative risk aversion parameter. The execution cost is determined from price impact functions. In particular, a linear price impact model is defined by the temporary impact matrix H and the permanent impact matrix Γ, which represent the expected price depression caused by trading assets at a unit rate. In this paper, we analyze the sensitivity of the optimal execution strategy to estimation errors in the impact matrices under a linear price impact model. We show that, instead of depending on H and Γ individually, the optimal execution strategy is determined by the combined impact matrix $\Theta = \frac{1}{\tau}\left(H + H^T\right) - \Gamma$, where τ is the time length between consecutive trades. We prove that the minimum expected execution cost strategy is the naive execution strategy, independent of perturbations, when the permanent impact matrix Γ is symmetric and the combined impact matrix Θ is positive definite. We provide upper bounds on the size of change in the optimal execution strategy in a general setting. These upper bounds are in terms of the changes in the impact matrices, the eigenvalues of a block tridiagonal matrix defined by Θ, the risk aversion parameter, and the covariance matrix. These upper bounds indicate that, when the covariance matrix is positive definite, a large risk aversion parameter reduces the sensitivity of the optimal execution strategy. Moreover, when the permanent impact matrix Γ and its perturbation are symmetric, the optimal execution strategy is asymptotically not sensitive to estimation errors when *either* the minimum eigenvalue of the covariance matrix *or* the minimum eigenvalue of Θ is large. In addition, our computational results confirm that the sensitivity of the optimal execution strategy to the perturbations decreases, when Γ and the perturbed permanent impact matrix are symmetric. Moreover, the change in the efficient frontier increases as the risk aversion parameter decreases for asymmetric perturbations. We consistently observe that imposing short selling constraints can reduce the sensitivity of the optimal execution strategy and the efficient frontier to the perturbations.

Keywords: price impact, execution cost problem, estimation error, sensitivity analysis

1 Introduction

Portfolio management of large trades is one of the most important problems in modern market microstructure theory (O'Hara (1998)). When an investor trades in large volumes, in addition to explicit costs such as brokerage, he faces some implicit execution costs. These costs mainly consist of liquidity costs and information effects transmitted by the size of the investor's own trade. Liquidity costs include the additional prices

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an investor pays for immediate execution of the trade (Focardi and Fabozzi (2004)); this is often called the *temporary price impact* and only affects the execution price at the moment of trading. Furthermore, the imbalance between supply and demand, due to the investor's trade, usually transmits some information to the market which can move the future market price. For example, selling a large block of some asset may suggest that the seller believes that the asset is overvalued. The effect of this information is often referred to as the *permanent price impact*. The total price impact is the accumulation of the temporary and permanent price impacts.

The distinction between temporary and permanent price impacts associated with large trades and their characteristics have been addressed broadly in the literature, e.g., see Huberman and Stanzl (2004) and the references herein. A common result in all of these studies is that the magnitude of the price impact is a function of the trading volume. This function is usually called the *price impact function* and has been widely used in the microstructure literature since the work of Kyle (1985). The expected price impact function is typically estimated through a linear or nonlinear regression based on the available historical data, e.g., see Huberman and Stanzl (2004), Almgren et al. (2005).

Due to the dependence of the price impact on the trading volume, portfolio managers usually split a large trade into some smaller partial orders, i.e., *packages*. They then submit these partial orders over several periods during a finite time horizon. Such a sequence of orders submitted during a number of periods is called an *execution strategy*. There are many possible execution strategies to complete a desired trade. Note that the execution cost associated with an execution strategy is typically random since the future execution price is uncertain. In this paper, we consider the *execution cost problem* which, for a given price impact function, yields an execution strategy which minimizes a weighted linear combination of the mean and the variance of the execution cost.

The execution cost problem shares a similar mathematical structure with the traditional multi-period mean-variance portfolio optimization problem when a transaction cost is associated with rebalancing the portfolio. In both problems, given some fixed number of investment periods and the initial state of the portfolio, the goal is to produce a sequence of trades that maximizes some expected utility of the final wealth. However, in the multi-period mean-variance portfolio optimization problem, the permanent price impact of the trade at each period on the subsequent prices is typically not considered. Moreover, the execution cost problem includes a specific constraint on the investor's position at the end of the time horizon, i.e., the investor's position in each asset at the end of the time horizon must be zero.

The similarity of the execution cost problem to the mean-variance portfolio optimization problem motivates the notion of an efficient frontier in the context of the execution cost problem. A feasible execution strategy is *efficient* if it has the least expected execution cost among all execution strategies with the same variance of the execution cost. The collection of efficient execution strategies form the *efficient frontier* of the execution strategy universe.

The sensitivity of mean-variance efficient portfolios to estimation errors in the expected returns and the covariance matrix has been widely noted in the literature, e.g., see Jobson and Korkie (1980), Kallberg and Ziemba (1984), Frost and Savarino (1988), Best and Grauer (1991), Broadie (1993), Chopra and Ziemba (1993), Chen and Zhao (2003). However, to the best of our knowledge, the sensitivity of the optimal execution strategy and efficient frontier to estimation errors in the price impact function has not been addressed yet. For the mean-variance portfolio optimization, the difficulty in accurate estimation of the expected rate of returns is well known. Simultaneous estimation of the expected temporary and permanent price impacts of concurrent trades is more complex and likely to be less accurate than estimation of the expected returns, e.g., see Torre (1997). Therefore, it is important to understand the sensitivity of an optimal execution strategy and the efficient frontier to any error in parameters of the price impact function. Recognizing the effect of estimation errors may provide more realistic expectations about the future performance of a chosen execution strategy.

A common approach to investigate the effect of estimation errors is to interpret the errors as perturbations to the data and to perform a sensitivity analysis on the optimal solution. Sensitivity discussions are essential in model validation. In this paper, we carry out a sensitivity analysis to study the effect of estimation errors in parameters of the price impact function. Our main goals here are to explore how different an optimal execution strategy obtained from the estimated price impact function is from the true optimal execution strategy, and how far away the obtained efficient frontier is from the true efficient frontier.

In this paper, we consider linear time-independent price impact functions in which price impacts are assumed to be proportional to the trading volume. Linear price impact functions are defined by the *temporary impact matrix* H and the *permanent impact matrix* Γ. These impact matrices are the expected price depression caused by trading assets at an unit rate. Linear price impact functions have been well-studied in the market microstructure literature, e.g., see Kyle (1985), Bertsimas and Lo (1998), Bertsimas et al. (1999), Almgren and Chriss (2000), Huberman and Stanzl (2004). Huberman and Stanzl (2004) demonstrate that nonlinear permanent price impact functions can give rise to the availability of a sequence of trades that generates infinite expected profits per unit of risk. In addition, although the type and size of the price impacts are different for buys and sells, and there is an asymmetry in the overall impact of buys and sells, e.g., see Holthausen et al. (1987), Chan and Lakonishok (1993), their mathematical models are similar. Hence, without loss of generality, our presentation in this paper assumes that the investor's goal is to liquidate blocks of assets. Our discussion is based on the price impact model in Almgren and Chriss (2000).

In this study, we assume that the covariance matrix is given and focus on the sensitivity of the optimal execution strategy and efficient frontier to perturbations in the impact matrices. We first show that, under the assumed linear price impact model, the optimal execution strategy depends on the *combined impact matrix* $\Theta = \frac{1}{\tau}\left(H + H^T\right) - \Gamma$, rather than H and Γ individually. Here τ is the time length between consecutive trades. This suggests that one may want to estimate Θ directly in order to determine an optimal execution strategy. In addition, we prove that when the permanent impact matrix is symmetric and the combined impact matrix is positive definite, a unique optimal execution strategy exists for any positive risk aversion parameter.

We discuss some cases in which the optimal execution strategy is insensitive to perturbations in the impact matrices. In particular, we prove that, for any symmetric permanent impact matrix and positive definite Θ, the naive execution strategy of liquidating an equal amount in each period minimizes the expected execution cost. Therefore, as long as the symmetry of the permanent impact matrix Γ is maintained, the minimum expected execution cost strategy is not sensitive to perturbations.

We then analyze the sensitivity of the optimal execution strategy when the risk aversion parameter is positive or the permanent impact matrix is asymmetric. Since the impact matrices appear both in the Hessian matrix and the linear coefficient of the quadratic objective function for the execution cost problem, the optimal execution strategy in general may be quite sensitive to their perturbations. We show that the optimal execution strategy is Lipschitz continuous in the impact matrices and provide upper bounds on the size of the change in the optimal execution strategy. These upper bounds are represented in terms of the change in the impact matrices and a magnification factor, which is essentially the Lipschitz constant. We also present upper bounds for the magnification factors. These upper bounds explicitly specify which factors may magnify the effect of estimation errors on the optimal execution strategy. For example, following the established upper bounds, it can be easily seen that the change in the optimal execution strategy decreases when a large risk aversion parameter is chosen. In general, upper bounds for the magnification factors depend on the eigenvalues of the block tridiagonal Hessian matrix defined by the covariance matrix, the impact matrices, and the risk aversion parameter. The upper bounds can be simplified when the permanent impact matrix and its perturbation are symmetric. Under these assumptions, for small perturbations, the magnification factor becomes small when the minimum eigenvalue of *either* the covariance matrix *or* the combined impact matrix Θ is large. When both of these minimum eigenvalues are small, the optimal execution strategy may be very sensitive to the estimation errors. These results implicitly evince that the optimal execution strategy for trading a single asset is expected to be less sensitive than the optimal strategy for trading portfolios.

We also illustrate the sensitivity of the efficient frontier to perturbations in the impact matrices through simulations. Our computational results demonstrate that, when short selling is prohibited, the optimal execution strategy and efficient frontier are less sensitive than the case when short selling is permitted. Indeed, when short selling is allowed, the efficient frontier can be quite sensitive to perturbations in the impact matrices. In particular, changes in the efficient frontier can become very significant for a small risk aversion parameter if perturbations in the permanent impact matrix are asymmetric. We also observe that, for the minimum variance execution cost strategies, estimation errors can lead to large variations in the expected execution cost. On the other hand, the variance of the execution cost for the minimum expected execution cost strategy can change significantly. Our sensitivity analysis is restricted to the execution cost problems and perturbations for which both the original problem and perturbed problems have unique optimal solutions.

The presentation of the paper is as follows. The mathematical formulation of the execution cost problem is described in §2. We discuss, in §3, the sensitivity of the optimal execution strategy to perturbations in the impact matrices and provide upper bounds on the size of its change. Simulations are carried out in §4 to illustrate the effect of perturbations in the impact matrices on the efficient frontier and optimal execution strategy. Concluding remarks are given in §5.

Throughout the paper, we use the following notations. Vectors and matrices are denoted respectively by lower and uppercase letters. We use I_k to represent a $k \times k$ identity matrix. We denote the matrix of all zeros with the appropriate dimension by 0. Throughout, e_k denotes a column vector with the appropriate dimension which is zero everywhere except that, at the kth entry, the value equals one. The subscripts of matrices show their dimensions. We use $A \succeq 0$ and $A \succ 0$ to denote (not necessarily symmetric) positive semidefiniteness and positive definiteness respectively. Moreover, for a given vector x, we use $x \geq 0$ to mean x has nonnegative elements. By $\|A\|_2$ and $\kappa_2(A)$, we mean the Euclidean norm and the condition number with respect to the Euclidean norm of the matrix A. For a symmetric matrix A, $\lambda_i(A)$ denotes the ith eigenvalue of A when the eigenvalues are numbered in a nondecreasing order. Moreover, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ stand for the maximum and minimum eigenvalues of A respectively. The Kronecker product of two matrices $A_{m\times n}$ and $B_{p\times q}$ is denoted by the $mp \times nq$ matrix $A \otimes B$. For the properties of the Kronecker product, a reader is referred to §4.5.5 in Golub and Loan (1996).

2 The Portfolio Execution Cost Problem

Assume that an investor plans to liquidate his holdings in m assets during N periods in the time horizon T, $t_0 = 0 < t_1 < \cdots < t_N = T$, where $\tau \stackrel{\text{def}}{=} t_k - t_{k-1} = \frac{T}{N}$ for $k = 1, 2, \ldots, N$. The investor's position at time t_k is denoted by the m-vector $x_k = (x_{1k}, x_{2k}, \ldots, x_{mk})^T$, where x_{ik} is the investor's holding in the ith asset at period k. The investor's initial position is $x_0 = \bar{S}$, and his final position x_N equals 0, which guarantees complete liquidation by time T. The difference between the positions of two successive periods $k-1$ and k is denoted by an m-vector $n_k = x_{k-1} - x_k$ for $k = 1, 2, \ldots, N$. Negative n_{ik} implies that the ith asset is bought between t_{k-1} and t_k. We refer to a sequence $\{n_k\}_{k=1}^N$ satisfying $\sum_{k=1}^N n_k = \bar{S}$ as an *execution strategy*. Similarly, $\{x_k\}_{k=0}^N$ with $x_N = 0$ is referred to as an *execution position*.

Let $\tilde{P}_k$ be the execution price of one unit of assets at time t_k for $k = 1, 2, \ldots, N$. The deterministic initial market price, before the trade begins, is denoted by P_0. Due to the price volatility, $\tilde{P}_k$ is not deterministic over the execution horizon. In this paper, we assume that the execution price $\tilde{P}_k$ is given by

$$\tilde{P}_k = P_{k-1} - h\left(\frac{n_k}{\tau}\right), \quad k = 1, 2, \ldots, N, \tag{1}$$

where the market price P_k evolves according to the discrete arithmetic random walk,

$$P_k = P_{k-1} + \tau^{1/2}\Sigma\xi_k - \tau g\left(\frac{n_k}{\tau}\right). \tag{2}$$

Here $\xi_k = (\xi_{1k}, \xi_{2k}, ..., \xi_{lk})^T$ represents an l-vector of independent standard normals and Σ is an $m \times l$ volatility matrix of the asset prices. The functions $g(.)$ and $h(.)$ measure the expected permanent price impact and temporary price impact respectively. In each interval $(t_{k-1}, t_k]$, let price impacts be proportional to the trading rate $v = \frac{n_k}{\tau}$. Then we have the following price impact model

$$\begin{aligned} g(v) &= \Gamma v, \\ h(v) &= Hv, \end{aligned} \tag{3}$$

where the m-by-m matrix Γ is the permanent impact matrix and H is the temporary impact matrix. The temporary impact matrix H is not necessarily symmetric but needs to be positive semidefinite in order to exclude arbitrage opportunities. Otherwise the existence of a vector $v \neq 0$ with $v^T Hv < 0$ suggests that trading at the rate v makes a net profit from instantaneous market impact, e.g., see Almgren and Chriss (2000).

The *execution cost* of the trade is often defined as $P_0^T\bar{S} - \sum_{k=1}^N n_k^T\tilde{P}_k$. Hence, the mean-variance formulation of the execution cost problem with the risk aversion parameter $\mu \geq 0$ is

$$\begin{aligned} \min_{n_1,n_2,\ldots,n_N} \quad & \mathbf{E}\left(P_0^T\bar{S} - \sum_{k=1}^N n_k^T\tilde{P}_k\right) + \mu\cdot\mathbf{Var}\left(P_0^T\bar{S} - \sum_{k=1}^N n_k^T\tilde{P}_k\right) \\ \text{s.t.} \quad & \sum_{k=1}^N n_k = \bar{S}, \\ & n_k \geq 0, \quad k = 1,2,\ldots,N, \end{aligned} \tag{4}$$

where $\mathbf{E}(\cdot)$ and $\mathbf{Var}(\cdot)$ denote the expectation and the variance of a random variable respectively. In terms of the execution positions $\{x_k\}_{k=0}^N$, the portfolio execution cost problem becomes

$$\begin{aligned} \min_{x_0,x_1,x_2,\ldots,x_N} \quad & \mathbf{E}\left(P_0^T\bar{S} - \sum_{k=1}^N (x_{k-1}-x_k)^T\tilde{P}_k\right) + \mu\cdot\mathbf{Var}\left(P_0^T\bar{S} - \sum_{k=1}^N (x_{k-1}-x_k)^T\tilde{P}_k\right) \\ \text{s.t.} \quad & x_0 = \bar{S}, \\ & x_N = 0, \\ & x_{k-1} \geq x_k, \quad k = 1,2,\ldots,N. \end{aligned} \tag{5}$$

A large value of μ corresponds to the investor's small tolerance to risk. These problems can be modified to reflect regulation constraints or the investor's preferences on the trading volumes. We refer to an optimal solution of Problem (4) and Problem (5) as an *optimal execution strategy* and *optimal execution position* respectively. In both of the aforementioned problems, the inequality constraints rule out short sales.

For any execution strategy $\{n_k\}_{k=1}^N$ and its associated execution position $\{x_k\}_{k=0}^N$, applying the price impact model (3) and the execution price dynamic model (1), we obtain

$$\sum_{k=1}^N n_k^T\tilde{P}_k = \bar{S}^TP_0 + \sum_{k=1}^N \tau^{\frac{1}{2}}x_k^T\Sigma\xi_k - \tau\sum_{k=1}^N x_k^Tg\left(\frac{n_k}{\tau}\right) - \sum_{k=1}^N n_k^Th\left(\frac{n_k}{\tau}\right). \tag{6}$$

Thus the variance of the execution cost equals

$$\mathbf{Var}\left(P_0^T\bar{S} - \sum_{k=1}^N n_k^T\tilde{P}_k\right) = \tau\sum_{k=1}^N x_k^TCx_k,$$

where the $m\times m$ symmetric positive semidefinite matrix C is the covariance matrix of asset prices, i.e., $C = \Sigma\Sigma^T$.

From (6), the expected execution cost can be expressed as below:

$$\begin{aligned} &\mathbf{E}\left(P_0^T\bar{S} - \sum_{k=1}^N n_k^T\tilde{P}_k\right) = \sum_{k=1}^N x_k^T\Gamma n_k + \sum_{k=1}^N n_k^TH\frac{n_k}{\tau} \\ &= \sum_{k=1}^N x_k^T\Gamma(x_{k-1}-x_k) + \sum_{k=1}^N \frac{1}{\tau}(x_k - x_{k-1})^T H(x_k - x_{k-1}) \\ &= \sum_{k=1}^N x_k^T\left(\frac{1}{\tau}H - \Gamma\right)x_k + \sum_{k=1}^N x_k^T\left(\Gamma - \frac{1}{\tau}H\right)x_{k-1} - \sum_{k=1}^N \frac{1}{\tau}x_{k-1}^THx_k + \sum_{k=2}^N \frac{1}{\tau}x_{k-1}^THx_{k-1} + \frac{1}{\tau}x_0^THx_0 \\ &= \frac{1}{\tau}\bar{S}^TH\bar{S} - \frac{1}{\tau}x_N^THx_N + \sum_{k=1}^N x_k^T\left(\frac{2}{\tau}H - \Gamma\right)x_k + \sum_{k=1}^N x_k^T\left(\Gamma - \frac{1}{\tau}\left(H+H^T\right)\right)x_{k-1}. \end{aligned} \tag{7}$$

Define

$$L \stackrel{\text{def}}{=} \frac{2}{\tau}H - \Gamma + \mu\tau C, \quad \text{and} \quad \Theta \stackrel{\text{def}}{=} \frac{1}{\tau}\left(H+H^T\right) - \Gamma. \tag{8}$$

Subsequently, we refer to Θ as the *combined impact matrix*. Clearly,

$$L + L^T = \frac{2}{\tau}(H + H^T) - (\Gamma + \Gamma^T) + 2\mu\tau C = (\Theta + \Theta^T) + 2\mu\tau C.$$

Using these notations, we obtain

$$\begin{aligned}
&\mathbf{E}\left(P_0^T\bar{S} - \sum_{k=1}^{N}(x_{k-1} - x_k)^T\tilde{P}_k\right) + \mu\cdot\mathbf{Var}\left(P_0^T\bar{S} - \sum_{k=1}^{N}(x_{k-1} - x_k)^T\tilde{P}_k\right)\\
&= \frac{1}{\tau}\bar{S}^T H\bar{S} - \frac{1}{\tau}x_N^T H x_N + \sum_{k=1}^{N} x_k^T L x_k - \sum_{k=1}^{N} x_k^T\Theta x_{k-1}\\
&= \frac{1}{\tau}\bar{S}^T H\bar{S} - \frac{1}{\tau}x_N^T H x_N + \sum_{k=1}^{N}\frac{1}{2}\left(x_k^T L x_k + x_k^T L^T x_k\right) - \sum_{k=1}^{N}\frac{1}{2}\left(x_k^T\Theta x_{k-1} + x_{k-1}^T\Theta^T x_k\right)\\
&= \frac{1}{\tau}\bar{S}^T H\bar{S} - \frac{1}{\tau}x_N^T H x_N + \sum_{k=1}^{N}\frac{1}{2}x_k^T\left(L + L^T\right)x_k - \sum_{k=1}^{N}\frac{1}{2}\left(x_k^T\Theta x_{k-1} + x_{k-1}^T\Theta^T x_k\right). \qquad (9)
\end{aligned}$$

To simplify the representation of the constraints in Problem (5), we introduce the sequence of square matrices $\{G_k\}_{k=1}^{+\infty}$, where $G_1 = (1)$ and

$$G_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad G_k = \begin{pmatrix} G_{k-1} & \begin{pmatrix} 0 \\ \cdots \\ 0 \\ -1 \end{pmatrix} \\ 0 & 1 \end{pmatrix} \quad \text{for } k \geq 2. \qquad (10)$$

Using the Kronecker product, eliminating the constant term $\frac{1}{\tau}\bar{S}^T H\bar{S}$ from the objective function, and explicitly imposing $x_N = 0$ and $x_0 = \bar{S}$, Problem (5) is reduced to the following problem:

$$\begin{array}{ll}
\min\limits_{z\in\mathbb{R}^{m(N-1)}} & \frac{1}{2}z^T W(H,\Gamma,\mu)z + b^T(H,\Gamma)z, \\
\text{s.t.} & (-e_1^T\otimes I_m)z \geq -\bar{S}, \\
 & (G_{N-1}\otimes I_m)z \geq 0.
\end{array} \qquad (11)$$

The $m(N-1)\times m(N-1)$ symmetric tridiagonal block matrix $W(H,\Gamma,\mu)$, and the $m(N-1)$-vectors $b(H,\Gamma)$ and z are defined as follows:

$$W(H,\Gamma,\mu) \stackrel{\text{def}}{=} \begin{pmatrix} L+L^T & -\Theta^T & 0 & \dots & 0 \\ -\Theta & L+L^T & -\Theta^T & \dots & 0 \\ 0 & -\Theta & L+L^T & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & L+L^T \end{pmatrix}, \quad b(H,\Gamma) \stackrel{\text{def}}{=} \begin{pmatrix} -\Theta\bar{S} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad z \stackrel{\text{def}}{=} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{pmatrix}.$$

Similar to the mean-variance portfolio optimization problem, the execution cost problem (11) is a quadratic programming problem. However, in contrast to the mean-variance portfolio optimization problem in which the expected return appears only in the linear term of the quadratic objective, in the execution cost problem (11) the impact matrices appear in both the quadratic term and the linear term of the objective function in a structured fashion. Therefore sensitivity analysis restricted to perturbations in the linear term of the quadratic objective function, e.g., see Best and Grauer (1991), is not applicable in this context. It is necessary to explicitly analyze the effect of estimation errors in the impact matrices for the execution cost problem.

The quadratic programming problem (11) is convex if and only if $W(H,\Gamma,\mu) \succeq 0$. When the variance of the execution cost is not considered, the minimum expected execution cost problem corresponds to Problem (11) with $\mu = 0$. In Problem (11), constraints appear only when short selling is not permitted, which implies $0 \le x_{N-1} \le \ldots \le x_2 \le x_1 \le \bar{S}$. When short selling is allowed, the execution cost problem is reduced to the following unconstrained quadratic minimization problem:

$$\min_{z\in\mathbb{R}^{m(N-1)}} \frac{1}{2} z^T W(H,\Gamma,\mu) z + b^T(H,\Gamma) z. \tag{12}$$

When $W(H,\Gamma,\mu)$ is not positive semidefinite, Problem (12) has no local minima. If $W(H,\Gamma,\mu)$ is positive semidefinite but singular, Problem (12) has either no solution or infinitely many solutions. Problem (12) has a unique minimizer if and only if $W(H,\Gamma,\mu) \succ 0$. The unique minimizer in this case is $z^* = -W^{-1}(H,\Gamma,\mu) b(H,\Gamma)$.

When short selling is prohibited, the set of feasible solutions of Problem (11),

$$\left\{ (x_1^T, ..., x_N^T)^T : \ \bar{S} \ge x_1, \, x_{k-1} \ge x_k \ \text{ for } \ k = 2,3,\ldots,N-1, \ \ x_{N-1} \ge 0 \right\},$$

is compact. Therefore, the Weierstrass' Theorem, e.g., see Bertsekas (1996), along with the continuity of the objective function of Problem (11), implies that Problem (11) has a global minimizer. Moreover, positive definiteness of $W(H,\Gamma,\mu)$ guarantees that the global minimizer is unique.

Since one expects a unique optimal execution strategy under a reasonable price impact model (whether short selling is permitted or not), assuming $W(H,\Gamma,\mu)$ is positive definite seems appropriate for a linear price impact model (3). This assumption guarantees that both Problems (11) and (12) have unique optimal solutions.

The representation of Problem (11) indicates that the optimal execution strategy only depends on the combined impact matrix $\Theta = \frac{1}{\tau}(H + H^T) - \Gamma$ rather than matrices H and Γ individually. This suggests that, in order to decrease estimation errors, one may want to estimate Θ directly. In particular, the estimation method in Almgren et al. (2005) can be modified to directly estimate Θ. When $\Theta = 0$, the expected execution cost of any chosen execution strategy equals the constant $\frac{1}{\tau}\bar{S}^T H \bar{S}$. Assume, in addition, that $C \succ 0$ and $\mu > 0$. Then $W(H,\Gamma,\mu) \succ 0$ and the unique minimizer of Problem (12) is $z^* = 0$, which remains optimal even when short selling is not permitted. The solution $z^* = 0$, i.e., the execution strategy $x_k^* = 0$ for $k = 1,2,\ldots,N$, corresponds to completely liquidating the portfolio in the first period between time t_0 and t_1.

In the following lemma, we show that when the permanent impact matrix Γ is symmetric, positive definiteness of Θ is a necessary and sufficient condition for the positive definiteness of $W(H,\Gamma,0)$. Symmetric permanent impact matrices include diagonal matrices, which have been frequently the focus of discussion in the literature on the execution cost problem, e.g., see Almgren and Chriss (2000), Almgren et al. (2005).

Lemma 2.1. *Let the permanent impact matrix Γ be symmetric. Then $W(H,\Gamma,0) \succeq 0$ ($W(H,\Gamma,0) \succ 0$) if and only if $\Theta \succeq 0$ ($\Theta \succ 0$).*

Proof. When Γ is symmetric, $\Theta = \Theta^T$. For any real $m(N-1)$-vector $h = (h_1^T, h_2^T, ..., h_{N-1}^T)^T$,

$$\begin{aligned} h^T W(H,\Gamma,0) h &= -\sum_{i=1}^{N-2} h_{i+1}^T \Theta h_i + \sum_{i=1}^{N-1} h_i^T \Theta h_i + \sum_{i=1}^{N-1} h_i^T \Theta h_i - \sum_{i=1}^{N-2} h_i^T \Theta h_{i+1} \\ &= h_1^T \Theta h_1 - \sum_{i=1}^{N-2} h_{i+1}^T \Theta (h_i - h_{i+1}) + \sum_{i=1}^{N-2} h_i^T \Theta (h_i - h_{i+1}) + h_{N-1}^T \Theta h_{N-1} \\ &= h_1^T \Theta h_1 + \sum_{i=1}^{N-2} (h_{i+1} - h_i)^T \Theta (h_{i+1} - h_i) + h_{N-1}^T \Theta h_{N-1}. \end{aligned} \tag{13}$$

If $\Theta \succeq 0$, each term in (13) is nonnegative. Thus $h^T W(H,\Gamma,0) h \ge 0$ and consequently $W(H,\Gamma,0) \succeq 0$. The other direction of the statement follows from the fact that 2Θ is a leading principle submatrix of $W(H,\Gamma,0)$. Therefore (symmetric) positive semidefiniteness of $W(H,\Gamma,0)$ implies $\Theta \succeq 0$. □

For any $\mu \geq 0$, $W(H,\Gamma,\mu) = 2\mu\tau I_{N-1} \otimes C + W(H,\Gamma,0)$. Hence, when the permanent impact matrix Γ is symmetric and $\Theta \succeq 0$, Lemma 2.1 implies that the matrix $W(H,\Gamma,\mu) \succeq 0$ for any $\mu \geq 0$.

Thus whether short selling is permitted or not, the uniqueness of the optimal execution strategy for any risk aversion parameter $\mu \geq 0$ is guaranteed when $\Theta \succ 0$ and $\Gamma = \Gamma^T$. In the next proposition, we show that under these assumptions, the execution strategy that minimizes the expected execution cost ($\mu = 0$) can be found explicitly. Surprisingly, it is independent of the values of the impact matrices.

Proposition 2.1. *Let the permanent impact matrix Γ be symmetric. Then, whether short selling is allowed or not, the naive execution strategy $n_k^* = \frac{\bar{S}}{N}$, $k = 1, 2, \ldots, N$, is the unique execution strategy that minimizes the expected execution cost if and only if $\Theta \succ 0$.*

Proof. The assumption $\Gamma^T = \Gamma$ implies $\Theta^T = \Theta$. Consequently $W(H,\Gamma,0) = (G_{N-1} + G_{N-1}^T) \otimes \Theta$. Firstly we consider the situation when short selling is allowed, i.e., Problem (12) with $\mu = 0$. A direct use of Lemma 2.1 implies that Problem (12) with $\mu = 0$ has a unique optimal solution if and only if $\Theta \succ 0$. Applying properties of the Kronecker product, this unique solution equals

$$z^* = -W^{-1}(H,\Gamma,0)b(H,\Gamma) = \left(\left(G_{N-1} + G_{N-1}^T\right)^{-1} \otimes \Theta^{-1}\right)\left(e_1 \otimes \Theta\bar{S}\right) = \left(\left(G_{N-1} + G_{N-1}^T\right)^{-1} e_1\right) \otimes \bar{S}.$$

Recall that $e_1 = (1, 0, \cdots, 0)^T$. Applying the explicit representation of the inverse of the tridiagonal matrix $G_{N-1} + G_{N-1}^T$, e.g., see Fonseca (2007), we have

$$\left(G_{N-1} + G_{N-1}^T\right)^{-1} e_1 = \left(\frac{N-1}{N}, \frac{N-2}{N}, \cdots, \frac{1}{N}\right)^T.$$

Hence, the minimum expected execution cost position is $\{x_k^*\}_{k=0}^N = \left\{\frac{(N-k)}{N}\bar{S}\right\}_{k=0}^N$, which corresponds to the naive execution strategy $n_k^* = \frac{\bar{S}}{N}$, $k = 1, \cdots, N$. Since this solution satisfies the constraints of Problem (11), it is also the unique minimum expected execution strategy when short selling is prohibited. ☐

Therefore, the minimum expected execution cost strategy is always the naive execution strategy of trading a constant number of shares per period, under the assumptions that the permanent impact matrix Γ is symmetric and Θ is positive definite. Thus, the minimum expected execution cost strategy is insensitive to any change in the impact matrices as long as the perturbation maintains the strict convexity of the objective function and the symmetry in the permanent impact matrix.

When short selling is permitted, the optimal execution strategy is the solution to a linear system with the coefficient matrix $W(H,\Gamma,\mu)$. Thus it is important to analyze the condition number of this matrix. Moreover, as we show subsequently in §3, sensitivity of the optimal execution strategy to perturbations in the impact matrices depends on the eigenvalues of $W(H,\Gamma,\mu)$. In the rest of this section, we analyze the condition number and the minimum eigenvalue of the matrix $W(H,\Gamma,\mu)$. We apply the following result from Kulkarni et al. (1999) in our discussion.

Lemma 2.2. *Let $N \geq 2$. Then the eigenvalues $\lambda_i\left(G_{N-1} + G_{N-1}^T\right) = 2\left(1 - \cos\left(\frac{i\pi}{N}\right)\right)$ for $i = 1, 2, \ldots, N-1$.*

A direct consequence of Lemma 2.2 is

$$\kappa_2\left(G_{N-1} + G_{N-1}^T\right) = \frac{\lambda_{\max}\left(G_{N-1} + G_{N-1}^T\right)}{\lambda_{\min}\left(G_{N-1} + G_{N-1}^T\right)} = \frac{1 - \cos\left(\frac{(N-1)\pi}{N}\right)}{1 - \cos\left(\frac{\pi}{N}\right)} = \frac{1 + \cos\left(\frac{\pi}{N}\right)}{1 - \cos\left(\frac{\pi}{N}\right)} = \cot^2\left(\frac{\pi}{2N}\right). \tag{14}$$

Since $W(H,\Gamma,\mu) = 2\mu\tau I_{N-1} \otimes C + W(H,\Gamma,0)$ and the matrices C and $W(H,\Gamma,0)$ are symmetric, the Courant-Fischer Theorem, e.g., see Theorem 8.1.2 in Golub and Loan (1996), implies that

$$\lambda_{\min}(W(H,\Gamma,\mu)) \geq 2\mu\tau\lambda_{\min}(C) + \lambda_{\min}(W(H,\Gamma,0)). \tag{15}$$

When Γ is symmetric, this lower bound can be stated explicitly in terms of the combined impact matrix Θ:

Corollary 2.1. *Let $N \geq 2$ and the permanent impact matrix Γ be symmetric. Then*

$$\lambda_{\min}\left(W(H,\Gamma,\mu)\right) \geq 2\mu\tau\lambda_{\min}(C) + 4\sin^2\left(\frac{\pi}{2N}\right)\lambda_{\min}(\Theta).$$

In addition, the equality holds when $\mu = 0$.

Proof. When Γ is symmetric, $W(H,\Gamma,0)$ can be represented as the Kronecker product of the matrices $\left(G_{N-1} + G_{N-1}^T\right)$ and Θ. Thus

$$\lambda_{\min}(W(H,\Gamma,0)) = \lambda_{\min}\left(G_{N-1} + G_{N-1}^T\right)\lambda_{\min}(\Theta) = 2\left(1 - \cos\left(\frac{\pi}{N}\right)\right)\lambda_{\min}(\Theta).$$

This result, along with inequality (15), completes the proof. □

In the next proposition, we investigate how $\kappa_2(W(H,\Gamma,\mu))$ depends on the condition number of the covariance matrix and the combined impact matrix Θ.

Proposition 2.2. *Let $W(H,\Gamma,0) \succ 0$ and $N \geq 2$. Then*

(a) $\kappa_2(W(H,\Gamma,0)) \geq \cot^2\left(\frac{\pi}{2N}\right)\kappa_2(\Theta + \Theta^T)$.

(b) If, in addition, Γ is symmetric, then

$$\kappa_2(W(H,\Gamma,0)) = \cot^2\left(\frac{\pi}{2N}\right)\kappa_2(\Theta). \tag{16}$$

(c) Assume $C \succ 0$. Then $\kappa_2(W(H,\Gamma,\mu)) \leq \kappa_2(C) + \kappa_2(W(H,\Gamma,0))$ for any $\mu \geq 0$.

Proof. (a) To prove part (a), we use the fact that the matrices $W(H,\Gamma,0)$ and $W(H,\Gamma^T,0)$ have identical eigenvalues. More precisely, $(r_1^T, r_2^T, ..., r_{N-1}^T)^T$ is an eigenvector of $W(H,\Gamma,0)$ associated with the eigenvalue λ if and only if $(r_{N-1}^T, r_{N-2}^T, ..., r_1^T)^T$ is an eigenvector of $W(H,\Gamma^T,0)$ for the same eigenvalue. In particular, we have

$$\lambda_{\max}\left(W(H,\Gamma,0)\right) = \lambda_{\max}\left(W(H,\Gamma^T,0)\right), \quad \lambda_{\min}\left(W(H,\Gamma,0)\right) = \lambda_{\min}\left(W(H,\Gamma^T,0)\right).$$

Now the corollary of the Courant-Fischer Theorem, e.g., see Theorem 8.1.5 in Golub and Loan (1996), along with the assumption $W(H,\Gamma,0) \succ 0$, results in

$$\frac{\lambda_{\max}\left(W(H,\Gamma,0) + W(H,\Gamma^T,0)\right)}{\lambda_{\min}\left(W(H,\Gamma,0) + W(H,\Gamma^T,0)\right)} \leq \frac{2\lambda_{\max}\left(W(H,\Gamma,0)\right)}{2\lambda_{\min}\left(W(H,\Gamma,0)\right)} = \kappa_2\left(W(H,\Gamma,0)\right).$$

Consequently $\kappa_2\left(W(H,\Gamma,0) + W(H,\Gamma^T,0)\right) \leq \kappa_2\left(W(H,\Gamma,0)\right)$. This inequality, along with the expression of $W(H,\Gamma,0) + W(H,\Gamma^T,0)$ as the Kronecker product of the matrices $\left(G_{N-1} + G_{N-1}^T\right)$ and $(\Theta + \Theta^T)$, yields

$$\kappa_2\left(W(H,\Gamma,0)\right) \geq \kappa_2\left(W(H,\Gamma,0) + W(H,\Gamma^T,0)\right) = \kappa_2\left(\left(G_{N-1} + G_{N-1}^T\right) \otimes \left(\Theta + \Theta^T\right)\right). \tag{17}$$

Thus

$$\kappa_2(W(H,\Gamma,0)) \geq \kappa_2\left(G_{N-1} + G_{N-1}^T\right)\kappa_2\left(\Theta + \Theta^T\right) = \cot^2\left(\frac{\pi}{2N}\right)\kappa_2(\Theta + \Theta^T), \tag{18}$$

which completes the proof of part (a).

(b) When Θ is symmetric, $W(H,\Gamma,\mu) = W(H,\Gamma^T,\mu)$. Therefore

$$\kappa_2\left(W(H,\Gamma,0) + W(H,\Gamma^T,0)\right) = \kappa_2\left(2W(H,\Gamma,0)\right) = \kappa_2(W(H,\Gamma,0)).$$

Hence, equality holds in (17) and (18), which completes the proof of part (b).

(c) To prove part (c), let $\mu \geq 0$ be given. Using $W(H,\Gamma,\mu) = 2\mu\tau I_{N-1} \otimes C + W(H,\Gamma,0)$, we have

$$\begin{aligned}\kappa_2\left(W(H,\Gamma,\mu)\right) &= \frac{\lambda_{\max}\left(W(H,\Gamma,0) + 2\mu\tau I_{N-1} \otimes C\right)}{\lambda_{\min}\left(W(H,\Gamma,0) + 2\mu\tau I_{N-1} \otimes C\right)} \\ &\leq \frac{\lambda_{\max}\left(W(H,\Gamma,0)\right)}{\lambda_{\min}\left(W(H,\Gamma,0)\right)} + \frac{\lambda_{\max}(2\mu\tau I_{N-1} \otimes C)}{\lambda_{\min}(2\mu\tau I_{N-1} \otimes C)} \\ &= \kappa_2(W(H,\Gamma,0)) + \kappa_2(C),\end{aligned} \tag{19}$$

where inequality (19) comes from the fact that $\lambda_{\min}(W(H,\Gamma,0)) > 0$ and $\lambda_{\min}(2\mu\tau I_{N-1} \otimes C) > 0$. □

Proposition 2.2 shows that the condition number of the matrix $W(H,\Gamma,\mu)$ can be large when the condition number of either the covariance matrix C or the Hessian matrix $W(H,\Gamma,0)$ is large. However, $\kappa_2(W(H,\Gamma,0))$ is at least as large as $\cot^2\left(\frac{\pi}{2N}\right)$ times the condition number of the matrix $(\Theta+\Theta^T)$. Proposition 2.2 also implies that, in the single asset trading, the condition number of the obtained matrix $W(H,\Gamma,0)$ depends only on the number of periods N.

In the next section, we investigate the sensitivity of optimal execution strategies to perturbations in the impact matrices H and Γ.

3 The Sensitivity of the Optimal Execution Strategy

In this section, we show that the optimal execution strategy is Lipschitz continuous in a given pair of impact matrices H and Γ with respect to the Euclidean norm. We then use an estimation of the Lipschitz constant to derive some upper bounds for the change in the optimal execution strategy due to perturbations in the impact matrices. These upper bounds are represented in terms of changes in the impact matrices and eigenvalues of the Hessian of the objective function. Such analysis indicates under what conditions the optimal execution strategy is insensitive to perturbations and when it may become very sensitive. Throughout, we denote perturbations in the temporary and permanent impact matrices as ΔH and $\Delta\Gamma$ respectively. In subsequent discussions, we assume that $W(H,\Gamma,\mu)\succ 0$. Therefore, for sufficiently small perturbations of ΔH and $\Delta\Gamma$, the matrix $W(H+\Delta H,\Gamma+\Delta\Gamma,\mu)$ is symmetric positive definite. Consequently the optimal execution strategy after perturbation remains unique.

Given the perturbed impact matrices $H+\Delta H$ and $\Gamma+\Delta\Gamma$, the perturbed execution cost problem (11) is

$$\begin{array}{ll} \min\limits_{z\in\mathbb{R}^{m(N-1)}} & \frac{1}{2}z^T W(H+\Delta H,\Gamma+\Delta\Gamma,\mu)z + b^T(H+\Delta H,\Gamma+\Delta\Gamma)z, \\ \text{s.t.} & (-e_1^T\otimes I_m)z \ge -\bar{S}, \\ & (G_{N-1}\otimes I_m)z \ge 0, \end{array} \tag{20}$$

where the matrix G_{N-1} is defined in (10). Problems (11) and (20) have the same set of feasible solutions. Applying the properties

$$\begin{array}{ll} b(H+\Delta H,\Gamma+\Delta\Gamma) = b(H,\Gamma)+\Delta b, & \Delta b \stackrel{\text{def}}{=} b(\Delta H,\Delta\Gamma), \\ W(H+\Delta H,\Gamma+\Delta\Gamma,\mu) = W(H,\Gamma,\mu)+\Delta W, & \Delta W \stackrel{\text{def}}{=} W(\Delta H,\Delta\Gamma,0), \end{array} \tag{21}$$

we may restate the objective function of Problem (20):

$$\frac{1}{2}z^T W(H,\Gamma,\mu)z + b^T(H,\Gamma)z + \frac{1}{2}z^T\Delta W z + \Delta b^T z. \tag{22}$$

Quantities ΔW and Δb are determined by $\Delta\Theta = \frac{1}{\tau}(\Delta H+\Delta H^T)-\Delta\Gamma$. Thus the optimal solution of Problem (20) and consequently the optimal execution strategy depends on the perturbation in the combined impact matrix $\Delta\Theta$ rather than ΔH or $\Delta\Gamma$ individually. Therefore, all of the perturbations in the impact matrices that produce the same $\Delta\Theta$ affect the optimal execution strategy identically. In particular, when $\Delta\Theta=0$, we have $\Delta W=0$ and $\Delta b=0$. Therefore, Problem (20) is equivalent to Problem (11) and their optimal solutions are identical. Hence the optimal execution strategy is insensitive to this special perturbation of the impact matrices ΔH and $\Delta\Gamma$, when $\Delta\Theta=0$.

Furthermore, as we discussed in Proposition 2.1, the minimum expected execution cost strategy is also insensitive to any perturbations in the impact matrices as long as the perturbed permanent impact matrix $\Gamma+\Delta\Gamma$ remains symmetric and $W(H+\Delta H,\Gamma+\Delta\Gamma,0)\succ 0$. Specifically, when trading a single asset and $\mu=0$, the optimal execution strategy is not sensitive to any changes in the impact matrices, assuming that the minimum expected execution cost problem has a unique optimal solution.

Therefore, in the aforementioned cases the optimal execution strategy and the variance of the execution cost remain the same. However, in both cases, the expected value of the corresponding execution cost changes as the impact matrices are perturbed. When $\Delta\Theta=0$, the change in the mean of the execution cost is $\frac{1}{\tau}\bar{S}^T \Delta H \bar{S}$. In the second case, in which $\mu = 0$, $\Theta = \Theta^T$ and $\Delta\Theta = \Delta\Theta^T$ are considered, the variation in the mean of the execution cost equals $\frac{1}{2}z^T \Delta W z + \Delta b^T z + \frac{1}{\tau}\bar{S}^T \Delta H \bar{S}$, where z is the solution of Problem (11) corresponding to the naive execution strategy. However, when $\Delta\Gamma$ is not symmetric, our simulation study in §4 shows that the changes in the mean and particularly the variance of the execution cost become very significant for small values of μ, especially when $\mu = 0$.

In the rest of this section, we analyze the sensitivity of the optimal execution strategy to more general perturbations in the impact matrices. Firstly, we note that the Euclidean distance between any two execution strategies $n^* = \{n_k^*\}_{k=1}^N$ and $\bar{n} = \{\bar{n}_k\}_{k=1}^N$ is related to the change between corresponding execution positions $x^* = \{x_k^*\}_{k=0}^N$ and $\bar{x} = \{\bar{x}_k\}_{k=0}^N$:

$$\begin{aligned} \|n^* - \bar{n}\|_2^2 = \sum_{k=1}^N \|n_k^* - \bar{n}_k\|_2^2 \quad &= \sum_{k=1}^N \left\|x_{k-1}^* - x_k^* - (\bar{x}_{k-1} - \bar{x}_k)\right\|_2^2 \\ &= \sum_{k=1}^N \left\|x_{k-1}^* - \bar{x}_{k-1}\right\|_2^2 + \sum_{k=1}^N \|x_k^* - \bar{x}_k\|_2^2 - 2\sum_{k=1}^N (x_{k-1}^* - \bar{x}_{k-1})^T (x_k^* - \bar{x}_k) \\ &\leq 2\sum_{k=1}^N \left\|x_{k-1}^* - \bar{x}_{k-1}\right\|_2^2 + 2\sum_{k=1}^N \|x_k^* - \bar{x}_k\|_2^2 = 4\sum_{k=1}^{N-1} \|x_k^* - \bar{x}_k\|_2^2 . \end{aligned}$$

This result can be summarized as

$$\|n^* - \bar{n}\|_2 \leq 2\,\|x^* - \bar{x}\|_2 . \tag{23}$$

We start our sensitivity discussion with the strategy in which short selling is permitted. In the following theorem, we exploit the explicit representation of the optimal solution of Problem (12) to determine the exact change in the optimal execution strategy. For notational simplicity, abbreviate $W(H, \Gamma, \mu)$ as W when there is no confusion.

Theorem 3.1. *Consider the execution cost problem* (12) *when short selling is allowed. Assume* $W(H, \Gamma, \mu) \succ 0$ *and* $W(H + \Delta H, \Gamma + \Delta\Gamma, \mu) \succ 0$. *Denote the unique optimal solutions of Problem* (12) *before and after perturbation with* z^* *and* $\bar{z}$ *respectively. Then*

$$z^* - \bar{z} = W^{-1}(H + \Delta H, \Gamma + \Delta\Gamma, \mu)\left(\Delta b - \Delta W W^{-1}(H, \Gamma, \mu) b(H, \Gamma)\right). \tag{24}$$

Furthermore, let $n^* = \{n_k^*\}_{k=1}^N$ *and* $\bar{n} = \{\bar{n}_k\}_{k=1}^N$ *be the optimal execution strategies corresponding to the optimal solutions* z^* *and* $\bar{z}$ *respectively. Then, there exists a magnification factor* $\vartheta > 0$ *such that:*

$$\|n^* - \bar{n}\|_2 \leq 2\|z^* - \bar{z}\|_2 \leq 2\vartheta \|\bar{S}\|_2 \left(1 + 4\sqrt{m}\vartheta\|\Theta\|_2\right) \|\Delta\Theta\|_2, \tag{25}$$

where $\vartheta \leq \frac{1}{\min\{\lambda_{\min}(W), \lambda_{\min}(W + \Delta W)\}}$.

Proof. Positive definiteness of W guarantees Problem (12) has the unique optimal solution $z^* = -W^{-1}b(H, \Gamma)$. Similarly, under the assumption $W + \Delta W \succ 0$, Problem (12), with the perturbed impact matrices $H + \Delta H$ and $\Gamma + \Delta\Gamma$, has a unique optimal solution, namely $\bar{z} = -(W + \Delta W)^{-1}(b(H, \Gamma) + \Delta b)$. Therefore

$$\begin{aligned} (W + \Delta W)(z^* - \bar{z}) \quad &= (W + \Delta W)[-W^{-1}b(H, \Gamma) - (-(W + \Delta W)^{-1}(b(H, \Gamma) + \Delta b))] \\ &= -b(H, \Gamma) - \Delta W W^{-1} b(H, \Gamma) + b(H, \Gamma) + \Delta b \\ &= -\Delta W W^{-1} b(H, \Gamma) + \Delta b, \end{aligned}$$

which proves (24). Thus

$$\begin{aligned} \|z^* - \bar{z}\|_2 \quad &= \left\|(W + \Delta W)^{-1}\left(\Delta b - \Delta W W^{-1} b(H, \Gamma)\right)\right\|_2 \\ &\leq \|(W + \Delta W)^{-1}\|_2 \left(\|\Delta b\|_2 + \|\Delta W\|_2 \|W^{-1}\|_2 \|b(H, \Gamma)\|_2\right). \end{aligned}$$

Since $W + \Delta W$ and W are symmetric positive definite, the above inequality is reduced to

$$\begin{aligned} \|z^* - \bar{z}\|_2 &\le \frac{1}{\lambda_{\min}(W + \Delta W)} \left(\|\Delta b\|_2 + \frac{\|\Delta W\|_2}{\lambda_{\min}(W)} \|b(H, \Gamma)\|_2 \right) &(26)\\ &\le \frac{1}{\lambda_{\min}(W + \Delta W)} \left(\|\Delta \Theta\|_2 + \frac{\|\Delta W\|_2}{\lambda_{\min}(W)} \|\Theta\|_2 \right) \|\bar{S}\|_2. &(27) \end{aligned}$$

Since ΔW is symmetric, $\|\Delta W\|_1 = \|\Delta W\|_\infty$ and $\|\Delta W\|_2 \le \sqrt{\|\Delta W\|_1 \|\Delta W\|_\infty} = \|\Delta W\|_1$. Thus

$$\begin{aligned} \|\Delta W\|_2 &\le \|\Delta \Theta\|_1 + \|\Delta \Theta + \Delta \Theta^T\|_1 + \|\Delta \Theta^T\|_1 \\ &\le 2\|\Delta \Theta\|_1 + 2\|\Delta \Theta^T\|_1 \\ &= 2\|\Delta \Theta\|_1 + 2\|\Delta \Theta\|_\infty \\ &\le 4\sqrt{m}\|\Delta \Theta\|_2. &(28) \end{aligned}$$

Substituting inequality (28) in (27) and using inequality (23) complete the proof of (25). □

Inequality (27) is valid for any unconstrained quadratic minimization problem with the Hessian matrix W and the linear coefficient $b(H, \Gamma)$. If the perturbation of the Hessian matrix ΔW is sufficiently small, relative to the change in the linear coefficient Δb, the upper bound is dominated by $\vartheta\|\Delta b\|$, which is linear in ϑ. This is particularly the case in the traditional mean-variance portfolio optimization since the covariance (Hessian) matrix can in general be estimated more accurately than the mean rate of return (linear coefficient). However, in the execution cost problem, the change in the combined impact matrix $\Delta\Theta$ appears in both the linear coefficient $b(H, \Gamma)$ and the Hessian matrix W. Therefore, when the magnification factor ϑ is sufficiently large, the upper bound is dominated by the term $\vartheta^2\|\Delta W\|_2$, which is quadratic in ε. Thus, the effect of estimation errors in the impact matrices can potentially be more significant than the effect of perturbations in the mean rate of return in the traditional mean-variance portfolio optimization.

The upper bound in (25) illustrates the main factors which can magnify the effect of estimation errors in the impact matrices on the optimal execution strategy. This effect is described through the magnification factor ϑ. When the upper bound of ϑ is small, the optimal execution strategy is not so sensitive to perturbations in the impact matrices. On the other hand, the optimal execution strategy may be sensitive to the perturbation $\Delta\Theta$ when this upper bound is large.

The provided upper bound for ϑ in Theorem 3.1 depends only on the minimum eigenvalues of W and $W + \Delta W$. When both of these eigenvalues are large, the magnification factor becomes small. Consequently the optimal execution strategy does not change significantly due to perturbations in the impact matrices. When $\mu > 0$ and $C \succ 0$, according to inequality (15), $\lambda_{\min}(W)$ (and similarly $\lambda_{\min}(W + \Delta W)$) increases as $\mu\lambda_{\min}(C)$ increases, which implies that the magnification factor ϑ becomes small. This result indicates that, when the risk aversion parameter is nonzero and $\lambda_{\min}(C)$ is large (or equivalently $\kappa_2(C)$ is small), the optimal execution strategy is not very sensitive to the perturbations. Furthermore, assuming $C \succ 0$, the variation in the optimal execution strategy due to the perturbations diminishes as $\mu \to +\infty$. This result is entirely expected; since as $\mu \to +\infty$ the objective function of Problem (12) is dominated by the variance of the execution cost which depends only on the covariance matrix C. In these two cases the investor may not need to be concerned about the effect of estimation errors in the impact matrices.

On the other hand, when the minimum eigenvalue of $C \succ 0$ is small, the influence of estimation errors on the optimal execution strategy may become more prominent for a small risk aversion parameter. This dependence of the effect of estimation errors on the risk aversion parameter is analogous to the traditional mean-variance portfolio optimization, e.g., see Chopra and Ziemba (1993).

When the permanent impact matrices Γ and $\Gamma + \Delta\Gamma$ are symmetric, using Corollary 2.1, the upper bound for ϑ presented in Theorem 3.1 can be stated in terms of the minimum eigenvalues of Θ and $\Theta + \Delta\Theta$.

Corollary 3.1. *Let the assumptions in Theorem 3.1 hold. In addition, assume that Γ and $\Delta\Gamma$ are symmetric, $\lambda_{\min}(\Theta) \ge 0$ and $\lambda_{\min}(\Theta + \Delta\Theta) \ge 0$. Then there exists a magnification factor $\vartheta_0 > 0$ such that*

$$\|n^* - \bar{n}\|_2 \le 2\|z^* - \bar{z}\|_2 \le \vartheta_0 \|\bar{S}\|_2 \left(1 + 2\sqrt{m}\vartheta_0\|\Theta\|_2\right) \|\Delta\Theta\|_2,$$

where

$$\vartheta_0 \leq \frac{1}{\mu\tau\lambda_{\min}(C) + 2\sin^2\left(\frac{\pi}{2N}\right)\min\{\lambda_{\min}(\Theta), \lambda_{\min}(\Theta+\Delta\Theta)\}}. \tag{29}$$

For sufficiently small perturbations, the upper bound of ϑ_0 in (29) becomes small if and only if *either* $\mu\lambda_{\min}(C)$ *or* $\lambda_{\min}(\Theta)$ is large. Either case results in a small sensitivity of the optimal execution strategy to perturbations. However, for a given positive risk aversion parameter, when both $\lambda_{\min}(C)$ and $\lambda_{\min}(\Theta)$ are small, the change in the optimal execution strategy can potentially be large relative to the perturbation $\Delta\Theta$. Note that, based on Proposition 2.1, the minimum expected execution cost strategy is insensitive to perturbations when matrices Γ and $\Gamma + \Delta\Gamma$ are symmetric and Θ and $\Theta + \Delta\Theta$ are positive definite.

Now consider the execution cost problem when short selling is not permitted. Denote the coefficient matrix of the short selling constraints in Problem (11) with A, i.e., $A = Y \otimes I_m$, where

$$Y \stackrel{\text{def}}{=} \begin{pmatrix} -e_1^T \\ G_{N-1} \end{pmatrix}. \tag{30}$$

The following property of the bidiagonal matrix G_{N-1} is proved in Fonseca (2007).

Proposition 3.1. *For every integer $k \geq 1$, $\lambda_i\left(G_k G_k^T\right) = 2\left(1 - \cos\left(\frac{2i-1}{2k+1}\pi\right)\right)$ for $i = 1, 2, \ldots, k$. Particularly, $\lambda_{\min}\left(G_k G_k^T\right) = 2\left(1 - \cos\left(\frac{\pi}{2k+1}\right)\right) = 4\sin^2\left(\frac{\pi}{4k+2}\right)$.*

Therefore, $\lambda_{\min}(G_k G_k^T)$ is a decreasing function of k, i.e., for $k \geq 1$, $\lambda_{\min}(G_k G_k^T) \geq \lambda_{\min}(G_{k+1}G_{k+1}^T)$. Moreover, $1 \geq \lambda_{\min}(G_k G_k^T) > 0$ and consequently the matrix $G_k G_k^T$ is symmetric positive definite.

Next, we present a property of the coefficient matrix of the binding constraints in Problem (11) at a feasible solution. Throughout, for a given subset J of the row indices of A, we let A_J denote the submatrix of A consisting of the rows with indices in J. Similarly, the submatrix of A consisting of those columns with indices in a subset J of column indices is denoted by A^J.

Lemma 3.1. *Consider the coefficient matrix A of the constraints in Problem (11), i.e., $A = Y \otimes I_m$, where Y is defined in (30). Let z^* be a feasible solution of Problem (11) and J be the set of indices of the binding constraints at z^*. Then*

$$\lambda_{\min}\left(A_J A_J^T\right) \geq \lambda_{\min}\left(G_{N-1}G_{N-1}^T\right) = 4\sin^2\left(\frac{\pi}{4N-2}\right). \tag{31}$$

Proof. Applying properties of the Kronecker product, we have

$$A_J A_J^T = (Y \otimes I_m)_J (Y \otimes I_m)_J^T = \left((Y \otimes I_m)(Y \otimes I_m)^T\right)_J^J = \left(\left(YY^T\right) \otimes I_m\right)_J^J. \tag{32}$$

Let M be the permutation matrix such that

$$[1, 2, \ldots, N, 1, 2, \ldots, N, \ldots, 1, 2, \ldots, N]M^T = [1, \ldots, 1, 2, \ldots, 2, \ldots, N, \ldots, N].$$

Therefore, corresponding to the index set J, we can find an index set J' such that

$$M\left(\left((YY^T) \otimes I_m\right)_J^J\right)M^T = \left(I_m \otimes \left(YY^T\right)\right)_{J'}^{J'}.$$

Thus

$$\begin{aligned} \lambda_{\min}\left(\left((YY^T) \otimes I_m\right)_J^J\right) &= \lambda_{\min}\left(M\left(\left((YY^T) \otimes I_m\right)_J^J\right)M^T\right) \\ &= \lambda_{\min}\left(\left(I_m \otimes \left(YY^T\right)\right)_{J'}^{J'}\right) \\ &\geq \min_{i=1,2,\ldots,m} \lambda_{\min}\left(Y_{(i)}Y_{(i)}^T\right). \end{aligned}$$

Here $Y_{(i)}$ is the submatrix of Y with the row indices equal to $j - N(i-1)$ where

$$j \in \left(J' \cap \{N(i-1)+1, ..., N(i-1)+N\}\right).$$

This result, along with equality (32), yields

$$\lambda_{\min}\left(A_J A_J^T\right) \geq \min_{i=1,2,\ldots,m} \lambda_{\min}\left(Y_{(i)} Y_{(i)}^T\right). \tag{33}$$

In the rest of the proof, we show that for every $i = 1, 2, \ldots, m$,

$$\lambda_{\min}\left(Y_{(i)} Y_{(i)}^T\right) \geq \lambda_{\min}\left(G_{N-1} G_{N-1}^T\right). \tag{34}$$

Denote the execution position associated with the feasible solution z^* with $\{x_k^*\}_{k=0}^N$. Note that, for each asset i, a positive number of shares must be sold in at least one of the periods. Thus, for every asset $i = 1, 2, \ldots, m$, there is at least some $j \in \{1, 2, ..., N\}$ so that the constraint corresponding to the jth row of Y is not active. For any asset i, there are two cases to consider: either $\bar{S}_i > x_{i1}^*$ or $\bar{S}_i = x_{i1}^*$. In the first case, $\bar{S}_i > x_{i1}^*$, the rows of $Y_{(i)}$ are a subset of the rows of G_{N-1}. Let $Y_{(-i)}$ denote the submatrix consisting of rows of G_{N-1} that are not in $Y_{(i)}$. We then have

$$\begin{aligned}
\lambda_{\min}\left(G_{N-1} G_{N-1}^T\right) &= \min_{z \neq 0} \frac{z^T\left(G_{N-1} G_{N-1}^T\right) z}{z^T z} \\
&= \min_{(z_1, z_2) \neq 0} \frac{z_1^T Y_{(i)} Y_{(i)}^T z_1 + z_2^T Y_{(-i)} Y_{(i)}^T z_1 + z_1^T Y_{(i)} Y_{(-i)}^T z_2 + z_2^T Y_{(-i)} Y_{(-i)}^T z_2}{z_1^T z_1 + z_2^T z_2} \\
&\leq \min_{(z_1, 0) \neq 0} \frac{z_1^T\left(Y_{(i)} Y_{(i)}^T\right) z_1}{z_1^T z_1} = \lambda_{\min}\left(Y_{(i)} Y_{(i)}^T\right),
\end{aligned} \tag{35}$$

which proves inequality (34).

Now consider the case $\bar{S}_i = x_{i1}^*$. In this case, there must be at least some $j \in \{2, 3, \ldots, N\}$, such that $x_{i(j-1)}^* > x_{ij}^*$. When $N = 2$, the second constraint must be inactive, which implies that the only row of $Y_{(i)}$ is the first row of Y. Therefore, $\lambda_{\min}\left(Y_{(i)} Y_{(i)}^T\right) = \lambda_{\min}\left(G_{N-1} G_{N-1}^T\right)$.

When $N \geq 3$, at least one of the rows of G_{N-1} corresponds to an inactive constraint. Let this row be the jth row of G_{N-1} where $j \in \{1, 2, \ldots, N-1\}$. When $j = N-1$, $Y_{(i)}$ does not include the last row of G_{N-1}. Since the matrix Y after eliminating its last row equals $-G_{N-1}^T$, $Y_{(i)}$ is a submatrix of $-G_{N-1}^T$. Thus similar to (35), we can show that

$$\lambda_{\min}\left(Y_{(i)} Y_{(i)}^T\right) \geq \lambda_{\min}\left(G_{N-1}^T G_{N-1}\right) = \lambda_{\min}\left(G_{N-1} G_{N-1}^T\right),$$

where the last equality comes from the fact that $G_{N-1}^T G_{N-1}$ and $G_{N-1} G_{N-1}^T$ have identical eigenvalues. This result proves inequality (34) for this case.

When $j \in \{1, 2, \ldots, N-2\}$, the rows of $Y_{(i)}$ are a subset of the rows of the following matrix

$$\begin{pmatrix} -G_j^T & 0 \\ 0 & G_{N-j-1} \end{pmatrix}.$$

Note that

$$\begin{pmatrix} -G_j^T & 0 \\ 0 & G_{N-j-1} \end{pmatrix} \begin{pmatrix} -G_j & 0 \\ 0 & G_{N-j-1}^T \end{pmatrix} = \begin{pmatrix} G_j^T G_j & 0 \\ 0 & G_{N-j-1} G_{N-j-1}^T \end{pmatrix}.$$

Therefore

$$\lambda_{\min}\left(Y_{(i)} Y_{(i)}^T\right) \geq \min\left\{\lambda_{\min}\left(G_j^T G_j\right), \lambda_{\min}\left(G_{N-j-1} G_{N-j-1}^T\right)\right\} \geq \lambda_{\min}\left(G_{N-1} G_{N-1}^T\right),$$

where the last inequality comes from the facts that $\lambda_{\min}\left(G_j^T G_j\right) = \lambda_{\min}\left(G_j G_j^T\right)$ and $\lambda_{\min}\left(G_j G_j^T\right)$ is a decreasing function of j. Thus, for every $i = 1, 2, \ldots, m$, inequality (34) holds. Applying inequalities (33), (34), and Proposition 3.1 completes the proof. $\square$

Our analysis for the sensitivity of the optimal execution strategy, when short selling is prohibited, is based on a result of Hager (1979). He proves that for a linearly constrained quadratic programming problem, under some conditions on the Hessian of the objective function and the Jacobian matrix of the binding constraints, both the optimal solution and the dual multipliers are Lipschitz continuous functions of the problem data. An estimate for the Lipschitz constant is discussed in §3 of Hager (1979); this result is summarized in the following theorem. Note that the upper bound presented in the following theorem is slightly tighter than the bound in Lemma 3.2 of Hager (1979); but the result essentially follows from the same proof.

Theorem 3.2. *Consider the following quadratic programming problem with the data $d = (Q, b, A, c)$*

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \frac{1}{2} x^T Q x + b^T x \\ s.t. \quad & Ax + c \le 0. \end{aligned} \tag{36}$$

Let $\mathcal{D}$ be a convex set of data so that for every $d \in \mathcal{D}$, the above problem has a unique optimal solution, denoted by $x(d)$, and a unique dual multiplier, denoted by $u(d)$. Let $J(d)$ be the set of indices corresponding to the binding constraints at $x(d)$. Assume that there exist some parameters $v_1 < +\infty$, $v_2 < +\infty$, $\beta > 0$, and $\alpha > 0$ so that for every $d = (Q, b, A, c)$ in $\mathcal{D}$:

(a) $\|Q\|_2 \le v_1$,

(b) $\left\|A^T_{J(d)}\right\|_2 \le v_2$,

(c) $\left\|A^T_{J(d)} u(d)\right\|_2 \ge \beta \|u(d)\|_2$,

(d) $x^T Q x \ge \alpha \|x\|_2^2$, *for every x such that $A_{J(d)} x = 0$.*

Then there exists a positive constant $\varrho < +\infty$ such that for every $d_1, d_2 \in \mathcal{D}$:

$$\begin{aligned} \|x(d_1) - x(d_2)\|_2 \le \quad & \varrho \left(\|b(d_1) - b(d_2)\|_2 + \|c(d_1) - c(d_2)\|_2 \right) + \varrho^2 \left(\max_{d \in \mathcal{D}} \|b(d)\|_2 + \max_{d \in \mathcal{D}} \|c(d)\|_2 \right) \\ & \left(\|Q(d_1) - Q(d_2)\|_2 + \|A(d_1) - A(d_2)\|_2 + \left\|A^T(d_1) - A^T(d_2)\right\|_2 \right), \end{aligned}$$

where $\varrho \le \frac{1}{\alpha} + \frac{1}{\beta}\left(\frac{v_1}{\alpha} + 1\right)\left(v_2 + \frac{v_2 v_1}{\beta} + 1\right)$.

In the execution cost problem, the impact matrices only appear in the objective function. Therefore, perturbations in the impact matrices do not affect the constraints of Problem (11). When the constraints in Problem (36) do not change for any $d \in \mathcal{D}$, a tighter upper bound for ϱ can be obtained as in the following corollary:

Corollary 3.2. *Let the assumptions in Theorem 3.2 hold. In addition, assume that $A(d)$ and $c(d)$ are constant on $\mathcal{D}$, i.e., $A(d) = A$ and $c(d) = c$ for every $d \in \mathcal{D}$. Then there exists a positive constant $\varrho_0 < +\infty$ such that for every $d_1, d_2 \in \mathcal{D}$*

$$\|x(d_1) - x(d_2)\|_2 \le \varrho_0 \|b(d_1) - b(d_2)\|_2 + \varrho_0^2 \left(\max_{d \in \mathcal{D}} \|b(d)\|_2 + \max\{1, \|Q\|_2\} \|c\|_2 \right) \|Q(d_1) - Q(d_2)\|_2 ,$$

where $\varrho_0 \le \frac{1}{\alpha} + \frac{1}{\beta \max\{1, \|Q\|_2\}}\left(\frac{v_1}{\alpha} + 1\right)\left(v_2 + \frac{v_2 v_1}{\beta \max\{1, \|Q\|_2\}} + 1\right)$.

Proof. Assume Problem (36) with the input data $d = (Q, b, A, c)$, and the corresponding parameters v_1, v_2, β, and α are given. Clearly Problem (36) and the following problem have the identical optimal solution:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \frac{1}{2} x^T \bar{Q} x + \bar{b}^T x \\ \text{s.t.} \quad & Ax + c \le 0, \end{aligned} \tag{37}$$

where $\bar{Q} = rQ$ and $\bar{b} = rb$ for any $r > 0$. Applying Theorem 3.2 to Problem (37) and using the fact that $A(d)$ and $c(d)$ are constant on $\mathcal{D}$, there exists a positive constant $\hat{\varrho} < +\infty$ such that for every $d_1, d_2 \in \mathcal{D}$

$$\begin{aligned}\|x(d_1) - x(d_2)\|_2 &\leq \hat{\varrho}\left(\left\|\bar{b}(d_1) - \bar{b}(d_2)\right\|_2 + \|c(d_1) - c(d_2)\|_2\right) + \hat{\varrho}^2\left(\max_{d\in\mathcal{D}}\|\bar{b}(d)\|_2 + \max_{d\in\mathcal{D}}\|c(d)\|_2\right)\\ &\quad \left(\left\|\bar{Q}(d_1) - \bar{Q}(d_2)\right\|_2 + \|A(d_1) - A(d_2)\|_2 + \left\|A^T(d_1) - A^T(d_2)\right\|_2\right)\\ &= \hat{\varrho}\,\|rb(d_1) - rb(d_2)\|_2 + \hat{\varrho}^2\left(\max_{d\in\mathcal{D}}\|rb(d)\|_2 + \|c\|_2\right)\|rQ(d_1) - rQ(d_2)\|_2\\ &= r\hat{\varrho}\,\|b(d_1) - b(d_2)\|_2 + r^2\hat{\varrho}^2\left(\max_{d\in\mathcal{D}}\|b(d)\|_2 + \frac{1}{r}\|c\|_2\right)\|Q(d_1) - Q(d_2)\|_2\end{aligned}$$

where $\hat{\varrho} \leq \frac{1}{r\alpha} + \frac{1}{\beta}\left(\frac{rv_1}{r\alpha} + 1\right)\left(v_2 + \frac{rv_2v_1}{\beta} + 1\right)$ or equivalently $r\hat{\varrho} \leq \frac{1}{\alpha} + \frac{r}{\beta}\left(\frac{v_1}{\alpha} + 1\right)\left(v_2 + \frac{rv_2v_1}{\beta} + 1\right)$. Defining $\varrho_0 = r\hat{\varrho}$ and substituting $r = \frac{1}{\max\{1, \|Q\|_2\}}$ complete the proof. □

The following theorem establishes an upper bound on the size of the change in the optimal execution strategy, due to perturbations in the impact matrices, when short selling is not permitted.

Theorem 3.3. *Assume for the given risk aversion parameter $\mu \geq 0$, $W(H, \Gamma, \mu) \succ 0$ and $W(H + \Delta H, \Gamma + \Delta\Gamma, \mu) \succ 0$. Denote the unique optimal solutions of Problems (11) and (20) with z^* and $\bar{z}$ respectively. Then there exists some $\varsigma > 0$ such that*

$$\|n^* - \bar{n}\|_2 \leq 2\,\|z^* - \bar{z}\|_2 \leq 2\varsigma\left\|\bar{S}\right\|_2\left(1 + 4\varsigma\sqrt{m}\left(\max\{1, \|W\|_2\} + \|\Theta\|_2 + \|\Delta\Theta\|_2\right)\right)\|\Delta\Theta\|_2, \tag{38}$$

where $n^* = \{n_k^*\}_{k=1}^N$ and $\bar{n} = \{\bar{n}_k\}_{k=1}^N$ are the optimal execution strategies associated with z^* and $\bar{z}$ respectively, and

$$\varsigma \leq \frac{1}{\underline{\lambda}}\left(1 + \frac{1}{2\sin^2\left(\frac{\pi}{4N-2}\right)}\left(\frac{\bar{\lambda} + \underline{\lambda}}{\max\{1, \lambda_{\max}(W)\}}\right)\left(\frac{\bar{\lambda}}{\max\{1, \lambda_{\max}(W)\}} + 3\sin\left(\frac{\pi}{4N-2}\right)\right)\right), \tag{39}$$

with $\bar{\lambda} = \max_{\eta\in[0,1]} \lambda_{\max}(W + \eta\Delta W)$ and $\underline{\lambda} = \min_{\eta\in[0,1]} \lambda_{\min}(W + \eta\Delta W)$.

Proof. For the given perturbations $\Delta\Gamma$ and ΔH of the impact matrices, define

$$\mathcal{D} \stackrel{\text{def}}{=} \{d(\eta) = (H + \eta\Delta H, \Gamma + \eta\Delta\Gamma) : \ \eta \in [0, 1]\}.$$

Clearly, $\mathcal{D}$ is a convex set. Since $W(H, \Gamma, \mu) \succ 0$ and $W(H + \Delta H, \Gamma + \Delta\Gamma, \mu) \succ 0$ for any $\eta \in [0, 1]$, we have

$$W + \eta\Delta W = W(H + \eta\Delta H, \Gamma + \eta\Delta\Gamma, \mu) = (1 - \eta)W(H, \Gamma, \mu) + \eta W(H + \Delta H, \Gamma + \Delta\Gamma, \mu) \succ 0.$$

Therefore, for any $\eta \in [0, 1]$, Problem (11) with the impact matrices $H + \eta\Delta H$ and $\Gamma + \eta\Delta\Gamma$ has a unique optimal solution, denoted as $z(\eta)$. For a given $\eta \in [0, 1]$, let $J(\eta)$ denote the set of indices corresponding to the binding constraints of Problem (11) at $z(\eta)$. Lemma 3.1 implies that $\lambda_{\min}\left(A_{J(\eta)}A_{J(\eta)}^T\right) > 0$ and consequently $A_{J(d)}A_{J(d)}^T$ is invertible. Thus the rows of $A_{J(d)}$ are linearly independent and Problem (11) has a unique dual multiplier $u(\eta)$.

Define

$$\bar{\lambda} \stackrel{\text{def}}{=} \max\{\lambda_{\max}(W + \eta\Delta W) : \eta \in [0, 1]\}, \quad \underline{\lambda} \stackrel{\text{def}}{=} \min\{\lambda_{\min}(W + \eta\Delta W) : \eta \in [0, 1]\}.$$

Since for any $\eta \in [0, 1]$, $W + \eta\Delta W$ is symmetric positive definite, $\bar{\lambda}$ and $\underline{\lambda}$ are positive. In addition,

$$\|W + \eta\Delta W\|_2 = \lambda_{\max}(W + \eta\Delta W) \leq \bar{\lambda} \quad \forall\eta \in [0, 1]. \tag{40}$$

Using the corollary of the Courant-Fischer Theorem, we have $\lambda_{\max}(W+\eta\Delta W) \le \lambda_{\max}(W) + \eta\lambda_{\max}(\Delta W)$, which implies $\bar{\lambda} < +\infty$. Furthermore, for any $\eta \in [0,1]$, $W + \eta\Delta W$ is symmetric. The Courant-Fischer Theorem yields

$$z^T(W+\eta\Delta W)z \ge \lambda_{\min}(W+\eta\Delta W)\|z\|_2^2 \ge \underline{\lambda}\|z\|_2^2 \quad \forall \eta \in [0,1]. \tag{41}$$

Applying the definitions of 1-norm, $\|.\|_1$, and ∞-norm, $\|.\|_\infty$, for the matrix $A^T_{J(\eta)}$, we get

$$\left\|A^T_{J(\eta)}\right\|_1 = \max_{i\in J(\eta)} \sum_{j=1}^{N-1} |a_{ij}| \le \max_{i=1,2,\ldots,N} \sum_{j=1}^{N-1} |a_{ij}| = \|A\|_\infty, \quad \forall \eta \in [0,1]$$

$$\left\|A^T_{J(\eta)}\right\|_\infty = \max_{j=1,2,\ldots,N-1} \sum_{i\in J(\eta)} |a_{ij}| \le \max_{j=1,2,\ldots,N-1} \sum_{i=1}^{N} |a_{ij}| = \|A\|_1, \quad \forall \eta \in [0,1],$$

where a_{ij} is the entry of A in the ith row and jth column. Hence

$$\left\|A^T_{J(\eta)}\right\|_2 \le \sqrt{\left\|A^T_{J(\eta)}\right\|_1 \left\|A^T_{J(\eta)}\right\|_\infty} \le \sqrt{\|A\|_1\|A\|_\infty} \le \sqrt{\|(Y\otimes I_m)\|_1\|(Y\otimes I_m)\|_\infty} \le 2,$$

where the last inequality follows from $\|Y\otimes I_m\|_\infty = 2$ and $\|Y\otimes I_m\|_1 = 2$. Therefore,

$$\left\|A^T_{J(\eta)}\right\|_2 \le 2 \quad \forall \eta \in [0,1]. \tag{42}$$

For any $\eta \in [0,1]$ and the associated unique optimal dual multiplier $u(\eta)$, the Courant-Fischer Theorem yields

$$\left\|A^T_{J(\eta)}u(\eta)\right\|_2^2 = u(\eta)^T A_{J(\eta)}A^T_{J(\eta)}u(\eta) \ge \lambda_{\min}\left(A_{J(\eta)}A^T_{J(\eta)}\right)\|u(\eta)\|_2^2 \ge \lambda_{\min}\left(G_{N-1}G^T_{N-1}\right)\|u(\eta)\|_2^2, \tag{43}$$

where the last inequality comes from Lemma 3.1. Hence

$$\left\|A^T_{J(\eta)}u(\eta)\right\|_2 \ge 2\sin\left(\frac{\pi}{4N-2}\right)\|u(\eta)\|_2 \quad \forall \eta \in [0,1]. \tag{44}$$

Inequalities (40), (41), (42), and (44) show that the assumptions of Theorem 3.2 are satisfied on the convex data set $\mathcal{D}$ for

$$v_1 \overset{\text{def}}{=} \bar{\lambda}, \quad v_2 \overset{\text{def}}{=} 2, \quad \beta \overset{\text{def}}{=} 2\sin\left(\frac{\pi}{4N-2}\right) > 0, \quad \alpha \overset{\text{def}}{=} \underline{\lambda} > 0.$$

Applying Corollary 3.2 to Problem (11), there exists some ς such that

$$\begin{aligned}
\|z^* - \bar{z}\|_2 &\le \varsigma\|\Delta b\|_2 + \varsigma^2\|\Delta W\|_2\left(\max_{\eta\in[0,1]}\|b(H+\eta\Delta H, \Gamma+\eta\Delta\Gamma)\|_2 + \max\{1,\|W\|_2\}\|\bar{S}\|_2\right)\\
&= \varsigma\|\Delta\Theta\bar{S}\|_2 + \varsigma^2\|\Delta W\|_2\left(\max_{\eta\in[0,1]}\|(\Theta+\eta\Delta\Theta)\bar{S}\|_2 + \max\{1,\|W\|_2\}\|\bar{S}\|_2\right)\\
&\le \varsigma\|\bar{S}\|_2\left(\|\Delta\Theta\|_2 + \varsigma\|\Delta W\|_2\left(\|\Theta\|_2 + \|\Delta\Theta\|_2 + \max\{1,\|W\|_2\}\right)\right),
\end{aligned}$$

where $\varsigma \le \frac{1}{\underline{\lambda}}\left(1 + \frac{1}{2\sin^2\left(\frac{\pi}{4N-2}\right)}\left(\frac{\bar{\lambda}+\underline{\lambda}}{\max\{1,\lambda_{\max}(W)\}}\right)\left(\frac{\bar{\lambda}}{\max\{1,\lambda_{\max}(W)\}} + 3\sin\left(\frac{\pi}{4N-2}\right)\right)\right)$. Applying inequality (28), i.e., $\|\Delta W\|_2 \le 4\sqrt{m}\|\Delta\Theta\|_2$, and inequality (23) completes the proof. $\square$

Theorem 3.3 provides an upper bound for the size of the change in the optimal execution strategy, when short selling is not permitted. For a given N, the upper bound of the magnification factor ς depends, at least asymptotically (as $\Delta W \to 0$), only on the eigenvalues of the Hessian matrix W.

Similar to Theorem 3.1, a small value of ς guarantees that the optimal execution strategy is not very sensitive to the perturbation in the combined impact matrix $\Delta\Theta$. As $\Delta W \to 0$, the term

$$\left(1+\frac{1}{2\sin^2\left(\frac{\pi}{4N-2}\right)}\left(\frac{\bar{\lambda}+\underline{\lambda}}{\max\{1,\lambda_{\max}(W)\}}\right)\left(\frac{\bar{\lambda}}{\max\{1,\lambda_{\max}(W)\}}+3\sin\left(\frac{\pi}{4N-2}\right)\right)\right), \tag{45}$$

is bounded by a constant which depends only on N. Therefore, for the fixed number of periods N, asymptotically (as $\Delta W \to 0$), the upper bound for the magnification factor ς is small when $\lambda_{\min}(W)$ is large.

The eigenvalues $\lambda_{\max}(W)$, $\underline{\lambda}$ and $\bar{\lambda}$ increase with the same rate as the risk aversion parameter μ increases, and consequently, all the terms in (45) are bounded as $\mu \to +\infty$. However, when $C \succ 0$, $\frac{1}{\underline{\lambda}}$ approaches zero as $\mu \to +\infty$. Therefore, as the risk aversion parameter μ increases, the upper bound for ς becomes small. Hence, when the covariance matrix is positive definite, $\|z^* - \bar{z}\|_2 \to 0$ as $\mu \to +\infty$. This result indicates that, similar to the case that short selling is allowed, the sensitivity of the optimal execution strategy to perturbations in the impact matrices diminishes as the risk aversion parameter μ increases. Furthermore, when the risk aversion parameter is positive and $\lambda_{\min}(C)$ is large (or equivalently $\kappa_2(C)$ is small), the optimal execution strategy is not very sensitive to the perturbations.

We can express the upper bound for the magnification factor ς provided in Theorem 3.3 in terms of the eigenvalues of C and Θ, when the permanent impact matrix Γ and its perturbation $\Delta\Gamma$ are symmetric. Under these assumptions, the Courant-Fischer Theorem can be applied, and we have

$$\begin{aligned}
\underline{\lambda} &= \min_{\eta\in[0,1]} \lambda_{\min}(W+\eta\Delta W) \geq 2\mu\tau\lambda_{\min}(C) + \lambda_{\min}(G_{N-1}+G_{N-1}^T)\min_{\eta\in[0,1]}\lambda_{\min}(\Theta+\eta\Delta\Theta), \qquad (46)\\
\bar{\lambda} &= \max_{\eta\in[0,1]} \lambda_{\max}(W+\eta\Delta W) \leq 2\mu\tau\lambda_{\max}(C) + \lambda_{\max}(G_{N-1}+G_{N-1}^T)\max_{\eta\in[0,1]}\lambda_{\max}(\Theta+\eta\Delta\Theta).
\end{aligned}$$

Applying these inequalities, the upper bound in (39) can be simplified as follows:

Corollary 3.3. *Let the assumptions in Theorem 3.3 hold. In addition, assume that the matrices Γ and $\Delta\Gamma$ are symmetric. Then there exists a magnification factor ς_0 such that*

$$\|n^* - \bar{n}\|_2 \leq 2\|z^*-\bar{z}\|_2 \leq \varsigma_0\|\bar{S}\|_2\left(1+2\varsigma_0\sqrt{m}\left(\max\{1,\|W\|_2\}+\|\Theta\|_2+\|\Delta\Theta\|_2\right)\right)\|\Delta\Theta\|_2,$$

where

$$\begin{aligned}
\varsigma_0 \;\leq\; & \left(\frac{1}{\mu\tau\lambda_{\min}(C)+2\sin^2\left(\frac{\pi}{2N}\right)\min\limits_{\eta\in[0,1]}\lambda_{\min}(\Theta+\eta\Delta\Theta)}\right) \qquad (47)\\
& \left(1+\frac{1}{2\sin^2\left(\frac{\pi}{4N-2}\right)}\left(\frac{\bar{\lambda}+\underline{\lambda}}{\max\{1,\lambda_{\max}(W)\}}\right)\left(\frac{\bar{\lambda}}{\max\{1,\lambda_{\max}(W)\}}+3\sin\left(\frac{\pi}{4N-2}\right)\right)\right).
\end{aligned}$$

Inequality (47) indicates that when the permanent impact matrices Γ and $\Gamma+\Delta\Gamma$ are symmetric, the magnification factor ς_0 asymptotically (as $\Delta W \to 0$) depends on $\mu\tau\lambda_{\min}(C)+2\sin^2\left(\frac{\pi}{2N}\right)\lambda_{\min}(\Theta)$. In this case, for a given positive risk aversion parameter, ς_0 becomes small when *either* the minimum eigenvalue of the covariance matrix *or* the minimum eigenvalue of the combined impact matrix Θ is large. However, when both $\mu\lambda_{\min}(C)$ and $\lambda_{\min}(\Theta)$ are small, the upper bound for ς_0 in (47) becomes large, which suggests pronounced sensitivity of the optimal execution strategy to estimation errors in the impact matrices.

Theorems 3.1 and 3.3 imply that for a given risk aversion parameter $\mu \geq 0$ and for every given pair of impact matrices H and Γ, where $W(H,\Gamma,\mu) \succ 0$, the unique optimal execution strategy is Lipschitz continuous at H and Γ with respect to the Euclidean norm.

Both upper bounds in inequalities (24) and (38) indicate that the change in the optimal execution strategy increases proportionally with respect to the size of the initial portfolio holding $\bar{S}$. Next, we precisely analyze the dependence of the optimal execution strategy on the initial portfolio holding.

Proposition 3.2. *Consider the execution cost problem (11) with the impact matrices H and Γ where $W(H,\Gamma,\mu) \succ 0$. Let z^* be the optimal solution with the initial portfolio holding $\bar{S}$. Then, for every $\alpha \geq 0$, αz^* is the optimal solution of Problem (11) with the initial portfolio holding $\alpha\bar{S}$.*

Proof. First note that z is a feasible solution of Problem (11) with the initial portfolio holding $\bar{S}$ if and only if αz is a feasible solution of Problem (11) with the initial portfolio holding $\alpha\bar{S}$. Since z^* is the optimal solution, for every feasible solution z of Problem (11), we have

$$\frac{1}{2}(z^*)^T W(H,\Gamma,\mu) z^* - (\Theta\bar{S})^T x_1^* \le \frac{1}{2} z^T W(H,\Gamma,\mu) z - (\Theta\bar{S})^T x_1,$$

where x_1^* and x_1 are execution positions in the first period corresponding to z^* and z respectively. Therefore, multiplying the above inequality by α^2, we get

$$\frac{1}{2}(\alpha z^*)^T W(H,\Gamma,\mu)(\alpha z^*) - (\alpha\Theta\bar{S})^T(\alpha x_1^*) \le \frac{1}{2}(\alpha z)^T W(H,\Gamma,\mu)(\alpha z) - (\alpha\Theta\bar{S})^T(\alpha x_1).$$

This result yields αz^* is the optimal solution of Problem (11) with the initial portfolio holding $\alpha\bar{S}$. □

A similar result holds for problem (12). Proposition 3.2 yields that, when $\bar{S}$ is multiplied by some nonnegative scalar α, the change in the optimal execution strategy is also multiplied by α. However, the coordinates (variance, mean) of the efficient frontier are multiplied by α^2. This illustrates the significant effect of estimation errors in the impact matrices on trades with large volumes.

In the next section, we use simulation to illustrate the sensitivity of the optimal execution strategy and the efficient frontier to perturbations in the impact matrices.

4 Computational Investigation

In this section, we use simulations to computationally investigate the influence of perturbations in the impact matrices on the optimal execution strategy and efficient frontier. The covariance matrix is assumed to be given. The simulations are done using MATLAB Version 6.5.

Consider an investor who holds a portfolio of three different assets with the initial holding $\bar{S}_i = 10^5$, $i = 1,2,3$. The goal is to liquidate the holdings in five days by trading daily, i.e., $T = 5$, $N = 5$, and $\tau = 1$. Let the true daily asset price covariance matrix be

$$C = \begin{pmatrix} 0.3246 & 0.0230 & 0.4204 \\ 0.0230 & 0.0499 & 0.0192 \\ 0.4204 & 0.0192 & 0.7641 \end{pmatrix} \times 1\%,$$

Note that $\lambda_{\min}(C) = 0.0005$. The price impact model (3) assumes that the price impacts are proportional to the trading rate. Assume that the median daily trading volume of each asset is one million shares. For the temporary impact matrix, we suppose that for each 10% of the daily volume traded, the price impact equals the daily variance. In addition, we assume that selling 20% of the daily volume incurs a permanent price depression equal to the daily variance. In other words,

$$H = \frac{C}{0.10 \times 10^6} = 10^{-5} C \ \ \$/\text{share}^2, \qquad \Gamma = \frac{C}{0.20 \times 10^6} = (0.5 \times 10^{-5}) C \ \ \$/\text{share}^2.$$

Note that $W(H,\Gamma,0) \succ 0$ and $\lambda_{\min}(W(H,\Gamma,0)) = 2.5960 \times 10^{-9}$. Throughout this section, we refer to H and Γ as the *true* impact matrices, and to the corresponding optimal execution strategy as the *true optimal execution strategy*.

In our simulation investigation, we assume that perturbations in the impact matrices have independent normal distributions. Specifically,

$$\Delta H = \rho \max\{\|He_i\|_\infty,\ i = 1,2,3\}\,\Phi, \quad \Delta\Gamma = \rho \max\{\|\Gamma e_i\|_\infty,\ i = 1,2,3\}\,\Psi, \tag{48}$$

where Φ and Ψ are 3×3 random matrices whose elements are independent zero-mean Gaussian random variables with unit variance. We use the `randn` command in MATLAB to generate Φ and Ψ. The parameter $\rho \in [0,1]$ indicates the size of the relative perturbation.

In order to ensure that the optimal execution strategy corresponding to the perturbed impact matrices, $H + \Delta H$ and $\Gamma + \Delta\Gamma$, is unique, we only consider perturbations with $W(H + \Delta H, \Gamma + \Delta\Gamma, 0) \succ 0$. We refer to the optimal execution strategy determined from a pair of perturbed impact matrices as the *estimated optimal execution strategy*. We use the convex quadratic optimization solver mskqpopt in the software package MOSEK Version 4.0 to compute the optimal execution strategies, both in the presence and absence of short selling constraints.

In §4.1, we illustrate the sensitivity of the optimal execution strategy to perturbations in the impact matrices and present some typical plots of true and estimated optimal execution strategies . The sensitivity of the efficient frontier is demonstrated in §4.2.

4.1 The Sensitivity of The Optimal Execution Strategy

In this section, we investigate the effect of the risk aversion parameter μ and no short selling constraints on the sensitivity of the optimal execution strategy. For illustration, we focus on the case when the risk aversion parameter $\mu = 0$, which corresponds to minimizing the expected execution cost, and $\mu = 10^{-5}$. Following Proposition 2.1, the true optimal execution strategy, which minimizes the expected execution cost, is the naive execution strategy $n_k = \frac{1}{5}\bar{S}$ for $k = 1, \ldots, 5$, since in our assumed setting Γ is symmetric and $\Theta \succ 0$. On the other hand, the perturbed impact matrices $H + \Delta H$ and $\Gamma + \Delta\Gamma$ from (48) are typically not symmetric. For each simulation study, a total of $M = 50$ simulations is used unless otherwise stated. In addition, a relative perturbation $\rho = 0.05$ is assumed.

Figure 1 plots the true optimal execution strategy when $\mu = 0$ (the naive execution strategy) against the optimal execution strategies from the perturbed impact matrices. The left plots are generated under the assumption that short selling is allowed. For the plots on the right, it is assumed that short selling is not permitted. Graphs in Figure 1 demonstrate that the optimal execution strategy in this case is quite sensitive to perturbations in the impact matrices. In addition, these plots illustrate that imposing short selling constraints on the problem significantly decreases the sensitivity of the optimal execution strategy to perturbations. Note that the range in the number of shares traded (vertical axis), when short selling is allowed, is much larger than the range after imposing short selling constraints.

For the risk aversion parameter $\mu = 10^{-5}$, the true optimal execution strategy and estimated optimal execution strategies associated with the same set of perturbed impact matrices are plotted in Figure 2. Similar to the previous case, the left plots are generated under the assumption that short selling is allowed. For the plots on the right, it is assumed that short selling is not permitted. Comparing Figure 2 with Figure 1, it is clear that, the sensitivity of the optimal execution strategy to perturbations in the impact matrices is decreased when the risk aversion parameter $\mu = 10^{-5}$. Moreover, in Figure 2, there is little difference in the sensitivity of the optimal execution strategy to perturbations whether short selling constraints are imposed or not.

In addition, we compute, for each asset, the ratio of the average difference, between the true and estimated optimal execution strategies, to the initial holding, i.e.,

$$\varepsilon_i(\mu) \stackrel{\text{def}}{=} \frac{1}{M\bar{S}_i} \sum_{\ell=1}^{M} \|n_i^{(\ell)} - n_i^*\|_1, \qquad i = 1, 2, 3,$$

where the vector $n_i^{(\ell)}$ is the estimated optimal execution cost strategy of the ith asset in the ℓth simulation. Table 1 presents the values of $\varepsilon_i(\mu)$ for various choices of μ. From Table 1, we observe that, whether short selling is allowed or not, the relative average error $\varepsilon_i(\mu)$ decreases as the risk aversion parameter μ increases. For example while the relative average error in asset 2 is 35.73191% for $\mu = 10^{-5}$, it becomes less than 0.1% for $\mu \geq 0.05$. This observation is consistent with our analytical result that the change in the optimal execution strategy decreases as the risk aversion parameter increases. Table 1 also confirms that the optimal execution strategy when short selling is prohibited is less sensitive than the one obtained when short selling is allowed. This difference is more striking for small values of μ. As μ increases, the difference between the two cases almost diminishes. While $\varepsilon_i(0)$ when short selling is allowed is almost twice as that when short selling is prohibited, the value of $\varepsilon_i(10^{-5})$ in both cases is almost identical.

Given $\Gamma = \Gamma^T$ in our example, Proposition 2.1 implies that, when perturbation in the permanent impact matrix satisfies $\Delta\Gamma^T = \Delta\Gamma$, the unique minimum expected execution cost strategy is the naive execution

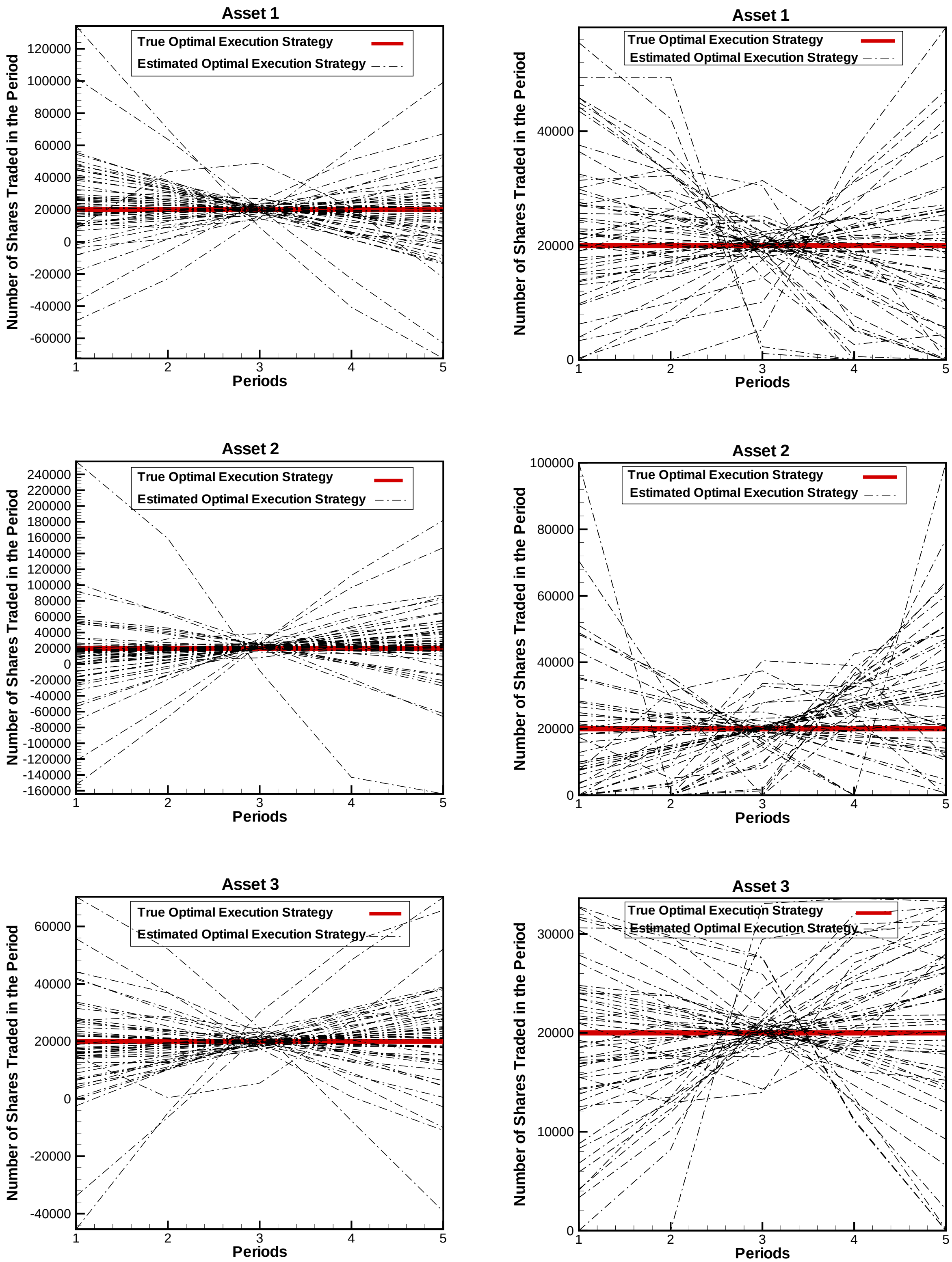


Figure 1: Optimal execution strategies for $\mu = 0$ with 5% relative perturbation ($\rho = 0.05$) for $M = 50$ simulations. For plots on the left, short selling is allowed. Short selling is prohibited for plots on the right.

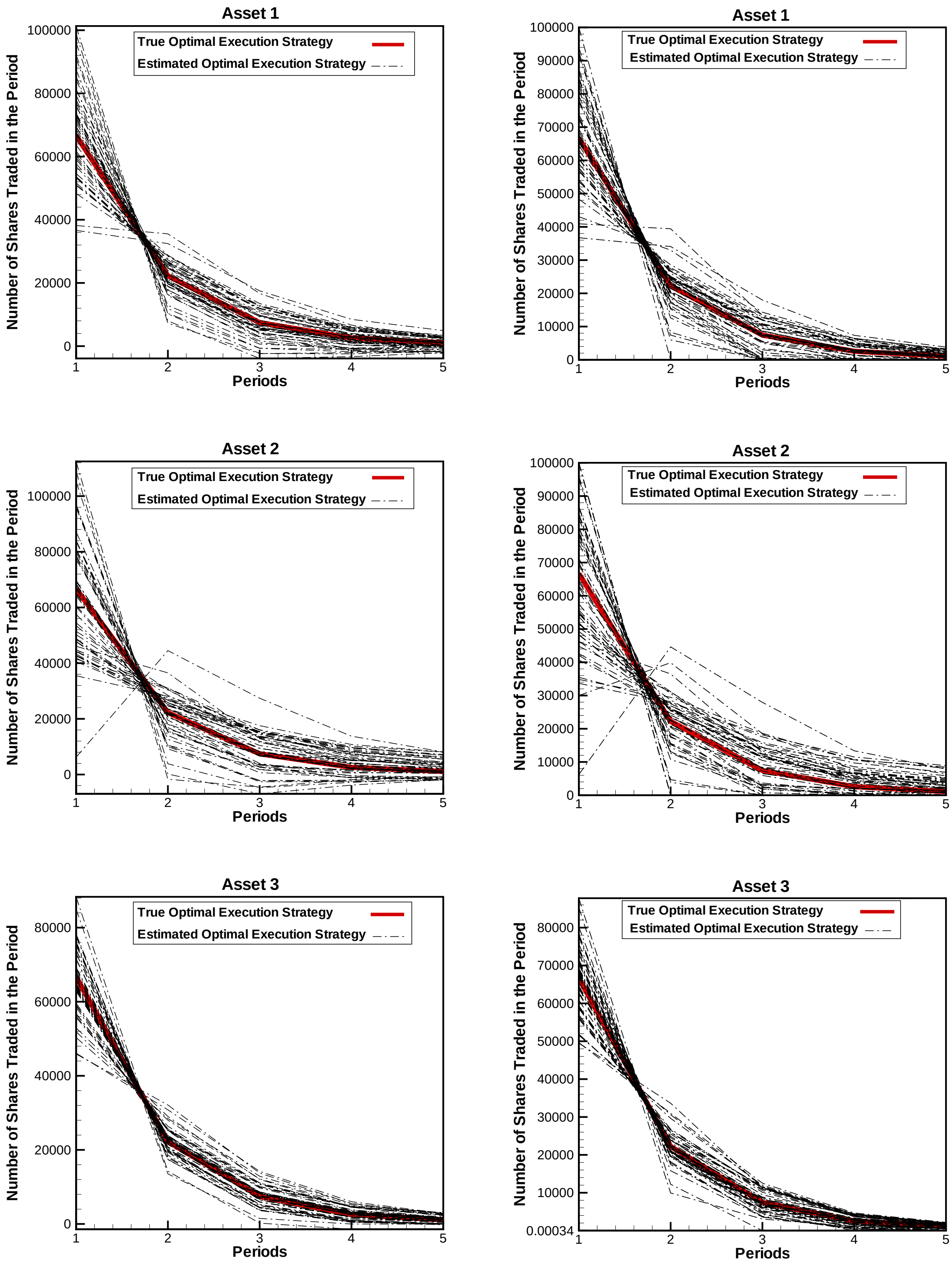


Figure 2: Optimal execution strategies for $\mu = 10^{-5}$ with 5% relative perturbation ($\rho = 0.05$) for $M = 50$ simulations. For plots on the left, short selling is allowed. Short selling is prohibited for plots on the right.

μ	Short selling is allowed			Short selling is prohibited		
	Asset 1	Asset 2	Asset 3	Asset 1	Asset 2	Asset 3
0	62.62835	105.79562	39.85288	38.62073	59.90888	24.29109
10^{-5}	23.86030	35.73191	14.20801	23.02855	35.47777	13.96043
0.05	0.01979	0.03288	0.01178	0.01907	0.03094	0.01147
0.5	0.00198	0.00329	0.00118	0.00191	0.00310	0.00115
1	0.00099	0.00165	0.00059	0.00095	0.00155	0.00057
100	0.00001	0.00003	0.00001	0.00001	0.00003	0.00000

Table 1: Relative average error $\varepsilon_i(\mu)$ (percentages) with 5% relative perturbation based on 50 simulations with general (likely asymmetric) perturbations in the permanent impact matrix.

μ	Short selling is allowed			Short selling is prohibited		
	Asset 1	Asset 2	Asset 3	Asset 1	Asset 2	Asset 3
0	9.51×10^{-10}	1.04×10^{-9}	5.10×10^{-10}	2.08×10^{-5}	1.78×10^{-4}	1.23×10^{-5}
10^{-5}	25.86364	42.39184	14.98491	22.66105	32.06339	13.13987
0.05	0.01991	0.03900	0.01188	0.01931	0.03182	0.01166
0.5	0.00199	0.00390	0.00119	0.00193	0.00319	0.00117
1	0.00100	0.00195	0.00059	0.00097	0.00159	0.00058
100	0.00001	0.00002	0.00001	0.00001	0.00003	0.00000

Table 2: Relative average error $\varepsilon_i(\mu)$ (percentage) with 5% relative perturbations based on 50 simulations with symmetric perturbations in the permanent impact matrix.

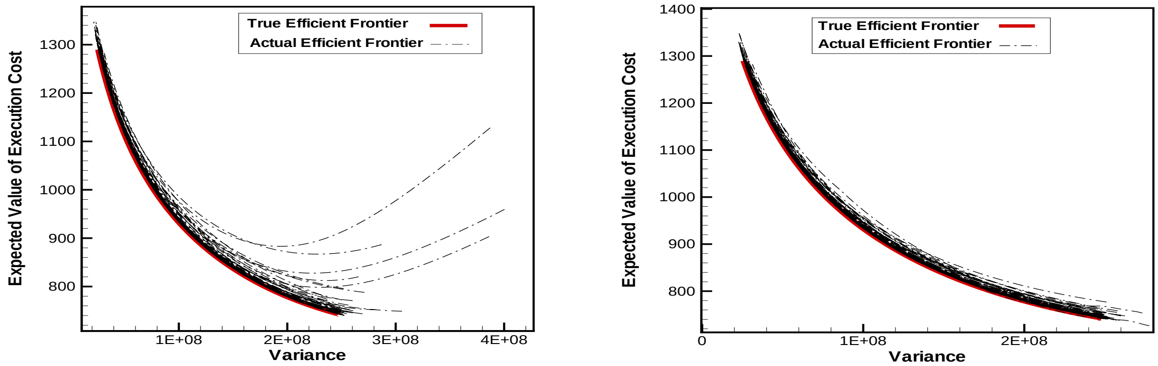
strategy. In our computation setting, the only possible violation is the asymmetric perturbation in the permanent impact matrix. This evinces the importance of maintaining symmetry in estimating the permanent impact matrix if it is known or assumed to be symmetric. To illustrate how restricting to symmetric perturbations affects the sensitivity of the optimal execution strategy, we compute $\varepsilon_i(\mu)$ for estimated optimal execution strategies with the perturbed impact matrices $H + \Delta H$ and $\Gamma + \frac{1}{2}(\Delta\Gamma + \Delta\Gamma^T)$, where ΔH and $\Delta\Gamma$ are determined according to (48). Table 2 presents these values.

Comparing Table 2 with Table 1, we observe that the relative average error $\varepsilon_i(0)$, with symmetric perturbations in the permanent impact matrix, is much smaller than the relative average error $\varepsilon_i(0)$ in Table 1 with asymmetric perturbations. However for $\mu \geq 10^{-5}$ there is little difference in the relative average errors. Thus, when the permanent impact matrix Γ is known to be symmetric and this property is maintained with its estimate, an investor who wants to minimize the expected execution cost need not to worry about the effect of estimation errors in the impact matrices on the optimal execution strategy.

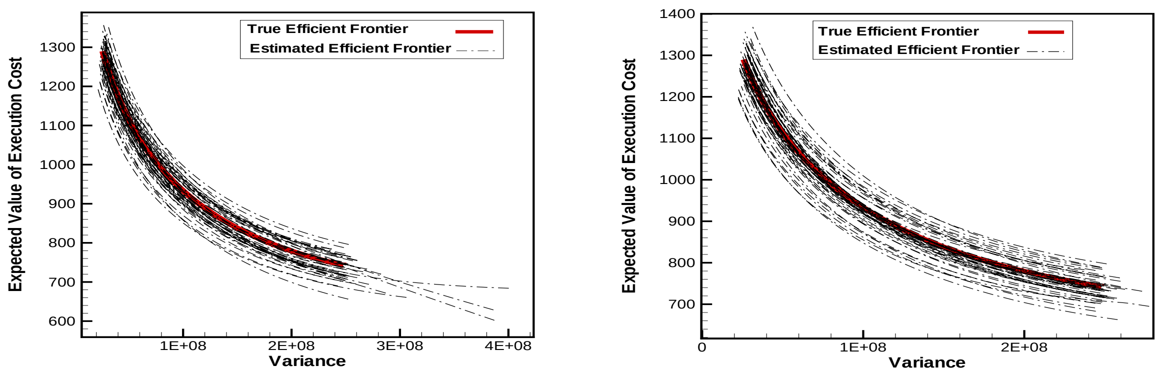
4.2 The Sensitivity of The Efficient Frontier

In this section, we illustrate the effect of perturbations in the impact matrices on the efficient frontier in the space of the variance and the expected execution cost. For a given pair of perturbed impact matrices $H + \Delta H$ and $\Gamma + \Delta\Gamma$, we compute the following three efficient frontiers of the optimal execution strategies for $\mu \in [0, 10^{-5}]$:

- The *true (efficient) frontier* is the efficient frontier computed from the true values of the impact matrices H and Γ.
- The *actual (efficient) frontier* is the curve of the true mean and variance of the execution cost of the optimal execution strategy determined from the perturbed impact matrices $H + \Delta H$ and $\Gamma + \Delta\Gamma$.



(a) Actual Frontiers versus the True Frontier



(b) Estimated Frontiers versus the True Frontier

Figure 3: Actual and estimated frontiers with 5% relative asymmetric perturbations for 50 simulations. Short selling is allowed for plots on the left. Short selling is prohibited for plots on the right.

The actual frontier depicts the true performance of estimated optimal execution strategies for various values of μ.

- The *estimated (efficient) frontier* is the efficient frontier based on the perturbed impact matrices $H + \Delta H$ and $\Gamma + \Delta\Gamma$.

The notions of the actual frontier and estimated frontier have been used in Broadie (1993) to investigate the effect of estimation errors in mean returns and the covariance matrix in the traditional mean-variance portfolio optimization. As mentioned in Broadie (1993), the estimated frontier is what appears to be the case based on estimated input data, but the actual frontier is what really occurs based on the true values of the data. Since the true values of the data are unknown, the true and actual frontiers are unobservable in practice. Note that actual frontiers can never be below the true efficient frontier as the execution cost problem is a minimization problem. However, the estimated frontiers can be either above or below the actual and true frontiers.

When perturbation in the permanent impact matrix is asymmetric, the effect of perturbations in the impact matrices on the efficient frontier is demonstrated in Figure 3. Figure 3 (a) illustrates deviations of actual frontiers from the true efficient frontier for $\mu \in [0, 10^{-5}]$. From the plot on the left generated under the assumption that short selling is allowed, it can be observed that large deviations of actual frontiers can occur, particularly when the risk aversion parameter is very small. Moreover, the lengths of the actual frontier from different simulations vary drastically; the lengths of some actual frontiers largely differ from the length of the true frontier. Figure 3 (a) also demonstrates that, similar to the sensitivity of the optimal execution strategy, the change in the efficient frontier decreases as μ increases. Therefore, an execution strategy that minimizes the variance of the execution cost can be estimated more accurately than the one that minimizes the mean of the execution cost. This is consistent with our theoretical results.

Comparing the right plot to the left plot in Figure 3 (a), deviations of the actual frontiers from the true frontier are significantly reduced in the right plot in which short selling is prohibited. There is also less variation in the length of the actual frontiers. Thus, imposing no short selling constraints significantly decreases the effect of estimation errors in impact matrices. Similar phenomena has been observed in the mean-variance portfolio optimization, e.g., see Frost and Savarino (1988), Best and Grauer (1991).

Figure 3 (b) depicts deviations of estimated frontiers from the true frontier. Comparing to plots in Figure 3 (a), there seems to be less difference in the deviations for different risk aversion parameters, whether short selling constraints are imposed or not. In addition, for a large risk aversion parameter, we observe larger deviations in the estimated frontiers than in the actual frontiers. On the other hand, deviations of estimated frontiers are smaller than those of actual frontiers for a small risk aversion parameter.

Let perturbed impact matrices be $H + \Delta H$ and $\Gamma + \frac{1}{2}(\Delta\Gamma + \Delta\Gamma^T)$ where ΔH and $\Delta\Gamma$ are defined in (48); thus the permanent impact matrix perturbation is symmetric. For these symmetric perturbations in the permanent impact matrix, Figure 4 illustrates that the differences between actual frontiers and the true efficient frontier are significantly reduced when the risk aversion parameter is small. In particular, when the risk aversion parameter is near zero, actual frontiers are very close to the true frontier. Maintaining symmetry does not seem to affect the sensitivity at the left end of the frontier for a large value of μ. In addition, we note that the assumption of symmetry in the permanent impact matrix has little effect in the estimated frontiers.

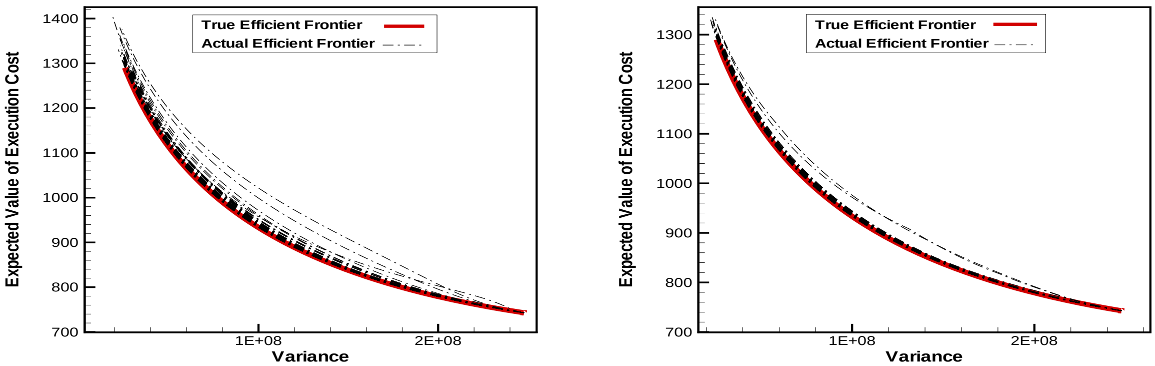
We now compare the sensitivity of the mean of the execution cost with the sensitivity of the variance of the execution cost. Figure 5 displays (mean,variance) points on the actual frontiers for $\mu = 0$ and $\mu = 10^{-5}$. The left plots are generated when short selling is permitted. For the plots on the right, short selling is prohibited. This figure suggests that, when $\mu = 0$, variation in the variance of the execution cost is relatively larger than the variation in the mean of the execution cost. For $\mu = 10^{-5}$, on the other hand, the relative variation in the mean is larger than variance of the execution cost.

To quantify the relative variations in the mean and variance of the actual execution cost, for a given risk aversion parameter μ and a relative perturbation ρ, we consider the following measure,

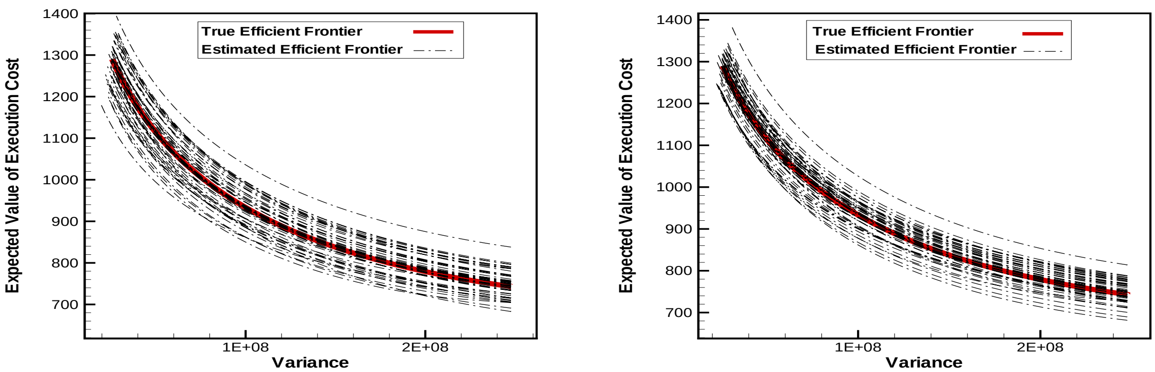
$$\varepsilon_{\text{var}}(\mu,\rho) \stackrel{\text{def}}{=} \frac{\max_{j=1}^{M} \text{var}_j(\mu) - \min_{j=1}^{M} \text{var}_j(\mu)}{|\text{true variance for } (\mu=0) - \text{ true variance for } (\mu=10^{-5})|},$$
$$\varepsilon_{\text{mean}}(\mu,\rho) \stackrel{\text{def}}{=} \frac{\max_{j=1}^{M} \text{mean}_j(\mu) - \min_{j=1}^{M} \text{mean}_j(\mu)}{|\text{true mean for } (\mu=0) - \text{true mean for } (\mu=10^{-5})|},$$

where $(\text{var}_j(\mu), \text{mean}_j(\mu))$ is the coordinate of the actual frontier for the jth simulation. Table 3 displays values of $\varepsilon_{\text{var}}(\mu,\rho)$ and $\varepsilon_{\text{mean}}(\mu,\rho)$ based on 50 simulations. These results illustrate that, whether short selling is allowed or not, the relative variation $\varepsilon_{\text{var}}(0,\rho)$ is larger than the relative variation in mean $\varepsilon_{\text{mean}}(0,\rho)$. On the other hand, the relative variation in variance $\varepsilon_{\text{var}}(10^{-5},\rho)$ is less than the relative variation in mean $\varepsilon_{\text{mean}}(10^{-5},\rho)$. In addition, Table 3 demonstrates that the difference in both mean and variance increases quickly and nonlinearly as the relative perturbation ρ increases. The fast increase is particularly prominent when short selling is permitted.

In summary, our computational investigation suggests that the effect of estimation errors in impact matrices on the execution strategy and efficient frontiers can be quite large in general. Moreover, the effect of these errors varies with the risk aversion parameter. For a large risk aversion parameter, the difference between the true frontier and the actual frontier is small. In addition, we consistently observe that imposing



(a) Actual Frontiers versus the True Frontier

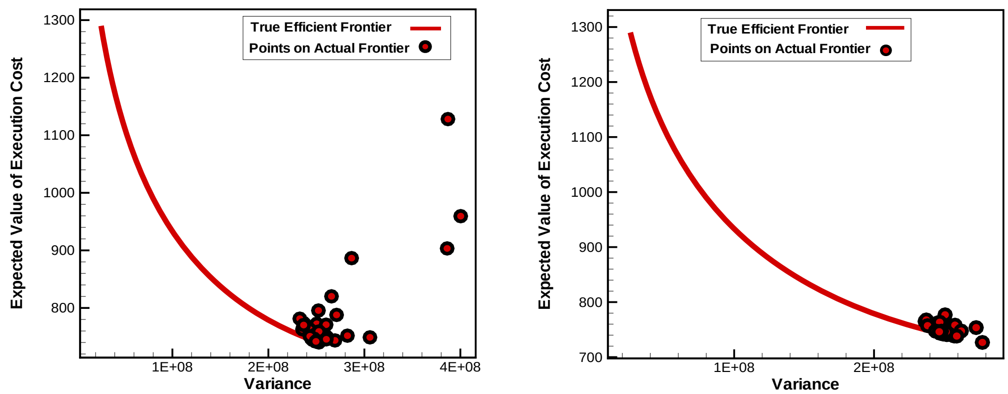


(b) Estimated Frontiers versus the True Frontier

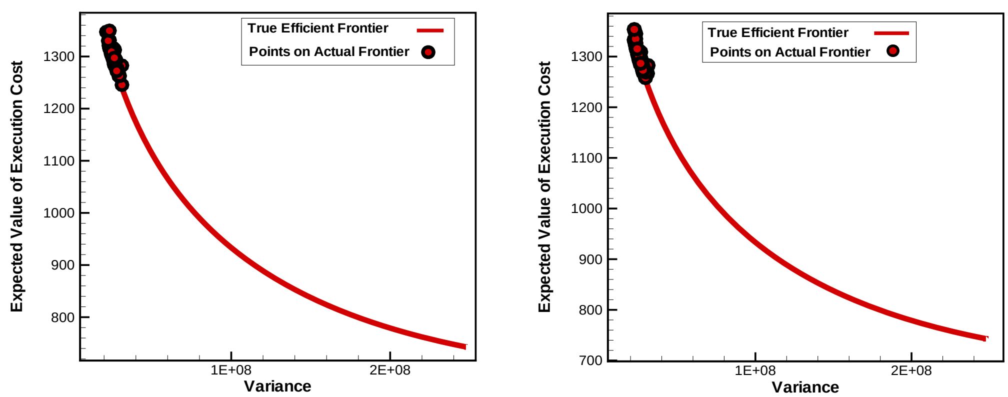
Figure 4: Actual and estimated frontiers with 5% relative symmetric perturbations in the permanent impact matrix for 50 simulations. Short selling is allowed for the plots on the left. Short selling is prohibited for plots on the right.

ρ	Short selling is allowed				Short selling is prohibited			
	$\mu = 0$		$\mu = 10^{-5}$		$\mu = 0$		$\mu = 10^{-5}$	
	ε_{var}	$\varepsilon_{\text{mean}}$	ε_{var}	$\varepsilon_{\text{mean}}$	ε_{var}	$\varepsilon_{\text{mean}}$	ε_{var}	$\varepsilon_{\text{mean}}$
0.02	0.16026	0.09950	0.01271	0.05570	0.04643	0.03685	0.02038	0.08873
0.05	0.75538	0.70920	0.04420	0.18996	0.18433	0.09185	0.03951	0.17412
0.08	3.83520	3.05705	0.05605	0.26419	0.29582	0.18370	0.06779	0.33724
0.10	45.16636	33.01414	0.08737	0.36430	0.39132	0.14567	0.06600	0.27287

Table 3: Relative variation in mean and variance of the execution cost: $\varepsilon_{\text{var}}(\mu, \rho)$ and $\varepsilon_{\text{mean}}(\mu, \rho)$.



(a) Risk Aversion Parameter $\mu = 0$



(b) Risk Aversion Parameter $\mu = 10^{-5}$

Figure 5: Points on the actual frontier for $\mu = 0$ and $\mu = 10^{-5}$ with 5% relative asymmetric perturbations for 50 simulations. Short selling is allowed for plots on the left. For plots on the right, short selling is prohibited.

short selling constraints decreases the effect of estimation errors on both the optimal execution strategy and the efficient frontier. Moreover, when appropriate, maintaining symmetry in the permanent impact matrix decreases the effect of estimation errors.

In our simulations, we have also noticed that it is possible for a small perturbation in impact matrices to make the Hessian matrix indefinite. As the number of assets grow, this issue becomes even more pronounced. When the Hessian matrix is indefinite, the execution cost problem (12), in which short selling is permitted, no longer has an optimal solution since the objective function becomes unbounded below. This is another evidence of potentially large sensitivity of the optimal execution strategy to perturbations in the impact matrices.

5 Concluding Remarks

Specification and estimation of the price impact function in the execution cost problem inevitably have errors. Therefore it is important to analyze how sensitive the optimal execution strategy and the efficient frontier are to these estimation errors. In this paper, we consider linear price impact functions and study the effect of perturbations in the parameters of the price impact function.

We first show that the optimal execution strategy is determined from the combined impact matrix $\Theta = \frac{1}{\tau}\left(H + H^T\right) - \Gamma$. Therefore one may want to estimate the combined impact Θ directly, rather than estimating the temporary and permanent impact matrices individually.

We discuss some cases in which the optimal execution strategy is insensitive to the estimation errors in the impact matrices. For example, the optimal execution strategy, which minimizes the expected execution cost, is the naive execution strategy as long as the permanent impact matrix and its perturbation are symmetric and the combined impact matrices Θ and $\Theta + \Delta\Theta$ are positive definite. In other words, when symmetry is maintained for the permanent impact matrices and positive definiteness is maintained for the combined impact matrices, the minimum expected execution cost strategy is not sensitive to changes in the impact matrices.

We prove that the optimal execution strategy is Lipschitz continuous in the impact matrices H and Γ with respect to the Euclidean norm. We provide upper bounds for the size of the change in the optimal execution strategy in terms of the change in the impact matrices and some magnification factor. In general, the magnification factor is defined by the minimum eigenvalue of the block tridiagonal Hessian matrix W. This matrix W is determined by the covariance matrix C, the combined impact matrix Θ, and the risk aversion parameter μ. From the established upper bounds, it can be concluded that the change in the optimal execution strategy diminishes as the risk aversion parameter increases. However, for a small risk aversion parameter, estimation errors may significantly affect the optimal execution strategy and efficient frontiers.

When the permanent impact matrix and its perturbation are symmetric, magnification factors can be explicitly expressed in terms of minimum eigenvalues of the covariance matrix C and Θ. Specifically, the magnification factor becomes small when either $\lambda_{\min}(C)$ or $\lambda_{\min}(\Theta)$ is large, assuming that $\mu > 0$ is fixed. In this case, the sensitivity of the optimal execution strategy to perturbations in the impact matrices is not as pronounced.

Our computational investigation confirms the importance of accurate specification of the impact matrices. We demonstrate that, in addition, maintaining symmetry of the permanent impact matrix also reduces the effect of estimation errors. Moreover, our simulations suggest that adding appropriate constraints, such as short selling constraints, can significantly alleviate the effect of estimation errors in the impact matrices. Consistent with our theoretical results, the computational investigation shows large sensitivity of the optimal execution strategy and the efficient frontier for a small risk aversion parameter μ, when the permanent impact matrix is asymmetric. Specially, this change becomes more significant in the absence of short selling constraints. This result also coincides with the observation that, for the traditional mean-variance portfolio optimization, the effect of estimation errors in the mean and covariance matrix reduces when short selling constraints are imposed on the problem.

In summary, our theoretical and computational results indicate that the optimal execution strategy can potentially be very sensitive to estimation errors in the impact matrices. This is particularly the case if the permanent impact matrix is asymmetric, the risk aversion parameter is small, and short selling is permitted.

For future research, it may be interesting to perform hypersensitivity analysis, e.g., see Churilov et al. (2004), on the structure of the price impact function. In addition, we assume, in this paper, that the covariance matrix is accurately given. It may also be important to compare the effect of estimation errors in the covariance matrix versus the effect of estimation errors in the parameters of the price impact function on the optimal execution cost strategy.

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