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# OPTIMAL CONSTANT-REBALANCED PORTFOLIO INVESTMENT STRATEGIES FOR DYNAMIC PORTFOLIO SELECTION

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In this paper we propose a variant of the continuous-time Markowitz mean-variance model by incorporating the Earnings-at-Risk measure in the portfolio optimization problem. Under the Black-Scholes framework, we obtain closed-form expressions for the optimal constant-rebalanced portfolio (CRP) investment strategy. We also derive explicitly the corresponding mean-EaR efficient portfolio frontier, which is a generalization of the Markowitz mean-variance efficient frontier.

*Keywords*: Dynamic portfolio optimization; earnings-at-risk; constant-rebalanced portfolios; Black-Scholes model.

## 1. Introduction

The seminar work of Markowitz [17] on the portfolio selection mean-variance efficient frontier has become the foundation of modern finance theory. This result has generated a proliferation of research. In term of its generations, two challenges can be identified from theoretical point of view. First is the extension of a classical single-period model to a multi-period or continuous-time model. A common approach along the dynamic mean-variance model is to focus on maximizing some time-additive utility of terminal wealth and/or consumption (see, e.g. [18, 19, 22, 23]). This technique however is intractable (see, e.g., [4]). Using embedding techniques, explicit-form optimal strategies of the dynamic mean-variance problems was finally solved in recent years by Li and Ng [13] and Zhou and Li [25], respectively, in the discrete-time and continuous-time frameworks.

The second challenge lies on the correct measure of risk. While there is no ambiguity on the definition of return, the measure of risk is more subjective. Consequently, many variants of risk measures have been proposed. These include absolute deviation, semi-variance, shortfall probability, safety-first, etc. Many of these measures are typically based on the notion of downside risk concepts such as the lower partial moments. More recently, risk measures such as the value at risk (VaR) [12], the coherence risk measure [1] and the limited expected loss [2] have been advocated.

Among these risk measures, VaR remains the most prominent risk measure in recent years and its importance continues to grow since regulators accept it as a benchmark for controlling market risk, despite several problems have been reported associated with such risk measure. In addition to quantifying the market risk, VaR has also been proposed as a measure of downside risk in the context of portfolio optimization. For example, Litterman [14, 15] and Lucas and Klassen [16] discussed the problems of portfolio optimization using VaR as a constraint. Emmer *et al.* [8, 9] defined a VaR-based related concept known as Capital-at-Risk (CaR) and demonstrated how to incorporate such measure in the portfolio optimization problem. In particular, by formulating the dynamic portfolio optimization as a constant-rebalanced portfolio (CRP) investment strategy, Emmer *et al.* [8, 9] derived analytically the solution to the portfolio optimization problem as well as the mean-CaR efficient portfolio frontier.

We now explain what is a CRP strategy. A CRP strategy is an investment strategy which ensures that the proportion of total wealth invested in each of the underlying securities is the same at any time point, regardless of the level of wealth. These strategies, also known as the constant mix strategies, are widely studied in the literature; see for example, Perold and Sharpe [20], Black and Perold [3], Cover [5] and Helmbold *et al.* [11]. There are a number of advantages of adopting CRP strategies. Merton [18, 19] showed that this form of strategies is optimal to the portfolio selection problems of maximizing expected utility with constant relative risk-aversion. Furthermore, these strategies are widely used in asset allocation practice (see, for example, [3, 20]). However, since such strategies may not be feedback strategies under general models, the optimal CRP strategy to our model or to the models in [8, 9] may not be a globally optimal feedback strategy. For further discussions of feedback controls and optimal feedback policies, see Fleming and Soner [10].

It should be emphasized that a CRP strategy is a dynamic investment strategy in that it requires trading over time. As the stock prices evolve randomly, one has to trade at every instant to ensure the fraction of wealth for each security remains constant. Note that this rebalancing requires selling an asset when its price rises relative to the other prices, and conversely, buying an asset when its price drops relative to the others. The example in Helmbold *et al.* [11] exemplifies the significance of a CRP strategy. To illustrate this, let us assume that there are only two securities. The first security is a riskless asset whose price never changes. The second asset is extremely volatile whose price doubles (its initial price) on even days and halves (its initial price) on odd days. Consequently, the price processes can be described by the sequences $\{1, 1, 1, \ldots\}$ and $\{\frac{1}{2}, 2, \frac{1}{2}, 2, \ldots\}$ for the first and second asset, respectively. By construction, strictly investing in either of these securities cannot increase its initial wealth by more than twice. On the other hand, a CRP with equal proportion invested in each of these assets will increase its wealth exponentially. To see this, let us recall that a CRP strategy rebalances the portfolio whenever these is a change in the underlying asset. For the portfolio which maintains an equal wealth in each security, the portfolio value decreases by a factor of $\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$ on odd day and increases by $\frac{1}{2} \times 1 + \frac{1}{2} \times 2 = \frac{3}{2}$ on even day. Hence with rebalancing, the investor's wealth grows by a factor of $\frac{3}{4} \times \frac{3}{2} = \frac{9}{8}$ after two consecutive trading days. It is easy to show that in general the wealth increases by a factor of $\left(\frac{9}{8}\right)^n$ after $2n$ trading days. Consequently it only takes twelve days to double the initial wealth.

While VaR-based measure has become a standard measure in quantifying market risk, it is important to note that as pointed out by Basak and Shapiro [2] and Vorst [24], using VaR as a constraint in portfolio optimization can induce some perverse incentives so that in some circumstances it can lead to an increase in risk taking. On the other hand when the downside risk is measured by the *expected shortfall* (e.g., tail-VaR), these perverse incentives disappear. This is also the key motivation for proposing the tail-VaR based Earning-at-Risk (EaR) measure in this paper in the context of portfolio optimization. This is in contrast to the model considered by Emmer *et al.* [8, 9] which uses a VaR-based Capital-at-Risk measure. Our proposed formulation of the portfolio optimization model in connection with the risk measure EaR is described in Sec. 2. Its analytical results are derived in Sec. 3. Section 4 provides a comparison between the mean-EaR portfolio model and the mean-variance model. Section 5 concludes the paper.

Note that both VaR and tail-VaR can be applied to risk with asymmetric distributions. In particular, VaR and tail-VaR as a risk measure for dependent risks and risks with heavy-tailed distributions have generated considerable interests in recent years; see for example, Embrechts *et al.* [6], Embrechts *et al.* [7] and references therein. In this paper, we focus on issue related to the portfolio selection policy. In the interest of obtaining analytic and tractable results, we confine our model to the popular Black-Scholes type geometric Brownian motion framework.

## 2. The Dynamic Portfolio Selection Model

In this section, we consider a dynamic portfolio selection model. Our framework involves (i) the Black-Scholes type financial market, (ii) a CRP investment strategy, and (iii) a mean-EaR trade-off. Consider a standard Black-Scholes type financial market in which $n+1$ assets (or securities) are traded continuously in the horizon $[0,T]$. For convenience, we index these assets by $i=0,1,\ldots,n$ with $i=0$ denotes the riskless bond whose price process $P_0(t)$ evolves according to the following (deterministic) ordinary differential equation:

$$dP_0(t) = P_0(t)rdt \quad \text{for } t \in [0,T], \;\; P_0(0) = 1, \tag{2.1}$$

where $r$ is a constant rate of interest. The remaining $n$ assets are risky stocks whose price processes $P_1(t),\ldots,P_n(t)$ governed by the following stochastic differential equations:

$$dP_i(t) = P_i(t)\left(b_i dt + \sum_{j=1}^{n} \sigma_{ij} dB_j(t)\right) \quad \text{for } t \in [0,T], \;\; i = 1,\ldots,n, \tag{2.2}$$

where $b = (b_1,\ldots,b_n)'$ is the vector of stock-appreciation rate, $\sigma = (\sigma_{ij})_{n\times n}$ is the matrix of stock-volatilities and $B(t) = (B_1(t),\ldots,B_n(t))'$ is a standard $n$-dimensional Brownian motion. Here $b$ and $\sigma$ are assumed to be constant in time. As usual, we further assume that $\sigma$ is invertible and that $b_i \geq r$ for all $i$.

Let $W^\pi(t)$ be the wealth at time $t$ for a given portfolio strategy $\pi(t) = (\pi_1(t),\ldots,\pi_n(t))' \in \mathbb{R}^n$, where $\pi_i(t)$ is the fraction of the wealth $W^\pi(t)$ invested in asset $i$ at time $t$. Then $\pi_0(t) = 1 - \pi(t)'\mathbf{1}$, where $\mathbf{1} = (1,\ldots,1)'$ is the vector whose components are all units. By definition, the number of units at time $t$ invested in riskless bond and risky assets are:

$$\left.\begin{aligned} N_0(t) &= W^\pi(t)(1-\pi(t)'\mathbf{1})/P_0(t), \\ N_i(t) &= W^\pi(t)\pi_i(t)/P_i(t), \quad i = 1,\ldots,n \end{aligned}\right\}. \tag{2.3}$$

Hence,

$$W^\pi(t) = \sum_{i=0}^{n} N_i(t)P_i(t). \tag{2.4}$$

Throughout the paper, we further assume that there is no transaction costs nor consumption and that the portfolio strategy $\pi(t)$ is self-financing. Thus

$$\begin{aligned} dW^\pi(t) &= \sum_{i=0}^{n} N_i(t)dP_i(t) \\ &= \left\{rN_0(t)P_0(t) + \sum_{i=1}^{n} b_i N_i(t)P_i(t)\right\}dt + \sum_{i=1}^{n} N_i(t)P_i(t)\sum_{j=1}^{n}\sigma_{ij}dB_j(t) \\ &= W^\pi(t)\left\{((1-\pi(t)'\mathbf{1})r + \pi(t)'b)dt + \pi(t)'\sigma dB(t)\right\}, \end{aligned} \tag{2.5}$$

with $W^\pi(0) = w > 0$ being the initial wealth of an investor.

As consistent with [8]–[11] and many others, in what follows we restrict ourselves to constant-rebalanced portfolio (CRP) strategies. As noted in the introduction, a CRP strategy implies that the portfolio $\pi(t)$ does not change over time; i.e., $\pi(t) = \pi = (\pi_1, \ldots, \pi_n)'$ for all $t \in [0, T]$. But such portfolio is continuously rebalanced in order to restore a constant proportion of wealth in each asset.

It follows from standard Itô integral and that $E[e^{sB_j(t)}] = e^{ts^2/2}$, where $E$ is the expectation operator, it is easy to establish the following explicit formulae for the wealth process $W^\pi(t)$ for all $t \in [0, T]$ (see, e.g., [8]):

$$W^\pi(t) = w \exp((\pi'(b - r\mathbf{1}) + r - \|\pi'\sigma\|^2/2)t + \pi'\sigma B(t)), \tag{2.6}$$

$$E[W^\pi(t)] = w \exp((\pi'(b - r\mathbf{1}) + r)t), \tag{2.7}$$

$$Var[W^\pi(t)] = w^2 \exp(2(\pi'(b - r\mathbf{1}) + r)t)[\exp(\|\pi'\sigma\|^2 t) - 1], \tag{2.8}$$

where $w$ denotes the initial wealth, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $Var$ is the variance operator.

We now use the notation $\rho(\pi)$ to denote the relevant risk measure over time horizon $T$ on a portfolio with position $\pi$ and initial wealth $w$. As discussed earlier there exists various representations of this measure. For instance, the popular $\alpha$-quantile based VaR ($\rho_0(\pi) := \rho(\pi)$) is defined as

$$Pr(W^\pi(T) \le \rho_0(\pi)) = \alpha, \tag{2.9}$$

where $Pr(\cdot)$ is the probability and $\alpha \in (0, 1)$. The tail-VaR, or more precisely the conditional tail expectation or the expected shortfall of $W^\pi(T)$ is defined as

$$\rho_1(\pi) := \rho(\pi) = E[W^\pi(T) | W^\pi(T) \le \rho_0(\pi)]. \tag{2.10}$$

Hence tail-VaR gives the expected severity of the shortfall given that the loss exceeds the corresponding VaR. See for example Artner *et al.* [1].

For a given risk measure $\rho(\pi)$, we formally define a general class of Earnings-at-Risk as follows:

**Definition 2.1.** The *Earnings-at-Risk (EaR)* of a CRP investment strategy $\pi$ relative to a risk measure $\rho$ is defined as the difference between the mean terminal wealth and its associated risk measure; i.e.,

$$EaR(\pi) := E[W^\pi(T)] - \rho(\pi). \tag{2.11}$$

Note that EaR depends on a chosen risk measure. In this paper, we confine the discussion of EaR by assuming tail-VaR as the relevant risk measure. Hence our subsequent reference to EaR refers to EaR relative to tail-VaR. Intuitively, tail-VaR is a more appealing risk measure than the quantile-VaR. Tail-VaR gives the average severity of the loss given that the loss exceeds its quantile-VaR. Quantile-VaR, on the other hand, only provides a probabilistic statement for which the loss exceeds the quantile-VaR. Additional theoretical justification of tail-VaR over quantile-VaR can be found in Artzner *et al.* [1].

Note that there are important distinctions between the proposed EaR and the Capital-at-Risk (CaR) considered by Emmer *et al.* [9]. CaR is defined as the difference between the terminal wealth of the pure bond (riskless) investment strategy and the risk measure $\rho(\pi)$. For a given risk measure $\rho(\pi)$, EaR measures risk relative to mean terminal wealth $E[W^{\pi}(T)]$ while CaR measures risk relative to pure bond investment strategy. The mean terminal wealth depends explicitly on the adopted investment strategy $\pi$ while the pure bond strategy is independent of $\pi$. EaR therefore provides a trade-off between investing in the portfolio with position $\pi$ and its expected shortfall as a result of adopting such an investment strategy. When formulated as an optimization problem, both the mean return and its risk measure are considered jointly. Hence it is a more relevant measure over CaR which only provides a trade-off between the risk-free investment and its associated risk measure. More critically, CaR is VaR based measure while our proposed EaR is tail-VaR based. As pointed in the introduction (see [2, 24]) the VaR-based measure can lead to undesirable situation in the context of portfolio optimization.

Let $z_\alpha$ be the $\alpha$-quantile of the standard normal distribution and $\Phi$ be the distribution function of a standard normal random variable. Since $\pi'\sigma B(T)/(\|\pi'\sigma\|\sqrt{T})$ is a standard normal random variable, it follows from (2.6), (2.9) and (2.10) that we can express explicitly the risk measures $\rho_0$ and $\rho_1$ as

$$\rho_0(\pi) = w\exp\big((\pi'(b - r\mathbf{1}) + r - \|\pi'\sigma\|^2/2)T + z_\alpha\|\pi'\sigma\|\sqrt{T}\big), \tag{2.12}$$

$$\rho_1(\pi) = w\exp\big((\pi'(b - r\mathbf{1}) + r)T\big)\frac{\Phi(z_\alpha - \|\pi'\sigma\|\sqrt{T})}{\alpha}, \tag{2.13}$$

respectively (see [9]). Therefore, a closed-form representation of $EaR(\pi)$ (relative to tail-VaR) is given by

$$EaR(\pi) = w\exp((\pi'(b - r\mathbf{1}) + r)T)\left(1 - \frac{\Phi(z_\alpha - \|\pi'\sigma\|\sqrt{T})}{\alpha}\right). \tag{2.14}$$

To avoid some subcases in the results of this paper, we make the following assumption.

**Assumption 2.1.** *The parameter $\alpha$ satisfies $\alpha < 0.5$, hence $z_\alpha < 0$.*

Our formulation of the dynamic portfolio selection model follows the classical Markowitz model. Recall that one approach of deriving the mean-variance efficient portfolio is to minimize the variance of the portfolio return for a given level of the expected portfolio return. Analogously, our optimization problem involves minimizing the EaR for a given level of the expected terminal wealth. In other words, we solve the following optimization problem:

$$(P) \qquad \min_{\pi\in\mathbb{R}^n} EaR(\pi) \quad \text{subject to} \quad E[W^{\pi}(T)] \geq C,$$

where $C$ is a predetermined minimum attainable expected terminal wealth $E[W^{\pi}(T)]$. We refer the above optimization problem as the mean-EaR problem.

Since the pure bond policy yields a deterministic terminal wealth of $w\exp(rT)$, it is natural to assume that the minimum expected wealth $C$ satisfies the following lower bound condition:

$$C \geq w\exp(rT). \tag{2.15}$$

## 3. Optimal Strategy and Efficient Frontier

In this section we derive analytically the best CRP investment strategy; i.e., the optimal solution to portfolio optimization problem $(P)$. As a by-product, we also obtain a closed-form expression for the corresponding mean-EaR efficient frontier.

To establish our result, we begin with the following property of Earnings-at-Risk:

**Proposition 3.1.**

(i) $\sup_{\pi\in\mathbb{R}^n} EaR(\pi) = \begin{cases} we^{rT} & \textit{if } b = r\mathbf{1}, \\ +\infty & \textit{otherwise.} \end{cases}$

(ii) $\min_{\pi\in\mathbb{R}^n} EaR(\pi) = 0$ *and the minimum is only attained at* $\pi = 0$.

**Proof.** (i) If $b = r\mathbf{1}$, the conclusions are obvious. Now we assume that $b \neq r\mathbf{1}$.

We rewrite expression (2.14) of $EaR$ in the following form:

$$EaR(\pi) = \begin{cases} we^{f(\pi)} & \text{if } \|\pi'\sigma\| > 0, \\ 0 & \text{if } \|\pi'\sigma\| = 0, \end{cases} \tag{3.1}$$

where

$$f(\pi) = (\pi'(b - r\mathbf{1}) + r)\,T + \ln\left(1 - \frac{\Phi(z_\alpha - \|\pi'\sigma\|\sqrt{T})}{\alpha}\right). \tag{3.2}$$

Now consider the following optimization problem

$$\max_{\pi}\ f(\pi) \quad \text{subject to } \|\pi'\sigma\| = \varepsilon, \tag{3.3}$$

for any given $\varepsilon > 0$. Over the (boundary of the) ellipsoid defined by the constraint in problem (3.3), the objective function is equivalent to

$$f(\pi) = (\pi'(b - r\mathbf{1}) + r)\,T + \ln\left(1 - \frac{\Phi(z_\alpha - \varepsilon\sqrt{T})}{\alpha}\right). \tag{3.4}$$

Hence, solving problem (3.3) is equivalent to solving the following problem

$$\max_{\pi}\ \pi'(b - r\mathbf{1}) \quad \text{subject to } \pi'(\sigma\sigma')\pi = \varepsilon^2. \tag{3.5}$$

Using the Lagrangian method, the unique optimal solution to this problem is given by

$$\pi^*_\varepsilon = \varepsilon\frac{(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|}, \tag{3.6}$$

with maximum value

$$f(\pi_\varepsilon^*) = \varepsilon\theta T + rT + \ln\left(1 - \frac{\Phi(z_\alpha - \varepsilon\sqrt{T})}{\alpha}\right). \tag{3.7}$$

In the above expression, we have $\theta = \|\sigma^{-1}(b - r\mathbf{1})\|$. Clearly,

$$\lim_{\varepsilon\to+\infty} f(\pi_\varepsilon^*) = +\infty, \tag{3.8}$$

which leads to second part of statement (i).

(ii) Statement (ii) follows directly from the assumption that the matrix $\sigma$ is invertible and the fact that $EaR(\pi) > 0 = EaR(0)$ for all $\pi \neq 0$ by (2.14). □



Proposition 3.1 implies that EaR attains a lower bound of zero for the pure bond strategy. It is bounded from above by $we^{rT}$ in a risk-neural market and unbounded above otherwise.

We now give the main result of this paper.

**Theorem 3.1.** *Assume that* $b \neq r\mathbf{1}$. *Then the unique optimal policy of problem* $(P)$ *is*

$$\pi^* = \varepsilon^* \frac{(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|}, \tag{3.9}$$

*where*

$$\varepsilon^* = \frac{\ln(C/w) - rT}{\|\sigma^{-1}(b - r\mathbf{1})\|T}. \tag{3.10}$$

*The corresponding expected terminal wealth is* $E[W^{\pi^*}(T)] = C$ *and Earnings-at-Risk is*

$$EaR(\pi^*) = C\left[1 - \frac{\Phi(z_\alpha - \varepsilon^*\sqrt{T})}{\alpha}\right]. \tag{3.11}$$

**Proof.** We first reformulate problem $(P)$ as follows:

$$(P) \qquad \begin{cases} \text{minimize} & w\exp((\pi'(b - r\mathbf{1}) + r)T)\left[1 - \frac{\Phi(z_\alpha - \|\pi'\sigma\|\sqrt{T})}{\alpha}\right] \\ \text{subject to} & w\exp((\pi'(b - r\mathbf{1}) + r)T) \geq C. \end{cases}$$

If $C = w\exp(rT)$, it follows from Proposition 3.1 (ii) that the pure bond policy $\pi^* = 0$ is a feasible solution to problem $(P)$, with the global minimal Earnings-at-Risk $EaR(\pi^*) = 0$. Hence, $\pi^* = 0$ is the unique optimal solution of $(P)$ which means that the conclusions asserted is true for this special case.

Now we assume that $C > w\exp(rT)$. The feasible set of the problem is

$$\Pi = \left\{\pi : (b - r\mathbf{1})'\pi T \geq \ln\frac{C}{w} - rT\right\}. \tag{3.12}$$

Given $\varepsilon \geq 0$, the intersection of $\Pi$ and the ellipsoid $\|\pi'\sigma\| = \varepsilon$ is

$$\Pi(\varepsilon) = \left\{\pi : \|\pi'\sigma\| = \varepsilon,\ (b - r\mathbf{1})'\pi T \geq \ln\frac{C}{w} - rT\right\}. \tag{3.13}$$

The hyperplane $(b - r\mathbf{1})'\pi T = \ln(C/w) - rT$ is tangent to the ellipsoid $\|\pi'\sigma\| = \varepsilon$ if and only if $\varepsilon\theta T = \ln(C/w) - rT$, that is $\varepsilon = \varepsilon^* := \frac{\ln(C/w)-rT}{\theta T} > 0$, where $\theta = \|\sigma^{-1}(b - r\mathbf{1})\|$. Consequently $\Pi(\varepsilon) = \emptyset$ if $\varepsilon < \varepsilon^*$ and hence $\Pi = \bigcup_{\varepsilon\geq\varepsilon^*}\Pi(\varepsilon)$. Thus problem $(P)$ is equivalent to the following bilevel optimization problem

$$(P') \quad \min_{\varepsilon\geq\varepsilon^*}\min_{\pi\in\Pi(\varepsilon)} w\exp((\pi'(b - r\mathbf{1}) + r)T)\left[1 - \frac{\Phi(z_\alpha - \varepsilon\sqrt{T})}{\alpha}\right].$$

For each fixed $\varepsilon \geq \varepsilon^*$, we solve the problem

$$\min_{\pi\in\Pi(\varepsilon)} w\exp((\pi'(b - r\mathbf{1}) + r)T)\left[1 - \frac{\Phi(z_\alpha - \varepsilon\sqrt{T})}{\alpha}\right], \tag{3.14}$$

or equivalently

$$\min_{\pi\in\Pi(\varepsilon)}(b - r\mathbf{1})'\pi T. \tag{3.15}$$

When $\varepsilon = \varepsilon^*$, the optimal solution is the unique tangent point $\pi^* = \varepsilon^*\frac{(\sigma\sigma')^{-1}(b-r\mathbf{1})}{\|\sigma^{-1}(b-r\mathbf{1})\|}$ of the hyperplane $(b - r\mathbf{1})'\pi T = \ln\frac{C}{w} - rT$ to the ellipsoid $\|\pi'\sigma\| = \varepsilon^*$, with $(b - r\mathbf{1})'\pi^*T = \varepsilon^*\theta T$. When $\varepsilon > \varepsilon^*$, $\min\{(b - r\mathbf{1})'\pi T : \pi \in \Pi(\varepsilon)\} = \ln\frac{C}{w} - rT = \varepsilon^*\theta T$, and every point on both the hyperplane $(b - r\mathbf{1})'\pi T = \ln\frac{C}{w} - rT$ and the ellipsoid $\|\pi'\sigma\| = \varepsilon$ is an optimal solution. Therefore, we can obtain the solution of problem $(P')$ by solving the problem

$$\min_{\varepsilon\geq\varepsilon^*} w\exp((\varepsilon^*\theta + r)T)\left[1 - \frac{\Phi(z_\alpha - \varepsilon\sqrt{T})}{\alpha}\right]. \tag{3.16}$$

Since the function $1 - \frac{1}{\alpha}\Phi(z_\alpha - \varepsilon\sqrt{T})$ is strictly increasing with respect to $\varepsilon$, the optimal $\varepsilon$ for the above problem is the unique $\varepsilon^*$. This completes the proof. □

We now make the following remarks:

- The analytic result in Theorem 3.1 provides an explicit relation between the optimal Earnings-at-Risk and the expected terminal wealth. Letting $\xi := E[W^{\pi^*}(T)]$, we have

$$EaR(\xi) = \xi\left[1 - \frac{1}{\alpha}\Phi\left(z_\alpha - \frac{\ln(\xi/w) - rT}{\|\sigma^{-1}(b - r\mathbf{1})\|\sqrt{T}}\right)\right] \quad \text{for } \xi \geq w\exp(rT). \tag{3.17}$$

  The above relationship is known as the efficient frontier in the mean-EaR space.

- Observe that the mean-EaR efficient frontier depends on the confident level $\alpha$. Smaller $\alpha$ is achieved at the expense of a higher EaR risk measure in order to maintain the same expected terminal wealth.
- Theorem 3.1 also implies that for a given level of expected terminal wealth, EaR of the best CRP investment strategy is decreasing in time horizon $T$. This is consistent with intuition.
- The above mean-EaR efficient frontier is obtained by solving the optimization problem (P). Equivalently, the same efficient frontier could have obtained by maximizing the mean terminal wealth for a given level of EaR; i.e.,

$$(\tilde{P}) \qquad \max_{\pi \in \mathbb{R}^n} \; E[W^\pi(T)] \quad \text{subject to} \;\; EaR(\pi) \leq \tilde{C},$$

  where $\tilde{C}$ is a given constant.

We now consider the following two examples to highlight our proposed model.

**Example 3.1.** In this example, we analyze the impact of using EaR measure in portfolio construction. We use the parameter values $n = 1, w = 1000, r = 0.05, \alpha = 0.05, \sigma = 0.2$. Then $z_\alpha = -1.65, \theta = 0.25$. Figure 1 demonstrates that the EaR of a pure stock policy is an increasing function over the time horizon $T, 0 < T \leq 5$. Furthermore, the stock with a higher appreciation rate ($b = 0.15$) yields higher value of EaR. This is to be expected since EaR increases with the stock's appreciation rate in a pure stock policy. To compare with the optimal CRP investment strategy we assume the minimum expected terminal wealth of $C = we^{rT} = 1284$, which is the return from investing in a pure bond policy over 5 year horizon. By construction, the EaR decreases monotonically over time to 0 in year 5, as confirmed in Fig. 1. It is interesting to note that to achieve the same expected wealth level of $C$, the EaR under the optimal CRP strategy is actually lower for stock with higher appreciation rate. Figure 2 indicates that in earlier years, the optimal portfolio always contains a short position in the bond as long as this is tolerated by the EaR measure. In particular, the cut-off level is 2.5 years for $b = 0.15$ and 1.7 for $b = 0.15$. Note also that in order to attain the same level of expected terminal wealth, the optimal portfolio for $b = 0.10$ is constructed at the expense of higher leveraging.

**Example 3.2.** In the last example, we considered the impact on the EaR and the optimal portfolios by fixing the expected wealth level. In this example, we held the upper bound of EaR constant and examine its effects on the expected terminal wealth. We use the same set of parameter values as in the last example except that we set EaR to be the respective EaR from the pure stock policy with $b = 0.10$ and 0.15. The results are depicted in Fig. 3 as a function of the time horizon $T, 0 < T \leq 5$, together with the expected terminal wealth from both pure stock strategy (with $b = 0.10$ and 0.15) and pure bond strategy ($r = 0.05$). The expected terminal wealth under these investment strategies increases with the time horizon. Observe that the optimal expected terminal wealth with $b = 0.10$ and 0.15 exceed the corresponding pure stock investment.

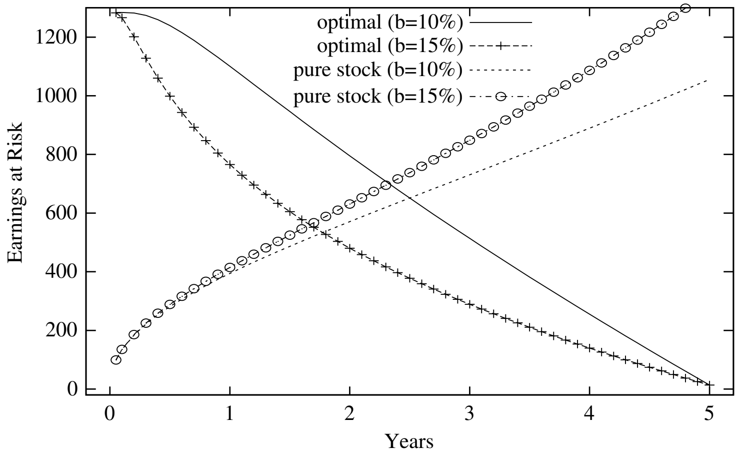


Fig. 1. EaR of the optimal CRP investment strategy and the pure stock strategy as functions of the time horizon $T, 0 < T \leq 5$, and for both $b = 0.10$ and $b = 0.15$.

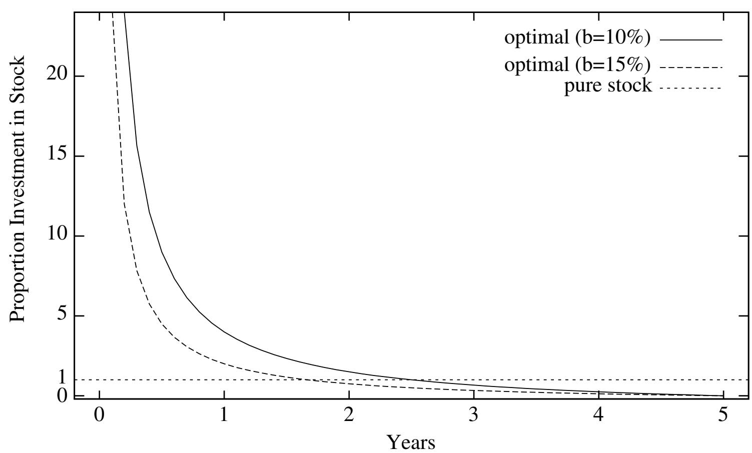


Fig. 2. Optimal portfolios and pure stock portfolio as functions of the time horizon $T, 0 < T \leq 5$, for both $b = 0.10$ and $b = 0.15$.



## 4. A Comparison with Mean-Variance Analysis

In this section we compare the proposed mean-EaR model to the classical portfolio selection mean-variance model. In particular, we consider the following mean-variance optimization problem:

$$(\hat{P}) \qquad \min_{\pi \in \mathbb{R}^n} \; Var[W^\pi(T)] \quad \text{subject to} \;\; E[W^\pi(T)] \geq C,$$

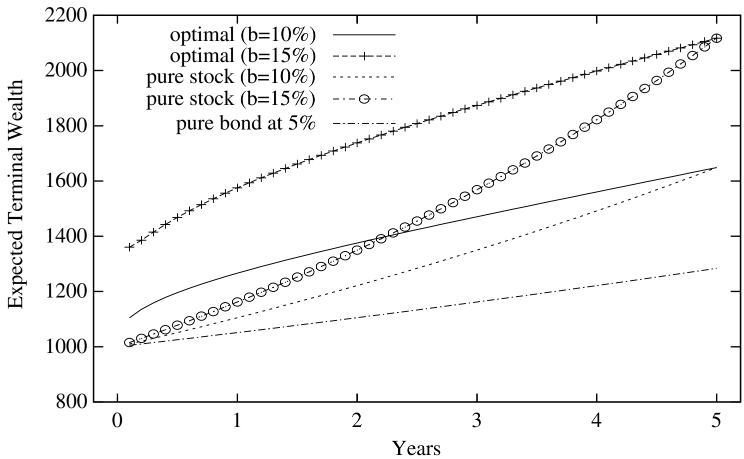


Fig. 3. Expected terminal wealth of different investment strategies as function of the time horizon $T, 0 < T \leq 5$.

where $C$, as in problem $(P)$, is the predetermined minimum level of the expected terminal wealth $E[W^\pi(T)]$ that satisfies condition (2.15).

The solution to the above optimization problem $(\hat{P})$ is summarized in the following theorem. We omit the proof since it is very similar to the proof of Theorem 3.1.

**Theorem 4.1.** *Assume that $b \neq r\mathbf{1}$. Then the unique optimal policy of problem $(\hat{P})$ is*

$$\pi^* = \varepsilon^* \frac{(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|}, \tag{4.1}$$

*where*

$$\varepsilon^* = \frac{\ln(C/w) - rT}{\|\sigma^{-1}(b - r\mathbf{1})\|T}. \tag{4.2}$$

*The corresponding expected terminal wealth is $E[W^{\pi^*}(T)] = C$ and variance*

$$Var[W^{\pi^*}(T)] = C^2 \left[ \exp(\varepsilon^{*2} T) - 1 \right]. \tag{4.3}$$

It follows immediately from the above result that the efficient frontier for the mean-variance problem in mean-variance space is given by

$$\nu = \xi^2 \left[ \exp\left( \frac{[\ln(\xi/w) - rT]^2}{\|\sigma^{-1}(b - r\mathbf{1})\|^2 T} \right) - 1 \right] \quad \text{for } \xi \geq w\exp(rT). \tag{4.4}$$

where $\nu := Var[W^{\pi^*}(T)]$ and $\xi := E[W^{\pi^*}(T)]$.

It should be pointed out that the mean-variance model considered by Emmer *et al.* [8] maximizes the expected terminal wealth for a given level of variance of the terminal wealth. Although they also obtained a solution that has the same representation as (4.1), the parameter $\varepsilon^*$ however was not obtained explicitly as in (4.2). In fact in their formulation (see [8, Proposition 2.9]), $\varepsilon^*$ is expressed as the unique positive solution to a nonlinear equation. Consequently, they did not obtain the mean-variance efficient frontier explicitly.

An interesting consequence of Theorems 3.1 and 4.1 is that for a given minimum level $C$ of the expected terminal wealth $E[W^\pi(T)]$, the optimal CRP investment strategies for both the mean-EaR and the mean-variance problems are equivalent, as indicated by (3.9) and (4.1). In fact, it can also be shown that similar optimal $\pi^*$ can also be obtained if we had considered the risk measure CaR as in the mean-CaR optimization problem. This implies all these risk measures yield similar optimal CRP investment strategies as long as the preselected level $C$ is identical.

The above observation also provides a linkage between the EaR and the variance of terminal wealth. For instance, suppose we fix the level of EaR. From the mean-EaR efficient frontier (3.17), we derive the highest attainable expected return and hence the optimal portfolio $\pi^*$ using (3.9). This in turn allows us to determine the corresponding minimum variance of terminal wealth using (4.3). Similarly, if the level of variance of terminal wealth is given, the mean-variance efficient frontier (4.4) can be used to obtain the corresponding expected terminal wealth and hence the minimum acceptable EaR using (3.11).

We now draw additional insights based on efficient frontiers (3.17) and (4.4) derived respectively from the mean-EaR and mean-variance problems:

(i) The global minimal EaR is zero and the minimum EaR portfolio strategy is the pure bond strategy. This is a consequence of Proposition 3.1. The global minimal variance is zero and the minimum variance portfolio strategy is the pure bond strategy.

(ii) Both EaR and $Var[W^{\pi^*}(T)]$ are strictly increasing functions of the expected terminal wealth, as to be expected.

(iii) For the mean-EaR frontier, EaR is a concave function of the expected terminal wealth if $\theta\sqrt{T} \leq |z_\alpha|$. If $\theta\sqrt{T} > |z_\alpha|$, EaR is convex in expected terminal wealth over interval $[w\exp(rT), w\exp(rT + (\theta\sqrt{T} - |z_\alpha|)\theta\sqrt{T})]$ and is concave for the range $(w\exp(rT + (\theta\sqrt{T} - |z_\alpha|)\theta\sqrt{T}), +\infty)$. This is in contrast to the mean-variance frontier whereby the variance is always a convex function of the expected terminal wealth. These facts imply that the marginal risk (variance) of the expected terminal wealth is always increasing on the mean-variance efficient frontier, while the marginal risk (EaR) is decreasing at least on part of the mean-EaR efficient frontier $(\max\{w\exp(rT), w\exp(rT+(\theta\sqrt{T}-|z_\alpha|)\theta\sqrt{T})\}, +\infty)$.

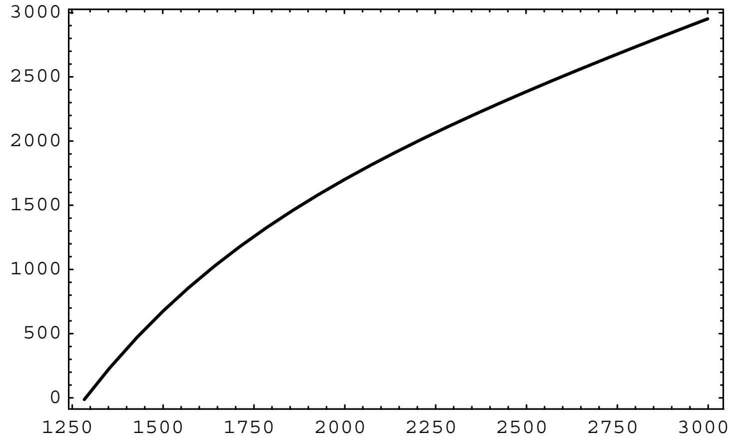


Fig. 4. Mean-EaR efficient frontier with the mean on the horizontal axis and the EaR on the vertical axis.

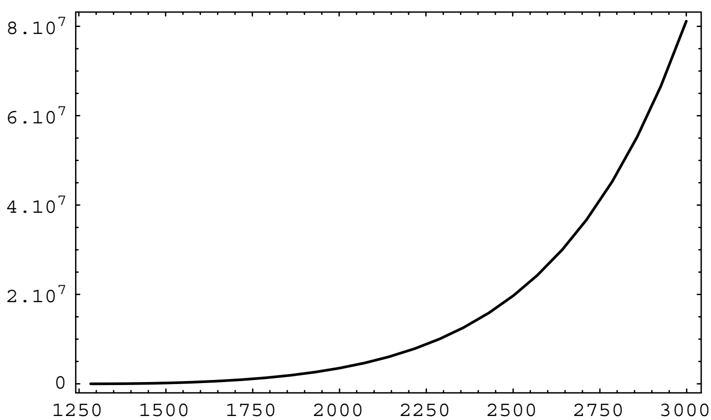


Fig. 5. Mean-Variance efficient frontier with the mean on the horizontal axis and the variance on the vertical axis.



To end this section, we consider an example to illustrate the difference between the mean-EaR and the mean-variance efficient frontiers.

**Example 4.1.** Let $n = 1, w = 1000, T = 5, r = 0.05, \alpha = 0.05, b = 0.1, \sigma = 0.2$. Using these parameters, the mean-EaR efficient frontier is depicted in Fig. 4 with the mean on the horizontal axis and the EaR on the vertical axis. Similarly, the mean-variance efficient frontier is plotted in Fig. 5 with the mean on the horizontal axis and the variance on the vertical axis. Clearly, the mean-EaR efficient frontier is increasing and concave while the mean-variance efficient frontier is increasing and convex.

## 5. Conclusion

In this paper, we derived closed-form solutions to mean-EaR and mean-variance dynamic portfolio optimization problems under the Black-Scholes setting. These results allow us to express explicitly the exact formulae for best CRP investment strategies and efficient frontiers and hence facilitate the calculation.

The approach of proving Theorem 3.1 and the idea in this paper also provide useful insights for other dynamic portfolio optimization problems. If other risk measure (as well as Safety-First type problems ( [21]) is appropriate other than the tail-VaR considered in our mean-EaR model, similar technique can be used to analyze the assumed model.

As indicated earlier, the CRP strategies lead to a variety of optimality properties in the context of portfolio optimization, though the optimal CRP strategy in our model may not be globally optimal in the set of all dynamic strategies. It will be of interest to generalize our result to other dynamic investment strategies other than the CRP policy. We leave this for future area of research.

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