Algorithms for Equilibrium Prices in Linear Market Models

Kurt Mehlhorn

Max-Planck-Institut für Informatik, Saarbrücken, Germany
mehlhorn@mpi-inf.mpg.de

Near the end of the 19th century, Leon Walrus [Wal74] and Irving Fisher [Fis91] introduced general market models and asked for the existence of equilibrium prices. Chapters 5 and 6 of [NRTV07] are an excellent introduction into the algorithmic theory of market models. In Walrus' model, each person comes to the market with a set of goods and a utility function for bundles of goods. At a set of prices, a person will only buy goods that give him maximal satisfaction.[1] The question is to find a set of prices at which the market clears, i.e., all goods are sold and all money is spent. Observe that the money available to an agent is exactly the money earned by selling his goods. Fisher's model is somewhat simpler. In Fisher's model every agent comes with a predetermined amount of money. Market clearing prices are also called *equilibrium prices*. Walrus and Fisher took it for granted that equilibrium prices exist. Fisher designed a hydromechanical computing machine that would compute the prices in a market with three buyers, three goods, and linear utilities [BS00].

In the 20th century it became clear that the existence of equilibrium prices requires rigorous proof. Arrow and Debreu [AD54] refined Walras' model and proved the existence of equilibrium prices for general convex utility functions. Their proof is non-constructive and uses a fixed-point theorem in a crucial way. The obvious next question for an algorithmicist is whether market clearing prices can be computed (efficiently)? We discuss the situation for linear markets.

In the linear Fisher market, there are n buyers and n goods. We assume for w.l.o.g that there is one unit of each good. The i-th buyer comes with a non-negative budget b_i. The utility for buyer i of receiving the full unit of good j is $u_{ij} \geq 0$. Let $x_{ij} \geq 0$ be the fraction of good j that is allocated to buyer i. Under this assigment and the assumption of linear additive utilities, the total utility of i is

$$\sum_j u_{ij} x_{ij}.$$

[1] Consider the case of linear additive utilites, i.e., two items of the same good give twice the utility of one item and utilities of different goods add. Assume that an agent values an item of good A twice as much as an item of good B. If the price of an item of A is less than twice the price of an item of B, the agent will only want A. If the price is more than twice, the agent will only want B. If the price is twice the price of B, the agent is indifferent and any combination of A and B is equally good. Linear utilities are a gross simplification.

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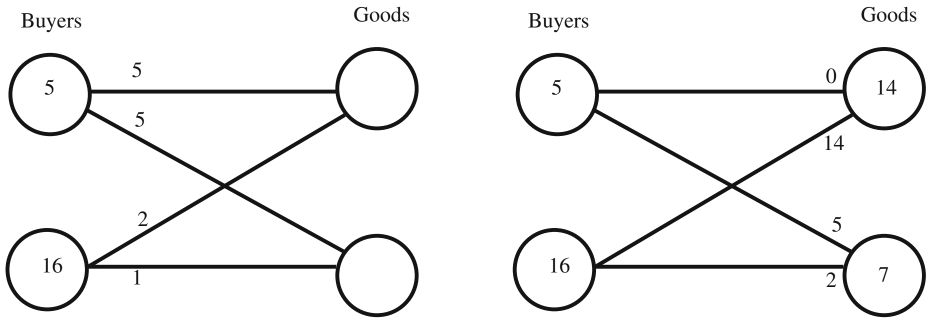

Fig. 1. A Fisher market: there are two buyers and two goods: the first buyer has a budget of five and the second buyer has a budget of 16 ($b_1 = 5$ and $b_2 = 16$). The first buyer draws a utility of 5 from both goods and the second buyer draws a utility of 2 from the first good and a utility of 1 from the second good ($u_{11} = u_{12} = 5$, $u_{21} = 2$, and $u_{22} = 1$). A solution is shown on the right. For price vector $p_1 = 14$ and $p_2 = 7$, the first buyer prefers the second good over the first good and therefore is only willing to spend money on the second good, and the second buyer is indifferent and hence is willing to spend money on both goods. For the allocation shown (the first good is allocated completely to the second buyer and the second good is split in the ratio 5:2), the market is in equilibrium.

Let p_j be the (to be determined) price of good j. Then the utility of good j for buyer i per unit of money is u_{ij}/p_j. Buyers spend their money only on goods that give them maximal utility per unit of money, i.e.,

$$x_{ij} > 0 \quad \Rightarrow \quad \frac{u_{ij}}{p_j} = \alpha_i = \max_j u_{ij}/p_j. \tag{1}$$

α_i is called the bang-per-buck for agent i at price vector p.

A price vector p is *market clearing* in the Fisher model if there is an allocation $x = (x_{ij})$ such that (1) and

$$\sum_i x_{ij} = 1 \quad \text{for all } j \qquad \text{good } j \text{ is completely sold} \tag{2}$$

$$\sum_j p_{ij} x_{ij} = b_i \quad \text{for all } i \qquad \text{buyer } i \text{ spends his complete budget} \tag{3}$$

hold. In the linear Arrow-Debreu market there is the additional constraint

$$b_i = p_i \quad \text{for all } i, \tag{4}$$

i.e., the i-th buyer is also the owner of the i-th good and his budget is precisely the revenue for this good. Figures 1 and 2 illustrate the market concepts.

Fisher's model is a special case of the Arrow-Debreu model. In the former model, each buyer comes with a budget and money has intrinsic value. In the latter model, money is only used for comparing goods. The former model reduces to the latter by introducing a $n + 1$-th good corresponding to money.

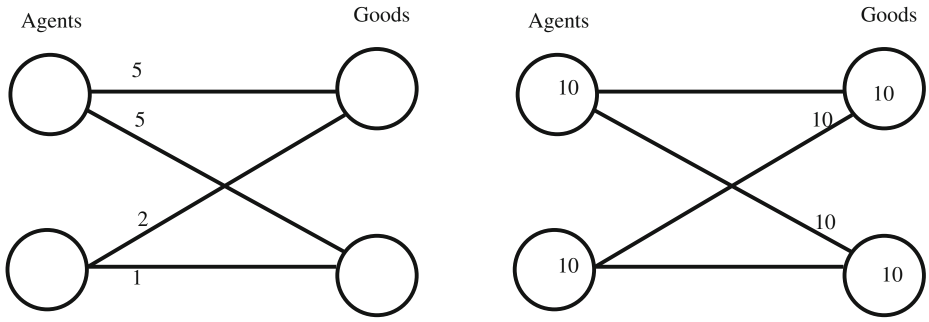

Fig. 2. An Arrow-Debreu market: there are two agents and two goods: the first agent owns good one and the second agent owns good two. As in Figure 1, the first buyer draws a utility of 5 from both goods and the second buyer draws a utility of 2 from the first good and a utility of 1 from the second good ($u_{11} = u_{12} = 5$, $u_{21} = 2$, and $u_{22} = 1$). A solution is shown on the right. For price vector $p_1 = 10$ and $p_2 = 10$, the first agent is indifferent between the goods and is willing to spend money on both goods. The second agent prefers the first good and is only willing to spend money on the first good.

In recent years, polynomial time algorithms were found for the computation of equilibrium prices in linear markets. Not surprisingly, Fisher's model was solved first. Already in 1958, Eisenberg and Gale [EG58] characterized equilibrium prices by a convex program. With the advent of the Ellipsoid method, the characterization became a polynomial time algorithm. In 2008, Devanur, Papadimitriou, Saberi, Vazerani [DPSV08] gave the first combinatorial algorithm. It computes market clearing prices by repeated price adjustments and maximum flow computations. A simpler and more efficient algorithm was found by Orlin [Orl10] in 2010. For the Arrow-Debreu market, the first polynomial time algorithms are due to Jain [Jai07] and Ye [Ye07]. Both algorithms give a characterization by a convex program and then use the Ellipsoid and interior point method, respectively, to solve the program. Duan and Mehlhorn [DM13] found a combinatorial algorithm in 2012.

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