# UNIVALENCE IN SIMPLICIAL SETS

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Abstract. We present an accessible account of Voevodsky's construction of a univalent universe of Kan fibrations.

Our goal in this note is to give a concise, self-contained account of the results of [Voe11, Section 5]: the construction of a homotopically universal small Kan fibration $\pi \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$; the proof that $\mathrm{U}_\alpha$ is a Kan complex; and the proof that $\pi$ is univalent.

We assume some background knowledge of the homotopy theory of simplicial sets, and category theory in general; [Hov99] and [ML98] are canonical and sufficient references. Other good sources include [May67], [GJ09], and [Joy09].

In Section 1, we construct $\pi \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$, and prove that it is a weakly universal $\alpha$-small Kan fibration. In Section 2, we prove further that the base $\mathrm{U}_\alpha$ is a Kan complex.

Section 3 is dedicated to constructing the fibration of weak equivalences between two fibrations over a common base. In Section 4 we define univalence for a general fibration, and prove our main theorem: that $\pi$ is univalent. Finally, in Section 5, we derive from this a statement of "homotopical uniqueness" for the universal property of $\mathrm{U}_\alpha$.

Overall, we largely follow Voevodsky's original presentation, with some departures: some proofs in Sections 2 and 4 are simplified based on a result of André Joyal ([Joy11, Lemma 0.2], cf. our Lemmas 17, 18); and Section 3 also is somewhat modified and reorganised.

A recurring theme throughout is that when a map is defined by a "right-handed" universal property, showing that it is a fibration (resp. trivial fibration) corresponds to showing that the objects it represents extend along trivial (resp. all) cofibrations.

An alternative construction of $\pi \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ can be found in [Str11], and an alternative proof of univalence in [Moe11].

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without whose constant support in many ways these notes would not exist. The first-named author would like to dedicate this paper to his mother.

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## 1. Representability of fibrations

**Definition 1.** Let $X$ be a simplicial set. A *well-ordered morphism* $f\colon Y \longrightarrow X$ is a pair consisting of a morphism into $X$ (also denoted by $f$) and a function assigning to each simplex $x \in X_n$ a well-ordering on the fiber $Y_x := f^{-1}(x) \subseteq Y_n$.

If $f\colon Y \longrightarrow X$, $f'\colon Y' \longrightarrow X$ are well-ordered morphisms into $X$, an *isomorphism* of well-ordered morphisms from $f$ to $f'$ is an isomorphism $Y \cong Y'$ over $X$ preserving the well-orderings on the fibers.

**Remark 2.** Since we require no compatibility conditions, there are infinitely many (specifically, $2^\omega$) well-orderings even on the map $1 \amalg 1 \longrightarrow 1$. The well-orderings are haphazard beasts, and not of intrinsic interest; they are essentially just a technical device to obtain Lemma 5.

**Proposition 3.** *Given two well-ordered sets, there is at most one isomorphism between them. Given two well-ordered morphisms over a common base, there is at most one isomorphism between them.*

*Proof.* The first statement is classical, and immediate by induction; the second follows from the first, applied in each fiber. □

**Definition 4.** Fix (once and for all) a regular cardinal $\alpha$. Say a map $f\colon Y \longrightarrow X$ is *$\alpha$-small* if each of its fibers $Y_x$ has cardinality $< \alpha$.

Given a simplicial set $X$ we define $\mathbf{W}_\alpha(X)$ to be the set of isomorphism classes of $\alpha$-small well-ordered morphisms $f\colon Y \longrightarrow X$. Given a morphism $t\colon X' \longrightarrow X$ we define $\mathbf{W}_\alpha(t)\colon \mathbf{W}_\alpha(X) \longrightarrow \mathbf{W}_\alpha(X')$ by $\mathbf{W}_\alpha(t) = t^*$ (the pullback functor). This gives a contravariant functor $\mathbf{W}_\alpha\colon \mathbf{sSets}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$.

**Lemma 5.** *$\mathbf{W}_\alpha$ preserves all limits.*

*Proof.* Suppose $F\colon \mathcal{I} \longrightarrow \mathbf{sSets}$ is some diagram, and $X = \operatorname{colim}_{\mathcal{I}} F$ is its colimit, with injections $\nu_i\colon F(i) \longrightarrow X$. We need to show that the canonical map $\mathbf{W}_\alpha(X) \longrightarrow \lim_{\mathcal{I}} \mathbf{W}_\alpha(F(i))$ is an isomorphism.

To see that it is surjective, suppose we are given $[f_i\colon Y_i \longrightarrow F(i)] \in \lim_{\mathcal{I}} \mathbf{W}_\alpha(F(i))$. For each $x \in X_n$, choose some $i$ and $\bar{x} \in F(i)$ with $\nu(\bar{x}) = x$, and set $Y_x := (Y_i)_{\bar{x}}$. By Proposition 3, this is well-defined up to *canonical* isomorphism, independent of the choices of representatives $i$, $\bar{x}$, $Y_i$, $f_i$. The total space of these fibers then defines a well-ordered morphism $f\colon Y \longrightarrow X$, with fibers smaller than $\alpha$, and with pullbacks isomorphic to $f_i$ as required.

For injectivity, suppose $f, f'$ are well-ordered morphisms over $X$, and $\nu_i^* f \cong \nu_i^* f'$ for each $i$. By Proposition 3, these isomorphisms agree on each fiber, so together give an isomorphism $f \cong f'$. □

Define the simplicial set $\mathrm{W}_\alpha$ by

$$\mathrm{W}_\alpha := \mathbf{W}_\alpha \circ \mathbf{y}^{\mathrm{op}} \colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Sets},$$

where $\mathbf{y}$ denotes the Yoneda embedding $\Delta \longrightarrow \mathbf{sSets}$.

**Lemma 6.** *The functor* $\mathbf{W}_\alpha$ *is representable, represented by* $\mathrm{W}_\alpha$.

*Proof.* Given $X \in \mathbf{sSets}$, we have isomorphisms, natural in $X$:

$$\begin{aligned} \mathbf{W}_\alpha(X) &\cong \mathbf{W}_\alpha(\operatorname{colim}_{\int X} \Delta[n]) \\ &\cong \lim_{\int X} \mathbf{W}_\alpha(\Delta[n]) \\ &\cong \lim_{\int X} (\mathrm{W}_\alpha)_n \\ &\cong \lim_{\int X} \mathbf{sSets}(\Delta[n], \mathrm{W}_\alpha) \\ &\cong \mathbf{sSets}(\operatorname{colim}_{\int X} \Delta[n], \mathrm{W}_\alpha) \\ &\cong \mathbf{sSets}(X, \mathrm{W}_\alpha). \end{aligned}$$

□

**Notation 7.** Given an $\alpha$-small well-ordered map $f \colon Y \longrightarrow X \in \mathbf{W}_\alpha(X)$, the corresponding map $X \longrightarrow \mathrm{W}_\alpha$ will be denoted by $\ulcorner f \urcorner$.

Applying the natural isomorphism above to the identity map $\mathrm{W}_\alpha \longrightarrow \mathrm{W}_\alpha$ gives a universal $\alpha$-small well-ordered simplicial set $\widetilde{\mathrm{W}}_\alpha \longrightarrow \mathrm{W}_\alpha$. Explicitly, $n$-simplices of $\widetilde{\mathrm{W}}_\alpha$ are pairs

$$(f \colon Y \longrightarrow \Delta[n], s \in f^{-1}(1_{[n]}))$$

i.e. the fiber of $\widetilde{\mathrm{W}}_\alpha$ over an $n$-simplex $\ulcorner f \urcorner \in \mathrm{W}_\alpha$ is exactly (an isomorphic copy of) the main fiber of $f$. So, by construction:

**Proposition 8.** *The canonical projection* $\widetilde{\mathrm{W}}_\alpha \longrightarrow \mathrm{W}_\alpha$ *is universal for* $\alpha$*-small well-ordered morphisms.*

**Corollary 9.** *The canonical projection* $\widetilde{\mathrm{W}}_\alpha \longrightarrow \mathrm{W}_\alpha$ *is weakly universal for* $\alpha$*-small morphisms of simplicial sets; that is, any such morphism can be given (not necessarily uniquely) as a pullback of the projection.*

*Proof.* By the well-ordering principle and the axiom of choice, one can well-order the fibers, and then use the universal property of $\mathrm{W}_\alpha$. □

**Definition 10.** Let $\mathbf{U}_\alpha \subseteq \mathbf{W}_\alpha$ (respectively, $\mathrm{U}_\alpha \subseteq \mathrm{W}_\alpha$) be the subobject consisting of $\alpha$-small well-ordered fibrations[1]; and define $\pi \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ as the pullback:

$$\begin{array}{ccc} \widetilde{\mathrm{U}}_\alpha & \longrightarrow & \widetilde{\mathrm{W}}_\alpha \\ {\scriptstyle \pi}\downarrow & \lrcorner & \downarrow \\ \mathrm{U}_\alpha & \hookrightarrow & \mathrm{W}_\alpha \end{array}$$

**Lemma 11.** *The map* $\pi \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ *is a fibration.*

[1] Here and throughout, by "fibration" we always mean "Kan fibration".

*Proof.* Consider a horn to be filled

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & \widetilde{\mathrm{U}}_\alpha \\ \big\downarrow & & \big\downarrow \pi \\ \Delta[n] & \xrightarrow[\ulcorner x \urcorner]{} & \mathrm{U}_\alpha \end{array}$$

for some $0 \leq k \leq n$. It factors through the pullback

$$\begin{array}{ccccc} \Lambda^k[n] & \longrightarrow & \bullet & \longrightarrow & \widetilde{\mathrm{U}}_\alpha \\ \big\downarrow & & \big\downarrow x & & \big\downarrow \pi \\ \Delta[n] & = & \Delta[n] & \xrightarrow[\ulcorner x \urcorner]{} & \mathrm{U}_\alpha \end{array}$$

where by the definition of $\mathrm{U}_\alpha$, $x$ is a fibration. Thus the left square admits a diagonal filler, and hence so does the outer rectangle. □

**Lemma 12.** *An $\alpha$-small well-ordered morphism $f\colon Y \longrightarrow X \in \mathbf{W}_\alpha(X)$ is a fibration if and only if $\ulcorner f \urcorner\colon X \longrightarrow \mathrm{W}_\alpha$ factors through $\mathrm{U}_\alpha$.*

*Proof.* For '⇒', assume that $f\colon Y \longrightarrow X$ is a fibration. Then the pullback of $f$ to any representable is certainly a fibration:

$$\begin{array}{ccc} \bullet & \longrightarrow & Y \\ x^* f \big\downarrow & & \big\downarrow f \\ \Delta[n] & \xrightarrow[x]{} & X. \end{array}$$

so $\ulcorner f \urcorner(x) = x^* f \in \mathrm{U}_\alpha$, and hence $\ulcorner f \urcorner$ factors through $\mathrm{U}_\alpha$.

Conversely, suppose $\ulcorner f \urcorner$ factors through $\mathrm{U}_\alpha$. Then we obtain:

$$\begin{array}{ccccc} Y & \longrightarrow & \widetilde{\mathrm{U}}_\alpha & \longrightarrow & \widetilde{\mathrm{W}}_\alpha \\ f \big\downarrow & & \big\downarrow \pi & & \big\downarrow \\ X & \longrightarrow & \mathrm{U}_\alpha & \hookrightarrow & \mathrm{W}_\alpha, \end{array}$$

where the lower composite is $\ulcorner f \urcorner$, and the outer rectangle and the right square are pullbacks. Hence so is the left square, so by Lemma 11 $f$ is a fibration. □

As an immediate consequence we obtain the following corollary.

**Corollary 13.** *The functor $\mathbf{U}_\alpha$ is representable, represented by $\mathrm{U}_\alpha$. The map $\pi\colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ is universal for $\alpha$-small well-ordered fibrations, and weakly universal for $\alpha$-small fibrations.*

## 2. Fibrancy of $\mathrm{U}_\alpha$

Our next goal is to prove the following theorem.

**Theorem 14.** *The simplicial set* $\mathrm{U}_\alpha$ *is a Kan complex.*

Before proceeding with the proof we will gather four useful lemmas. The first two, on the theory of *minimal fibrations*, come originally from [Qui68] and [BGM59]. Since these two lemmas contain all that we need to know about minimal fibrations, we treat the notion as a black box, and refer the interested reader to [May67] for more.

**Lemma 15** (Quillen's Lemma, [Qui68])**.** *Any fibration* $f\colon Y \longrightarrow X$ *may be factored as* $f = pg$*, where* $p$ *is a minimal fibration and* $g$ *is a trivial fibration.*

**Lemma 16** ([BGM59, III.5.6]; see also [May67, Cor. 11.7])**.** *Suppose* $X$ *is contractible, with* $x_0 \in X$*, and* $p\colon Y \longrightarrow X$ *is a minimal fibration with fiber* $F := Y_{x_0}$*. Then there is an isomorphism*

$$\begin{array}{ccc} Y & \xrightarrow{\;g\;} & F \times X \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & X & \end{array}$$

*over* $X$*.*

For the last outstanding lemma, the proof we give is due to André Joyal, somewhat simpler than Voevodsky's original proof. We include details here since the original [Joy11] is not currently publicly available. For this, and again for Theorem 28 below, we make crucial use of exponentiation along cofibrations; so we pause first to establish some facts about this.

**Lemma 17** (Cf. [Joy11, Lemma 0.2])**.** *Suppose* $i\colon A \longrightarrow B$ *is a cofibration. Let* $i_*$ *and* $i_!$ *denote respectively the right and the left adjoint to the pullback functor* $i^*\colon \mathbf{sSets}/B \longrightarrow \mathbf{sSets}/A$*. Then:*

1. $i_*\colon \mathbf{sSets}/A \longrightarrow \mathbf{sSets}/B$ *preserves trivial fibrations;*
2. *the counit* $i^* i_* \longrightarrow 1_{\mathbf{sSets}/A}$ *is an isomorphism;*
3. *if* $p\colon E \longrightarrow A$ *is* $\alpha$*-small, then so is* $i_* p$*.*

*Proof.*

1. By adjunction, since $i^*$ preserves cofibrations.
2. Since $i$ is mono, $i^* i_! \cong 1_{\mathbf{sSets}/A}$; so by adjointness, $i^* i_* \cong 1_{\mathbf{sSets}/A}$.
3. For any $n$-simplex $x\colon \Delta[n] \longrightarrow B$, we have $(i_* p)_x \cong \mathrm{Hom}_{\mathbf{sSets}/B}(i^* x, p)$. As a subobject of $\Delta[n]$, $i^* x$ has only finitely many non-degenerate simplices, so $(i_* p)_x$ injects into a finite product of fibers of $p$ and is thus of size $< \alpha$. □

**Lemma 18** ([Joy11, Lemma 0.2])**.** *If* $t\colon Y \longrightarrow X$ *is a trivial fibration and* $j\colon X \longrightarrow X'$ *is a cofibration, then there exists a trivial fibration* $t'\colon Y' \longrightarrow X'$

*and a pullback square of the form:*

$$\begin{array}{ccc} Y & \dashrightarrow & Y' \\ {\scriptstyle t}\downarrow & \lrcorner & \vdots\,{\scriptstyle t'} \\ X & \xhookrightarrow[j]{} & X'. \end{array}$$

*If $t$ is $\alpha$-small, then $t'$ may be chosen to also be.*

*Proof.* Take $(Y', t') := j_*(Y, t)$. By part 1 of Lemma 17, this is a trivial fibration; by part 2, $j^*Y' \cong Y$; and by part 3, it is small. $\square$

We are now ready to prove that $\mathrm{U}_\alpha$ is a Kan complex.

*Proof of Theorem 14.* We need to show that we can extend any horn in $\mathrm{U}_\alpha$ to a simplex:

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & \mathrm{U}_\alpha \\ \big\downarrow & \nearrow & \\ \Delta[n] & & \end{array}$$

By Corollary 13, such a horn corresponds to an $\alpha$-small well-ordered fibration $q\colon Y \longrightarrow \Lambda^k[n]$. To extend $\ulcorner q \urcorner$ to a simplex, we just need to construct an $\alpha$-small fibration $Y'$ over $\Delta[n]$ which restricts on the horn to $Y$:

$$\begin{array}{ccc} Y & \dashrightarrow & Y' \\ {\scriptstyle q}\downarrow & \lrcorner & \vdots\,{\scriptstyle q'} \\ \Lambda^k[n] & \hookrightarrow & \Delta[n]. \end{array}$$

By the axiom of choice one can then extend the well-ordering of $q$ to $q'$, so the map $\ulcorner q' \urcorner\colon \Delta[n] \longrightarrow \mathrm{U}_\alpha$ gives the desired simplex.

By Quillen's Lemma, we can factor $q$ as

$$Y \xrightarrow{q_t} Y_0 \xrightarrow{q_m} \Lambda^k[n],$$

where $q_t$ is a trivial fibration and $q_m$ is a minimal fibration. Both are still $\alpha$-small: each fiber of $q_t$ is a subset of a fiber of $q$, and since a trivial fibration is onto, each fiber of $q_m$ is a quotient of a fiber of $q$.

By Lemma 16, we have an isomorphism $Y_0 \cong F \times \Lambda^k[n]$, and hence a pullback diagram:

$$\begin{array}{ccc} Y_0 & \hookrightarrow & F \times \Delta[n] \\ \big\downarrow & \lrcorner & \big\downarrow \\ \Lambda^k[n] & \hookrightarrow & \Delta[n] \end{array}$$

By Lemma 18, we can then complete the upper square in the following diagram, with both right-hand vertical maps $\alpha$-small fibrations:

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ {\scriptstyle q_t}\downarrow & \lrcorner & \downarrow \\ Y_0 & \hookrightarrow & F \times \Delta[n] \\ {\scriptstyle q_m}\downarrow & \lrcorner & \downarrow \\ \Lambda^k[n] & \hookrightarrow & \Delta[n] \end{array} .$$

Since $\alpha$ is regular, the composite of the right-hand side is again $\alpha$-small; so we are done. □

## 3. Representability of weak equivalences

To define univalence, we first need to construct the *object of weak equivalences* between fibrations $p_1 \colon E_1 \longrightarrow B$ and $p_2 \colon E_2 \longrightarrow B$ over a common base. In other words, we want an object representing the functor sending $(X, f) \in \mathbf{sSets}/B$ to the set $\mathrm{Eq}_X(f^*E_1, f^*E_2)$. As we did for $\mathbf{U}_\alpha$, we proceed in two steps, first exhibiting it as a subfunctor of a functor more easily seen (or already known) to be representable.

For the remainder of the section, fix fibrations $E_1$, $E_2$ as above over a base $B$. Since **sSets** is locally Cartesian closed, we can construct the exponential object between them:

**Definition 19.** Let $\mathrm{Hom}_B(E_1, E_2) \longrightarrow B$ denote the internal hom from $E_1$ to $E_2$ in $\mathbf{sSets}/B$.

Then for any $X$, a map $X \longrightarrow \mathrm{Hom}_B(E_1, E_2)$ corresponds to a map $f \colon X \longrightarrow B$, together with a map $u \colon f^*E_1 \longrightarrow f^*E_2$ over $X$.

Together with the Yoneda lemma, this implies the explicit description: an $n$-simplex of $\mathrm{Hom}_B(E_1, E_2)$ is a pair

$$(b \colon \Delta[n] \longrightarrow B, u \colon b^*E_1 \longrightarrow b^*E_2).$$

**Lemma 20.** $\mathrm{Hom}_B(E_1, E_2) \longrightarrow B$ *is a Kan fibration.*

*Proof.* The functor $(-) \times_B E_1 \colon \mathbf{sSets}/B \longrightarrow \mathbf{sSets}/B$ preserves trivial cofibrations (since **sSets** is right proper); so its right adjoint $\mathrm{Hom}_B(E_1, -)$ preserves fibrant objects. □

Within $\mathrm{Hom}_B(E_1, E_2)$, we now want to construct the subobject of weak equivalences.

**Lemma 21.** *Let $f \colon E_1 \longrightarrow E_2$ be a weak equivalence over $B$, and suppose $g \colon B' \longrightarrow B$. Then the induced map between pullbacks $g^*E_1 \longrightarrow g^*E_2$ is a weak equivalence.*

*Proof.* The pullback functor $g^* \colon \mathbf{sSets}/B \longrightarrow \mathbf{sSets}/B'$ preserves trivial fibrations; so by Ken Brown's Lemma [Hov99, Lemma 1.1.12], it preserves all weak equivalences between fibrant objects. □

Thus, weak equivalences from $E_1$ to $E_2$ form a subfunctor of the functor of maps from $E_1$ to $E_2$. To show that this is representable, we need just to show:

**Lemma 22.** *Let $f\colon E_1 \longrightarrow E_2$ be a morphism over $B$. If for each simplex $b\colon \Delta[n] \longrightarrow B$ the induced map $f_b\colon b^*E_1 \longrightarrow b^*E_2$ is a weak equivalence, then $f$ is a weak equivalence.*

*Proof.* Without loss of generality, $B$ is connected; otherwise, apply the result over each connected component separately. Take some vertex $b\colon \Delta[0] \longrightarrow B$, and set $F_i := b^*E_i$.

Now $\pi_0(f)$ factors as $\pi_0(E_1) \cong \pi_0(F_1) \xrightarrow{\pi_0(f_b)} \pi_0(F_2) \cong \pi_0(E_2)$, so is an isomorphism, since by hypothesis $\pi_0(f_b)$ is. Similarly, for any vertex $e\colon \Delta[0] \longrightarrow F_1$, we have by the long exact sequence for a fibration:

$$\begin{array}{ccccccccc}
\pi_{n+1}(B,b) & \longrightarrow & \pi_n(F_1,e) & \longrightarrow & \pi_n(E_1,e) & \longrightarrow & \pi_n(B,b) & \longrightarrow & \pi_{n-1}(F_1,e) \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \pi_n(f_b)} & & \downarrow{\scriptstyle \pi_n(f)} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \pi_{n-1}(f_b)} \\
\pi_{n+1}(B,b) & \longrightarrow & \pi_n(F_2,f(e)) & \longrightarrow & \pi_n(E_2,f(e)) & \longrightarrow & \pi_n(B,b) & \longrightarrow & \pi_{n-1}(F_2,f(e))
\end{array}$$

Each $\pi_n(f_b)$ is an isomorphism, so by the Five Lemma, so is each $\pi_n(f)$. Thus $f$ is a weak equivalence. □

**Definition 23.** Let $\mathrm{Eq}_B(E_1, E_2)$ be the simplicial subset of $\mathrm{Hom}_B(E_1, E_2)$ consisting of the $n$-simplices of the form:

$$(b\colon \Delta[n] \longrightarrow B, w\colon b^*E_1 \longrightarrow b^*E_2)$$

such that $w$ is a weak equivalence. (By Lemma 21, this indeed defines a simplicial subset.)

From Lemma 22, we immediately have:

**Corollary 24.** *Let $(f, u)\colon X \longrightarrow \mathrm{Hom}_B(E_1, E_2)$. Then $u$ is a weak equivalence if and only if $(f, u)$ factors through $\mathrm{Eq}_B(E_1, E_2)$.*

*Thus, maps $X \longrightarrow \mathrm{Eq}_B(E_1, E_2)$ correspond to pairs of maps*

$$(f\colon X \longrightarrow B, w\colon f^*E_1 \longrightarrow f^*E_2),$$

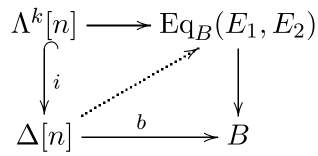
*where $w$ is a weak equivalence.* □

While Lemma 22 was stated just as required by representability, its proof actually gives a slightly stronger statement:

**Lemma 25.** *Let $f\colon E_1 \longrightarrow E_2$ be a morphism over $B$. If for some vertex $b\colon \Delta[0] \longrightarrow B$ in each connected component the map of fibers $f_b\colon b^*E_1 \longrightarrow b^*E_2$ is a weak equivalence, then $f$ is a weak equivalence.* □

**Corollary 26.** *The map $\mathrm{Eq}_B(E_1, E_2) \longrightarrow B$ is a fibration.*

*Proof.* Suppose we wish to fill a square:



By the universal property of $\mathrm{Eq}_B(E_1, E_2)$ this corresponds to showing that we can extend a weak equivalence $w\colon i^*b^*E_1 \longrightarrow i^*b^*E_2$ over $\Lambda^k[n]$ to a weak equivalence $\overline{w}\colon b^*E_1 \longrightarrow b^*E_2$ over $\Delta[n]$.

By Lemma 20, we can certainly find some map $\overline{w}$ extending $w$. But then since $\Delta[n]$ is connected, Lemma 25 implies that $\overline{w}$ is a weak equivalence. □

## 4. Univalence

Let $p\colon E \longrightarrow B$ be a fibration. We then have two fibrations over $B \times B$, given by pulling back $E$ along the projections. Call the object of weak equivalences between these $\mathrm{Eq}(E) := \mathrm{Eq}_{B\times B}(\pi_1^*E, \pi_2^*E)$. Concretely, simplices of $\mathrm{Eq}(E)$ are triples

$$(b_1, b_2 \in B_n,\ w\colon b_1^*E \longrightarrow b_2^*E).$$

By Corollary 24, a map $f\colon X \longrightarrow \mathrm{Eq}(E)$ corresponds to a pair of maps $f_1, f_2\colon X \longrightarrow B$ together with a weak equivalence $f_1^*E \longrightarrow f_2^*E$ over $X$. In particular, there is a diagonal map $\delta\colon B \longrightarrow \mathrm{Eq}(E)$, corresponding to the triple $(1_B, 1_B, 1_E)$, and sending a simplex $b \in B_n$ to the triple $(b, b, 1_{E_b})$.

There are also source and target maps $s, t\colon \mathrm{Eq}(E) \longrightarrow B$, given by the composites $\mathrm{Eq}(E) \longrightarrow B \times B \xrightarrow{\pi_i} B$, sending $(b_1, b_2, w)$ to $b_1$ and $b_2$ respectively. These are both retractions of $\delta$; and by Corollary 26, if $B$ is fibrant then they are moreover fibrations.

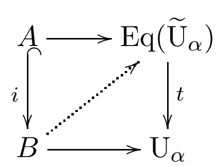
**Definition 27.** A fibration $p\colon E \longrightarrow B$ is called *univalent* if $\delta\colon B \longrightarrow \mathrm{Eq}(E)$ is a weak equivalence.

Since $\delta$ is always a monomorphism (thanks to its retractions), this is equivalent to saying that $B \longrightarrow \mathrm{Eq}(E) \longrightarrow B \times B$ is a (trivial cofibration, fibration) factorisation of the diagonal $\Delta\colon B \longrightarrow B \times B$, i.e. that $\mathrm{Eq}(E)$ is a *path object* for $B$.

**Theorem 28.** *The fibration $\pi\colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ is univalent.*

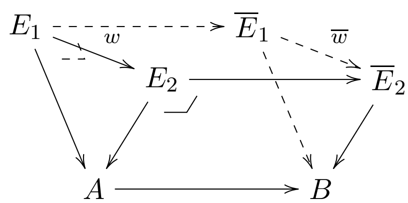
*Proof.* We will show that $t$ is a trivial fibration. Since it is a retraction of $\delta$, this implies by 2-out-of-3 that $\delta$ is a weak equivalence.

So, we need to fill a square



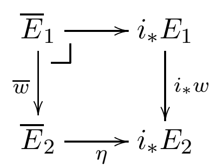
where $i\colon A \hookrightarrow B$ is a cofibration.

By the universal properties of $\mathrm{U}_\alpha$ and $\mathrm{Eq}(\widetilde{\mathrm{U}}_\alpha)$, these data correspond to a weak equivalence $w\colon E_1 \longrightarrow E_2$ between small well-ordered fibrations over $A$, and an extension $\overline{E}_2$ of $E_2$ to a small, well-ordered fibration over $B$; and a filler corresponds to an extension $\overline{E}_1$ of $E_1$, together with a weak equivalence $\overline{w}$ extending $w$:




As usual, it is sufficient to construct this first without well-orderings on $\overline{E}_2$; these can then always be chosen so as to extend those of $E_2$.

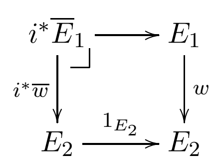
Recalling Lemmas 17–18, we define $\overline{E}_1$ and $\overline{w}$ as the pullback




in $\mathbf{sSets}/B$, where $\eta$ is the unit of $i^* \dashv i_*$ at $\overline{E}_2$. To see that this construction works, it remains to show:

(a) $i^*\overline{E}_1 \cong E_1$ in $\mathbf{sSets}/A$, and under this, $i^*\overline{w}$ corrsponds to $w$;
(b) $\overline{E}_1$ is small over $B$;
(c) $\overline{E}_1$ is a fibration over $B$, and $\overline{w}$ is a weak equivalence.

For (a), pull the defining diagram of $\overline{E}_1$ back to $\mathbf{sSets}/A$; by Lemma 17 part 2, we get a pullback square




in $\mathbf{sSets}/A$, giving the desired isomorphism.

For (b), Lemma 17 part 3 gives that $i_*E_1$ is $\alpha$-small over $B$, so $\overline{E}_1$ is a subobject of a pullback of $\alpha$-small maps.

For (c), note first that by factoring $w$, we may reduce to the cases where it is either a trivial fibration or a trivial cofibration.

In the former case, by Lemma 17 part 1 $i_*w$ is also a trivial fibration, and hence so is $\overline{w}$; so $\overline{E}_1$ is fibrant over $\overline{E}_2$, hence over $B$.

In the latter case, $E_1$ is then a deformation retract of $E_2$ over $A$; we will show that $\overline{E}_1$ is also a deformation retract of $\overline{E}_2$ over $B$. Let $H\colon E_2 \times \Delta[1] \longrightarrow E_2$ be a deformation retraction of $E_2$ onto $E_1$. We want some homotopy $\overline{H}\colon \overline{E}_2 \times \Delta[1] \longrightarrow \overline{E}_2$ extending $H$ on $E_2 \times \Delta[1]$, $1_{\overline{E}_1} \times \Delta[1]$ on $\overline{E}_1 \times \Delta[1]$, and $1_{\overline{E}_2}$ on $\overline{E}_2 \times \{0\}$. Since these three maps agree on the intersections of their domains, this is exactly an instance of the homotopy lifting extension property, i.e. a square-filler

$$\begin{array}{ccc} (E_2 \times \Delta[1]) \cup (\overline{E}_1 \times \Delta[1]) \cup (\overline{E}_2 \times \{0\}) & \xrightarrow{H \cup 1 \cup 1} & \overline{E}_2 \\ \big\downarrow & \overset{\overline{H}}{\nearrow} & \big\downarrow \\ \overline{E}_2 \times \Delta[1] & \longrightarrow & B \end{array}$$

which exists since the left-hand map is a trivial cofibration.

For $\overline{H}$ to be a deformation retraction, we need to see that $\overline{H}_{\{1\}}\colon \overline{E}_2 \longrightarrow \overline{E}_2$ factors through $\overline{E}_1$. By the definition of $\overline{E}_1$, a map $f\colon X \longrightarrow \overline{E}_2$ over $b\colon X \longrightarrow B$ factors through $\overline{E}_1$ just if the pullback $i^*f\colon i^*X \longrightarrow E_2$ factors through $E_1$. In the case of $\overline{H}_{\{1\}}$, the pullback is by construction $i^*(\overline{H}_{\{1\}}) = (i^*\overline{H})_{\{1\}} = H_{\{1\}}\colon E_2 \longrightarrow E_2$, which factors through $E_1$ since $H$ was a deformation retraction onto $E_1$.

So $\overline{w}$ embeds $\overline{E}_1$ as a deformation retract of $\overline{E}_2$ over $B$; thus $\overline{E}_1$ is a fibration over $B$ and $\overline{w}$ a weak equivalence, as desired. □

## 5. Uniqueness in the universal property of $\mathbf{U}_\alpha$

Finally, as promised, we will give a uniqueness statement for the representation of a small fibration as a pullback of $\pi\colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$: we show that the space of such representations is contractible.

Let $p\colon E \longrightarrow B$ be any fibration. We define a functor

$$\mathbf{P}_p\colon \mathbf{sSets}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$$

taking $\mathbf{P}_p(X)$ to be the set of pairs of a map $f\colon X \times B \longrightarrow \mathrm{U}_\alpha$, and a weak equivalence $w\colon X \times E \longrightarrow f^*\widetilde{\mathrm{U}}_\alpha$ over $X \times B$; equivalently, the set of squares

$$\begin{array}{ccc} X \times E & \xrightarrow{f'} & \widetilde{\mathrm{U}}_\alpha \\ {\scriptstyle X \times p}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X \times B & \xrightarrow[f]{} & \mathrm{U}_\alpha \end{array}$$

such that the induced map $X \times E \longrightarrow f^*\widetilde{\mathrm{U}}_\alpha$ is a weak equivalence. Lemma 21 ensures that this is functorial in $X$, by pullback.

**Lemma 29.** *The functor* $\mathbf{P}_p$ *is representable, represented by the simplicial set* $(\mathrm{P}_p)_n := \mathbf{P}_p(\Delta[n])$.

*Proof.* Let $\mathbf{Q}_p(X)$ be the set of all commutative squares $(f, f')$ from $p$ to $\widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$; we know that $\mathbf{Q}_p$ is represented by $\mathrm{Q}_p := E^{\widetilde{\mathrm{U}}_\alpha} \times_{E^{\mathrm{U}_\alpha}} B^{\mathrm{U}_\alpha}$.

Now, $\mathbf{P}_p$ is a subfunctor of $\mathbf{Q}_p$. By Lemma 22, an element $(f, f') \in \mathbf{Q}_p(X)$ lies in $\mathbf{P}_p(X)$ if and only if for each $x\colon \Delta[n] \longrightarrow X$, the pullback $x^*(f, f')$ lies in $\mathbf{P}_p(X)$; that is, if its representing map $X \longrightarrow \mathrm{Q}_p$ factors through $\mathrm{P}_p$. □

**Proposition 30.** *Let $p$ be an $\alpha$-small fibration. Then $\mathrm{P}_p$ is contractible.*

*Proof.* By Corollary 13, take some map $\ulcorner p \urcorner \colon B \longrightarrow \mathrm{U}_\alpha$ such that $E \cong \ulcorner p \urcorner^* \widetilde{\mathrm{U}}_\alpha$.

Now, for any $X$, maps $X \longrightarrow \mathrm{P}_p$ correspond by definition to pairs of maps $f \colon X \times B \longrightarrow \mathrm{U}_\alpha$, $w \colon X \times E \longrightarrow f^* \widetilde{\mathrm{U}}_\alpha$. But $X \times E \cong (\ulcorner p \urcorner \cdot \pi_2)^* \widetilde{\mathrm{U}}_\alpha$ over $X$; so such pairs also correspond to maps $\bar{f} \colon X \times B \longrightarrow \mathrm{Eq}(\widetilde{\mathrm{U}}_\alpha)$ such that $s \cdot \bar{f} = \ulcorner p \urcorner \cdot \pi_2 \colon X \times B \longrightarrow \mathrm{U}_\alpha$.

From this, we conclude that $\mathrm{P}_p \longrightarrow 1$ is a trivial fibration: filling a square

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{P}_p \\ \downarrow & \nearrow & \downarrow \\ X & \longrightarrow & 1 \end{array}$$

corresponds to filling the square

$$\begin{array}{ccc} Y \times B & \longrightarrow & \mathrm{Eq}(\widetilde{\mathrm{U}}_\alpha) \\ \downarrow & \nearrow & \downarrow s \\ X \times B & \xrightarrow{\ulcorner p \urcorner \cdot \pi_2} & \mathrm{U}_\alpha \end{array}$$

but if $Y \longrightarrow X$ is a cofibration, then so is $Y \times B \longrightarrow X \times B$; and by univalence, $s$ is a trivial fibration; so a filler exists. □

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