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Multi-objective portfolio selection model with fuzzy random returns and a compromise approach-based genetic algorithm

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ABSTRACT

This paper addresses the multi-objective portfolio selection model with fuzzy random returns for investors by studying three criteria: return, risk and liquidity. In addition, securities historical data, experts' opinions and judgements and investors' different attitudes are considered in the portfolio selection process, such that the investor's individual preference is reflected by an optimistic–pessimistic parameter λ. To avoid the difficulty of evaluating a large set of efficient solutions and to ensure the selection of the best solution, a compromise approach-based genetic algorithm has been designed to solve the proposed model. In addition, a numerical example is presented to illustrate the proposed algorithm.

1. Introduction

Modern portfolio selection theory originated from the pioneering research work of Markowitz's mean–variance model [35]. Based on the mean–variance model, many scholars proposed model extensions by assuming the securities' rates of return were random variables and thus only used historical data to describe the securities future rates of return. However, in addition to random uncertainty, there are many non-probability factors in the securities market that cannot be resolved using probability theory. With the introduction of fuzzy set theory [50,51], some authors have developed fuzzy portfolio selection models (cf. [6,14,17,18,25,28,47,48,52,2] and the references therein). These authors recognized the existence of fuzziness in the securities market but ignored other categories of uncertainty because only fuzzy uncertainty is reflected in the research.

In a complicated financial market, some variables can exhibit random uncertainty properties and others can exhibit fuzzy uncertainty properties. Because random uncertainty and fuzzy uncertainty are often combined in a real-world setting, the portfolio selection process must simultaneously consider twofold uncertainty. Katagiri and Ishii [19] first assumed securities' rates of returns were fuzzy random variables and proposed a portfolio selection model based on possibility theory and a chance-constrained model in stochastic programming. Smimou et al. [41] presented a method for the derivation of the attainable efficient frontier in the presence of fuzzy information in data. Li and Xu [26] proposed the λ-mean variance portfolio selection model based on fuzzy random theory. Yoshida [49] discussed a value-at-risk portfolio model of randomness and fuzziness to derive its analytical solution. Lacagnina and Pecorella [24] developed a multistage stochastic soft constraints fuzzy program with the goal of capturing both uncertainty and imprecision as well as to re-solving a portfolio management issue.

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Expected return and risk are two fundamental factors in portfolio selection. However, explicit return and risk cannot capture all relevant information for an investment decision. Therefore, criteria for portfolio selection problems, in addition to the standard expected return and variance, have become more popular in recent years [14]. Steuer et al. [43] discussed portfolio selection for investors using a multi-objective stochastic programming problem. Parra et al. [36] proposed a portfolio selection model with the three criteria (return, risk and liquidity) and resolved the model using a fuzzy goal programming approach. Fang et al. [11] presented a portfolio rebalancing model with three criteria (return, risk and liquidity) based on fuzzy decision theory. Gupta et al. [14] studied a semi-absolute deviation portfolio selection model, intended for investors' that incorporates five criteria (short-term return, long-term return, dividend, risk and liquidity). However, realistic constraints are not considered in the above-cited works. Because of the existence of realistic constraints, it is difficult to resolve constrained multi-objective portfolio selection models using traditional multi-objective programming algorithms. Some authors use evolutionary algorithms to resolve constrained multi-objective portfolio optimization models. Ehrgott et al. [10] used a genetic algorithm to optimize a mixed-integer (due to the constraints used) multi-objective portfolio optimization problem with objectives aggregated through user-specified utility functions. Subbu et al. [46] presented a hybrid evolutionary algorithm that integrated genetic algorithms with linear programming for a portfolio design issume with multiple measures for risk and return.

In this paper, we propose a constrained multi-objective portfolio selection model with fuzzy random returns for investors. This model includes three criteria (return, risk and liquidity) and a compromise approach-based genetic algorithm designed to obtain a compromised portfolio strategy. The model has the ability to introduce expert opinion and judgment (fuzzy information) into the portfolio selection process and to obtain a satisfactory personal portfolio selection in accordance with the attitudes of the different investors'. The rest of the paper is organized as follows. In Section 2, definitions for fuzzy random variable, fuzzy expectation and variance of fuzzy random variables are briefly introduced. In Section 3, we use the λ-average value of the fuzzy expectation of portfolio to quantify the return, the variance to quantify risk, and the crisp possibilistic mean value of the turnover rate portfolio to quantify portfolio liquidity. Then, a constrained multi-objective portfolio selection model with fuzzy random returns is proposed. In Section 4, to avoid the difficulty of evaluation and the selection of the best solution from the efficient frontier or its discretized representation, a compromise approach-based genetic algorithm has been designed to resolve the proposed model and to obtain a compromised portfolio strategy. An example is given in Section 5 to illustrate the proposed model and algorithm, and concluding remarks are given in Section 6.

2. Preliminaries

A fuzzy number $\widetilde{X}$ is described as any fuzzy subset of the real line $\mathbb{R}$, whose membership function $\mu_{\widetilde{X}} : \mathbb{R} \to [0,1]$ satisfies the following conditions:

(i) $\widetilde{X}$ is normal, i.e., there exists an $x \in \mathbb{R}$ such that $\mu_{\widetilde{X}}(x) = 1$;
(ii) $\mu_{\widetilde{X}}$ is quasi-concave, i.e., $\mu_{\widetilde{X}}(\lambda x + (1-\lambda)y) \leqslant \min\{\mu_{\widetilde{X}}(x), \mu_{\widetilde{X}}(y)\}$, for all $\lambda \in [0,1]$;
(iii) $\mu_{\widetilde{X}}$ is upper semi-continuous, i.e., $\{x \in \mathbb{R} | \mu_{\widetilde{X}}(x) \leqslant \alpha\}$ is a closed set, for all $\alpha \in [0,1]$; and
(iv) the closure of the set $\{x \in \mathbb{R} | \mu_{\widetilde{X}}(x) > 0\}$ is a compact set.

An α-level set of $\widetilde{X}$ is defined by $\widetilde{X}_\alpha = \{x \in \mathbb{R} | \mu_{\widetilde{X}}(x) \geqslant \alpha\}$ if $\alpha > 0$ and $\widetilde{X}_\alpha = \mathrm{cl}\{x \in \mathbb{R} | \mu_{\widetilde{X}}(x) > 0\}$ (the closure of the support of $\widetilde{X}$) if $\alpha = 0$. It is well known that if $\widetilde{X}$ is a fuzzy number, then $\widetilde{X}_\alpha = [\widetilde{X}_\alpha^-, \widetilde{X}_\alpha^+]$ is a compact subset of $\mathbb{R}$ for all $\alpha \in [0,1]$.

The concept of fuzzy random variable, which was first introduced by Kwakernaak [23], applies to a situation when randomness and fuzziness appear simultaneously.

Definition 1. ([37]) Let $(\Omega, \mathscr{A}, P)$ be a probability space, where $\mathscr{A}$ is a σ-field of Ω and P is a non-atomic probability measure. A mapping $\widetilde{\widetilde{X}} : \Omega \to \mathscr{F}_c(\mathbb{R})$ is said to be a fuzzy random variable if the set-valued function $\widetilde{\widetilde{X}}_\alpha : \Omega \to \mathscr{K}_c(\mathbb{R})$ such that $\widetilde{\widetilde{X}}_\alpha(w) = (\widetilde{\widetilde{X}}(w))_\alpha = \{x \in \mathbb{R} | \mu_{\widetilde{\widetilde{X}}(w)}(x) \geqslant \alpha\}$ for all $w \in \Omega$ is $\mathscr{A}$-measurable for all $\alpha \in [0,1]$, where $\mathscr{F}_c(\mathbb{R})$ denotes the set of all fuzzy numbers, and $\mathscr{K}_c(\mathbb{R})$ denotes the class of all non-empty bounded closed intervals.

As shown in [33], if $\widetilde{\widetilde{X}}$ is a fuzzy random variable, the left endpoint $(\widetilde{\widetilde{X}}(w))_\alpha^-$ and the right endpoint $(\widetilde{\widetilde{X}}(w))_\alpha^+$ of the α-level sets of $\widetilde{\widetilde{X}}(w)$ are real-valued random variables for all $\alpha \in (0,1]$.

Example 1. Let L, R:$[0,1] \to [0,1]$ be continuous and strictly decreasing functions with $R(0) = L(0) = 1$ and $R(1) = L(1) = 0$. A fuzzy random variable $\widetilde{\widetilde{X}}$ characterized by the membership function

$$\mu_{\widetilde{\widetilde{X}}(w)}(x) = \begin{cases} L\left(\frac{a(w)-x}{\alpha}\right), & \text{if } a(w) - \alpha \leqslant x \leqslant a(w), \\ R\left(\frac{x-a(w)}{\beta}\right), & \text{if } a(w) < x \leqslant a(w) + \beta, \end{cases} \quad \forall w \in \Omega,$$

is called an L–R type fuzzy random variable, where random variable $a(w)$ is the center value and positive real numbers α and β are the left width and right width of the fuzzy number $\widetilde{\widetilde{X}}(w), w \in \Omega$, respectively. For simplicity, $\widetilde{\widetilde{X}}$ is denoted by $\widetilde{\widetilde{X}}(w) = (a(w), \alpha, \beta)_{LR},\ w \in \Omega$ (see Fig. 1).

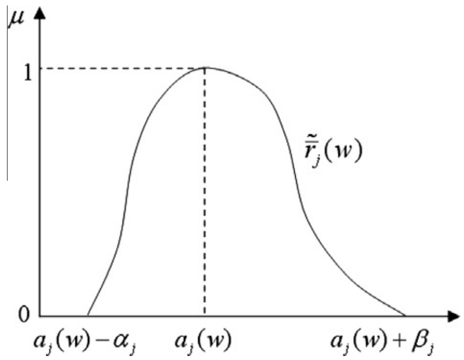

Fig. 1. L–R type fuzzy random variable $\tilde{\tilde{r}}_j$.

Moreover, if $L(x) = R(x) = 1 - x$, an L–R type fuzzy random variable $\tilde{\tilde{X}}$ is called a triangular fuzzy random variable, which is denoted by $\tilde{\tilde{X}}(w) = (a(w) - \alpha, a(w), a(w) + \beta)$ for all $w \in \Omega$.

For given fuzzy random variables $\tilde{\tilde{X}}, \tilde{\tilde{Y}}$ and real number $\lambda \in \mathbb{R}$, the addition and scalar multiplication on the fuzzy random variables are defined as

$$(\tilde{\tilde{X}} \tilde{+} \tilde{\tilde{Y}})(w) = \tilde{\tilde{X}}(w) \oplus \tilde{\tilde{Y}}(w), \quad \forall w \in \Omega, \tag{2.1}$$

$$(\lambda \tilde{\tilde{X}})(w) = \lambda \tilde{\tilde{X}}(w), \quad \forall w \in \Omega, \tag{2.2}$$

where "$\tilde{+}$" denotes the sum operation between the fuzzy random variables, and "$\oplus$" denotes the sum operation between the fuzzy numbers. If the fuzzy random variable $\tilde{\tilde{Y}}$ degenerates into a fuzzy number, denoted by $\tilde{Y}$, then from (2.1) $(\tilde{\tilde{X}} \tilde{+} \tilde{Y})(w) = \tilde{\tilde{X}}(w) \oplus \tilde{Y}$ for $w \in \Omega$.

In this paper it is assumed that the fuzzy random variable $\tilde{\tilde{X}}$ is square integrable, i.e.

$$E\|\tilde{\tilde{X}}\|_2^2 = E(\|\tilde{\tilde{X}}(w)\|_2^2) = \int_\Omega \int_0^1 |s_{\tilde{\tilde{X}}(w)}(\alpha, u)|^2 d\alpha \, P(dw) < \infty,$$

where $s_{\tilde{\tilde{X}}(w)}(\alpha, u)$ is the unique support function corresponding to the fuzzy set $\tilde{\tilde{X}}(w) \in \mathscr{F}_c(\mathbb{R})$ [21]. This function implies that expectation and variance of the fuzzy random variables always exist.

Definition 2. [37] Let $(\Omega, \mathscr{A}, P)$ be a complete probability space. The expectation $E(\tilde{\tilde{X}})$ of an integrable bounded fuzzy random variable $\tilde{\tilde{X}}$ is defined as a fuzzy number such that

$$\left(E(\tilde{\tilde{X}})\right)_\alpha = \int_\Omega \tilde{\tilde{X}}_\alpha dP = \left\{ \int_\Omega f(w) dP(w) : f \in L^1(P), f(w) \in \tilde{\tilde{X}}_\alpha(w) \text{ a.s. } [P] \right\} \tag{2.3}$$

for $\alpha \in (0,1]$, where $L^1(P)$ denotes the set of all functions $f : \Omega \to \mathbb{R}$ that are integrable with respect to P and $\int_\Omega \tilde{\tilde{X}}_\alpha dP$ is the Aumann integral of $\tilde{\tilde{X}}_\alpha$ in Ω.

The α-level sets of $E(\tilde{\tilde{X}})$ can be rewritten as the following compact convex intervals:

$$(E(\tilde{\tilde{X}}))_\alpha = \left[(E(\tilde{\tilde{X}}))_\alpha^-, (E(\tilde{\tilde{X}}))_\alpha^+\right] = \left[\int_\Omega (\tilde{\tilde{X}}(w))_\alpha^- dP(w), \int_\Omega (\tilde{\tilde{X}}(w))_\alpha^+ dP(w)\right] \tag{2.4}$$

for $\alpha \in (0,1]$ (see [32]).

Definition 3. [12] Let $\tilde{\tilde{X}}$ and $\tilde{\tilde{Y}}$ be square integrable fuzzy random variables. Then the covariance $\mathrm{Cov}(\tilde{\tilde{X}}, \tilde{\tilde{Y}})$ and the variance $\mathrm{Var}(\tilde{\tilde{X}})$ are defined as follows:

$$\mathrm{Cov}(\tilde{\tilde{X}}, \tilde{\tilde{Y}}) = \frac{1}{2} \int_0^1 \left[\mathrm{Cov}\left(\tilde{\tilde{X}}_\alpha^-, \tilde{\tilde{Y}}_\alpha^-\right) + \mathrm{Cov}\left(\tilde{\tilde{X}}_\alpha^+, \tilde{\tilde{Y}}_\alpha^+\right)\right] d\alpha, \tag{2.5}$$

$$\mathrm{Var}(\tilde{\tilde{X}}) = \mathrm{Cov}(\tilde{\tilde{X}}, \tilde{\tilde{X}}) = \frac{1}{2} \int_0^1 \left[\mathrm{Var}\left(\tilde{\tilde{X}}_\alpha^-\right) + \mathrm{Var}\left(\tilde{\tilde{X}}_\alpha^+\right)\right] d\alpha, \tag{2.6}$$

where $\tilde{\tilde{X}}_\alpha^-$ and $\tilde{\tilde{Y}}_\alpha^-$ (resp., $\tilde{\tilde{X}}_\alpha^+$ and $\tilde{\tilde{Y}}_\alpha^+$) are the left endpoints (resp., the right endpoints) of the α-level sets of $\tilde{\tilde{X}}(w)$ and $\tilde{\tilde{Y}}(w)$ for all $\alpha \in (0,1]$.

Lemma 1. *Let $\tilde{\tilde{X}}$ and $\tilde{\tilde{Y}}$ be square integrable fuzzy random variables, and $\lambda, \gamma \in \mathbb{R}$. Then*

(i) $E(\lambda\tilde{\tilde{X}} \pm \gamma\tilde{\tilde{Y}}) = \lambda E(\tilde{\tilde{X}}) \pm \gamma E(\tilde{\tilde{Y}})$;
(ii) $\mathrm{Var}(\lambda\tilde{\tilde{X}} + u) = \lambda^2\mathrm{Var}(\tilde{\tilde{X}})$;
(iii) $\mathrm{Var}(\tilde{\tilde{X}} + \tilde{\tilde{Y}}) = \mathrm{Var}(\tilde{\tilde{X}}) + \mathrm{Var}(\tilde{\tilde{Y}}) + 2\mathrm{Cov}(\tilde{\tilde{X}}, \tilde{\tilde{Y}})$;
(iv) $\mathrm{Cov}(\lambda\tilde{\tilde{X}} + u, \gamma\tilde{\tilde{Y}} + v) = \lambda\gamma\mathrm{Cov}(\tilde{\tilde{X}}, \tilde{\tilde{Y}})$, where u, v are fuzzy numbers, $\lambda\gamma \geqslant 0$.

3. Constrained multi-objective portfolio selection model with fuzzy random returns

In additon to return and risk, liquidity is also one of the main concerns for investors when making decisions. In this section, de-fuzzification of the expectation of portfolio return, risk and liquidity are discussed. A constrained multi-objective portfolio selection model with fuzzy random returns for investors is proposed, and this approach considers the aforementioned three criteria.

It is assumed that investors allocate their wealth among n securities offering fuzzy random rates of return. Let $\tilde{\tilde{r}}_j$ be the future return rates of the jth securities and x_j be the proportion of wealth invested in the jth securities, $j = 1, 2, \ldots, n$. Then, the future return rate of the portfolio is denoted by $\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j$. From (2.1) and (2.2) it can be determined that $\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j$ is a fuzzy random variable.

3.1. De-fuzzification of the expectation of the portfolio return

Lemma 2. *Let the future return rates of the jth securities $\tilde{\tilde{r}}_j$ be a L–R type fuzzy random variable, characterized by the membership function*

$$\mu_{\tilde{\tilde{r}}_j(w)}(x) = \begin{cases} L\left(\frac{a_j(w)-x}{\alpha_j}\right), & \text{if } a_j(w) - \alpha_j \leqslant x_j \leqslant a_j(w), \\ R\left(\frac{x-a_j(w)}{\beta_j}\right), & \text{if } a_j(w) < x_j \leqslant a_j(w) + \beta_j, \end{cases} \quad w \in \Omega, \quad j = 1, 2, \ldots, n, \tag{3.7}$$

where the center value is $a_j \sim \mathcal{N}(E(a_j), \sigma_j^2)$, α_j and β_j are the left width and right width of the fuzzy number $a_j(w)$, respectively. Then, the expectation of $\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j$ is an L–R type fuzzy number, i.e.

$$E\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right) = \left(\sum_{j=1}^{n}E(a_j)x_j, \sum_{j=1}^{n}\alpha_j x_j, \sum_{j=1}^{n}\beta_j x_j\right)_{LR}. \tag{3.8}$$

Proof. If L and R are strictly decreasing continuous functions on $[0,1]$, it can be verified that

$$(\tilde{\tilde{r}}_j(w))_\alpha = \left[(\tilde{\tilde{r}}_j(w))_\alpha^-, (\tilde{\tilde{r}}_j(w))_\alpha^+\right] = \left[a_j(w) - L^{-1}(\alpha)\alpha_j, a_j(w) + R^{-1}(\alpha)\beta_j\right], \tag{3.9}$$

for each $\alpha \in [0,1]$, $j = 1, 2, \ldots, n$. From Definition 2 the α-level sets of the expectation of $\tilde{\tilde{r}}_j$ are obtained:

$$\left(E(\tilde{\tilde{r}}_j)\right)_\alpha = \left[E((\tilde{\tilde{r}}_j(w))_\alpha^-), E((\tilde{\tilde{r}}_j(w))_\alpha^+)\right] = \left[E(a_j) - L^{-1}(\alpha)\alpha_j, E(a_j) + R^{-1}(\alpha)\beta_j\right],$$

which implies the expectation of $\tilde{\tilde{r}}_j$ is an L–R type fuzzy number, i.e.

$$E(\tilde{\tilde{r}}_j) = (E(a_j), \alpha_j, \beta_j)_{LR}, \quad j = 1, 2, \ldots, n. \tag{3.10}$$

By Lemma 1 (i) and (3.10), we obtain that the expectation of $\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j$ is

$$E\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right) = \sum_{j=1}^{n}E(\tilde{\tilde{r}}_j)x_j = \left(\sum_{j=1}^{n}E(a_j)x_j, \sum_{j=1}^{n}\alpha_j x_j, \sum_{j=1}^{n}\beta_j x_j\right)_{LR}.$$

Then, the result of this Lemma holds and the proof is completed. The homogeneous expectation assumption is one of the basic assumptions of Markowitz's mean–variance model. This model assumes that all investors share the same expected returns, predicted variances, and predicted co-variances regarding securities' rates of future return. This is clearly not the case. In fact, it is almost a hallmark of investors to specialize in different prognostications [44]. In [26], it is first proposed to use a λ-average ranking method [4] to convert $E\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right)$ into a crisp number, then the investors' different expectations securities' rates of future return can be incorporated into a portfolio selection model. Here, the same method is adopted to convert the fuzzy number $E\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right)$ into a crisp number to reflect each investor's attitude regarding securities returns. □

Definition 4. [4] Let $\tilde{A}$ be a fuzzy number with α-level sets $\left[\tilde{A}_\alpha^-, \tilde{A}_\alpha^+\right]$. The λ-average value of $\tilde{A}$ is defined as

$$V_S^\lambda(\tilde{A}) = \int_0^1 \left[\lambda\tilde{A}_\alpha^+ + (1-\lambda)\tilde{A}_\alpha^-\right]dS(\alpha), \tag{3.11}$$

where parameter $\lambda(\in[0,1])$ is a decision-maker's subjective optimism–pessimism degree and S is an additive measure on $(0,1]$ that determines the weight or importance associated with different α-level sets.

In what follows, it is assumed that all α-level sets have the same importance. Then the integral in (3.11) is calculated with respect to $d\alpha$ in the case of continuous membership functions. In this case $V^{\lambda}(\widetilde{A})$ is used instead of $V_S^{\lambda}(\widetilde{A})$, i.e.

$$V^{\lambda}(\widetilde{A}) = \int_0^1 \left[\lambda \widetilde{A}_{\alpha}^{+} + (1-\lambda)\widetilde{A}_{\alpha}^{-}\right] d\alpha. \tag{3.12}$$

From (3.12), the λ-average value of the portfolio expectation is

$$\begin{aligned} V^{\lambda}\left(E\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right)\right) &= \int_0^1 \left\{\lambda\left[E(a_j) + R^{-1}(\alpha)\beta_j\right] + (1-\lambda)\left[E(a_j) - L^{-1}(\alpha)\alpha_j\right]\right\} d\alpha \\ &= \sum_{j=1}^{n}\left[E(a_j) + \lambda_j R^0 \beta_j - (1-\lambda_j) L^0 \alpha_j\right] x_j, \end{aligned} \tag{3.13}$$

where $R^0 = \int_0^1 R^{-1}(\alpha)d\alpha$ and $L^0 = \int_0^1 L^{-1}(\alpha)d\alpha$. If $\tilde{\tilde{r}}_j$'s are triangular fuzzy random variables, then $R^0 = L^0 = \int_0^1 (1-\alpha)d\alpha = 1/2$ and

$$V^{\lambda}\left(E\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right)\right) = \sum_{j=1}^{n}\left[E(a_j) + \lambda_j \beta_j/2 - (1-\lambda_j)\alpha_j/2\right] x_j.$$

As discussed in [4], parameter λ_j reflects the investor's subjective degree of optimism for the future returns of jth securities, where $j = 1, 2, \ldots, n$. For an aggressive and completely optimistic investor, λ_j should be set to 1, and for a conservative and completely pessimistic investor, λ_j should be set to 0. When λ_j varies between $(0,1)$, the investor's optimism–pessimism attitude can be reflected in (3.13). Here the de-fuzzified value of $E\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right)$, i.e., $V^{\lambda}\left(E\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right)\right)$ is used to measure the expected portfolio rate of return.

3.2. Variance of the portfolio return

The variance of the portfolio return $\mathrm{Var}\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right)$ is discussed in this subsection to characterize investment risk.

Lemma 3. *The variance of the portfolio return* $\mathrm{Var}\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right)$ *is equal to the variance of random variable* $\sum_{j=1}^{n} a_j x_j$, *i.e.*

$$\mathrm{Var}\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right) = \mathrm{Var}\left(\sum_{j=1}^{n} a_j x_j\right).$$

Proof. From (3.9), it can be verified that

$$\begin{aligned} \mathrm{Var}((\tilde{\tilde{r}}_j(w))_{\alpha}^{-}) &= \mathrm{Var}\left(a_j(w) - L^{-1}(\alpha)\alpha_j\right) = \mathrm{Var}(a_j), \\ \mathrm{Var}((\tilde{\tilde{r}}_j(w))_{\alpha}^{+}) &= \mathrm{Var}\left(a_j(w) + R^{-1}(\alpha)\beta_j\right) = \mathrm{Var}(a_j). \end{aligned}$$

It follows from (2.5) and (2.6) that

$$\begin{aligned} \mathrm{Var}(\tilde{\tilde{r}}_j) &= \frac{1}{2}\int_0^1 \left[\mathrm{Var}((\tilde{\tilde{r}}_j(w))_{\alpha}^{-}) + \mathrm{Var}((\tilde{\tilde{r}}_j(w))_{\alpha}^{+})\right] d\alpha = \mathrm{Var}(a_j), \\ \mathrm{Cov}(\tilde{\tilde{r}}_i, \tilde{\tilde{r}}_j) &= \frac{1}{2}\int_0^1 \left[\mathrm{Cov}\left(a_i - L^{-1}(\alpha)\alpha_i, a_j - L^{-1}(\alpha)\alpha_j\right)\right. \\ &\quad \left. + \mathrm{Cov}\left(a_i + R^{-1}(\alpha)\beta_i, a_j + R^{-1}(\alpha)\beta_j\right)\right] d\alpha \\ &= \mathrm{Cov}(a_i, a_j). \end{aligned}$$

By Lemma 1 (iii) the following result is obtained:

$$\begin{aligned} \mathrm{Var}\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right) &= \sum_{j=1}^{n}\mathrm{Var}(\tilde{\tilde{r}}_j x_j) + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\mathrm{Cov}(\tilde{\tilde{r}}_i x_i, \tilde{\tilde{r}}_j x_j) \\ &= \sum_{j=1}^{n}\mathrm{Var}(a_j)x_j^2 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\mathrm{Cov}(a_i, a_j)x_i x_j \\ &= \mathrm{Var}\left(\sum_{j=1}^{n} a_j x_j\right). \end{aligned}$$

Thus, the proof is completed. □

3.3. Characterization of portfolio liquidity and its de-fuzzification

Liquidity has been measured as the possibility of converting an investment into cash without any significant loss in value. In the stock market, many stocks have poor liquidity, and agents often cannot sell or purchase them quickly. Generally, investors prefer to choose securities with better liquidity. Here, the measurement of liquidity is based on a security's turnover rate. It is known that a security's turnover rate cannot accurately be predicted in the stock market; therefore, fuzzy set theory can play a very significant role in studying this imprecision. In this paper, trapezoidal fuzzy numbers are used to denote the turnover rates of securities. The historical turnover rates of a security over a long period of time were collected, and it was found that most of the data were concentrated around an interval, denoted by [a%, b%], from the historical security turnover rates histogram. Ignoring the historical turnover rates located in [0, a%) and (b%, +∞], a simple method can be used to determine a rough estimation for the security's turnover rate by assuming it is a trapezoidal fuzzy number.

The crisp possibilistic mean value of a fuzzy number $\widetilde{A}$ is defined as [5]

$$M(\widetilde{A}) = \int_0^1 \alpha\left(\widetilde{A}_\alpha^- + \widetilde{A}_\alpha^+\right)d\alpha, \tag{3.14}$$

where $\widetilde{A}_\alpha^-$ and $\widetilde{A}_\alpha^+$ are the left width and the right width of the α-level sets, respectively. Because the crisp possibilistic mean value of the continuous possibility distribution is consistent with the extension principle, the crisp possibilistic mean value of the turnover rate is adopted to measure portfolio liquidity.

Lemma 4. *Let the trapezoidal fuzzy number $\tilde{l}_j = (b_j, c_j, \delta_j, \gamma_j)$ be the turnover rate of the jth securities with the following membership function:*

$$\mu_{\tilde{l}_j}(t) = \begin{cases} 1 - \frac{b_j - t}{\delta_j}, & \text{if } b_j - \delta_j \leqslant t < b_j, \\ 1, & \text{if } b_j \leqslant t < c_j, \\ 1 - \frac{t - c_j}{\gamma_j}, & \text{if } c_j \leqslant t < c_j + \gamma_j, \\ 0 & \text{otherwise}, \end{cases} \qquad j = 1, 2, \cdots, n, \tag{3.15}$$

where b_j is the lower modal value, c_j is the upper modal value, δ_j is the left width and γ_j is the right width of $\tilde{l}_j$ (see Fig. 2). Then the possibilistic mean value of the turnover rate associated with portfolio x is

$$M(\tilde{l}(x)) = \sum_{j=1}^n \left[\frac{(b_j + c_j)}{2} + \frac{(\gamma_j - \delta_j)}{6}\right]x_j.$$

Proof. It follows from (3.14) that the possibilistic mean value of $\tilde{l}_j$, denoted by $M(\tilde{l}_j)$, is

$$\begin{aligned} M(\tilde{l}_j) &= \int_0^1 \alpha\left[(\tilde{l}_j)_\alpha^- + (\tilde{l}_j)_\alpha^+\right]d\alpha = \int_0^1 \alpha[(b_j - (1-\alpha)\delta_j) + (c_j + (1-\alpha)\gamma_j)]d\alpha \\ &= \int_0^1 \alpha(b_j + c_j)d\alpha + \int_0^1 \alpha(1-\alpha)(\gamma_j - \delta_j)d\alpha = \frac{b_j + c_j}{2} + \frac{\gamma_j - \delta_j}{6}. \end{aligned}$$

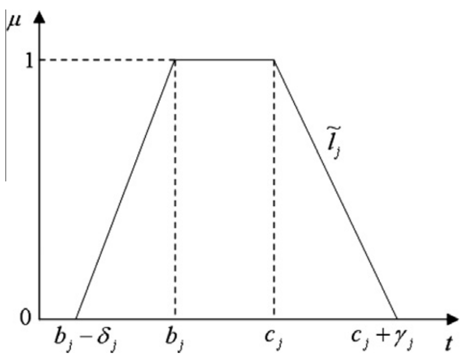

Fig. 2. Fuzzy liquidity $\tilde{l}_j$.

Then the possibilistic mean value of the turnover rate associated with portfolio x is

$$M(\tilde{l}(x)) = \sum_{j=1}^{n}\left[\frac{(b_j + c_j)}{2} + \frac{(\gamma_j - \delta_j)}{6}\right]x_j.$$

Thus, this calculation completes the proof. □

3.4. Proposed model

Based on the above discussion, if an investor wants to maximize the investment's expected return rate, minimize the risk, and maximize the portfolio liquidity, the selection scenario can be modeled as follows:

$$\begin{aligned}
&\max \quad V^{\lambda}\left(E\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right)\right)\\
&\min \quad \mathrm{Var}\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right)\\
&\max \quad M(\tilde{l}(x))\\
&\text{s.t.} \quad \begin{cases}\sum_{j=1}^{n}\mathrm{sign}(x_j) \leqslant K,\\ x_j \geqslant d_j, \text{if} x_j > 0,\\ x_j = y_j \cdot e_j, y_j \in \mathbf{N},\\ \sum_{j=1}^{n}x_j = 1, \quad x_j \geqslant 0, \quad j = 1,2,\ldots,n.\end{cases}
\end{aligned} \tag{3.16}$$

According to Lemmas 2–4, model (3.16) can be rewritten as

$$\begin{aligned}
&\max \quad \sum_{j=1}^{n}[E(a_j) + \lambda_j\beta_j/2 - (1-\lambda_j)\alpha_j/2]x_j\\
&\min \quad \mathrm{Var}\left(\sum_{j=1}^{n}a_j x_j\right)\\
&\max \quad \sum_{j=1}^{n}\Big[(b_j + c_j)/2 + (\gamma_j - \delta_j)/6\Big]x_j\\
&\text{s.t.} \quad \begin{cases}\sum_{j=1}^{n}\mathrm{sign}(x_j) \leqslant K,\\ x_j \geqslant d_j, \quad \text{if} x_j > 0,\\ x_j = y_j \cdot e_j, y_j \in \mathbf{N},\\ \sum_{j=1}^{n}x_j = 1, \quad x_j \geqslant 0, \quad j = 1,2,\cdots,n.\end{cases}
\end{aligned} \tag{3.17}$$

For simplicity, the feasible region of model (3.16) and (3.17) is denoted as X. In addition, a capital budget constraint, a no short-selling constraint, and three realistic constraints (i.e., cardinality, buy-in threshold and round-lots constraints), are also considered in the proposed model (3.16).

(i) The cardinality constraint $\sum_{j=1}^{n}\mathrm{sign}(x_j) \leqslant K$ restricts the maximum number of securities included in the portfolio, where sign $(\cdot)$ is the sign function defined as

$$\mathrm{sign}(x) = \begin{cases} 1, & \text{if } x > 0,\\ 0, & \text{if } x = 0,\\ -1, & \text{if } x < 0,\end{cases}$$

and K denotes the maximum allowable number of securities in the portfolio.

(ii) The buy-in threshold constraint, $x_j \geqslant d_j$, if $x_j > 0$, restricts the minimum amounts d_j that are to be purchased in the portfolio, $j = 1, 2, \ldots, n$.

(iii) The round-lots constraint, $x_j = y_j \cdot e_j$, $y_j \in \mathbf{N}$, restricts the smallest volume e_j that can be purchased of each security, where $\mathbf{N}$ denotes the set of all nonnegative integers, $j = 1, 2, \ldots, n$.

In this paper, it is assumed the future rates of return of the securities are the L–R type of fuzzy random variables, which assist in obtaining the crisp equivalent for the objective functions of model (3.16) and avoid complicated computations. Due to the difficulties in obtaining the crisp equivalent for the objective functions with general fuzzy random variables, it is possible to use fuzzy random simulation techniques proposed in [30] to evaluate the objective function values of the model (3.16).

4. Compromise approach-based genetic algorithm

Genetic algorithms (GAs), a type of stochastic algorithm that was inspired by natural evolution, can perform well in different types of optimization problems because they do not require the extra properties that are often associated with optimization problems, such as differentiability and continuity of the objective functions. GAs were proposed by Holland [16] in 1975, and since then, GAs have been well developed by authors such as Koza [22], Liu [31], Gen and Cheng [13]. In recent years, multi-objective evolutionary algorithms have played an important role in solving multi-objective optimization. Several multi-objective optimization algorithms, as given in various studies [42,9,54,53,20,8,27,1,38,40,55], have been proposed and successfully applied to a number of real-world, multi-objective optimization problems.

For the multi-objective portfolio optimization model, researchers plan to generate an efficient frontier [15] or its discretized representation [3,7,29,34,39,45]. However, generating an efficient frontier or its discretized representation for a constrained multi-objective portfolio selection model with more than three criteria is not an easy task. Furthermore, evaluating and selecting the best puts a considerable cognitive burden on investors. To avoid the above difficulties, a compromise approach-based genetic algorithm has been designed to resolve the proposed model (3.16) and to obtain a compromised portfolio strategy.

4.1. Compromise approach

To construct a regret function for the compromise approach, the ideal point and the anti-ideal point of model (3.16) are first computed. Consider the following two mathematical programming problems:

$$\max_{x\in X} V^{\lambda}\left(E\left(\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j\right)\right), \tag{4.18}$$

$$\min_{x\in X} V^{\lambda}\left(E\left(\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j\right)\right), \tag{4.19}$$

where X is the feasible region for the models. Furthermore, the following four mathematical programming problems are also considered.

$$\min_{x\in X} \mathrm{Var}\left(\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j\right), \tag{4.20}$$

$$\max_{x\in X} \mathrm{Var}\left(\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j\right), \tag{4.21}$$

$$\max_{x\in X} M(\tilde{l}(x)), \tag{4.22}$$

$$\min_{x\in X} M(\tilde{l}(x)). \tag{4.23}$$

Here R^* (resp. R_*, V^*, V_*, L^* and L_*) is used to denote the optimal values for problem (4.18) (resp. (4.19)–(4.22) and (4.23)). Then, the ideal point of model (3.16) is denoted by $Z^* = (R^*, V^*, L^*)$ and the anti-ideal point $Z_* = (R_*, V_*, L_*)$.

For each feasible solution $x \in X$, the regret function $r(x,p)$ ($p \geqslant 1$) is defined by the weighted L_p-norm,

$$r(x,p) := \left(w_1^p \left| \frac{V^{\lambda}\left(E\left(\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j\right)\right) - R^*}{R_* - R^*} \right|^p + w_2^p \left| \frac{\mathrm{Var}\left(\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j\right) - V_*}{V^* - V_*} \right|^p + w_3^p \left| \frac{M(\tilde{l}(x)) - L^*}{L_* - L^*} \right|^p \right)^{1/p}, \tag{4.24}$$

where $p > 0$, $w_1, w_2, w_3 \geqslant 0$ and $w_1 + w_2 + w_3 = 1$. It is assumed that decision-makers can provide the values for the weights w_1, w_2 and w_3 using their experience to assign different degrees of importance to the three objective functions in (4.24). Then, model (3.16) is converted into a single objective programming problem

$$\min_{x\in X} r(x,p). \tag{4.25}$$

However, it is often difficult to obtain optimal solutions for the above linear or nonlinear mixed integer programming models (4.18)–(4.23) with complicated constraints. Therefore, proxy ideal points and proxy anti-ideal points are used to replace the actual ideal points $Z^* = (R^*, V^*, L^*)$ and the actual anti-ideal points $Z_* = (R_*, V_*, L_*)$) appearing in (4.24), respectively. The proxy ideal point and proxy anti-ideal point are the ideal point and anti-ideal point not corresponding to a problem but corresponding to a generation. Take problem (4.18) and (4.19) as examples. Let P denote the set of the current population. Suppose that the largest objective function value of problem (4.18) in the current generation P is R^{P*}, and the smallest objective function value of problem (4.19) in the current generation P is R_{P*}, i.e.

$$R^{P*} = \max_{x\in P} V^{\lambda}\left(E\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right)\right), \quad R_{P*} = \min_{x\in P} V^{\lambda}\left(E\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right)\right).$$

Then, the proxy ideal point R^{P*} is used to replace the actual ideal point R^*, and the proxy anti-ideal point R_{P*} is used to replace the actual anti-ideal point R_*. Similarly, V^{P*}, V_{P*}, L^{P*} and L_{P*} are used to replace V^*, V_*, L^* and L_*, respectively, such that

$$V^{P*} = \min_{x\in P} \mathrm{Var}\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right), \qquad V_{P*} = \max_{x\in P} \mathrm{Var}\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right),$$

$$L^{P*} = \max_{x\in P} M(\tilde{l}(x)), \quad L_{P*} = \min_{x\in P} M(\tilde{l}(x)).$$

Let $Z^{P*} = (R^{P*}, V^{P*}, L^{P*})$ and $Z_{P*} = (R_{P*}, V_{P*}, L_{P*})$ be the proxy ideal point and the proxy anti-ideal point, respectively, of model (3.16) in the current generation P.

4.2. Genetic algorithm

The steps for the proposed compromise approach-based genetic algorithm are listed as follows:

(1) Representation structure: Model (3.16) is a multi-objective programming problem pertaining to the continuous decision vector x. Thus, a vector $x = (x_1, x_2, \ldots, x_n)$ satisfying the constraint condition is randomly chosen as a chromosome $V = (v_1, v_2, \ldots, v_n)$ to represent a solution to the optimization problem, such that the genes $v_1, v_2, \ldots, v_n$ are restricted in the interval $[0,1]$.

(2) Handling the constraints: Randomly generate a point from the hypercube $[0,1]^n$ and test its feasibility. If the point satisfies the constraints of problem (3.16), i.e., $V \in X$, it is accepted as a chromosome. Otherwise, the repair mechanisms are used to guarantee that the point satisfies the constraints of (3.16):
 (i) Keep the K largest values of x_j and set all other x_j to zero.
 (ii) Perform the following normalization technique $x_j' = x_j/\sum_{j=1}^{n} x_j$ to ensure that the random points satisfy the constraint $\sum_{j=1}^{n} x_j = 1$.
 (iii) To satisfy buy-in constraints, set all x_j below their given buy-in thresholds d_j to zero after applying the maximum number of asset repair mechanisms and normalization techniques. The normalization technique is performed again.
 (iv) To meet the round-lot constraints the algorithm rounds x_j to the next round-lot level, $x'_j = x_j - (x_j \bmod e_j)$, after cardinality repair, buy-in repair and normalization are applied. The remainder of the rounding process, $\sum_{j=1}^{n}(x_j \bmod e_j)$, is expended in quantities of e_j for those x_j that had the largest values for $x_j \bmod e_j$ until all of the remainder is disbursed.

(3) Initializing process: Initially, the feasible chromosomes $V_1, V_2, \cdots, V_{N_{pop}}$ are determined by repeating the above process N_{pop} times, given N_{pop} is the number of chromosomes.

(4) Evaluation function: The regret value of each chromosome V is calculated by

$$r_P(x,p) = \left(w_1^p \left| \frac{V^{\lambda}(E(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j)) - R^{P*}}{R_{P*} - R^{P*}} \right|^p + w_2^p \left| \frac{\mathrm{Var}\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right) - V_{P*}}{V^{P*} - V_{P*}} \right|^p + w_3^p \left| \frac{M(\tilde{l}(x)) - L^{P*}}{L_{P*} - L^{P*}} \right|^p \right)^{1/p},$$

where the weights w_1, w_2 and w_3 are provided by decision-makers using their experience to assign different degrees of importance to the three above objective functions. Then the fitness function for each chromosome is computed by

$$eval(x) = \frac{r^{\max} - r_P(x,p) + \varepsilon}{r^{\max} - r^{\min} + \varepsilon}, \tag{4.26}$$

where ε is a random number in $(0,1)$, $r^{\max}$ and $r^{\min}$ are the maximum and minimum regret values, respectively, in the current generation P.

(5) Selection process: The selection process is based on spinning the roulette wheel N_{pop} times. Each time a single chromosome for a new population is selected as follows. First, calculate the cumulative probability q_i for each chromosome x^i

$$q_0 = 0, \quad q_i = \sum_{j=1}^{i} eval(x^j), \quad i = 1, 2, \cdots, N_{pop}.$$

Generate a random number r in $[0, q_{N_{pop}}]$, and select the ith chromosome x^i such that $q_{i-1} < r \leqslant q_i$ and $1 \leqslant i \leqslant N_{pop}$. Repeat the above process N_{pop} times until N_{pop} copies of chromosomes are obtained.

(6) Crossover operation: Generate a random number c from the open interval (0,1). The chromosome x^i is selected as a parent, provided that $c < P_c$, where parameter P_c is the probability of the crossover operation. Repeat this process N_{pop} times. $P_c \cdot N_{pop}$ chromosomes are expected to be selected to undergo the crossover operation. Applying the crossover operator to the two parents x^1 and x^2 will produce two children y^1 and y^2 as follows:

$$y^1 = cx^1 + (1-c)x^2, \quad y^2 = cx^2 + (1-c)x^1.$$

Repair strategies are used to guarantee the feasibility of the two children.

(7) Mutation operation: Similar to the crossover process, chromosome x^i is selected as a parent to undergo the mutation operation, provided that random number $m < P_m$, where parameter P_m is the probability of the mutation operation. $P_m \cdot N_{pop}$ chromosomes are expected to be selected after repeating the mutation process N_{pop} times. Suppose that x is chosen as a parent. Then, a mutation direction $\mathbf{d} \in \mathbb{R}^n$ is chosen randomly, and a random positive number M is generated as a step. Therefore, the repair strategies are used to guarantee the feasibility of $x + M\mathbf{d}$.

The proposed compromise approach-based genetic algorithm process is listed as follows:

Step 0: Input the parameters N_{pop}, P_c, P_m and w_1, w_2, w_3 and p.
Step 1: Initialize N_{pop} chromosomes and convert to feasible equivalents.
Step 2: Compute the fitness of each chromosome according to the regret value.
Step 3: Select the chromosomes by spinning the roulette wheel N_{pop} times.
Step 4: Update the chromosomes using crossover and mutation operations and use the repair mechanism to guarantee the feasibility of the offspring.
Step 5: Repeat the second to fourth steps for a given number of cycles.
Step 6: Take the best chromosome as the compromise solution for the proposed constrained multi-objective portfolio selection model (3.16).

5. Numerical example

In this section, a numerical example is given to illustrate the proposed model and show the effectiveness of the proposed compromise approach-based genetic algorithm.

Assume that an investor chooses 30 stocks from the Shanghai Stock Exchange for his or her investment and that the future securities' return rates are triangular fuzzy random variables $\tilde{\bar{r}}_j(w) = (a_j(w) - \alpha_j, a_j(w), a_j(w) + \beta_j)$, $j = 1, 2, \ldots, 30$, $w \in \Omega$. The historical data for the 30 stocks from January 2005 to January 2008, are collected to determine the expectation vector for the random return vector $(a_1, a_2, \ldots, a_{30})$ and its covariance matrix $V = (\mathrm{Cov}\,(a_i, a_j))_{30\times 30}$. Table 1 shows the expected value of random variables a_j, the left width α_j and the right width β_j of $\tilde{\bar{r}}_j$, $j = 1, 2, \ldots, 30$. Here, it is assumed that an investment expert could provide α_j and β_j based on his or her experience.

The turnover rates for the stocks are assumed to be trapezoidal fuzzy numbers, denoted by $\tilde{l}_j = (b_j, c_j, \delta_j, \gamma_j)$, $j = 1, 2, \ldots, 30$. Next, we explain how to set the values for the four parameters of $\tilde{l}_j$, i.e., b_j, c_j, δ_j and γ_j based on the histograms of the turnover rates of these stocks. For example, examine the 26th stock. Daily turnover rates are collected from the above-mentioned historical data. Among the 719 daily turnover rates, most are concentrated in the interval [0%, 4%], which is subdivided into twenty smaller intervals for every 0.2% unit. Fig. 3 shows the histogram for the historical turnover rates of the sample stock. From Fig. 3 it can be observed that the daily turnover rates located in [0.2%, 0.4%], [0.4%, 0.6%], [0.6%, 0.8%], [0.8%, 1.0%], [1.0%, 1.2%], [1.2%, 1.4%], [1.4%, 1.6%] and [1.6%, 1.8%] clearly reoccur more frequently than those rates in other intervals. Therefore, the left endpoint of the tolerance interval is set at b_j = 0.2% and the right endpoint of the tolerance interval is c_j = 1.8%. The left width is set at δ_j = 0.1%, and the right width is set at γ_j = 0.8%. Similarly, the future turnover rates of all 30 stocks are deter-

Table 1
The expectation of a_j, the left width α_j and the right width β_j of $\tilde{\bar{r}}_j$ (%).

Stock	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$E(a_j)$	0.45	1.42	1.73	0.61	1.39	1.16	1.01	1.87	0.44	2.35	0.45	1.42	1.73	0.61	1.39
α_j	0.15	0.22	0.23	0.10	0.19	0.16	0.21	0.17	0.04	0.35	0.08	0.11	0.06	0.18	0.13
β_j	0.15	0.08	0.27	0.39	0.71	0.64	0.09	0.53	0.36	0.25	0.22	0.39	0.14	0.12	0.37
Stock	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$E(a_j)$	1.16	1.01	1.87	0.44	2.35	0.45	1.42	1.73	0.61	1.39	1.16	1.01	1.87	0.44	2.35
α_j	0.15	0.11	0.16	0.07	0.10	0.26	0.14	0.39	0.21	0.22	0.51	0.11	0.61	0.42	0.19
β_j	0.10	0.19	0.54	0.33	0.30	0.24	0.26	0.11	0.09	0.28	0.09	0.39	0.29	0.58	0.61

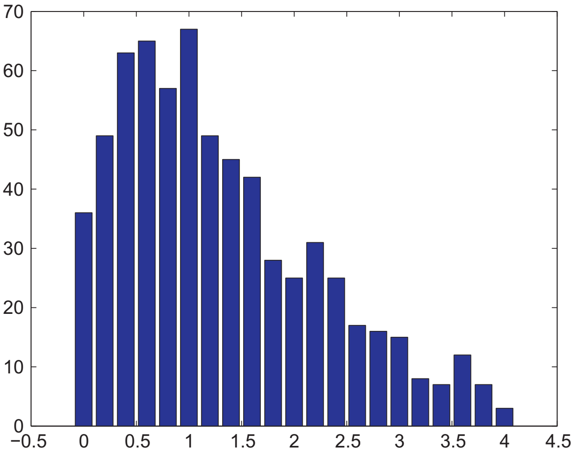

Fig. 3. The histogram of the historical turnover rates of the stock.

mined to be trapezoidal fuzzy numbers according to the histograms for the historical turnover rates of each stock, which are presented in Table 2.

In this example, the maximum number of securities is 20, the minimum amount to be purchased of each stock is 0.01, i.e., $d_j = 0.01$, and the smallest volume purchased of each stock is 0.0001, i.e., $e_j = 0.0001$. The compromise approach-based genetic algorithm is performed on a personal computer utilizing the following parameters: the population size is 300; the probability of crossover $P_c = 0.9$ and the probability of mutation $P_m = 0.1$; the parameter $p = 2$ in the regret function $r(x,p)$; and relatively important parameters of the three criteria (expected return, risk and liquidity), which are $w_1 = 0.4$, $w_2 = 0.4$ and $w_3 = 0.2$ in the regret function $r(x,2)$, respectively.

For an investor with a completely optimistic attitude toward all future stock returns, the compromised portfolio strategy is given in Table 3 after performing the proposed algorithm with 200 generations. The corresponding compromised objective function values are 0.0187, 0.0042 and 0.0072. Fig. 4 shows the regret values for the best chromosome in each generation when $\lambda = 1$.

For an investor with a completely pessimistic attitude toward all future stock returns and after performing the proposed algorithm with 200 generations, the compromised portfolio strategy is given in Table 4. The corresponding objective function values are 0.0138, 0.0035 and 0.0069. Fig. 5 shows the regret values for the best chromosome in each generation when $\lambda = 0$.

Table 2
The trapezoidal fuzzy turnover rates of all stocks (%).

Stock	Future turnover rate	Stock	Future turnover rate	Stock	Future turnover rate
1	(0.2, 1.2, 0.1, 0.6)	11	(0, 0.6, 0, 0.1)	21	(0.2, 1.2, 0.1, 0.4)
2	(0.6, 2, 0.2, 1)	12	(0.2, 0.8, 0.1, 0.4)	22	(0.1, 1, 0.1, 0.3)
3	(0.2, 1, 0.1, 0.2)	13	(0.4, 1, 0.2, 0.4)	23	(0.2, 1.6, 0.1, 0.3)
4	(0.4, 1.4, 0.2, 0.4)	14	(0.2, 0.8, 0.1, 0.3)	24	(0.1, 1.8, 0.1, 0.5)
5	(0.2, 1, 0.1, 0.2)	15	(0.2, 1, 0.1, 0.4)	25	(0.1, 1, 0.1, 0.3)
6	(0.2, 1.2, 0.1, 0.4)	16	(0.2, 1.4, 0.2, 0.4)	26	(0.2, 1.8, 0.1, 0.8)
7	(0.2, 0.8, 0.1, 0.2)	17	(0.2, 1.2, 0.1, 0.2)	27	(0.1, 0.8, 0, 0.3)
8	(0.2, 1.8, 0.1, 0.2)	18	(0.2, 0.8, 0.1, 0.4)	28	(0.1, 1, 0, 0.3)
9	(0.2, 1, 0.1, 0.6)	19	(0.2, 1, 0, 0.2)	29	(0.2, 1.8, 0.1, 0.8)
10	(0, 0.6, 0, 0.2)	20	(0.1, 1, 0, 0.2)	30	(0.2, 1.8, 0.1, 0.8)

Table 3
The compromised portfolio strategy for completely optimistic investor ($\lambda = 1$).

Stock	1	2	3	4	5	6	7	8	9	10
Proportions	0.0680	0.0561	0.0609	0	0	0	0	0.0342	0.0345	0
Stock	11	12	13	14	15	16	17	18	19	20
Proportions	0.0272	0	0	0.0659	0.0699	0	0.0683	0.0698	0.0543	0.0736
Stock	21	22	23	24	25	26	27	28	29	30
Proportions	0.0527	0.0624	0	0	0	0	0.0557	0.0489	0.0722	0.0254

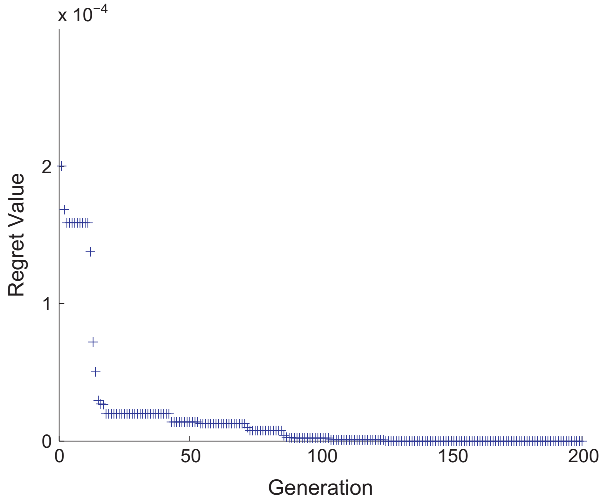

Fig. 4. Regret values for the best chromosome in each generation ($\lambda = 1$).

Table 4
The compromised portfolio strategy for completely pessimistic investor ($\lambda = 0$).

Stock	1	2	3	4	5	6	7	8	9	10
Proportions	0.0446	0.0277	0	0	0.0595	0.0667	0.0367	0.0325	0	0.0538
Stock	11	12	13	14	15	16	17	18	19	20
Proportions	0.0387	0.0555	0.0517	0.0599	0.0535	0.0434	0.0550	0.0528	0.0758	0.0411
Stock	21	22	23	24	25	26	27	28	29	30
Proportions	0	0	0	0.0595	0	0	0.0401	0	0	0.0515

For an investor with a neutral attitude toward all future stock returns, the compromised portfolio strategy is given in Table 5 after performing the proposed algorithm with 200 generations. The corresponding objective function values are 0.0169, 0.0041 and 0.0075. Fig. 6 shows the regret values for the best chromosome in each generation when $\lambda = 0.5$.

Also, the histograms for the compromised portfolio strategies when $\lambda = 1$, $\lambda = 0$ and $\lambda = 0.5$ are given in Fig. 7.

The investor can obtain more compromised portfolio strategies by changing the value of the optimistic–pessimistic parameter λ in $V^{\lambda}\left(E\left(\sum_{j=1}^{n}\tilde{\tilde{r}}_j x_j\right)\right)$.

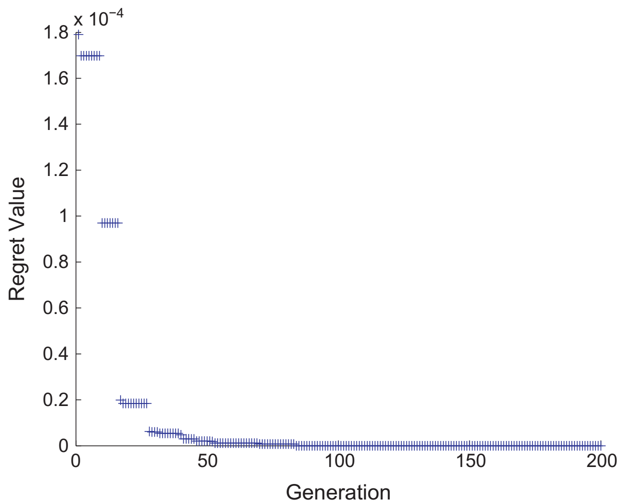

Fig. 5. Regret values for the best chromosome in each generation ($\lambda = 0$)

Table 5
The compromised portfolio strategy for neutral investor ($\lambda = 0.5$).

Stock	1	2	3	4	5	6	7	8	9	10
Proportions	0.0794	0.0679	0	0.0604	0.0231	0	0	0.0229	0.0636	0
Stock	11	12	13	14	15	16	17	18	19	20
Proportions	0.0502	0	0	0	0.0786	0.0211	0.0818	0.0413	0.0285	0.0921
Stock	21	22	23	24	25	26	27	28	29	30
Proportions	0.0681	0.0678	0	0	0	0	0	0.0728	0.0804	0

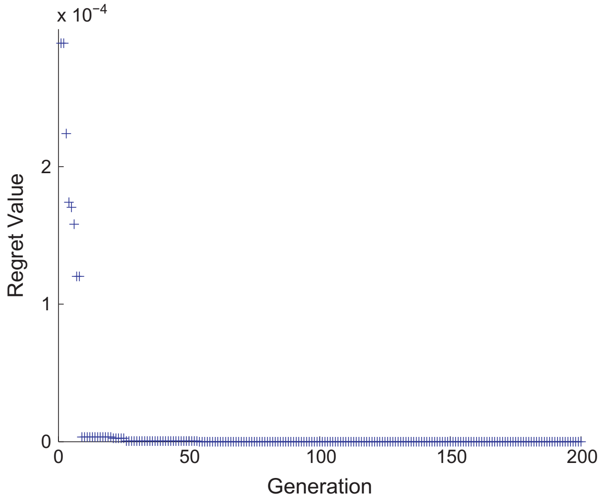

Fig. 6. Regret values for the best chromosome in each generation ($\lambda = 0.5$)

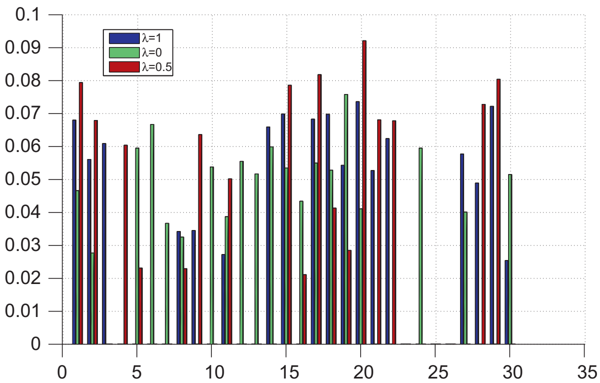

Fig. 7. Comparison between the histograms of the compromised portfolio strategies when $\lambda = 1$, $\lambda = 0$ and $\lambda = 0.5$.

6. Conclusions

In this paper, a constrained multi-objective portfolio selection model with fuzzy random returns is proposed after quantifying the return, risk and liquidity of a portfolio. To avoid the difficulty of evaluating a large set of efficient solutions and to ensure that the best solution is selected, a compromise approach-based genetic algorithm was designed to resolve the proposed model and consequently obtain a compromised portfolio strategy. In addition, a numerical example was presented to illustrate this modeling concept and to demonstrate and the effectiveness of the proposed algorithm.

In comparison to former multi-objective portfolio selection models, the proposed constrained multi-objective portfolio selection model can capture twofold uncertainty. In addition, the portfolio selection process incorporates historical securities data, expert judgment and experience, and the investors' subjective attitudes about securities' future returns. By varying the optimistic–pessimistic parameter λ, an investor can build his or her multi-objective portfolio selection model to obtain the corresponding compromised portfolio strategy. Therefore, the homogeneous expectation assumption is no longer needed in our proposed portfolio selection model. Moreover, the computational results show that the compromise approach-based genetic algorithm is a feasible and effective means of obtaining a compromised solution. This proposed compromise approach-based genetic algorithm can be generalized to other multi-objective programming models with non-smooth characteristics.

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