THE SIMPLICIAL MODEL OF UNIVALENT FOUNDATIONS

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ABSTRACT. In this paper, we construct and investigate a model of the Univalent Foundations of Mathematics in the category of simplicial sets.

To this end, we first give a new technique for constructing models of the type theory, using universes to obtain coherence. We then construct a (weakly) universal Kan fibration, and use it to exhibit a model in simplicial sets. Lastly, we introduce the Univalence Axiom, in several equivalent formulations, and show that it holds in our model.

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Introduction

The Univalent Foundations programme is a new proposed approach to foundations of mathematics, originally suggested by the third-named author in [Voe06] (closely related to independent work of Awodey, Warren, and collaborators [AW09]), building on the systems of dependent type theory developed by Martin-Löf and others.

A major motivation for earlier work with such logical systems has been their well-suitedness to computer implementation. One notable example is the Coq proof assistant, based on the Calculus of Inductive Constructions (a closely related dependent type theory), which has shown itself feasible for large-scale formal verification of mathematics, with developments including formal proofs of the Four-Colour Theorem [Gon08] and the Feit-Thompson (Odd Order) Theorem [G^+12].

One feature of dependent type theory which has previously remained comparatively unexploited, however, is its richer treatment of equality. In traditional foundations, equality carries no information beyond its truth-value: if two things are equal, they are equal in at most one way. This is fine for equality between elements of discrete sets; but it is unnatural for elements of higher-dimensional categories, or of spaces. In particular, it is at odds with the informal mathematical practice of treating isomorphic (and sometimes more weakly equivalent) objects as equal; which is why this usage must be so often disclaimed as an abuse of language, and kept rigorously away from formal statements, even though it is so appealing.

In dependent type theory, equalities can carry information: two things may be equal in multiple ways. So the basic objects—the *types*—may behave not just like discrete sets, but more generally like higher groupoids (with equalities being morphisms in the groupoid), or spaces (with equalities being paths in the space). And, crucially, this is the *only* equality one can talk about within the logical system: one cannot ask whether elements of a type are "equal on the nose", in the classical sense. The logical language only allows one to talk about properties and constructions which respect its equality.

The *Univalence Axiom*, introduced by the third-named author, strengthens this characteristic. In classical foundations one has sets of sets, or classes of sets, and uses these to quantify over classes of structures. Similarly, in type theory, types of types—*universes*—are a key feature of the language. The Univalence Axiom states that equality between types, as elements of a universe, is the same as equivalence between them, as types. It formalises the practice of treating equivalent structures as completely interchangeable; it ensures that one can only talk about properties of types, or more general structures, that respect such equivalence. In sum, it helps solidify the idea of types as some kind of spaces, in the homotopy-theoretic sense; and more practically—its original motivation—it provides for free many theorems (transfer along equivalences, naturality with respect to these, and so on) which must otherwise be re-proved by hand for each new construction.

The main import of this paper is to justify the intuition outlined above, of types as spaces. Specifically, we focus on simplicial sets, a well-studied model for spaces in homotopy theory; we construct a model of type theory in the category **sSets** of simplicial sets, and show that it satisfies the Univalence Axiom. (For comparison with other familiar notions: simplicial sets present the same homotopy theory as topological spaces; and form the basis of one of the most-studied models for higher groupoids.)

It also follows from this model that the Univalent Foundations are consistent, provided that the classical foundations we use are (precisely, ZFC together with the existence of two strongly inaccessible cardinals, or equivalently two Grothendieck universes).

This paper therefore includes a mixture of logical and homotopy-theoretic ingredients; however, we have aimed to separate the two wherever possible. Good background references for the logical parts include [NPS90], a general introduction to the type theory; [Hof97], for the categorical semantics; and [ML84], the *locus classicus* for the logical rules. For the homotopy-theoretic aspects, [GJ09] and [Hov99] are both excellent and sufficient references. Finally, for the category-theoretic language used throughout, [ML98] is canonical.

Organisation. In Section 1 we consider general techniques for constructing models of type theory. After setting out (in Section 1.1) the specific type theory that we will consider, we review (Section 1.2) some fundamental facts about its semantics in contextual categories, following [Str91]. In Section 1.3, we use universes to construct contextual categories, and hence models of (the structural core of) type theory; and in Section 1.4, we use categorical constructions on the universe to model the logical constructions of type theory. Together, these present a new solution to the *coherence problem* for modelling type theory (cf. [Hof95b]).

In Section 2, we turn towards constructing a model in the category of simplicial sets. Sections 2.1 and 2.2 are dedicated to the construction and investigation of a (weakly) universal Kan fibration (a "universe of Kan complexes"); in Section 2.3 we use this universe to apply the techniques of Section 1, giving a model of the full type theory in simplicial sets.

Section 3 is devoted to the Univalence Axiom. We formulate univalence first in type theory (Section 3.1), then directly in homotopy-theoretic terms (Section 3.2), and show that these definitions correspond under the simplicial model (Section 3.3). In Section 3.4, we show that the universal Kan fibration is univalent, and hence that the Univalence Axiom holds in the simplicial model. Finally, in Section 3.5 we discuss an alternative formulation of univalence, shedding further light on the universal property of the universe.

We should mention here that this paper is based in large part on the ongoing unpublished manuscript [Voe12] of the third-named author.

Related work. While the present paper discusses just models of the Univalent Foundations, the major motivation for this is the actual development of mathematics within these foundations. The work on this so far exists mainly in unpublished but publicly-accessible form: most substantially in [Voe] (for an introduction to which, see [PW12]), but also in [AGMLV11], [HoTa], and [HoTb].

Earlier work on homotopy-theoretic models of type theory can be found in [HS98], [AW09], [War11]. Other current and recent work on such models includes [GvdB11], [AK11], and [Shu12]. Other general coherence theorems, for comparison with the results of Section 1, can be found in [Hof95b] and [LW12]. Univalence in homotopy-theoretic settings is also considered in [Moe11] and [GK12]. (These references are, of course, far from exhaustive.)

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1. Models from Universes

In this section, we set up the machinery which we will use, in later sections, to model type theory in simplicial sets. The type theory we consider, and some of the technical machinery we use, are standard; the main original contribution is a new technique for solving the so-called coherence problem, using universes.

1.1. **The type theory under consideration.** Formally, the type theory we will work with is a slight variant of Martin-Löf's Intensional Type Theory, as presented in e.g. [ML84]. The rules of this theory are given in full in Appendix A; briefly, it is a dependent type theory, taking as basic constructors $\mathsf{\Pi}$-, $\mathsf{\Sigma}$-, Id-, and W-types, $\mathsf{0}$, $\mathsf{1}$, $+$, and a universe à la Tarski.

A related theory of particular interest is the Calculus of Inductive Constructions, on which the Coq proof assistant is based ([Wer94]). CIC differs from Martin-Löf type theory most notably in its very general scheme for inductive definitions, and in its treatment of universes. We do not pursue the question of how our model might be adapted to CIC, but for some discussion and comparison of the two systems, see [PM96], [Bar12], and [Voe12, 6.2].

One abuse of notation that we should mention: we will sometimes write e.g. $A(x)$ or $t(x, y)$ to indicate free variables on which a term or type may depend, so that we can later write $A(g(z))$ to denote the substitution $[g(z)/x]A$ more readably. Note however that the variables explicitly shown need not actually appear; and there may also always be other free variables in the term, not explicitly displayed.

1.2. **Contextual categories.** The syntax of type theory, while convenient for working in, is rather difficult to directly construct models of: doing so involves much bureaucracy. Instead, we construct models categorically, via the notion of a *contextual category*:

Definition 1.2.1 (Cartmell [Car78, Sec. 2.2], Streicher [Str91, Def. 1.2])**.** A *contextual category* $\mathcal{C}$ consists of the following data:

(1) a category $\mathcal{C}$;
(2) a grading of objects as $\mathrm{Ob}\,\mathcal{C} = \coprod_{n:\mathbb{N}} \mathrm{Ob}_n\,\mathcal{C}$;
(3) an object $1 \in \mathrm{Ob}_0\,\mathcal{C}$;
(4) maps $\mathrm{ft}_n : \mathrm{Ob}_{n+1}\,\mathcal{C} \longrightarrow \mathrm{Ob}_n\,\mathcal{C}$ (whose subscripts we usually suppress);
(5) for each $X \in \mathrm{Ob}_{n+1}\,\mathcal{C}$, a map $p_X : X \longrightarrow \mathrm{ft}\,X$ (the *canonical projection* from X);
(6) for each $X \in \mathrm{Ob}_{n+1}\,\mathcal{C}$ and $f : Y \longrightarrow \mathrm{ft}(X)$, an object $f^*(X)$ and a map $q(f, X) : f^*(X) \longrightarrow X$;

such that:

(1) 1 is the unique object in $\mathrm{Ob}_0(\mathcal{C})$;
(2) 1 is a terminal object in $\mathcal{C}$;
(3) for each $X \in \mathrm{Ob}\,\mathcal{C}$ and $f : Y \longrightarrow \mathrm{ft}(X)$, we have $\mathrm{ft}(f^*X) = Y$, and the square

$$\begin{array}{ccc} f^*X & \xrightarrow{q(f,X)} & X \\ {\scriptstyle p_{f^*X}}\downarrow & \lrcorner & \downarrow{\scriptstyle p_x} \\ Y & \xrightarrow{f} & \mathrm{ft}(X) \end{array}$$

is a pullback (the *canonical pullback* of X along f); and
(4) these canonical pullbacks are strictly functorial: that is, for $X \in \mathrm{Ob}_{n+1}\,\mathcal{C}$, $1^*_{\mathrm{ft}\,X}X = X$ and $q(1_{\mathrm{ft}\,X}, X) = 1_X$; and for $X \in \mathrm{Ob}_{n+1}\,\mathcal{C}$, $f : Y \longrightarrow \mathrm{ft}\,X$ and $g : Z \longrightarrow Y$, we have $(fg)^*(X) = g^*(f^*(X))$ and $q(fg, x) = q(f, X)q(g, f^*X)$.

Remark 1.2.2. Note that these may be seen as models of a multi-sorted essentially algebraic theory ([AR94, 3.34]), with sorts indexed by $\mathbb{N} + \mathbb{N} \times \mathbb{N}$.

This definition is best understood in terms of its prototypical example:

Example 1.2.3. Let $\mathbf{T}$ be any type theory. Then there is a contextual category $\mathcal{C}(\mathbf{T})$, described as follows:

- $\mathrm{Ob}_n\,\mathcal{C}(\mathbf{T})$ consists of the contexts $[x_1{:}A_1,\ \ldots,\ x_n{:}A_n]$ of length n, up to definitional equality and renaming of free variables;

- maps of $\mathcal{C}(\mathbf{T})$ are *context morphisms*, or *substitutions*, considered up to definitional equality and renaming of free variables. That is, a map
$$f\colon [x_1{:}A_1,\ \ldots,\ x_n{:}A_n] \longrightarrow [y_1{:}B_1,\ \ldots,\ y_m{:}B_m(y_1,\ldots,y_{m-1})]$$
is represented by a sequence of terms
$$x_1{:}A_1,\ \ldots,\ x_n{:}A_n \vdash f_1 : B_1$$
$$\vdots$$
$$x_1{:}A_1,\ \ldots,\ x_n{:}A_n \vdash f_m : B_m(f_1,\ \ldots,\ f_{m-1})$$
and two such maps $[f_i]$, $[g_i]$ are equal just if for each i,
$$x_1{:}A_1,\ \ldots,\ x_n{:}A_n \vdash f_i = g_i\colon B_i;$$
- composition is given by substitution, and the identity $\Gamma \longrightarrow \Gamma$ by the variables of Γ, considered as terms;
- 1 is the empty context [ ];
- $\mathrm{ft}[x_1{:}A_1,\ \ldots,\ x_{n+1}{:}A_{n+1}] = [x_1{:}A_1,\ \ldots,\ x_n{:}A_n]$;
- for $\Gamma = [x_1{:}A_1,\ \ldots,\ x_{n+1}{:}A_{n+1}]$, the map $p_\Gamma\colon \Gamma \longrightarrow \mathrm{ft}\,\Gamma$ is the *dependent projection* context morphism
$$(x_1,\ldots,x_n)\colon [x_1{:}A_1,\ \ldots,\ x_{n+1}{:}A_{n+1}] \longrightarrow [x_1{:}A_1,\ \ldots,\ x_n{:}A_n],$$
simply forgetting the last variable of Γ;
- for contexts
$$\Gamma = [x_1{:}A_1,\ \ldots,\ x_{n+1}{:}A_{n+1}(x_1,\ldots,x_n)],$$
$$\Gamma' = [y_1{:}B_1,\ \ldots,\ y_m{:}B_m(y_1,\ldots,y_{m-1})],$$
and a map $f = [f_i(\vec{y})]_{i\leq n}\colon \Gamma' \longrightarrow \mathrm{ft}\,\Gamma$, the pullback $f^*\Gamma$ is the context
$$[y_1{:}B_1,\ \ldots,\ y_m{:}B_m(y_1,\ldots,y_{m-1}),\ y_{m+1}{:}A_{n+1}(f_1(\vec{y}),\ldots,f_n(\vec{y}))],$$
and $q(\Gamma, f)\colon f^*\Gamma \longrightarrow \Gamma$ is the map
$$[f_1,\ \ldots, f_n,\ y_{n+1}].$$

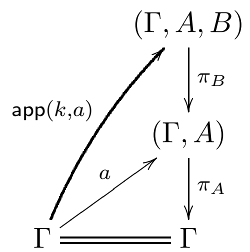
Note that terms $\Gamma \vdash t : A$ of $\mathbf{T}$ may be recovered from $\mathcal{C}(\mathbf{T})$, up to definitional equality, as sections of the projection $p_{[\Gamma,\ x:A]}\colon [\Gamma,\ x{:}A] \longrightarrow \Gamma$. For this reason, when working with contextual categories, we will often write just "sections" to refer to sections of dependent projections.

We will also use several other notations deserving of particular comment. For an object Γ, we will write e.g. (Γ, A) to denote an arbitrary object with $\mathrm{ft}(\Gamma, A) = \Gamma$, and will then write the dependent projection $p_{(\Gamma,A)}$ simply as p_A; similarly, (Γ, A, B), and so on. Similarly, we will write f^* not only for the canonical pullbacks of appropriate objects, but also the pullbacks of maps between them.

The plain definition of a contextual category corresponds precisely to the basic judgements and structural rules of dependent type theory. Similarly, each logical rule or type- or term-constructor corresponds to certain extra structure on a contextual category. We make this correspondence precise in Theorem 1.2.9 below, once we have set up the appropriate definitions.

Definition 1.2.4. A Π-*type structure* on a contextual category $\mathcal{C}$ consists of the following data:

(1) for each $(\Gamma, A, B) \in \mathrm{Ob}_{n+2}\,\mathcal{C}$, an object $(\Gamma, \Pi(A, B)) \in \mathrm{Ob}_{n+1}\,\mathcal{C}$;
(2) for each section $b\colon (\Gamma, A) \longrightarrow (\Gamma, A, B)$ of a dependent projection p_A, a morphism $\lambda(b)\colon \Gamma \longrightarrow (\Gamma, \Pi(A, B))$;
(3) for each pair of sections $k\colon \Gamma \longrightarrow (\Gamma, \Pi(A, B))$ and $a\colon \Gamma \longrightarrow (\Gamma, A)$, a section $\mathsf{app}(k, a)\colon \Gamma \longrightarrow (\Gamma, A, B)$ such that the following diagram commutes:



(4) such that for $a\colon \Gamma \longrightarrow (\Gamma, A)$ and $b\colon (\Gamma, A) \longrightarrow (\Gamma, A, B)$ we have $\mathsf{app}(\lambda(b), a) = b \cdot a$;
(5) and moreover, all the above operations are stable under substitution: for any morphism $f\colon \Delta \longrightarrow \Gamma$, we have

$$(\Delta, f^*\Pi(A, B)) = (\Delta, \Pi(f^*A, f^*B)),$$
$$\lambda(f^*b) = f^*\lambda(b), \qquad \mathsf{app}(f^*k, f^*a) = f^*(\mathsf{app}(k, a)).$$

Similarly, all the other logical rules of Appendix A may be routinely translated into structure on a contextual category; see [Hof97, 3.3] for more details and discussion.

Example 1.2.5. If $\mathbf{T}$ is a type theory with Π-types, then $\mathcal{C}(\mathbf{T})$ carries an evident Π-type structure; similarly for Σ-types and the other constructors of Sections A.2 and A.3.

Remark 1.2.6. Note that all of these structures, like the definition of contextual categories themselves, are essentially algebraic in nature.

Definition 1.2.7. A map $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ of contextual categories, or *contextual functor*, consists of a functor $\mathcal{C} \longrightarrow \mathcal{D}$ between underlying categories, respecting the gradings, and preserving (on the nose) all the structure of a contextual category.

Similarly, a map of contextual categories with Π-type structure, Σ-type structure, etc., is a contextual functor preserving the additional structure.

Remark 1.2.8. These are exactly the maps given by considering contextual categories as essentially algebraic structures.

We are now equipped to state precisely the sense in which the structures defined above correspond to the appropriate syntactic rules:

Theorem 1.2.9. *Let* $\mathbf{T}$ *be the type theory given by just the structural rules of Section A.1. Then* $\mathcal{C}(\mathbf{T})$ *is the initial contextual category.*

Similarly, let $\mathbf{T}$ *be the type theory given by the structural rules, plus any combination of the logical rules of Sections A.2, A.3. Then* $\mathcal{C}(\mathbf{T})$ *is initial among contextual categories with the appropriate extra structure.*

Proof. This is essentially the Correctness Theorem, p.181 (Chapter 3) of [Str91], with a different selection of logical constructors. □

This justifies the definition:

Definition 1.2.10. A *model* of dependent type theory with any selection of the logical rules of Section A.2 is a contextual category equipped with the structure corresponding to the chosen rules.

1.3. **Contextual categories from universes.** The major difficulty in constructing models of type theory is the so-called *coherence problem*: the requirement for pullback to be strictly functorial, and for the logical structure to commute strictly with it. In most natural categorical situations, operations on objects commute with pullback only up to isomorphism, or even more weakly; and for constructors with weak universal properties, operations on maps (corresponging for example to the Id-ELIM rule) may also fail to commute with pullback. Hofmann [Hof95b] gives a construction which solves the issue for Π- and Σ-types, but Id-types in particular remain problematic with this method. Other methods exist for certain specific categories ([HS98], [War08]), but are not applicable to the present case.

In order to obtain coherence for our model, we thus give a new construction based on *universes* (not necessarily the same as universes in the type-theoretic sense, though the two may sometimes coincide).

Definition 1.3.1. Let $\mathcal{C}$ be a category. A *universe* in $\mathcal{C}$ is an object U together with a morphism $p\colon \tilde{U} \longrightarrow U$, and for all $f\colon X \longrightarrow U$ a pullback square

$$\begin{array}{ccc} (X;f) & \xrightarrow{Q(f)} & \tilde{U} \\ {\scriptstyle P_{(X,f)}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & U. \end{array}$$

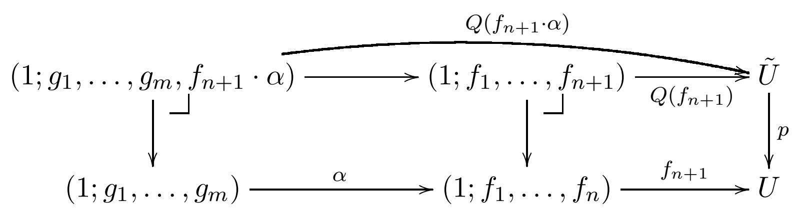
The intuition here is that the map p represents the generic family of types over the universe U.

By abuse of notation, we often refer to the universe simply as U, with p and the chosen pullbacks understood.

Given a map $f\colon Y \longrightarrow X$, we will often write $\ulcorner f \urcorner$ (or $\ulcorner Y \urcorner$, if f is understood) for a map $X \longrightarrow U$ such that $f \cong P_{(X,\ulcorner f \urcorner)}$. Also, for a sequence of maps $f_1\colon X \longrightarrow U$, $f_2\colon (X;f_1) \longrightarrow U$, etc., we write $(X;f_1,\ldots,f_n)$ for $(\ldots(X;f_1);\ldots);f_n)$.

Definition 1.3.2. Given a category $\mathcal{C}$, together with a universe U and a terminal object 1, we define a contextual category $\mathcal{C}_U$ as follows:

- $\mathrm{Ob}_n\, \mathcal{C}_U := \{\ (f_1, \ldots, f_n) \mid f_1 \colon 1 \longrightarrow U,\ f_{i+1} \colon (1; f_1, \ldots, f_i) \longrightarrow U\ \}$
- $\mathcal{C}_U((f_1, \ldots, f_n), (g_1, \ldots, g_m)) := \mathcal{C}((1; f_1, \ldots, f_n), (1; g_1, \ldots, g_n))$
- $1_{\mathcal{C}_U} := ()$, the empty sequence.
- $\mathrm{ft}(f_1, \ldots, f_{n+1}) := (f_1, \ldots, f_n)$;
- the projection $p_{(f_1,\ldots,f_{n+1})}$ is the map $P_{(X,f_{n+1})}$ provided by the universe structure on U;
- given $(f_1, \ldots, f_{n+1})$ and a map $\alpha \colon (g_1, \ldots, g_m) \longrightarrow (f_1, \ldots, f_n)$ in $\mathcal{C}_U$, the canonical pullback $\alpha^*(f_1, \ldots, f_{n+1})$ in $\mathcal{C}_U$ is given by $(g_1, \ldots, g_m, f_{n+1} \cdot \alpha)$, with projection induced by $Q(f_{n+1} \cdot \alpha)$:



Proposition 1.3.3.

(1) These data define a contextual category $\mathcal{C}_U$.
(2) This contextual category is well-defined up to canonical isomorphism given just $\mathcal{C}$ and $p \colon \tilde{U} \longrightarrow U$, independently of the choice of pullbacks.

Proof. Routine computation. □

Justified by the second part of this proposition, we will not explicitly consider the choices of pullbacks when we construct the universe in the category **sSets** of simplicial sets.

As an aside, let us note that every small contextual category arises in this way:

Proposition 1.3.4. *Let $\mathcal{C}$ be a small contextual category. Consider the universe U in the presheaf category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Sets}]$ given by*

$$U(X) = \{Y \mid \mathrm{ft}\, Y = X\}$$

$$\tilde{U}(X) = \{(Y, s) \mid \mathrm{ft}\, Y = X,\ s \text{ a section of } p_Y\},$$

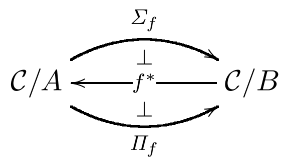
with the evident projection map, and any choice of pullbacks.

Then $[\mathcal{C}^{\mathrm{op}}, \mathbf{Sets}]_U$ is isomorphic, as a contextual category, to $\mathcal{C}$.

Proof. Straightforward, with liberal use of the Yoneda lemma. □

1.4. **Logical structure on universes.** Given a universe U in a category $\mathcal{C}$, we want to know how to equip $\mathcal{C}_U$ with various logical structure—Π-types, Σ-types, and so on. For general $\mathcal{C}$, this is rather fiddly; but when $\mathcal{C}$ is locally cartesian closed (as in our case of interest), it is more straightforward, since local cartesian closedness allows us to construct and manipulate "objects of U-contexts", and hence to construct objects representing the premises of each rule.

In working with locally cartesian closed structure, given a map $f\colon A \longrightarrow B$, we will denote the pullback functor and its adjoints as:



Also, the intended map $A \longrightarrow B$ is often clearly determined by the object A, as some sort of associated projection; in such a case, we will write Σ_A, Π_A for the functors arising from this map.

An alternative notation for locally cartesian closed categories is their internal logic, *extensional* dependent type theory [See84]. While this language is convenient and powerful, we avoid it due to the difficulties of working clearly with two logical languages in parallel.

Returning to the question at hand, first consider Π-types. We know that dependent products exist in $\mathcal{C}$; so informally, we need only to ensure that U (considered as a universe of types) is closed under such products. Specifically, given a type A in U (that is, a map $\ulcorner A \urcorner\colon X \longrightarrow U$), and a dependent family of types B over A, again from U (i.e. a map $\ulcorner B \urcorner\colon A := (X; \ulcorner A \urcorner) \longrightarrow U$), the product $\Pi_A B$ of these families in the slice $\mathcal{C}/X$ should again "live in U"; that is, there should be a map $\ulcorner \Pi(A,B) \urcorner\colon X \longrightarrow U$ such that $(X; \ulcorner \Pi(A,B) \urcorner) \cong \Pi_A B$. Moreover, we need this construction to be natural in X.

To obtain the naturality, we cannot simply provide this structure for each X and A, B individually. Instead, there is an object $U^{\Pi\text{-FORM}}$ representing such pairs (A,B), and a generic such pair $(A_{\text{gen}}, B_{\text{gen}})$ based on $U^{\Pi\text{-FORM}}$. It is sufficient to define Π in this generic case $X = U^{\Pi\text{-FORM}}$; the construction then extends to other X by precomposition, and as such, is automatically natural in X.

Precisely:

Definition 1.4.1. Given a universe U in a lccc $\mathcal{C}$, define

$$U^{\Pi\text{-FORM}} := \Sigma_U \Pi_{\tilde{U}} (U \times \tilde{U}).$$

Pulling back $\tilde{U}$ along the projection $U^{\Pi\text{-FORM}} \longrightarrow U$ induces an object A_{gen} over U; similarly, pulling back $\tilde{U}$ along the counit

$$A_{\text{gen}} = \tilde{U} \times_U \Pi_{\tilde{U}} (U \times \tilde{U}) \longrightarrow U \times \tilde{U} \longrightarrow U$$

induces a second object B_{gen}.

Moreover, the universal properties of the LCCC structure ensures that $B_{\text{gen}} \longrightarrow A_{\text{gen}} \longrightarrow U^{\Pi\text{-FORM}}$ are *generic*; that is, every other $B \longrightarrow A \longrightarrow \Gamma$ with maps $\Gamma \longrightarrow U$, $A \longrightarrow U$ exhibiting A and B as pullbacks of $\tilde{U}$ arises uniquely by precomposition and pullback along a map $\ulcorner(A, B)\urcorner \colon \Gamma \longrightarrow U^{\Pi\text{-FORM}}$.

(In the internal language of $\mathcal{C}$ as an LCCC, $U^{\Pi\text{-FORM}}$ may be written as $[\![A{:}U,\ B{:}[\tilde{U}_A, U]]\!]$, which can be seen as an internalisation of the premises of Π-FORM.)

Definition 1.4.2. A Π-*structure* on a universe U in a lccc $\mathcal{C}$ consists of a map

$$\Pi \colon U^{\Pi\text{-FORM}} \longrightarrow U.$$

whose realisation is a dependent product for the generic dependent family of types; that is, such that the square

$$\begin{array}{ccc} \Pi_{A_{\mathrm{gen}}} B_{\mathrm{gen}} & \longrightarrow & \tilde{U} \\ \downarrow & & \downarrow \\ U^{\Pi\text{-FORM}} & \xrightarrow{\ \Pi\ } & U \end{array}$$

is a pullback.

The approach used here gives a template which we follow for all the other constructors, with extra subtleties entering the picture just in the cases of Id-types and (type-theoretic) universes, since these structures do not arise from strict category-theoretic constructions.

Definition 1.4.3. A Σ-*structure* on a universe U in a lccc $\mathcal{C}$ consists of a map

$$\Sigma \colon U^{\Sigma\text{-FORM}} := \Sigma_U \Pi_{\tilde{U}} (U \times \tilde{U}) \longrightarrow U$$

whose realisation is a dependent sum for the generic dependent family of types; that is, such that the square such that the square

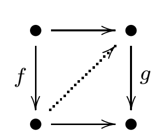
$$\begin{array}{ccc} \Sigma_{A_{\mathrm{gen}}} B_{\mathrm{gen}} & \longrightarrow & \tilde{U} \\ \downarrow & & \downarrow \\ U^{\Sigma\text{-FORM}} & \xrightarrow{\ \Sigma\ } & U \end{array}$$

is a pullback.

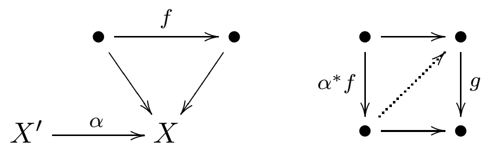
Id-structure requires a few auxiliary definitions.

Definition 1.4.4.

- Given maps f, g in $\mathcal{C}$, say f is *(weakly) orthogonal* to g if any commutative square from f to g has some diagonal filler:



- If f lies in some slice $\mathcal{C}/X$, say moreover that f is *stably orthogonal to g over X* if every pullback of f along a map $\alpha\colon X' \longrightarrow X$ is orthogonal to g.



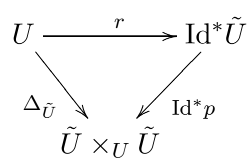
As shown in [GG08], the rules for Id-types can be understood roughly as follows. In a model where dependent types are interpreted as fibrations, the identity type over a type A is a factorisation of the diagonal $\Delta_A\colon A \longrightarrow A \times A$ as a stable trivial cofibration, followed by a fibration. Here, by a stable trivial cofibration, we mean a map which is stably orthogonal to all fibrations, over $A \times A$.

In our case, the "fibrations" are just the pullbacks of p; so it is sufficient to ask that the first map in the factorisation is stably orthogonal to p. Moreover, it is sufficient to construct this factorisation for the generic type $1_U\colon U \longrightarrow U$.

Definition 1.4.5 (Warren)**.** Id-*structure* on a universe consists of maps

$$\mathrm{Id}\colon U^{\mathsf{Id}\text{-}\mathrm{FORM}} = \tilde{U} \times_U \tilde{U} \longrightarrow U, \qquad r\colon U \longrightarrow \mathrm{Id}^*\tilde{U}$$

such that the triangle



commutes, and r is stably orthogonal to p over $\tilde{U} \times_U \tilde{U}$.

Definition 1.4.6. W-*structure* on a universe consists of a map

$$\mathrm{W}\colon U^{\mathsf{W}\text{-}\mathrm{FORM}} := \Sigma_U \Pi_{\tilde{U}}(U \times \tilde{U}) \longrightarrow U$$

such that $\mathrm{W}^*\tilde{U}$ is an initial algebra for the polynomial endofunctor of $\mathcal{C}/U$ specified by $A_{\mathrm{gen}} \longrightarrow B_{\mathrm{gen}}$, i.e. the endofunctor

$$\mathcal{C}/U^{\mathsf{W}\text{-}\mathrm{FORM}} \xrightarrow{p^*_{B_{\mathrm{gen}}} p^*_{A_{\mathrm{gen}}}} \mathcal{C}/B_{\mathrm{gen}} \xrightarrow{\Pi_{B_{\mathrm{gen}}}} \mathcal{C}/A_{\mathrm{gen}} \xrightarrow{\Sigma_{A_{\mathrm{gen}}}} \mathcal{C}/U^{\mathsf{W}\text{-}\mathrm{FORM}} .$$

(For details on polynomial endofunctors and their algebras, see [MP00], [GH04].)

Definition 1.4.7. $\mathbf{0}$-*structure* on U consists of a map $\mathbf{0}\colon 1 \longrightarrow U$ such that $\mathbf{0}^*\tilde{U} \cong 0$.

(By analogy with the preceding definitions, one might refer to 1 here as $U^{\mathsf{0}\text{-}\mathrm{FORM}}$, and similarly in the next two definitions. We choose not to do so simply for the sake of readability.)

Definition 1.4.8. $\mathbf{1}$-*structure* on U consists of a map $\mathbf{1}\colon 1 \longrightarrow U$ such that $\mathbf{1}^*\tilde{U} \cong 1$.

Definition 1.4.9. $+$-*structure* on U consists of a map $+\colon U \times U \longrightarrow U$, such that $+^*\tilde{U} \cong \pi_1^*\tilde{U} + \pi_2^*\tilde{U}$.

Finally, we consider the structure on U needed to give a universe (in the type-theoretic sense) in $\mathcal{C}_U$. Here, for the first time, we need to consider a nested pair of universes, since the internal universe of $\mathcal{C}_U$ must be some smaller universe U_0 in $\mathcal{C}$.

Definition 1.4.10. An *internal universe* (U_0, i) in U consists of arrows

$$u_0\colon 1 \longrightarrow U \qquad\qquad i\colon U_0 = u_0^*\tilde{U} \longrightarrow U.$$

Given these, i induces by pullback a universe structure $(p_0, \tilde{U}_0, \ldots)$ on U_0. We say that U_0 is closed under $\mathsf{\Pi}$-types in U if U_0 carries a Π-structure Π_0, commuting with i in the sense that the square

$$\begin{array}{ccc} U_0^{\mathsf{\Pi}\text{-FORM}} & \xrightarrow{i^{\mathsf{\Pi}\text{-FORM}}} & U^{\mathsf{\Pi}\text{-FORM}} \\ {\scriptstyle \Pi_0}\downarrow & & \downarrow{\scriptstyle \Pi} \\ U_0 & \xrightarrow{\quad i \quad} & U \end{array}$$

commutes (where the top map is induced by the evident functoriality of $U^{\mathsf{\Pi}\text{-FORM}}$ in U).

Similarly, we say that U_0 is closed under $\mathsf{\Sigma}$-types (resp. Id-types, etc.) if it carries Σ-structure Σ_0 (resp. an Id-structure (Id_0, r_0), etc.) commuting with i.

With these structures defined, we can now prove that they are fit for purpose:

Theorem 1.4.11. *Π-structure (resp. Σ-structure, etc.) structure on a universe U induces $\mathsf{\Pi}$-type structure (resp. $\mathsf{\Sigma}$-type structure, etc.) on $\mathcal{C}_U$.*

An internal universe (U_0, i) in U closed under any combination of $\mathsf{\Pi}$-types, $\mathsf{\Sigma}$-types, etc., induces a universe à la Tarski in $\mathcal{C}_U$ closed under the corresponding constructors.

Proof. This proof is esentially a routine verification; we give the case of $\mathsf{\Pi}$-types in full, and leave the rest mostly to the reader.

In a nutshell, the constructor $\mathsf{\Pi}$ is induced by the map Π; and the constructors λ and app are induced by the corresponding lccc structure in $\mathcal{C}$.

Precisely, we treat the rules of $\mathsf{\Pi}$-types (corresponding to the components of the desired $\mathsf{\Pi}$-type structure) one at a time.

($\mathsf{\Pi}$-FORM): The premises

$$\Gamma \vdash A \ \mathsf{type} \qquad \Gamma,\ x{:}A \vdash B \ \mathsf{type}$$

in $\mathcal{C}_U$ correspond to data in $\mathcal{C}$ of the form

$$\begin{array}{ccc} A & \longrightarrow & \tilde{U} \\ \downarrow & \lrcorner & \downarrow \\ \Gamma & \xrightarrow{\ulcorner A \urcorner} & U \end{array} \qquad \begin{array}{ccc} B & \longrightarrow & \tilde{U} \\ \downarrow & \lrcorner & \downarrow \\ A & \xrightarrow{\ulcorner B \urcorner} & U \end{array}$$

and hence to a map

$$(\ulcorner A \urcorner, \ulcorner B \urcorner) \colon \Gamma \longrightarrow U^{\mathsf{\Pi}\text{-FORM}}.$$

Then the composite $\Pi \cdot (\ulcorner A \urcorner, \ulcorner B \urcorner)$ gives a type $\Gamma \longrightarrow U$ which we take as $\mathsf{\Pi}(A, B)$. By construction, this is stable under substitution along any map $f \colon \Delta \longrightarrow \Gamma$, since substitution in $\mathcal{C}_U$ is again just composition in $\mathcal{C}$.

($\mathsf{\Pi}$-INTRO): Besides Γ, A, B as before, we have an additional premise

$$\Gamma,\ x{:}A \vdash t : B(x).$$

This is by definition a map $1_A \longrightarrow B$ in $\mathcal{C}/A$, corresponding by adjunction to a map $\hat{t} \colon 1_\Gamma \longrightarrow \mathit{\Pi}_A B$ in $\mathcal{C}/\Gamma$. But

$$\begin{aligned} \mathit{\Pi}_A B &\cong (\ulcorner A \urcorner, \ulcorner B \urcorner)^* \mathit{\Pi}_{A_{\text{gen}}} B_{\text{gen}} \\ &\cong (\ulcorner A \urcorner, \ulcorner B \urcorner)^* \Pi^* \tilde{U} \\ &\cong (\Pi \cdot (\ulcorner A \urcorner, \ulcorner B \urcorner))^* \tilde{U} \end{aligned}$$

so $\hat{t}$ corresponds to a section of $\mathsf{\Pi}(A, B)$ over Γ, which we take as $\lambda(t)$.

Stability under substitution follows by the uniqueness in the universal property of $\mathit{\Pi}_A B$.

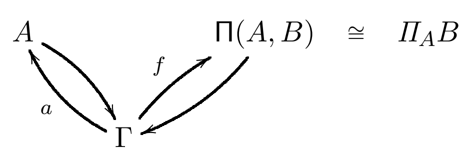
We could alternatively have defined λ more analogously to $\mathsf{\Pi}$, by representing the premises as a single map $(\ulcorner A \urcorner, \ulcorner B \urcorner, t) \colon \Gamma \longrightarrow U^{\mathsf{\Pi}\text{-INTRO}} := \mathit{\Sigma}_{U^{\mathsf{\Pi}\text{-FORM}}} \mathit{\Pi}_{A_{\text{gen}}} B_{\text{gen}}$; then taking the transpose of the generic term t_{gen} over $U^{\mathsf{\Pi}\text{-INTRO}}$; and then pulling this back along $(\ulcorner A \urcorner, \ulcorner B \urcorner, t)$. In fact, thanks to the uniqueness in the universal property of $\mathit{\Pi}_{A_{\text{gen}}} B_{\text{gen}}$, that would give the same result as the present, more straightforward, definitition. However, the alternative definition has the advantage that its stability under substitution follows simply from properties of pullbacks; this becomes important for Id-types, whose universal property lacks a uniqueness condition.

($\mathsf{\Pi}$-APP): The premises now are

$$\Gamma \vdash A \ \mathsf{type} \qquad \Gamma,\ x{:}A \vdash B \ \mathsf{type}$$

$$\Gamma \vdash f : \mathsf{\Pi}(A, B) \qquad \Gamma \vdash a : A$$

corresponding to Γ, A, B as before, plus sections



Together, these give a section over Γ of $\mathit{\Pi}_A B \times_\Gamma A$; so composing this with the evaluation map $\mathrm{ev}_{A,B}$ of $\mathit{\Pi}_A B$ gives a map $\Gamma \longrightarrow B$ lifting a, which we take to be $\mathsf{app}(f, a)$.

(Π-COMP): here, we have premises Γ, A, B, t as in Π-INTRO, and a as in Π-APP; and we have formed $\mathsf{app}(\lambda(t), a)$ as prescribed above. So, unwinding the isomorphism $\Pi(A, B) \cong \mathit{\Pi}_A B$ used in each case,

$$\begin{aligned} \mathsf{app}(\lambda(t), a) &= \mathrm{ev}_{A,B} \cdot (\hat{t}, a) \\ &= t \cdot a \end{aligned}$$

as desired, by the usual rules of LCCCs.

This completes the proof for Π-structures.

As indicated above, the remaining constructors are for the most part entirely analogous. The only subtlety is in the case for the Id-ELIM rule. In this case, there are two ways that one could define the appropriate structure: one can either pull back to each specific context and then choose liftings, or choose a lifting in the universal context and then pull it back (as discussed following the Π-INTRO case above). The second of these is the correct choice: the first is not generally stable under substitution. (For other constructors, this distinction does not arise, since their strict categorical universal properties canonically determine the maps involved.) □

2. The Simplicial Model

In this section, we construct (for any inaccessible cardinal α) a Kan fibration $p_\alpha \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$, weakly universal among Kan fibrations with α-small fibers, and investigate the key properties of U_α and p_α. In particular, we show that U_α is a Kan complex, and carries the various logical structures defined in Section 1.4. Together, these yield our first main goal: a model of type theory in **sSets**, with an internal universe.

2.1. A universe of Kan complexes.

In constructing a universe U_α intended to represent α-small Kan fibrations, one might expect (by the Yoneda lemma) to simply define $(\mathrm{U}_\alpha)_n$ as the set of α-small fibrations over $\Delta[n]$. This definition has two problems: firstly, it gives not a set, but a proper class; and secondly, it is not strictly functorial, since pullback is functorial only up to isomorphism.

Some extra technical device is therefore needed to resolve these issues. Several possible solutions exist; we take the approach of passing to isomorphism classes, having first added well-orderings to the mix so that fibrations have no non-trivial automorphisms (without which the crucial Lemma 2.1.4 would fail). We should emphasise, however, that this is the sole reason for introducing the well-orderings: they are of no intrinsic interest (and are indeed occasionally something of an inconvenience).

Definition 2.1.1. Let X be a simplicial set. A *well-ordered morphism* $f \colon Y \longrightarrow X$ is a pair consisting of a morphism into X (also denoted by f)

and a function assigning to each simplex $x \in X_n$ a well-ordering on the fiber $Y_x := f^{-1}(x) \subseteq Y_n$.

If $f \colon Y \longrightarrow X$, $f' \colon Y' \longrightarrow X$ are well-ordered morphisms into X, an *isomorphism* of well-ordered morphisms from f to f' is an isomorphism $Y \cong Y'$ over X preserving the well-orderings on the fibers.

Proposition 2.1.2. *Given two well-ordered sets, there is at most one isomorphism between them. Given two well-ordered morphisms over a common base, there is at most one isomorphism between them.*

Proof. The first statement is classical, and immediate by induction; the second follows from the first, applied in each fiber. □

Definition 2.1.3. Fix (for the remainder of this and the following section) a regular cardinal α. Say a map of simplicial sets $f \colon Y \longrightarrow X$ is *α-small* if each of its fibers Y_x has cardinality $< \alpha$.

Given a simplicial set X, define $\mathbf{W}_\alpha(X)$ to be the set of isomorphism classes of α-small well-ordered morphisms $Y \longrightarrow X$; together with the pullback action $\mathbf{W}_\alpha(f) := f^* \colon \mathbf{W}_\alpha(X) \longrightarrow \mathbf{W}_\alpha(X')$, for $f \colon X' \longrightarrow X$, this gives a contravariant functor $\mathbf{W}_\alpha \colon \mathbf{sSets}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$.

Lemma 2.1.4. $\mathbf{W}_\alpha$ *preserves all limits:* $\mathbf{W}_\alpha(\mathrm{colim}_i\, X_i) \cong \lim_i \mathbf{W}_\alpha(X_i)$.

Proof. Suppose $F \colon \mathcal{I} \longrightarrow \mathbf{sSets}$ is some diagram, and $X = \mathrm{colim}_{\mathcal{I}}\, F$ is its colimit, with injections $\nu_i \colon F(i) \longrightarrow X$. We need to show that the canonical map $\mathbf{W}_\alpha(X) \longrightarrow \lim_{\mathcal{I}} \mathbf{W}_\alpha(F(i))$ is an isomorphism.

To see that it is surjective, suppose we are given $[f_i \colon Y_i \longrightarrow F(i)] \in \lim_{\mathcal{I}} \mathbf{W}_\alpha(F(i))$. For each $x \in X_n$, choose some i and $\bar{x} \in F(i)$ with $\nu(\bar{x}) = x$, and set $Y_x := (Y_i)_{\bar{x}}$. By Proposition 2.1.2, this is well-defined up to *canonical* isomorphism, independent of the choices of representatives i, $\bar{x}$, Y_i, f_i. The total space of these fibers then defines a well-ordered morphism $f \colon Y \longrightarrow X$, with fibers of size $< \alpha$, and with pullbacks isomorphic to f_i as required.

For injectivity, suppose f, f' are well-ordered morphisms over X, and $\nu_i^* f \cong \nu_i^* f'$ for each i. By Proposition 2.1.2, these isomorphisms must agree on each fiber, so together give an isomorphism $f \cong f'$. □

Define the simplicial set W_α by

$$\mathrm{W}_\alpha := \mathbf{W}_\alpha \cdot \mathbf{y}^{\mathrm{op}} \colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Sets},$$

where $\mathbf{y}$ denotes the Yoneda embedding $\Delta \longrightarrow \mathbf{sSets}$.

Lemma 2.1.5. *The functor* $\mathbf{W}_\alpha$ *is representable, represented by* W_α.

Proof. Given $X \in \mathbf{sSets}$, we have isomorphisms, natural in X:

$$\begin{aligned}
\mathbf{W}_\alpha(X) &\cong \mathbf{W}_\alpha(\mathrm{colim}_{\int X} \Delta[n]) \\
&\cong \lim_{\int X} \mathbf{W}_\alpha(\Delta[n]) \\
&\cong \lim_{\int X} (\mathrm{W}_\alpha)_n \\
&\cong \lim_{\int X} \mathbf{sSets}(\Delta[n], \mathrm{W}_\alpha) \\
&\cong \mathbf{sSets}(\mathrm{colim}_{\int X} \Delta[n], \mathrm{W}_\alpha) \\
&\cong \mathbf{sSets}(X, \mathrm{W}_\alpha).
\end{aligned}$$

(Here $\int X$ denotes the category of elements of X.) □

Notation 2.1.6. Given an α-small well-ordered map $f\colon Y \longrightarrow X \in \mathbf{W}_\alpha(X)$, the corresponding map $X \longrightarrow \mathrm{W}_\alpha$ will be denoted by $\ulcorner f \urcorner$.

Applying the natural isomorphism above to the identity map $\mathrm{W}_\alpha \longrightarrow \mathrm{W}_\alpha$ gives a universal α-small well-ordered simplicial set $\widetilde{\mathrm{W}}_\alpha \longrightarrow \mathrm{W}_\alpha$. Explicitly, n-simplices of $\widetilde{\mathrm{W}}_\alpha$ are pairs

$$(f\colon Y \longrightarrow \Delta[n], s \in f^{-1}(1_{[n]}))$$

i.e. the fiber of $\widetilde{\mathrm{W}}_\alpha$ over an n-simplex $\ulcorner f \urcorner \in \mathrm{W}_\alpha$ is exactly (an isomorphic copy of) the main fiber of f. So, by construction:

Proposition 2.1.7. *The canonical projection $\widetilde{\mathrm{W}}_\alpha \longrightarrow \mathrm{W}_\alpha$ is universal for α-small well-ordered morphisms.*

Corollary 2.1.8. *The canonical projection $\widetilde{\mathrm{W}}_\alpha \longrightarrow \mathrm{W}_\alpha$ is weakly universal for α-small morphisms of simplicial sets; that is, any such morphism can be given (not necessarily uniquely) as a pullback of the projection.*

Proof. By the well-ordering principle and the axiom of choice, one can well-order the fibers, and then use the universal property of W_α. □

Definition 2.1.9. Let $\mathbf{U}_\alpha \subseteq \mathbf{W}_\alpha$ (respectively, $\mathrm{U}_\alpha \subseteq \mathrm{W}_\alpha$) be the subobject consisting of α-small well-ordered fibrations[1]; and define $p_\alpha\colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ as the pullback:

$$\begin{array}{ccc}
\widetilde{\mathrm{U}}_\alpha & \longrightarrow & \widetilde{\mathrm{W}}_\alpha \\
{\scriptstyle p_\alpha}\downarrow & \lrcorner & \downarrow \\
\mathrm{U}_\alpha & \hookrightarrow & \mathrm{W}_\alpha
\end{array}$$

Lemma 2.1.10. *The map $p_\alpha\colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ is a fibration.*

[1]Here and throughout, by "fibration" we always mean "Kan fibration".

Proof. Consider a horn to be filled

$$\begin{CD} \Lambda^k[n] @>>> \widetilde{\mathrm{U}}_\alpha \\ @VVV @VV{p_\alpha}V \\ \Delta[n] @>{\ulcorner x \urcorner}>> \mathrm{U}_\alpha \end{CD}$$

for some $0 \leq k \leq n$. It factors through the pullback

$$\begin{CD} \Lambda^k[n] @>>> \bullet @>>> \widetilde{\mathrm{U}}_\alpha \\ @VVV @VV{x}V @VV{p_\alpha}V \\ \Delta[n] @= \Delta[n] @>{\ulcorner x \urcorner}>> \mathrm{U}_\alpha \end{CD}$$

where by the definition of U_α, x is a fibration. Thus the left square admits a diagonal filler, and hence so does the outer rectangle. □

Lemma 2.1.11. *An α-small well-ordered morphism $f \colon Y \longrightarrow X \in \mathbf{W}_\alpha(X)$ is a fibration if and only if $\ulcorner f \urcorner \colon X \longrightarrow \mathrm{W}_\alpha$ factors through U_α.*

Proof. For '⇒', assume that $f \colon Y \longrightarrow X$ is a fibration. Then the pullback of f to any representable is certainly a fibration:

$$\begin{CD} \bullet @>>> Y \\ @V{x^*f}VV @VV{f}V \\ \Delta[n] @>{x}>> X. \end{CD}$$

so $\ulcorner f \urcorner(x) = x^* f \in \mathrm{U}_\alpha$, and hence $\ulcorner f \urcorner$ factors through U_α.

Conversely, suppose $\ulcorner f \urcorner$ factors through U_α. Then we obtain:

$$\begin{CD} Y @>>> \widetilde{\mathrm{U}}_\alpha @>>> \widetilde{\mathrm{W}}_\alpha \\ @V{f}VV @VV{p_\alpha}V @VVV \\ X @>>> \mathrm{U}_\alpha @>>> \mathrm{W}_\alpha, \end{CD}$$

where the lower composite is $\ulcorner f \urcorner$, and the outer rectangle and the right square are by construction pullbacks. Hence so is the left square; so by Lemma 2.1.10 f is a fibration. □

As an immediate consequence we obtain the following corollary.

Corollary 2.1.12. *The functor $\mathbf{U}_\alpha$ is representable, represented by U_α. The map $p_\alpha \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ is universal for α-small well-ordered fibrations, and weakly universal for α-small fibrations.*

2.2. **Kan fibrancy of the universe.** The previous section provides the main ingredients needed to use U_α as a universe in the sense of Section 1, and hence to give a model of the core type theory. However, to give additionally a type-theoretic universe within that model, we need to show that each U_α itself can be seen as an *type* of the model; in other words, that it is Kan. The main goal of this section is therefore to prove the following theorem.

Theorem 2.2.1. *The simplicial set* U_α *is a Kan complex.*

Before proceeding with the proof we will gather four useful lemmas. The first two concern *minimal fibrations*, which for the present purposes are a technical device whose details, beyond these two lemmas, are unimportant.

Lemma 2.2.2 (Quillen's Lemma, [Qui68])**.** *Any fibration* $f\colon Y \longrightarrow X$ *may be factored as* $f = pg$, *where* p *is a minimal fibration and* g *is a trivial fibration.*

Lemma 2.2.3 ([BGM59, III.5.6]; see also [May67, Cor. 11.7])**.** *Suppose* X *is contractible, with* $x_0 \in X$, *and* $p\colon Y \longrightarrow X$ *is a minimal fibration with fiber* $F := Y_{x_0}$. *Then there is an isomorphism over* X*:*

$$\begin{array}{ccc} Y & \xrightarrow{\;g\;} & F \times X \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & X. & \end{array}$$

For Lemma 2.2.5, the proof we give is due to André Joyal; we include details here since the original [Joy11] is not currently publicly available. For this, and again for Theorem 3.4.1 below, we make crucial use of exponentiation along cofibrations; so we pause first to establish some facts about this.

Lemma 2.2.4 (Cf. [Joy11, Lemma 0.2])**.** *For any map* $i\colon A \longrightarrow B$,

1. $\Pi_i\colon \mathbf{sSets}/A \longrightarrow \mathbf{sSets}/B$ *preserves trivial fibrations;*

and if moreover i *is a cofibration, then:*

2. the counit $i^*\Pi_i \longrightarrow 1_{\mathbf{sSets}/A}$ *is an isomorphism;*
3. if $p\colon E \longrightarrow A$ *is* α*-small, then so is* $\Pi_i p$.

Proof.

1. By adjunction, since i^* preserves cofibrations.
2. Since i is mono, $i^*\Sigma_i \cong 1_{\mathbf{sSets}/A}$; so by adjointness, $i^*\Pi_i \cong 1_{\mathbf{sSets}/A}$.
3. For any n-simplex $x\colon \Delta[n] \longrightarrow B$, we have $(\Pi_i p)_x \cong \mathrm{Hom}_{\mathbf{sSets}/B}(x, \Pi_i p) \cong \mathrm{Hom}_{\mathbf{sSets}/B}(i^*x, p)$. As a subobject of $\Delta[n]$, i^*x has only finitely many non-degenerate simplices, so $(\Pi_i p)_x$ injects into a finite product of fibers of p and is thus of size $< \alpha$. □

Lemma 2.2.5 ([Joy11, Lemma 0.2])**.** *Trivial fibrations extend along cofibrations. That is, if* $t\colon Y \longrightarrow X$ *is a trivial fibration and* $j\colon X \longrightarrow X'$ *is a* *cofibration, then there exists a trivial fibration $t' \colon Y' \longrightarrow X'$ and a pullback square of the form:*

$$\begin{array}{ccc} Y & \dashrightarrow & Y' \\ {\scriptstyle t}\downarrow & \lrcorner & \downarrow{\scriptstyle t'} \\ X & \overset{j}{\hookrightarrow} & X'. \end{array}$$

Moreover, if t is α-small, then t' may be chosen to also be.

Proof. Take $t' := \Pi_j t$. By part 1 of Lemma 2.2.4, this is a trivial fibration; by part 2, $j^* Y' \cong Y$; and by part 3, it is α-small. □

We are now ready to prove that U_α is a Kan complex.

Proof of Theorem 2.2.1. We need to show that we can extend any horn in U_α to a simplex:

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & \mathrm{U}_\alpha \\ \downarrow & \nearrow & \\ \Delta[n] & & \end{array}$$

By Corollary 2.1.12, such a horn corresponds to an α-small well-ordered fibration $q \colon Y \longrightarrow \Lambda^k[n]$. To extend $\ulcorner q \urcorner$ to a simplex, we just need to construct an α-small fibration Y' over $\Delta[n]$ which restricts on the horn to Y:

$$\begin{array}{ccc} Y & \dashrightarrow & Y' \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle q'} \\ \Lambda^k[n] & \hookrightarrow & \Delta[n]. \end{array}$$

By the axiom of choice one can then extend the well-ordering of q to q', so the map $\ulcorner q' \urcorner \colon \Delta[n] \longrightarrow \mathrm{U}_\alpha$ gives the desired simplex.

By Quillen's Lemma, we can factor q as

$$Y \xrightarrow{q_t} Y_0 \xrightarrow{q_m} \Lambda^k[n],$$

where q_t is a trivial fibration and q_m is a minimal fibration. Both are still α-small: each fiber of q_t is a subset of a fiber of q, and since a trivial fibration is onto, each fiber of q_m is a quotient of a fiber of q.

By Lemma 2.2.3, we have an isomorphism $Y_0 \cong F \times \Lambda^k[n]$, and hence a pullback diagram:

$$\begin{array}{ccc} Y_0 & \hookrightarrow & F \times \Delta[n] \\ \downarrow & \lrcorner & \downarrow \\ \Lambda^k[n] & \hookrightarrow & \Delta[n] \end{array}$$

By Lemma 2.2.5, we can then complete the upper square in the following diagram, with both right-hand vertical maps α-small fibrations:

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ {\scriptstyle q_t}\downarrow & \lrcorner & \downarrow \\ Y_0 & \hookrightarrow & F \times \Delta[n] \\ {\scriptstyle q_m}\downarrow & \lrcorner & \downarrow \\ \Lambda^k[n] & \hookrightarrow & \Delta[n] \end{array}$$

Since α is regular, the composite of the right-hand side is again α-small; so we are done. □

2.3. **Modelling type theory in simplicial sets.**

To prove that U_α carries the structure to model type theory, we will need a couple of further lemmas; firstly, that taking dependent products preserves fibrations:

Lemma 2.3.1. *Suppose $Z \xrightarrow{q} Y \xrightarrow{p} X$ are fibrations. Then the dependent product $\Pi_p q$ is a fibration over X.*

Proof. The pullback functor $p^* \colon \mathbf{sSets}/X \longrightarrow \mathbf{sSets}/Y$ preserves trivial cofibrations (since **sSets** is right proper and cofibrations are monomorphisms); so its right adjoint Π_p preserves fibrant objects. □

Secondly, to model Id-types, we will require well-behaved fibered path objects. The construction below may be found in [War08, Thm. 2.25]; we recall it in more elementary terms, which will be useful to us later.

Definition 2.3.2. Given a fibration $p \colon E \longrightarrow B$, define the *fibered path object* $\mathrm{P}_B(E)$ as the pullback

$$\begin{array}{ccc} \mathrm{P}_B(E) & \longrightarrow & E^{\Delta[1]} \\ \downarrow & \lrcorner & \downarrow {\scriptstyle p^{\Delta[1]}} \\ B & \xrightarrow{c} & B^{\Delta[1]}, \end{array}$$

the object of paths in E that are constant in B.

The "constant path" map $c \colon E \longrightarrow E^{\Delta[1]}$ factors through $\mathrm{P}_B(E)$; call the resulting map $r_p \colon E \longrightarrow \mathrm{P}_B(E)$. There are also evident source and taget maps $s_p, t_p \colon \mathrm{P}_B(E) \longrightarrow E$. (On all of these maps, we will omit the subscripts when they are clear from context.)

Proposition 2.3.3. *For any fibration $p \colon E \longrightarrow B$, the maps*

$$E \xrightarrow{r} \mathrm{P}_B(E) \xrightarrow{(s,t)} E \times_B E$$

give a factorisation of the diagonal map $\Delta_p \colon E \longrightarrow E \times_B E$ over B as a (trivial cofibration, fibration); and this is stable over B *in that the pullback along any $B' \longrightarrow B$ is again such a factorisation.*

Proof. It is clear that these maps give a factorisation of Δ_p over B. To see that they are a trivial cofibration and a fibration respectively, consider the pullback construction of $\mathrm{P}_B(E)$ via two intermediate stages:

$$\begin{array}{ccc}
\mathrm{P}_B(E) & \longrightarrow & E^{\Delta[1]} \\
{\scriptstyle (s,t)}\downarrow & \lrcorner & \downarrow{\scriptstyle (s,p^{\Delta[1]},t)} \\
E \times_B E & \longrightarrow & E \times_B B^{\Delta[1]} \times_B E \\
{\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle (\pi_1,\pi_2)} \\
E & \longrightarrow & E \times_B B^{\Delta[1]} \\
\downarrow & \lrcorner & \downarrow \\
B & \xrightarrow{\ c\ } & B^{\Delta[1]}
\end{array}$$

Now (s,t) is certainly a fibration, since it is a pullback of the map $E^{\Delta[1]} \longrightarrow E \times_B B^{\Delta[1]} \times_B E \cong E^{1+1} \times_{B^{1+1}} B^{\Delta[1]}$, which is a fibration by the monoidal model category axioms [Hov99, Lemma 4.2.2(3)], applied to the cofibration $1+1 \longrightarrow \Delta[1]$ and the fibration p.

Similarly, the source map $s\colon \mathrm{P}_B(E) \longrightarrow E$ is a trivial fibration, since it is a pullback of $E^{\Delta[1]} \longrightarrow E^1 \times_{B^1} B^{\Delta[1]}$, which is one by the monoidal model category axioms. But s is a retraction of r, so r is a weak equivalence (by 2-out-of-3) and a monomorphism, so is a trivial cofibration as desired.

Finally, stability of these properties under pullback follows immediately from the stability (up to isomorphism) of the construction itself: for any $f\colon B' \longrightarrow B$, there is a canonical isomorphism $\mathrm{P}_{B'}(f^*E) \cong f^*\mathrm{P}_B(E)$, commuting with the maps r, s, t. □

We are now fully equipped for the main result of the present section:

Theorem 2.3.4. *Let α be an inaccessible cardinal. Then U_α carries* Π-, Σ-, Id-, W-, **1**-, **0**-, *and* +-*structure.*

Moreover, if $\beta < \alpha$ is also inaccessible, then U_β gives an internal universe in U_α closed under all these constructors.

Proof. (Π-structure): Given a pair of α-small fibrations $Z \xrightarrow{q} Y \xrightarrow{p} X$, the dependent product $\Pi_p q$ in $\mathbf{sSets}/X$ is again a fibration, by Lemma 2.3.1; it is also α-small, since α is regular.

Hence by Corollary 2.1.12, the universal dependent product over $\mathrm{U}_\alpha{}^{\Pi\text{-FORM}}$ is representable as the pullback of $\widetilde{\mathrm{U}}_\alpha$ along some map $\Pi\colon \mathrm{U}_\alpha{}^{\Pi\text{-FORM}} \longrightarrow \mathrm{U}_\alpha$, giving the desired Π-structure.

(Σ-structure): Similarly, given α-small fibrations $Z \xrightarrow{q} Y \xrightarrow{p} X$, the composite $p \cdot q$ is again an α-small fibration. So the universal dependent sum over $\mathrm{U}_\alpha{}^{\Sigma\text{-FORM}}$ is representable by some map $\Sigma\colon \mathrm{U}_\alpha{}^{\Sigma\text{-FORM}} \longrightarrow \mathrm{U}_\alpha$.

(Id-structure): Given any α-small fibration $p\colon Y \longrightarrow X$, consider the factorisation of the diagonal Δ_p as $Y \xrightarrow{r} \mathrm{P}_X(Y) \xrightarrow{(s,t)} Y \times_X Y$. The fibration

(s, t) is easily seen to be α-small; and by Proposition 2.3.3, r is stably orthogonal to (s, t) over X.

Applying this construction to $p_\alpha \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ itself gives the desired Id-structure on U_α.

(W-structure): Given α-small fibrations $Z \xrightarrow{q} Y \xrightarrow{p} X$, the initial algebra $W_q \longrightarrow X$ for the induced polynomial endofunctor on $\mathbf{sSets}/X$ may be obtained as a transfinite colimit of iterations of the endofunctor; it can be shown from this description that it is again an α-small fibration [MvdB12].

($\mathbf{0}$-structure), ($\mathbf{1}$-structure), ($+$-structure): these cases are straightforward.

(Internal universe.) Since $\beta < \alpha$, U_β is itself α-small; and by Theorem 2.2.1, it is Kan. So U_β is representable as the pullback of $\widetilde{\mathrm{U}}_\alpha$ along some $u_\beta \colon 1 \longrightarrow \mathrm{U}_\alpha$. Moreover, there is a natural inclusion $i \colon \mathrm{U}_\beta \longrightarrow \mathrm{U}_\alpha$, with $\widetilde{\mathrm{U}}_\alpha[\beta] \cong i^* \widetilde{\mathrm{U}}_\alpha$ by construction. Together these give the desired internal universe (u_β, i).

Finally, to see that (u_β, i) is closed under the appropriate constructors in i, note that for each of Π, Σ, and Id as constructed above, the image of the composite with i lies again in U_β, and hence factors through i; for instance, in the case of Π,

$$\begin{array}{ccc} {\mathrm{U}_\beta}^{\Pi\text{-FORM}} & \xrightarrow{i^{\Pi\text{-FORM}}} & {\mathrm{U}_\alpha}^{\Pi\text{-FORM}} \\ \Big\downarrow \Pi & & \Big\downarrow \Pi \\ \mathrm{U}_\beta & \xrightarrow{\quad i \quad} & \mathrm{U}_\alpha \end{array}$$

(Note that while we do already have Π-structure (and so on) on U_β as constructed in the first parts of this theorem, those choices of the structure do not automatically commute with i.) □

Corollary 2.3.5. *Let $\beta < \alpha$ be inaccessible cardinals. Then there is a model of dependent type theory in $\mathbf{sSets}_{\mathrm{U}_\alpha}$ with all the logical constructors of Section A.2, and a universe (given by U_β) closed under these constructors.* □

We can now interpret the syntax of type theory as an internal language in **sSets**, writing $[\![\mathcal{J}]\!]$ for the interpretation of any judgement $\mathcal{J}$. In doing so, we will make several systematic abuses of notation. Firstly, referring in the syntax to fibrations, we will write E rather than $\ulcorner E \urcorner$, and so on, whenever some choice of name $\ulcorner E \urcorner \colon B \longrightarrow \mathrm{U}_\alpha$ for the fibration is understood; and conversely, referring to the interpretation of a type $\Gamma \vdash T \ \mathsf{type}$, we use $[\![T]\!]$ to refer to the fibration over $[\![\Gamma]\!]$ given by pulling back $\widetilde{\mathrm{U}}_\alpha$ along the literal interpretation $[\![\Gamma \vdash T \ \mathsf{type}]\!] \colon [\![\Gamma]\!] \longrightarrow \mathrm{U}_\alpha$.

As a first characteristic of the model, we note that both of the extra principles on equality of functions hold.

Proposition 2.3.6. *The η-rule and functional extensionality rules of Section A.4 hold in the simplicial model.*

Proof. The η-rule follows immediately from our use of categorical exponentials to interpret Π-types, by the uniqueness in the categorical universal property.

For functional extensionality, Garner [Gar09] shows that it holds just if each product of identity types,

$$f, g{:}\Pi_{x:A}B(x) \vdash \Pi_{x:A}\mathsf{Id}_{B(x)}(\mathsf{app}(f,x), \mathsf{app}(g,x)) \ \mathsf{type}$$

admits the structure given by the rules for the identity type on the corresponding product types,

$$f, g{:}\Pi_{x:A}B(x) \vdash \mathsf{Id}_{\Pi_{x:A}B(x)}(f,g) \ \mathsf{type}.$$

So it is enough to show that for any pair of (α-small, well-ordered) fibrations $Z \xrightarrow{q} Y \xrightarrow{p} X$, given by names

$$\ulcorner Y \urcorner \colon X \longrightarrow \mathrm{U}_\alpha, \qquad \ulcorner Z \urcorner \colon Y \longrightarrow \mathrm{U}_\alpha,$$

the interpretation of the product of identity types

$$[\![\Pi_{x:Y}\mathsf{Id}_{Z(x)}(\mathsf{app}(f,x), \mathsf{app}(g,x))]\!] \cong \mathit{\Pi}_p(\mathrm{P}_Y Z),$$

gives a suitably stable path object for the interpretation of the product types,

$$[\![\mathsf{Id}_{\Pi_{x:Y}Z(x)}(f,g)]\!] \cong \mathit{\Pi}_p Z.$$

For this, it is clear that $\mathit{\Pi}_p(s,t) \colon \mathit{\Pi}_p(\mathrm{P}_Y Z) \longrightarrow \mathit{\Pi}_p(Z \times_Y Z) \cong \mathit{\Pi}_p Z \times_X \mathit{\Pi}_p Z$ is a fibration, since $\mathit{\Pi}_p$ preserves fibrations (Lemma 2.3.1). Similarly, $\mathit{\Pi}_p r_q \colon \mathit{\Pi}_p Z \longrightarrow \mathit{\Pi}_p(\mathrm{P}_Y Z)$ is a cofibration since $\mathit{\Pi}_p$ preserves monomorphisms; and it is a weak equivalence, since $\mathit{\Pi}_p$ preserves trivial fibrations (Lemma 2.2.5), and so the retraction $\mathit{\Pi}_p s_q \colon \mathit{\Pi}_p(\mathrm{P}_Y Z) \longrightarrow \mathit{\Pi}_p Z$ is again a trivial fibration. Finally, the by the Beck-Chevalley condition in an LCCC, the entire construction is stable under pullback in X, as required. □

It now remains only to show that the Univalence Axiom holds in this model.

3. Univalence

In this section, we will introduce the Univalence Axiom, and show that it holds in the simplicial model.

The proof of this involves both simplicial and type-theoretic components; we keep these separate, as far as possible. First of all (Section 3.1), we define univalence type-theoretically and state the Univalence Axiom; next, we define an analogous simplicial concept of univalence (Section 3.2); we then show that via the simplicial model, the two notions coincide (Section 3.3). Finally, in Section 3.4, we prove our main theorem: that U_α is univalent (using the simplicial sense), and hence that the Univalence Axiom holds in the simplicial model of type theory. Lastly, in Section 3.5, we discuss an alternative formulation of simplicial univalence, and so obtain an up-to-homotopy uniqueness statement for the weak universal property of U_α.

3.1. **Type-theoretic univalence.** To state the univalence axiom, we first need to define a few basic notions in the type theory.

Definition 3.1.1 (Joyal)**.** Let $f\colon A \longrightarrow B$ be a function in some context Γ, i.e. $\Gamma \vdash f : [A, B]$ (where the function type $[A, B]$ is defined using $\mathsf{\Pi}$, as described in Section A.2).

- A *left homotopy inverse* for f is a function $g\colon B \longrightarrow A$, together with a homotopy $g \cdot f \simeq 1_A$. Formally, we define the type $\mathsf{LInv}(f)$ of left homotopy inverses to f:

$$\Gamma \vdash \mathsf{LInv}(f) := \Sigma_{g:[B,A]} \mathsf{\Pi}_{x:A} \mathsf{Id}_A(g(f(x)), x) \ \mathsf{type}$$

- Analogously, we define the type $\mathsf{RInv}(f)$ of *right homotopy inverses*:

$$\Gamma \vdash \mathsf{RInv}(f) := \Sigma_{g:[B,A]} \mathsf{\Pi}_{y:B} \mathsf{Id}_B(f(g(y)), y) \ \mathsf{type}$$

- We say $f\colon A \longrightarrow B$ is a *homotopy isomorphism* (or more briefly, an h-isomorphism) if it is equipped with both a left and a right inverse:

$$\Gamma \vdash \mathsf{isHIso}(f) := \mathsf{LInv}(f) \times \mathsf{RInv}(f) \ \mathsf{type}$$

- For any types A and B, we thus have the type of h-isomorphisms from A to B:

$$\Gamma \vdash \mathsf{HIso}(A, B) := \Sigma_{f:[A,B]} \mathsf{isHIso}(f)$$

It may perhaps be surprising that we use homotopy isomorphisms rather than the more familiar homotopy equivalences, with a single two-sided homotopy inverse. The reason is that while a map carries either structure if and only if it carries the other, the type, or object, of such structures on a map is different. In particular, the analogue of Lemma 3.3.4 for homotopy equivalences does not hold; for further discussion of these issues, see Appendix B.

Example 3.1.2. For any type B, the identity function on B is canonically an h-isomorphism.

Suppose now that A is any type, and $x\colon A \vdash B(x)$ type a family of types over A. By the identity elimination rule, we can derive

$$x, y{:}A,\ u{:}\mathsf{Id}_A(x, y) \vdash w_{x,y,u} : \mathsf{HIso}(B(x), B(y)).$$

This can equivalently be seen as a map

$$x, y{:}A, \vdash w_{x,y} : [\mathsf{Id}_A(x, y), \mathsf{HIso}(B(x), B(y))].$$

Definition 3.1.3. We say the family $B(x)$ is *univalent* if for each x, y, the map $w_{x,y}$ is itself a homotopy isomorphism:

$$\vdash \mathsf{isUnivalent}(x{:}A.B(x)) := \mathsf{\Pi}_{x,y:A} \mathsf{isHIso}(w_{x,y}).$$

Axiom 3.1.4. *The* Univalence Axiom*, for a given type-theoretic universe U, is the statement that the canonical family* El *of types over U is univalent.*

Informally, the Univalence Axiom says that just as elements of the universe correspond to types, so equalities in the universe correspond to equivalences between types. In particular, since every statement or construction must respect propositional equality, the Univalence Axiom stipulates that the language can never distinguish between equivalent types.

3.2. **Simplicial univalence.** To define a simplicial notion of univalence, we first need to construct the *object of weak equivalences* between fibrations $p_1 \colon E_1 \longrightarrow B$ and $p_2 \colon E_2 \longrightarrow B$ over a common base. In other words, we want an object representing the functor sending $(X, f) \in \mathbf{sSets}/B$ to the set $\mathrm{Eq}_X(f^*E_1, f^*E_2)$. As we did for $\mathbf{U}_\alpha$, we proceed in two steps, first exhibiting it as a subfunctor of a functor more easily seen (or already known) to be representable.

For the remainder of the section, fix fibrations E_1, E_2 as above over a base B. Since **sSets** is locally Cartesian closed, we can construct the exponential object between them:

Definition 3.2.1. Let $\mathbf{Hom}_B(E_1, E_2) \longrightarrow B$ denote the internal hom from E_1 to E_2 in $\mathbf{sSets}/B$.

Then for any X, a map $X \longrightarrow \mathbf{Hom}_B(E_1, E_2)$ corresponds to a map $f \colon X \longrightarrow B$, together with a map $u \colon f^*E_1 \longrightarrow f^*E_2$ over X.

Together with the Yoneda lemma, this implies the explicit description: an n-simplex of $\mathbf{Hom}_B(E_1, E_2)$ is a pair

$$(b \colon \Delta[n] \longrightarrow B, u \colon b^*E_1 \longrightarrow b^*E_2).$$

Lemma 3.2.2. $\mathbf{Hom}_B(E_1, E_2) \longrightarrow B$ *is a Kan fibration.*

Proof. Follows immediately from Lemma 2.3.1, since the exponential is a special case of dependent products. □

Within $\mathbf{Hom}_B(E_1, E_2)$, we now want to construct the subobject of weak equivalences.

Lemma 3.2.3. *Let $f \colon E_1 \longrightarrow E_2$ be a weak equivalence over B, and suppose $g \colon B' \longrightarrow B$. Then the induced map between pullbacks $g^*E_1 \longrightarrow g^*E_2$ is a weak equivalence.*

Proof. The pullback functor $g^* \colon \mathbf{sSets}/B \longrightarrow \mathbf{sSets}/B'$ preserves trivial fibrations; so by Ken Brown's Lemma [Hov99, Lemma 1.1.12], it preserves all weak equivalences between fibrant objects. □

Thus, weak equivalences from E_1 to E_2 form a subfunctor of the functor of maps from E_1 to E_2. To show that this is representable, we need just to show:

Lemma 3.2.4. *Let $f \colon E_1 \longrightarrow E_2$ be a morphism over B. If for each simplex $b \colon \Delta[n] \longrightarrow B$ the induced map $f_b \colon b^*E_1 \longrightarrow b^*E_2$ is a weak equivalence, then f is a weak equivalence.*

Proof. Without loss of generality, B is connected; otherwise, apply the result over each connected component separately. Take some vertex $b \colon \Delta[0] \longrightarrow B$, and set $F_i := b^* E_i$.

Now $\pi_0(f)$ factors as $\pi_0(E_1) \cong \pi_0(F_1) \xrightarrow{\pi_0(f_b)} \pi_0(F_2) \cong \pi_0(E_2)$, so is an isomorphism, since by hypothesis $\pi_0(f_b)$ is. Similarly, for any vertex $e \colon \Delta[0] \longrightarrow F_1$, we have by the long exact sequence for a fibration:

$$\begin{array}{ccccccccc}
\pi_{n+1}(B,b) & \longrightarrow & \pi_n(F_1,e) & \longrightarrow & \pi_n(E_1,e) & \longrightarrow & \pi_n(B,b) & \longrightarrow & \pi_{n-1}(F_1,e) \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \pi_n(f_b)} & & \downarrow{\scriptstyle \pi_n(f)} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \pi_{n-1}(f_b)} \\
\pi_{n+1}(B,b) & \longrightarrow & \pi_n(F_2,f(e)) & \longrightarrow & \pi_n(E_2,f(e)) & \longrightarrow & \pi_n(B,b) & \longrightarrow & \pi_{n-1}(F_2,f(e))
\end{array}$$

Each $\pi_n(f_b)$ is an isomorphism, so by the Five Lemma, so is each $\pi_n(f)$. Thus f is a weak equivalence. □

Definition 3.2.5. Take $\mathbf{Eq}_B(E_1, E_2)$ to be the subobject of $\mathbf{Hom}_B(E_1, E_2)$ consisting of all n-simplices

$$(b \colon \Delta[n] \longrightarrow B, w \colon b^* E_1 \longrightarrow b^* E_2)$$

such that w is a weak equivalence. (By Lemma 3.2.3, this indeed defines a simplicial subset.)

From Lemma 3.2.4, we immediately have:

Corollary 3.2.6. *Let $(f, u) \colon X \longrightarrow \mathbf{Hom}_B(E_1, E_2)$. Then u is a weak equivalence if and only if (f, u) factors through $\mathbf{Eq}_B(E_1, E_2)$.*

Thus, maps $X \longrightarrow \mathbf{Eq}_B(E_1, E_2)$ correspond to pairs of maps

$$(f \colon X \longrightarrow B, w \colon f^* E_1 \longrightarrow f^* E_2),$$

where w is a weak equivalence. □

While Lemma 3.2.4 was stated just as required by representability, its proof actually gives a slightly stronger statement:

Lemma 3.2.7. *Let $f \colon E_1 \longrightarrow E_2$ be a morphism over B. If for some vertex $b \colon \Delta[0] \longrightarrow B$ in each connected component the map of fibers $f_b \colon b^* E_1 \longrightarrow b^* E_2$ is a weak equivalence, then f is a weak equivalence.* □

Corollary 3.2.8. *The map $\mathbf{Eq}_B(E_1, E_2) \longrightarrow B$ is a fibration.*

Proof. Suppose we wish to fill a square:

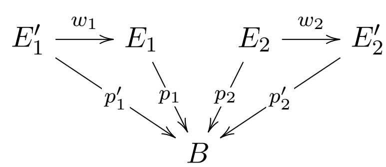
$$\begin{array}{ccc}
\Lambda^k[n] & \longrightarrow & \mathbf{Eq}_B(E_1, E_2) \\
{\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow \\
\Delta[n] & \xrightarrow{\ b\ } & B
\end{array}$$

By the universal property of $\mathbf{Eq}_B(E_1, E_2)$ this corresponds to showing that we can extend a weak equivalence $w \colon i^* b^* E_1 \longrightarrow i^* b^* E_2$ over $\Lambda^k[n]$ to a weak equivalence $\overline{w} \colon b^* E_1 \longrightarrow b^* E_2$ over $\Delta[n]$.

By Lemma 3.2.2, we can certainly find some map $\overline{w}$ extending w. But then since $\Delta[n]$ is connected, Lemma 3.2.7 implies that $\overline{w}$ is a weak equivalence. □

While on the subject, we collect a proposition which is not required for the definition of univalence, but which will be useful later:

Proposition 3.2.9. *If E_1, E_1', E_2, E_2' are fibrations over a common base B, and $w_1 \colon E_1' \longrightarrow E_1$, $w_2 \colon E_2 \longrightarrow E_2'$ are weak equivalences over B, then the induced map $\mathbf{Eq}_B(w_1, w_2) \colon \mathbf{Eq}_B(E_1, E_2) \longrightarrow \mathbf{Eq}_B(E_1', E_2')$ is a weak equivalence.*



Proof. As weak equivalences between fibrations, w_1 and w_2 are fibered homotopy equivalences over B. Choosing fibered homotopy inverses v_1, v_2 for w_1 and w_2 respectively gives a homotopy inverse $\mathbf{Hom}_B(v_1, v_2)$ for $\mathbf{Hom}_B(w_1, w_2) \colon \mathbf{Hom}_B(E_1, E_2) \longrightarrow \mathbf{Hom}_B(E_1', E_2')$. But by Lemma 3.2.7, the image of a homotopy in $\mathbf{Hom}$ whose endpoints lie in $\mathbf{Eq}$ must lie entirely in $\mathbf{Eq}$; so the restriction $\mathbf{Eq}_B(v_1, v_2)$ gives a homotopy inverse for $\mathbf{Eq}_B(w_1, w_2)$, as desired. □

We are now ready to define univalence.

Let $p \colon E \longrightarrow B$ be a fibration. We then have two fibrations over $B \times B$, given by pulling back E along the projections. Call the object of weak equivalences between these $\mathbf{Eq}(E) := \mathbf{Eq}_{B \times B}(\pi_1^* E, \pi_2^* E)$. Concretely, simplices of $\mathbf{Eq}(E)$ are triples

$$(b_1, b_2 \in B_n,\ w \colon b_1^* E \longrightarrow b_2^* E).$$

By Corollary 3.2.6, a map $f \colon X \longrightarrow \mathbf{Eq}(E)$ corresponds to a pair of maps $f_1, f_2 \colon X \longrightarrow B$ together with a weak equivalence $f_1^* E \longrightarrow f_2^* E$ over X. In particular, there is a "diagonal" map $\delta_E \colon B \longrightarrow \mathbf{Eq}(E)$ corresponding to the triple $(1_B, 1_B, 1_E)$, sending a simplex $b \in B_n$ to the triple $(b, b, 1_{E_b})$.

There are also source and target maps $s, t \colon \mathbf{Eq}(E) \longrightarrow B$, given by the composites $\mathbf{Eq}(E) \longrightarrow B \times B \xrightarrow{\pi_i} B$, sending (b_1, b_2, w) to b_1 and b_2 respectively. These are both retractions of δ; and by Corollary 3.2.8, if B is fibrant then they are moreover fibrations.

Definition 3.2.10. A fibration $p \colon E \longrightarrow B$ is *univalent* if the diagonal map $\delta_E \colon B \longrightarrow \mathbf{Eq}(E)$ is a weak equivalence.

Since δ_E is always a monomorphism (thanks to its retractions), this is equivalent to saying that $B \longrightarrow \mathbf{Eq}(E) \longrightarrow B \times B$ is a (trivial cofibration,

fibration) factorisation of the diagonal $\Delta_B \colon B \longrightarrow B \times B$, i.e. that $\mathbf{Eq}(E)$ is a *path object* for B.

We conclude this section with a few examples, and non-examples, of univalent fibrations.

Examples 3.2.11.

(1) The canonical map $X \longrightarrow 1$ is univalent if and only if the space of homotopy auto-equivalences of X is contractible.
(2) The identity map $X \longrightarrow X$ is univalent if and only if X is either empty or contractible. In particular, the identity map $1+1 \longrightarrow 1+1$ is *not* univalent: it has two fibers which are equivalent, over points that are not connected by any path.
(3) Any fibration weakly equivalent to a univalent fibration is itself univalent (essentially, by Proposition 3.2.9).

3.3. **Equivalence of type-theoretic and simplicial univalence.** Having defined the type-theoretic and simplicial notions of univalence, we now wish to show that they coincide. As ever, we make essential use of representability; in particular, we work with the interpretations of type-theoretic notions entirely via their universal properties. With this in view, we need to define what are represented by the interpretations of LInv, isHIso, etc.

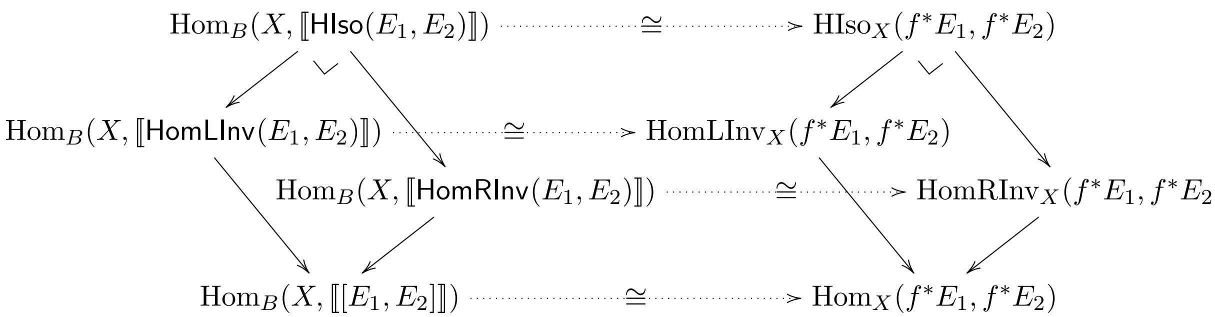
Definition 3.3.1. Let $p_1 \colon E_1 \longrightarrow B$, $p_2 \colon E_2 \longrightarrow B$ be fibrations over a common base (as in Definition 3.2.1).

Define $\mathrm{HomLInv}_B(E_1, E_2)$ to be the set of *maps with a left homotopy inverse* from X to Y, i.e. triples (f, g, H), where $f \colon E_1 \longrightarrow E_2$ and $g \colon E_2 \longrightarrow E_1$ are maps over B, and H is a fibred homotopy from $g \cdot f$ to 1_{E_1}, defined using the fibred path space $\mathrm{P}_B(E_1)$ (as used for the Id-structure in the proof of Theorem 2.3.4).

Similarly, define $\mathrm{HomRInv}_B(E_1, E_2)$ to consist of triples (f, g, H), where f, g are as before, and H is now a fibred homotopy from $f \cdot g$ to 1_{E_2}, defined using $\mathrm{P}_B(E_2)$.

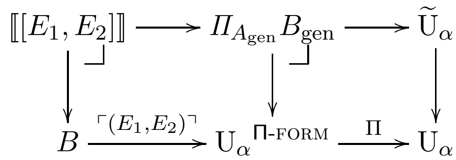
Finally, these both come with evident projections to $\mathrm{Hom}_B(E_1, E_2)$; define $\mathrm{HIso}_B(E_1, E_2) := \mathrm{HomLInv}_B(E_1, E_2) \times_{\mathrm{Hom}_B(E_1, E_2)} \mathrm{HomRInv}_B(E_1, E_2)$.

Lemma 3.3.2. *Let B, E_1, E_2 be as above; additionally, suppose they are given by names $\ulcorner B \urcorner \colon 1 \longrightarrow \mathrm{U}_\alpha$, $\ulcorner E_i \urcorner \colon B \longrightarrow \mathrm{U}_\alpha$. Then for any $f \colon X \longrightarrow B$, there are horizontal isomorphisms as in the diagram below, making the diagram commute, and natural in (X, f).*

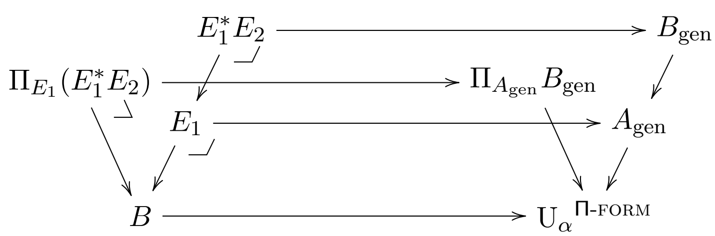


Proof. This is essentially a routine verification; we prove just the first case, that of $[\![[E_1, E_2]]\!]$. For this, we need to produce a natural isomorphism $\mathrm{Hom}_B(X, [\![[E_1, E_2]]\!]) \cong \mathrm{Hom}_X(f^*E_1, f^*E_2)$; in other words, to show that $[\![[E_1, E_2]]\!]$ is the exponential between E_1 and E_2 in $\mathbf{sSets}/B$.

Recall that by definition, $[\![[E_1, E_2]]\!]$ is constructed as the pullback of $\widetilde{\mathrm{U}}_\alpha$ along $\Pi \cdot \ulcorner (E_1, E_2) \urcorner \colon B \longrightarrow \mathrm{U}_\alpha$:



$[\![[E_1, E_2]]\!]$ is thus a pullback of the dependent product of the universal pair of fibrations over $\mathrm{U}_\alpha{}^{\Pi\text{-FORM}}$, and so by the Beck-Chevalley condition is a dependent product for the pullbacks of these fibrations along $\ulcorner (E_1, E_2) \urcorner$. But these pullbacks are isomorphic to E_1, $E_1 \times_B E_2$, by the two pullbacks lemma and the construction of A_{gen}, B_{gen} as pullbacks of $\widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$.



So $[\![[E_1, E_2]]\!]$ is the dependent product of $E_1 \times_B E_2 \longrightarrow E_1$ along $E_1 \longrightarrow B$; but this is exactly the usual construction of exponentials in slices from dependent products [Joh02, A1.5.2]. □

We also note, from the proof of the preceding lemma:

Corollary 3.3.3. *There is a natural isomorphism over B:*

$$[\![[E_1, E_2]]\!] \cong \mathbf{Hom}_B(E_1, E_2).$$

□

Following this, we take $\mathbf{HIso}_B(E_1, E_2) := [\![\mathsf{HIso}_B(E_1, E_2)]\!]$, and define **HomLInv**, **HomRInv** similarly.

Lemma 3.3.4. *The map* $\mathbf{HIso}_B(E_1, E_2) \longrightarrow \mathbf{Hom}_B(E_1, E_2)$ *factors through* $\mathbf{Eq}_B(E_1, E_2)$*; and the resulting map* $\mathbf{HIso}_B(E_1, E_2) \longrightarrow \mathbf{Eq}_B(E_1, E_2)$ *is a trivial fibration.*

Proof. The given map $\mathbf{HIso}_B(E_1, E_2) \longrightarrow [E_1, E_2] \cong \mathbf{Hom}_B(E_1, E_2)$ corresponds, under the isomorphisms of Lemma 3.3.2, to the maps on hom-sets

$$(1) \qquad \begin{aligned} \mathrm{Hom}_B(X, \mathbf{HIso}_B(E_1, E_2)) &\cong \mathrm{HIso}_X(f^*E_1, f^*E_2) \\ &\longrightarrow \mathrm{Hom}_X(f^*E_1, f^*E_2) \\ &\cong \mathrm{Hom}_B(X, \mathbf{Hom}_B(E_1, E_2)) \end{aligned}$$

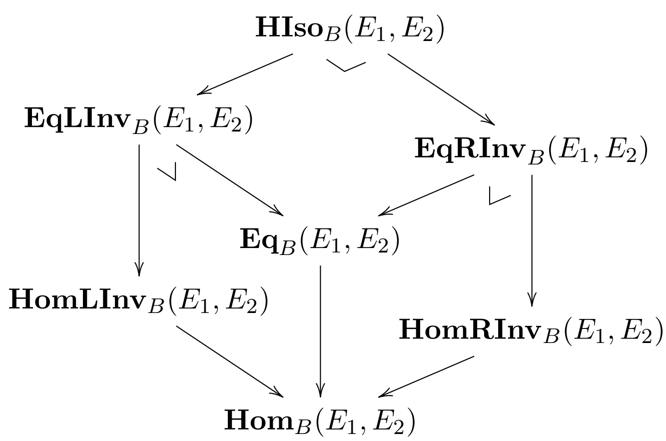
where the middle map just forgets the chosen homotopy inverses of an h-isomorphism. But since any map admitting both homotopy inverses is a weak equivalence, the natural map

$$\mathrm{HIso}_X(f^*E_1, f^*E_2) \longrightarrow \mathrm{Hom}_X(f^*E_1, f^*E_2)$$

factors through $\mathrm{Eq}_X(f^*E_1, f^*E_2)$; so by Yoneda, $\mathbf{HIso}_B(E_1, E_2) \longrightarrow [E_1, E_2] \cong \mathbf{Hom}_B(E_1, E_2)$ factors through $\mathbf{Eq}_B(E_1, E_2)$.

Thus, we obtain the desired map $\mathbf{HIso}_B(E_1, E_2) \longrightarrow \mathbf{Eq}_B(E_1, E_2)$, corresponding to the forgetful function $\mathrm{HIso}_X(f^*E_1, f^*E_2) \longrightarrow \mathrm{Eq}_X(f^*E_1, f^*E_2)$.

Combining this with the left-hand pullback square in Lemma 3.3.2, we can consider $\mathbf{HIso}_B(E_1, E_2)$ as the pullback:



where **EqLInv**, **EqRInv** are defined by the pullbacks above, and represent weak equivalences equipped with a left (resp. right) homotopy inverse. To show that the map $\mathbf{HIso}_B(E_1, E_2) \longrightarrow \mathbf{Eq}_B(E_1, E_2)$ is a trivial fibration, it thus suffices to show that the maps

$$\begin{aligned} \mathbf{EqLInv}_B(E_1, E_2) &\longrightarrow \mathbf{Eq}_B(E_1, E_2) \\ \mathbf{EqRInv}_B(E_1, E_2) &\longrightarrow \mathbf{Eq}_B(E_1, E_2) \end{aligned}$$

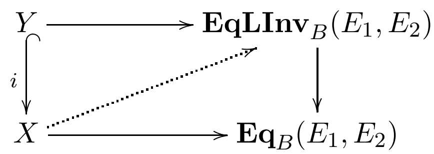
are each trivial fibrations.

Lemma 3.3.5. *For B, E_1, E_2 as above, the map*

$$\mathbf{EqLInv}_B(E_1, E_2) \longrightarrow \mathbf{Eq}_B(E_1, E_2)$$

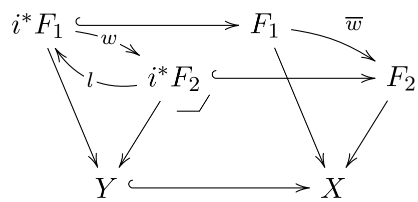
is a trivial fibration. Equivalently, left homotopy inverses to equivalences between fibrant objects extend along cofibrations.

Proof. For $\mathbf{EqLInv}_B(E_1, E_2) \longrightarrow \mathbf{Eq}_B(E_1, E_2)$, we need to find a filler for any diagram of the form



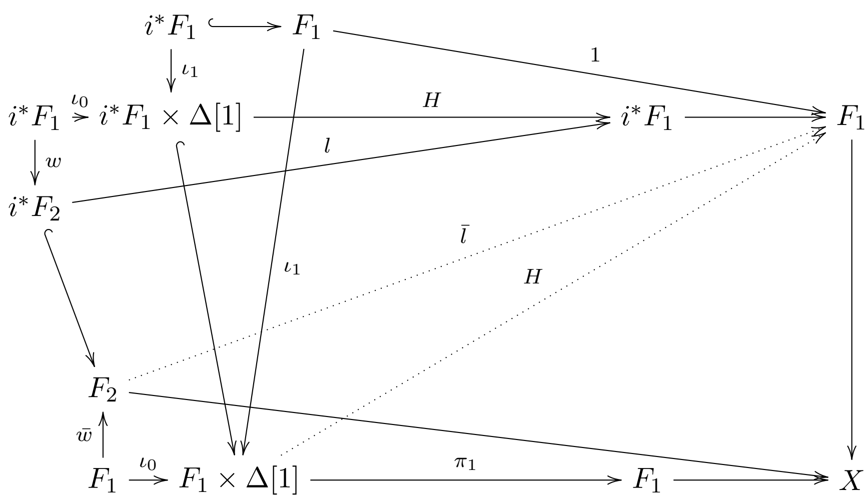
where $i\colon Y \hookrightarrow X$ is a cofibration.

Writing f for the induced map $X \longrightarrow B$ and F_i for f^*E_i, this square corresponds (by the universal properties of **Eq** and **EqLInv**) to a weak equivalence $\bar{w}\colon F_1 \longrightarrow F_2$, and a fibered left homtopy inverse to $w := i^*\bar{w}$; that is, $l\colon i^*F_2 \longrightarrow i^*F_1$, and a homotopy $H\colon l \cdot w \simeq 1_{i^*F_1}$, all fibered over Y:



A filler then corresponds to a fibered left homotopy inverse $(\bar{l}, \bar{H})$ to $\bar{w}$, extending (l, H).

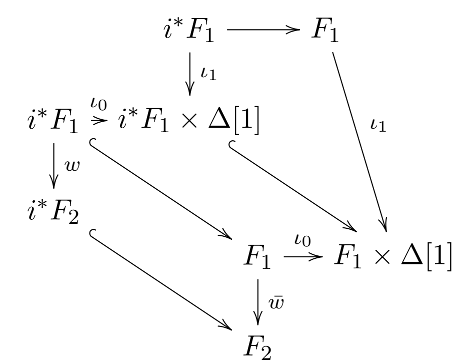
These data and desiderata may be summed up in a single commuting diagram:



Replacing the sub-diagrams on the left by their colimits, we see that we seek precisely a diagonal filler for an associated square:

$$\begin{array}{ccc} i^*F_2 +_{i^*F_1} (i^*F_1 \times \Delta[1]) +_{i^*F_1} F_1 & \longrightarrow & F_1 \\ \downarrow & \nearrow & \downarrow \\ F_2 +_{F_1} (F_1 \times \Delta[1]) & \longrightarrow & X \end{array}$$

So since $F_1 \longrightarrow X$ is a fibration, we just need to show that the left-hand map of pushouts, induced by



is a trivial cofibration. For convenience, call this map t.

To see that t is a weak equivalence, consider it in the square

$$\begin{array}{ccc} (i^*F_1 \times \Delta[1]) +_{i^*F_1} F_1 & \longrightarrow & F_1 \times \Delta[1] \\ \downarrow & & \downarrow \\ i^*F_2 +_{i^*F_1} ((i^*F_1 \times \Delta[1]) +_{i^*F_1} F_1) & \xrightarrow{t} & F_2 +_{F_1} (F_1 \times \Delta[1]). \end{array}$$

The top map is a trivial cofibration by the pushout-product property; the vertical maps are pushouts of w and $\bar{w}$ along cofibrations, so are also weak equivalences; and so by 2-out-of-3, t is a weak equivalence.

On that other hand, to see that t is a cofibration, consider it as induced by maps t_0, t_1 as in:

$$\begin{array}{ccc} i^*F_1 & \longrightarrow & F_1 \\ \downarrow & & \downarrow{\scriptstyle t_1} \\ i^*F_2 +_{i^*F_1} (i^*F_1 \times \Delta[1]) & \xrightarrow{t_0} & F_2 +_{F_1} (F_1 \times \Delta[1]). \end{array}$$

Here t_0 is isomorphic to the inclusion

$$i^*(F_2 +_{F_1} (F_1 \times \Delta[1])) \hookrightarrow F_2 +_{F_1} (F_1 \times \Delta[1])$$

(since pulling back preserves products and pushouts), so is mono. Next, i_0 and i_1 have disjoint images, so t_1 is also mono. Finally, the intersection of the images of t_0 and t_1 is exactly the image of i^*F_1; so t, as the induced map from $(i^*F_2 +_{i^*F_1} (i^*F_1 \times \Delta[1])) +_{i^*F_1} F_1$, is mono as desired.

Thus t is a trivial cofibration, completing the proof of the lemma. □

Lemma 3.3.6. *For B, E_1, E_2 as above, the map*

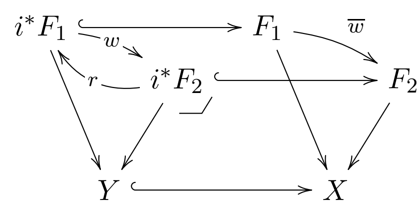
$$\mathbf{EqRInv}_B(E_1, E_2) \longrightarrow \mathbf{Eq}_B(E_1, E_2)$$

is a trivial fibration. Equivalently, right homotopy inverses to equivalences between fibrant objects extend along cofibrations.

Proof. We must provide lifts against any cofibration $i\colon Y \hookrightarrow X$:

$$\begin{array}{ccc} Y & \longrightarrow & \mathbf{EqRInv}_B(E_1, E_2) \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow \\ X & \longrightarrow & \mathbf{Eq}_B(E_1, E_2) \end{array}$$

Analogously to the previous lemma, and again writing $f\colon X \longrightarrow B$, $F_i := f^*E_1$, the square corresponds to a weak equivalence $\bar{w}\colon F_1 \longrightarrow F_2$ over X together with a fibered right homotopy inverse to $w := i^*\bar{w}$, i.e. $r\colon i^*F_2 \longrightarrow i^*F_1$ and a homotopy $H\colon w \cdot r \simeq 1_{i^*F_2}$ over Y;



and a filler corresponds to a fibered right homotopy inverse $(\bar{r}, \bar{H})$ for $\bar{w}$, extending (r, H).

Again, putting these conditions together, we see that they correspond to filling another square:

$$\begin{array}{ccccc} i^*F_2 & \xrightarrow{(r,H)} & i^*F_1 \times_{i^*F_2} \mathrm{P}_Y(i^*F_2) \longrightarrow & & F_1 \times_{F_2} \mathrm{P}_X F_2 \\ \downarrow & & {\scriptstyle (\bar{r},\bar{H})} \nearrow & & \downarrow {\scriptstyle \mathrm{ev}_1 \cdot \pi_1} \\ F_2 & & \xrightarrow{\quad 1 \quad} & & F_2 \end{array}$$

where the pullbacks are just the fibered mapping path spaces.

$$\begin{array}{ccc} i^*F_1 \times_{i^*F_2} \mathrm{P}_Y(i^*F_2) & \longrightarrow & \mathrm{P}_Y(i^*F_2) \\ \downarrow & \lrcorner & \downarrow {\scriptstyle \mathrm{ev}_0} \\ i^*F_1 & \xrightarrow{w} & i^*F_2 \end{array} \qquad \begin{array}{ccc} F_1 \times_{F_2} \mathrm{P}_X F_2 & \longrightarrow & \mathrm{P}_Y F_2 \\ \downarrow & \lrcorner & \downarrow {\scriptstyle \mathrm{ev}_0} \\ F_1 & \xrightarrow{\bar{w}} & F_2 \end{array}$$

Now $i^*F_2 \hookrightarrow F_2$ is certainly a cofibration; so to provide the filler, it suffices to show that the right-hand map is a trivial fibration. As the target map from a mapping path space, it is certainly a fibration. To see that it is a weak equivalence, consider the triangle

$$\begin{array}{ccc} F_1 & \xrightarrow{x \mapsto (x, c_{\bar{w}x})} & F_1 \times_{F_2} \mathrm{P}_X F_2 \\ & \searrow^{\bar{w}} & \downarrow \mathrm{ev}_0 \\ & & F_2 \end{array}$$

The top map is the inclusion of a deformation retraction, so is a weak equivalence; so by 2-out-of-3, the source map ev_0 is a weak equivalence. But ev_1 is homotopic to ev_0, so is also a weak equivalence, as required. □

Putting these two lemmas together concludes the proof of Lemma 3.3.4: **HIso** is trivially fibrant over **Eq**. □

Theorem 3.3.7. *Let B be a Kan complex, $p\colon E \longrightarrow B$ a fibration; choose some names $\ulcorner B \urcorner\colon 1 \longrightarrow \mathrm{U}_\alpha$, $\ulcorner E \urcorner\colon B \longrightarrow \mathrm{U}_\alpha$ for these. Then E is simplicially univalent if and only if the type* $\mathsf{isUnivalent}(E)$ *is inhabited in the model.*

Proof. By definition, $p\colon E \longrightarrow B$ is type-theoretically univalent when there exists a section of the type $[\![x_1, x_2{:}B \vdash \mathsf{isHIso}(w_{x_1,x_2})\ \mathsf{type}]\!]$ over $B \times B$ (where w_{x_1,x_2} is as in Definition 3.1.3). By Lemma 3.3.2 this is equivalent to the map $w_E = [\![x_1, x_2{:}B,\ \mathsf{Id}_B(x_1, x_2) \vdash w_{x_1,x_2}(p) : \mathsf{HIso}(E(x_1), E(x_2))]\!]$ admitting the structure of a homotopy isomorphism, or equivalently being a weak equivalence.

$$\begin{array}{ccc} [\![x_1, x_2{:}B, p{:}\mathsf{Id}_B(x_1, x_2)]\!] & \xrightarrow{w_E} & [\![x_1, x_2{:}B, f{:}\mathsf{HIso}(E(x_1), E(x_2))]\!] \\ & \searrow \quad \swarrow & \\ & B \times B & \end{array}$$

By Lemma 3.3.2, we may fit w_E into the following diagram.

$$\begin{array}{ccccc} B & \xrightarrow{r_B} & \mathbf{P}(B) & \xrightarrow{w_E} & \mathbf{HIso}_{B\times B}(\pi_1^* E, \pi_2^* E) \\ & & & & \downarrow \\ & & & & \mathbf{Eq}_{B\times B}(\pi_1^* E, \pi_2^* E) \\ & & & & \downarrow \\ & & & & \mathbf{Hom}_{B\times B}(\pi_1^* E, \pi_2^* E) \\ & & & & \downarrow \\ & \Delta_B \searrow & & \searrow & B \times B \end{array}$$

Then by the Id-COMP rule applied to the definition of w_{x_1,x_2}, the overall composite map $B \longrightarrow \mathbf{Hom}_{B\times B}(\pi_1^* E, \pi_2^* E)$ is the interpretation of the function $[\![x{:}B \vdash \lambda y{:}E(x).\, y : [E(x), E(x)]]\!]$, corresponding under the universal property of **Hom** to $(\Delta_B, 1_E)$. So the composite $B \longrightarrow \mathbf{Eq}_{B\times B}(\pi_1^* E, \pi_2^* E)$ is the map δ_E of Definition 3.2.10: by definition, E is univalent precisely if δ_E

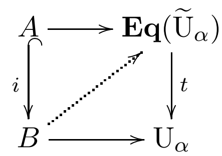
is a weak equivalence. But by 2-out-of-3 and Lemma 3.3.4, δ_E is a weak equivalence if and only if w_E is; so we are done. □

3.4. **Univalence of the simplicial universes.**

Theorem 3.4.1. *The fibration $p_\alpha \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ is univalent.*

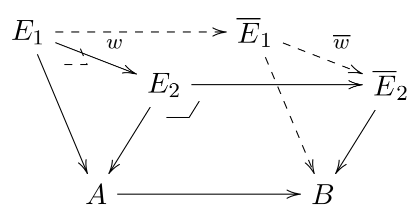
Proof. We will show that the target map $t \colon \mathbf{Eq}(\widetilde{\mathrm{U}}_\alpha) \longrightarrow U$ is a trivial fibration. Since t is a retraction of $\delta_{\widetilde{\mathrm{U}}_\alpha}$, this implies by 2-out-of-3 that $\delta_{\widetilde{\mathrm{U}}_\alpha}$ is a weak equivalence.

So, we need to fill a square



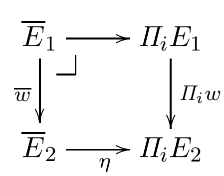
where $i \colon A \hookrightarrow B$ is a cofibration.

By the universal properties of U_α and $\mathbf{Eq}(\widetilde{\mathrm{U}}_\alpha)$, these data correspond to a weak equivalence $w \colon E_1 \longrightarrow E_2$ between α-small well-ordered fibrations over A, and an extension $\overline{E}_2$ of E_2 to an α-small, well-ordered fibration over B; and a filler corresponds to an extension $\overline{E}_1$ of E_1, together with a weak equivalence $\overline{w}$ extending w:



As usual, it is sufficient to construct this first without well-orderings on $\overline{E}_2$; these can then always be chosen so as to extend those of E_2.

Recalling Lemmas 2.2.4–2.2.5, we define $\overline{E}_1$ and $\overline{w}$ as the pullback



in $\mathbf{sSets}/B$, where η is the unit of $i^* \dashv \Pi_i$ at $\overline{E}_2$. To see that this construction works, it remains to show:

(a) $i^*\overline{E}_1 \cong E_1$ in $\mathbf{sSets}/A$, and under this, $i^*\overline{w}$ corrsponds to w;
(b) $\overline{E}_1$ is α-small over B;
(c) $\overline{E}_1$ is a fibration over B, and $\overline{w}$ is a weak equivalence.

For (a), pull the defining diagram of $\overline{E}_1$ back to $\mathbf{sSets}/A$; by Lemma 2.2.4 part 2, we get a pullback square

$$\begin{array}{ccc} i^*\overline{E}_1 & \longrightarrow & E_1 \\ {\scriptstyle i^*\overline{w}}\downarrow & \lrcorner & \downarrow{\scriptstyle w} \\ E_2 & \xrightarrow{1_{E_2}} & E_2 \end{array}$$

in $\mathbf{sSets}/A$, giving the desired isomorphism.

For (b), Lemma 2.2.4 part 3 gives that $\Pi_i E_1$ is α-small over B, so $\overline{E}_1$ is a subobject of a pullback of α-small maps.

For (c), note first that by factoring w, we may reduce to the cases where it is either a trivial fibration or a trivial cofibration.

In the former case, by Lemma 2.2.4 part 1 $\Pi_i w$ is also a trivial fibration, and hence so is $\overline{w}$; so $\overline{E}_1$ is fibrant over $\overline{E}_2$, hence over B.

In the latter case, E_1 is then a deformation retract of E_2 over A; we will show that $\overline{E}_1$ is also a deformation retract of $\overline{E}_2$ over B. Let $H\colon E_2 \times \Delta[1] \longrightarrow E_2$ be a deformation retraction of E_2 onto E_1. We want some homotopy $\overline{H}\colon \overline{E}_2 \times \Delta[1] \longrightarrow \overline{E}_2$ extending H on $E_2 \times \Delta[1]$, $1_{\overline{E}_1} \times \Delta[1]$ on $\overline{E}_1 \times \Delta[1]$, and $1_{\overline{E}_2}$ on $\overline{E}_2 \times \{0\}$. Since these three maps agree on the intersections of their domains, this is exactly an instance of the homotopy lifting extension property, i.e. a square-filler

$$\begin{array}{ccc} (E_2 \times \Delta[1]) \cup (\overline{E}_1 \times \Delta[1]) \cup (\overline{E}_2 \times \{0\}) & \xrightarrow{H \cup 1 \cup 1} & \overline{E}_2 \\ \big\downarrow & \overset{\overline{H}}{\nearrow} & \big\downarrow \\ \overline{E}_2 \times \Delta[1] & \longrightarrow & B \end{array}$$

which exists since the left-hand map is a trivial cofibration.

For $\overline{H}$ to be a deformation retraction, we need to see that $\overline{H}_{\{1\}}\colon \overline{E}_2 \longrightarrow \overline{E}_2$ factors through $\overline{E}_1$. By the definition of $\overline{E}_1$, a map $f\colon X \longrightarrow \overline{E}_2$ over $b\colon X \longrightarrow B$ factors through $\overline{E}_1$ just if the pullback $i^*f\colon i^*X \longrightarrow E_2$ factors through E_1. In the case of $\overline{H}_{\{1\}}$, the pullback is by construction $i^*(\overline{H}_{\{1\}}) = (i^*\overline{H})_{\{1\}} = H_{\{1\}}\colon E_2 \longrightarrow E_2$, which factors through E_1 since H was a deformation retraction onto E_1.

So $\overline{w}$ embeds $\overline{E}_1$ as a deformation retract of $\overline{E}_2$ over B; thus $\overline{E}_1$ is a fibration over B and $\overline{w}$ a weak equivalence, as desired. □

Remark 3.4.2. One can prove, within the type theory, that the Univalence Axiom together with the Π-η rule implies functional extensionality; see [Voe], [BL10] [Gam11] for details. So we could have omitted functional extensionality from Proposition 2.3.6, and instead deduced it here as a corollary of the Univalence Axiom.

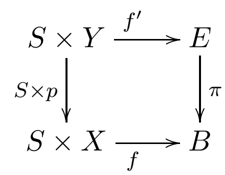
3.5. **Univalence and pullback representations.** We are now ready to give a uniqueness statement for the representation of an α-small fibration as a pullback of $p_\alpha \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$: we define the space of such representations, and show that it is contractible.

In fact, we work a bit more generally. For any fibrations q, p, we define a space $\mathrm{P}_{q,p}$ of representations of p as a pullback of q; and we show that p is univalent exactly when for any q, $\mathrm{P}_{q,p}$ is either empty or contractible.

Let $p \colon E \longrightarrow B$ and $q \colon Y \longrightarrow X$ be fibrations. We define a functor

$$\mathbf{P}_{q,p} \colon \mathbf{sSets}^{\mathrm{op}} \longrightarrow \mathbf{Sets},$$

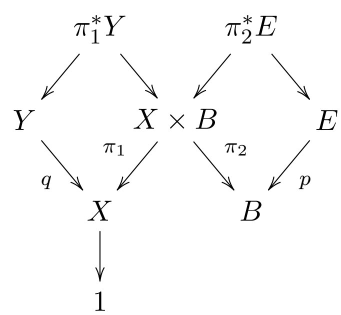
setting $\mathbf{P}_{q,p}(S)$ to be the set of pairs of a map $f \colon S \times X \longrightarrow B$, and a weak equivalence $w \colon S \times E \longrightarrow f^*E$ over $S \times X$; equivalently, the set of squares



such that the induced map $S \times Y \longrightarrow f^*E$ is a weak equivalence. Lemma 3.2.3 ensures that this is functorial in S, by pullback.

Lemma 3.5.1. *The functor* $\mathbf{P}_{q,p}$ *is representable, represented by the object*

$$\mathrm{P}_{q,p} := \Pi_X \Sigma_{\pi_1} \mathbf{Eq}_{X \times B}(\pi_1^* Y, \pi_2^* E).$$



Proof. For any S, we have:

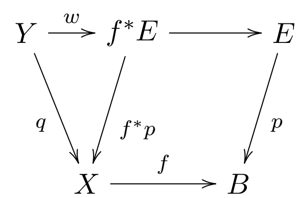
$$\begin{aligned}
&\mathrm{Hom}(S, \Pi_X \Sigma_{\pi_1} \mathbf{Eq}_{X \times B}(\pi_1^* Y, \pi_2^* E)) \\
&\quad \cong \mathrm{Hom}_X(X \times S, \Sigma_{\pi_1} \mathbf{Eq}_{X \times B}(\pi_1^* Y, \pi_2^* E)) \\
&\quad \cong \{(\hat{f}, \hat{w}) \mid \hat{f} \colon X \times S \longrightarrow X \times B \text{ over } X, \\
&\qquad\qquad \hat{w} \colon X \times S \longrightarrow \mathbf{Eq}_{X \times B}(\pi_1^* Y, \pi_2^* E) \text{ over } X \times B\} \\
&\quad \cong \{(f, w) \mid f \colon X \times S \longrightarrow B,\ w \colon Y \times S \longrightarrow f^*E \text{ w.e. over } X \times S\} \\
&\quad \cong \mathbf{P}_{q,p}(S) \qquad \square
\end{aligned}$$

Remark 3.5.2. By Yoneda, we see from this that $(\mathrm{P}_{q,p})_n \cong \mathbf{P}_{q,p}(\Delta[n])$.

Theorem 3.5.3. *Let* $p \colon E \longrightarrow B$ *be a fibration. Then* p *is univalent if and only if for every fibration* $q \colon Y \longrightarrow X$, $\mathrm{P}_{q,p}$ *is either empty or contractible.*[2]

[2] Constructively-minded readers might prefer to phrase this as: if $\mathrm{P}_{q,p}$ is inhabited, then it is contractible.

Proof. First, suppose that p is univalent. Take any q such that $\mathrm{P}_{q,p}$ is non-empty; then we have some map $1 \longrightarrow \mathrm{P}_{q,p}$, corresponding to a square



We claim that $\mathrm{P}_{q,p} \longrightarrow 1$ is a trivial fibration, and hence $\mathrm{P}_{q,p}$ is contractible. Π-functors preserve trivial fibrations (since their left adjoints, pullback, preserve cofibrations), so it is enough to show that

$$\mathbf{Eq}_{X\times B}(\pi_1^*Y, \pi_2^*E) \longrightarrow X \times B \xrightarrow{\pi_1} X$$

is a trivial fibration.

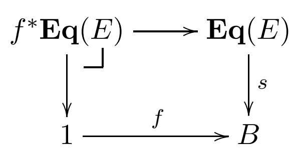
For this, first note that w, as a weak equivalence between fibrations, is a homotopy equivalence over X, so induces a homotopy equivalence

$$(w \cdot -)\colon \mathbf{Eq}_{X\times B}((\pi_1^*(f^*E), \pi_2^*E) \longrightarrow \mathbf{Eq}_{X\times B}(\pi_1^*Y, \pi_2^*E).$$

So it is enough to show that $\mathbf{Eq}_{X\times B}((\pi_1^*(f^*E), \pi_2^*E) \longrightarrow X \times B \xrightarrow{\pi_1} X$ is a trivial fibration; but this follows since it is the pullback along f of the source map $\mathbf{Eq}(E) = \mathbf{Eq}_{B\times B}(\pi_1^*E, \pi_2^*E) \longrightarrow B \times B \xrightarrow{\pi_1} B$, which is a trivial fibration since p is univalent.

Conversely, suppose that for every fibration q, $\mathrm{P}_{q,p}$ is either empty or contractible; now, we wish to show p univalent. For this, it is enough to show that the source map $s\colon \mathbf{Eq}(E) \longrightarrow B$ is a trivial fibration, which will hold if each of its fibers is contractible.

So, take some $f\colon 1 \longrightarrow B$, and consider the fiber $f^*\mathbf{Eq}(E)$. By the universal property of $\mathbf{Eq}(E)$, this is isomorphic to $\mathrm{P}_{f^*p,p}$; and it is certainly non-empty, containing the pair $(f, 1_{f^*E})$; so by assumption, it is contractible, as desired.



□

Corollary 3.5.4. *For any α-small fibration q, the simplicial set P_{q,p_α} of representations of q as a pullback of p_α is contractible.*

Appendix A. Rules of Martin-Löf Type Theory

Our presentation of the structural rules is based largely on [Hof97], which also includes a full construction of the syntax. Our selection of logical rules, and in particular our treatment of the universe, follows [ML84].

We take as basic the judgement forms

$$\Gamma \vdash A \ \mathsf{type} \qquad \Gamma \vdash A = A' \ \mathsf{type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash a = a' : A.$$

We treat contexts as a derived judgement, $\vdash \Gamma \ \mathsf{cxt}$.

A.1. **Structural Rules.** The structural rules of the type theory are (where $\mathcal{J}$ may be any the conclusion of any of the judgement forms):

$$\frac{\vdash \Gamma,\ x{:}A,\ \Delta \ \mathsf{cxt}}{\Gamma,\ x{:}A,\ \Delta \vdash x : A}\ \mathsf{Vble} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma,\ x{:}A,\ \Delta \vdash \mathcal{J}}{\Gamma,\ \Delta[a/x] \vdash \mathcal{J}[a/x]}\ \mathsf{Subst}$$

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma,\ \Delta \vdash \mathcal{J}}{\Gamma,\ x{:}A,\ \Delta \vdash \mathcal{J}}\ \mathsf{Wkg}$$

Definitional equality (also known as syntactic or judgemental equality):

$$\frac{\Gamma \vdash A \ \mathsf{type}}{\Gamma \vdash A = A \ \mathsf{type}} \qquad \frac{\Gamma \vdash A = B \ \mathsf{type}}{\Gamma \vdash B = A \ \mathsf{type}}$$

$$\frac{\Gamma \vdash A = B \ \mathsf{type} \qquad \Gamma \vdash B = C \ \mathsf{type}}{\Gamma \vdash A = C \ \mathsf{type}} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash a = a : A} \qquad \frac{\Gamma \vdash a = b : A}{\Gamma \vdash b = a : A}$$

$$\frac{\Gamma \vdash a = b : A \qquad \Gamma \vdash b = c : A}{\Gamma \vdash a = c : A} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash A = B \ \mathsf{type}}{\Gamma \vdash a : B}$$

$$\frac{\Gamma \vdash a = b : A \qquad \Gamma \vdash A = B \ \mathsf{type}}{\Gamma \vdash a = b : B}$$

Additionally, in the logical rules below, we assume rules stating that each constructor preserves definitional equality in each of its arguments; for instance, along with the Π-INTRO rule, we assume the rule

$$\frac{\begin{array}{c}\Gamma \vdash A = A' \ \mathsf{type} \\ \Gamma,\ x{:}A \vdash B(x) = B'(x) \ \mathsf{type} \qquad \Gamma,\ x{:}A \vdash b(x) = b'(x) : B(x)\end{array}}{\Gamma \vdash \lambda x{:}A.b(x) = \lambda x{:}A'.b'(x) : \Pi_{x:A} B(x)}\ \Pi\text{-INTRO-EQ}$$

A.2. **Logical Constructors.**

Π-types. (Dependent products; dependent function types).

$$\frac{\Gamma,\ x{:}A \vdash B(x) \ \mathsf{type}}{\Gamma \vdash \Pi_{x:A} B(x) \ \mathsf{type}}\ \Pi\text{-FORM}$$

$$\frac{\Gamma,\ x{:}A \vdash B(x) \ \mathsf{type} \qquad \Gamma,\ x{:}A \vdash b(x) : B(x)}{\Gamma \vdash \lambda x{:}A.b(x) : \Pi_{x:A} B(x)}\ \Pi\text{-INTRO}$$

$$\frac{\Gamma \vdash f{:}\Pi_{x:A}B(x) \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{app}(f,a) : B(a)}\ \Pi\text{-APP}$$

$$\frac{\Gamma,\ x{:}A \vdash B(x)\ \mathsf{type} \qquad \Gamma,\ x{:}A \vdash b(x) : B(x) \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{app}(\lambda x{:}A.b(x), a) = b(a) : B(a)}\ \Pi\text{-COMP}$$

As a special case of this, when B does not depend on x, we obtain the ordinary function type $[A, B] := \Pi_{x:A}B$.

Σ-types. (Dependent sums; disjoint sums.)

$$\frac{\Gamma \vdash A\ \mathsf{type} \qquad \Gamma,\ x{:}A \vdash B(x)\ \mathsf{type}}{\Gamma \vdash \Sigma_{x:A}B(x)\ \mathsf{type}}\ \Sigma\text{-FORM}$$

$$\frac{\Gamma \vdash A\ \mathsf{type} \qquad \Gamma,\ x{:}A \vdash B(x)\ \mathsf{type}}{\Gamma,\ x{:}A,\ y{:}B(x) \vdash \mathsf{pair}(x,y) : \Sigma_{x:A}B(x)}\ \Sigma\text{-INTRO}$$

$$\frac{\Gamma,\ z{:}\Sigma_{x:A}B(x) \vdash C(z)\ \mathsf{type} \qquad \Gamma,\ x{:}A,\ y{:}B(x) \vdash d(x,y) : C(\mathsf{pair}(x,y))}{\Gamma,\ z{:}\Sigma_{x:A}B(x) \vdash \mathsf{split}_d(z) : C(z)}\ \Sigma\text{-ELIM}$$

$$\frac{\Gamma,\ z{:}\Sigma_{x:A}B(x) \vdash C(z)\ \mathsf{type} \qquad \Gamma,\ x{:}A,\ y{:}B(x) \vdash d(x,y) : C(\mathsf{pair}(x,y))}{\Gamma,\ x{:}A,\ y{:}B(x) \vdash \mathsf{split}_d(\mathsf{pair}(x,y)) = d(x,y) : C(\mathsf{pair}(x,y))}\ \Sigma\text{-COMP}$$

Again, the special case where B does not depend on x is of particular interest: this gives the cartesian product $A \times B := \Sigma_{x:A}B$.

Id-types. (Identity types, equality types.)

$$\frac{\Gamma \vdash A\ \mathsf{type}}{\Gamma,\ x,y{:}A \vdash \mathsf{Id}_A(x,y)\ \mathsf{type}}\ \mathsf{Id}\text{-FORM} \qquad \frac{\Gamma \vdash A\ \mathsf{type}}{\Gamma,\ x{:}A \vdash \mathsf{refl}_A(x) : \mathsf{Id}_A(x,x)}\ \mathsf{Id}\text{-INTRO}$$

$$\frac{\Gamma,\ x,y{:}A,\ u{:}\mathsf{Id}_A(x,y) \vdash C(x,y,u)\ \mathsf{type} \qquad \Gamma,\ z{:}A \vdash d(z) : C(z,z,\mathsf{refl}_A(z))}{\Gamma,\ x,y{:}A,\ u{:}\mathsf{Id}_A(x,y) \vdash J_{z.d}(x,y,u) : C(x,y,u)}\ \mathsf{Id}\text{-ELIM}$$

$$\frac{\Gamma,\ x,y{:}A,\ u{:}\mathsf{Id}_A(x,y) \vdash C(x,y,u)\ \mathsf{type} \qquad \Gamma,\ z{:}A \vdash d(z) : C(z,z,r(z))}{\Gamma,\ x{:}A \vdash J_{z.d}(x,x,\mathsf{refl}_A(x)) = d(x) : C(x,x,\mathsf{refl}_A(x))}\ \mathsf{Id}\text{-COMP}$$

W**-types.** (Types of well-founded trees; free term algebras.)

$$\frac{\Gamma,\ x{:}A \vdash B(x)\ \mathsf{type}}{\Gamma \vdash \mathsf{W}_{x:A}B(x)\ \mathsf{type}}\ \mathsf{W}\text{-FORM}$$

$$\frac{\Gamma,\ x{:}A \vdash B(x)\ \mathsf{type}}{\Gamma,\ x{:}A,\ y{:}[B(x), \mathsf{W}_{u:A}B(u)] \vdash \mathsf{sup}(x,y) : \mathsf{W}_{u:A}B(u)}\ \mathsf{W}\text{-INTRO}$$

$$\frac{\begin{array}{c}\Gamma,\ w{:}\mathsf{W}_{x:A}B(x) \vdash C(w)\ \mathsf{type} \\ \Gamma,\ x{:}A,\ y{:}[B(x), \mathsf{W}_{u:A}B(u)],\ z{:}\Pi_{u:B(x)}C(\mathsf{app}(y,u)) \\ \hfill \vdash d(x,y,z) : C(\mathsf{sup}(x,y))\end{array}}{\Gamma,\ w{:}\mathsf{W}_{x:A}B(x) \vdash \mathsf{wrec}_d(w) : C(w)}\ \mathsf{W}\text{-ELIM}$$

$$\frac{\begin{array}{c}\Gamma,\ w{:}\mathsf{W}_{x:A}B(x) \vdash C(w)\ \mathsf{type} \\ \Gamma,\ x{:}A,\ y{:}[B(x), \mathsf{W}_{u:A}B(u)],\ z{:}\Pi_{u:B(x)}C(\mathsf{app}(y,u)) \\ \hfill \vdash d(x,y,z) : C(\mathsf{sup}(x,y))\end{array}}{\begin{array}{l}\Gamma,\ x{:}A,\ y{:}[B(x), \mathsf{W}_{u:A}B(u)] \vdash \mathsf{wrec}_d(\mathsf{sup}(x,y)) \\ \qquad = d(x,y,\lambda u{:}B(x).\mathsf{wrec}_d(\mathsf{app}(y,u))) : C(\mathsf{sup}(x,y))\end{array}}\ \mathsf{W}\text{-COMP}$$

$\mathsf{0}$**.** (Empty type.)

$$\frac{}{\vdash \mathsf{0}\ \mathsf{type}}\ \mathsf{0}\text{-FORM} \qquad \text{(No introduction rules.)}$$

$$\frac{\Gamma,\ x{:}\mathsf{0} \vdash C(x)\ \mathsf{type}}{\Gamma,\ x{:}\mathsf{0}\ \vdash \mathsf{case}(x) : C(x)}\ \mathsf{0}\text{-ELIM} \qquad \text{(No computation rules.)}$$

$\mathsf{1}$**.** (Unit type, singleton type.)

$$\frac{}{\vdash \mathsf{1}\ \mathsf{type}}\ \mathsf{1}\text{-FORM} \qquad \frac{}{\vdash * : \mathsf{1}}\ \mathsf{1}\text{-INTRO}$$

$$\frac{\Gamma,\ x{:}\mathsf{1} \vdash C(x)\ \mathsf{type} \qquad \Gamma \vdash d : C(*)}{\Gamma,\ x{:}\mathsf{1} \vdash \mathsf{rec}_d(x) : C(x)}\ \mathsf{1}\text{-ELIM}$$

$$\frac{\Gamma,\ x{:}\mathsf{1} \vdash C(x)\ \mathsf{type} \qquad \Gamma \vdash d : C(*)}{\Gamma \vdash \mathsf{rec}_d(*) = d : C(*)}\ \mathsf{1}\text{-COMP}$$

$+$**-types.** (Binary disjoint sums.)

$$\frac{\Gamma \vdash A\ \mathsf{type} \qquad \Gamma \vdash B\ \mathsf{type}}{\Gamma \vdash A + B\ \mathsf{type}}\ +\text{-FORM}$$

$$\frac{\Gamma \vdash A\ \mathsf{type} \qquad \Gamma \vdash B\ \mathsf{type}}{\Gamma,\ x{:}A \vdash \mathsf{inl}(x) : A + B}\ +\text{-INTRO 1.}$$

$$\frac{\Gamma \vdash A\ \mathsf{type} \qquad \Gamma \vdash B\ \mathsf{type}}{\Gamma,\ y{:}B \vdash \mathsf{inr}(y) : A+B}\ \text{+-INTRO 2.}$$

$$\frac{\begin{array}{c}\Gamma,\ z{:}A+B \vdash C(z)\ \mathsf{type}\\ \Gamma,\ x{:}A \vdash d_0(x) : C(\mathsf{inl}(x)) \qquad \Gamma,\ y{:}B \vdash d_1(y) : C(\mathsf{inr}(y))\end{array}}{\Gamma,\ z{:}A+B \vdash \mathsf{case}_{d_0,d_1}(z) : C(z)}\ \text{+-ELIM}$$

$$\frac{\begin{array}{c}\Gamma,\ z{:}A+B \vdash C(z)\ \mathsf{type}\\ \Gamma,\ x{:}A \vdash d_0(x) : C(\mathsf{inl}(x)) \qquad \Gamma,\ y{:}B \vdash d_1(y) : C(\mathsf{inr}(y))\end{array}}{\Gamma,\ x{:}A \vdash \mathsf{case}_{d_0,d_1}(\mathsf{inl}(x)) = d_0(x) : C(\mathsf{inl}(x))}\ \text{+-COMP 1.}$$

$$\frac{\begin{array}{c}\Gamma,\ z{:}A+B \vdash C(z)\ \mathsf{type}\\ \Gamma,\ x{:}A \vdash d_0(x) : C(\mathsf{inl}(x)) \qquad \Gamma,\ y{:}B \vdash d_1(y) : C(\mathsf{inr}(y))\end{array}}{\Gamma,\ y{:}B \vdash \mathsf{case}_{d_0,d_1}(\mathsf{inr}(y)) = d_1(y) : C(\mathsf{inr}(y))}\ \text{+-COMP 2.}$$

A.3. **Universes.** A universe within the theory may be closed under some or all of the logical constructors of the theory; we include below the rules corresponding to all of the constructors given above.

$$\frac{}{\vdash \mathsf{U}\ \mathsf{type}} \qquad \frac{}{\Gamma,\ x{:}\mathsf{U} \vdash \mathsf{El}(x)\ \mathsf{type}}$$

$$\frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma,\ x{:}\mathsf{El}(a) \vdash b(x) : \mathsf{U}}{\Gamma \vdash \boldsymbol{\pi}(a, x.b(x)) : \mathsf{U}}$$

$$\frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma,\ x{:}\mathsf{El}(a) \vdash b(x) : \mathsf{U}}{\Gamma \vdash \mathsf{El}(\boldsymbol{\pi}(a, x.b(x))) = \Pi_{x:\mathsf{El}(a)} \mathsf{El}(b(x))\ \mathsf{type}}$$

$$\frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma,\ x{:}\mathsf{El}(a) \vdash b(x) : \mathsf{U}}{\Gamma \vdash \boldsymbol{\sigma}(a, x.b(x)) : \mathsf{U}}$$

$$\frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma,\ x{:}\mathsf{El}(a) \vdash b(x) : \mathsf{U}}{\Gamma \vdash \mathsf{El}(\boldsymbol{\sigma}(a, x.b(x))) = \Sigma_{x:\mathsf{El}(a)} \mathsf{El}(b(x))\ \mathsf{type}}$$

$$\frac{\Gamma \vdash a, b : \mathsf{U}}{\Gamma \vdash a + b : \mathsf{U}} \qquad \frac{\Gamma \vdash a, b : \mathsf{U}}{\Gamma \vdash \mathsf{El}(a+b) = \mathsf{El}(a) + \mathsf{El}(b)\ \mathsf{type}}$$

$$\frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma \vdash b, c : \mathsf{El}(a)}{\Gamma \vdash \mathsf{id}_A(b, c) : \mathsf{U}} \qquad \frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma \vdash b, c : \mathsf{El}(a)}{\Gamma \vdash \mathsf{El}(\mathsf{id}_a(b, c)) = \mathsf{Id}_{\mathsf{El}(a)}(b, c)\ \mathsf{type}}$$

$$\frac{}{\vdash \mathsf{z} : \mathsf{U}} \qquad \frac{}{\vdash \mathsf{El}(\mathsf{z}) = 0\ \mathsf{type}} \qquad \frac{}{\vdash \mathsf{o} : \mathsf{U}} \qquad \frac{}{\vdash \mathsf{El}(\mathsf{o}) = 1\ \mathsf{type}}$$

$$\frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma,\ x{:}\mathsf{El}(a) \vdash b(x) : \mathsf{U}}{\Gamma \vdash \mathsf{w}(a, x.b(x)) : \mathsf{U}}$$

$$\frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma,\ x{:}\mathsf{El}(a) \vdash b(x) : \mathsf{U}}{\Gamma \vdash \mathsf{El}(\mathsf{w}(a, x.b(x)) = \mathsf{W}_{x:\mathsf{El}(a)}\mathsf{El}(b(x))\ \mathsf{type}}$$

A.4. **Further rules.** The rules above are somewhat weak in their treatment of the equality of functions. To this end, some further rules are often adopted: the *η-rule* for Π-types, and the *functional extensionality* rule(s). Our formulation of the latter is taken from [Gar09]; see also [Hof95a].

$$\frac{\Gamma \vdash f : \Pi_{x:A}B(x)}{\Gamma \vdash \eta(f) : f = \lambda x{:}A.\mathsf{app}(f,x) : \Pi_{x:A}B(x)}\ \Pi\text{-}\eta$$

$$\frac{\Gamma \vdash f, g : \Pi_{x:A}B(x) \qquad \Gamma \vdash h : \Pi_{x:A}\mathsf{Id}_{B(x)}(\mathsf{app}(f,x), \mathsf{app}(g,x))}{\Gamma \vdash \mathsf{ext}(f,g,h) : \mathsf{Id}_{\Pi_{x:A}B(x)}(f,g)}\ \Pi\text{-EXT}$$

$$\frac{\Gamma, x{:}A \vdash b : B(x)}{\Gamma \vdash \mathsf{ext\text{-}comp}(x.b) : \mathsf{Id}_{\Pi_{x:A}B(x)} (\mathsf{ext}(\lambda x{:}A.b, \lambda x{:}A.b, \lambda x{:}A.\mathsf{refl} b), \mathsf{refl}(\lambda x{:}A.b))}\ \Pi\text{-EXT-COMP-PROP}$$

Appendix B. Type-Theoretic Equivalences

In this appendix, we survey several possible definitions of *equivalence* between types. We discuss four different notions, and show they are all interderivable, while three of the four are moreover equivalent in the stronger sense that the types of them are equivalent (in any of these four senses!). In particular, this justifies the use of HIso in our formulation of univalence, as compared to WEq as used elsewhere, and explains why we do not use the perhaps more familiar HEq.

For brevity, we omit most proofs, and leave even a few definitions as sketches; for full details, see [Voe, Generalities] or [HoTa, `Equiv.v`].

Throughout, we work informally within the type theory as presented in Appendix A. In particular, we assume functional extensionality without comment. We do not, however, assume the Univalence Axiom.

Definition B.0.1. A *contraction* on a type consists of an element $x{:}X$, and a map c giving for each $y{:}X$ a path $c(y){:}\mathsf{Id}(y,x)$. A type is *contractible* if it equipped with a contraction. (It can be shown that if a contraction on a type exists, it is unique[3], justifying the treatment of contractibility as a property rather than extra structure.)

$$\mathsf{isContr}(X) := \Sigma_{x:X}\Pi_{y:X}\mathsf{Id}(x,y).$$

[3] *Unique* within the type theory is understood in terms of the identity types, so becomes up-to-homotopy unique in the interpretation.

Definition B.0.2. Given a function $f\colon X \longrightarrow Y$ and a point $y{:}Y$, the *homotopy fiber* $\mathsf{hFib}(f,y)$ is the type of points $x{:}X$ together with a path $p{:}\mathsf{Id}(f(x),y)$:

$$\mathsf{hFib}(f,y) := \Sigma_{x:X}\mathsf{Id}(f(x),y).$$

Definition B.0.3. A function $f\colon X \longrightarrow Y$ is a *weak equivalence* if each of its homotopy fibers is contractible; i.e.

$$\mathsf{isWEq}(f) := \Pi_{y:Y}\mathsf{isContr}(\mathsf{hFib}(f,y)) \qquad \mathsf{WEq}(X,Y) := \Sigma_{f\colon [X,Y]}\mathsf{isWEq}(f)$$

Definition B.0.4. A *homotopy* between two maps $f, g\colon X \longrightarrow Y$ is a map giving, for each $x{:}X$, a path $h(x){:}\mathsf{Id}(f(x),g(x))$:

$$\mathsf{Homot}(f,g) := \Pi_{x:X}\mathsf{Id}(f(x),g(x)).$$

Definition B.0.5. A (two-sided) *homotopy inverse* for $f\colon X \longrightarrow Y$ is a function $g\colon Y \longrightarrow X$, together with homotopies $\eta\colon 1_X \longrightarrow g\cdot f$, $\varepsilon\colon f\cdot g \longrightarrow 1_Y$.

A *homotopy equivalence* $(f,g,\eta,\varepsilon)\colon X \longrightarrow Y$ is a map together with a homotopy inverse.

$$\mathsf{HEq}(X,Y) := \Sigma_{f:[X,Y]}\Sigma_{g:[Y,X]}(\Pi_{x:X}\mathsf{Id}(x,g(f(x)))) \times (\Pi_{y:Y}\mathsf{Id}(f(g(y)),y)).$$

We also recall, from Definition 3.1.1, the definitions of left and right homotopy inverses, and of homotopy-isomorphisms:

Proposition B.0.6. *A* left homotopy inverse *for $f\colon X \longrightarrow Y$ is a function $g\colon Y \longrightarrow X$, together with a homotopy $\eta\colon 1_X \simeq g\cdot f$. Similarly, a* right homotopy inverse *is a function $g\colon Y \longrightarrow X$ together with a homotopy $\varepsilon\colon f\cdot g \simeq 1_y$.*

A homotopy isomorphism $(f,g_l,\eta,g_r,\varepsilon)\colon X \longrightarrow Y$ *is a map together with left and right homotopy inverses.*

The type of these objects, we denote by $\mathsf{LInv}(f)$, $\mathsf{RInv}(f)$, *and* $\mathsf{HIso}(X,Y)$.

The fourth notion of equivalence requires a few more auxiliary definitions to fully state—specifically, various actions of maps on paths, compositions of homotopies, and the like. We omit here the definitions of these; they are straightforward, and may be found in the references given above.

Definition B.0.7. An *adjoint homotopy inverse* $(g,\eta,\varepsilon,\alpha)$ for a function $f\colon X \longrightarrow Y$ is a homotopy inverse, together with an additional homotopy α witnessing one "triangle identity", $(f\eta)\circ(\varepsilon f) \simeq 1_f$. An *adjoint homotopy equivalence* is a map together with an adjoint homotopy inverse.

We denote the types of adjoint inverses and adjoint equivalences from X to Y by $\mathsf{AdjInv}(f)$, $\mathsf{AdjEq}(X,Y)$ respectively.

Proposition B.0.8. *For any types X, Y and function $f\colon X \longrightarrow Y$, each of the following implies all of the others:*

1. *f is a weak equivalence;*
2. *f may be equipped with a (two-sided) homotopy inverse;*
3. *f may be equipped with left and right homotopy inverses;*
4. *f may be equipped with an adjoint homotopy inverse.*

Sketch proof. The implications (4) ⇒ (2) ⇒ (3) are trivial; and (1) ⇒ (4) is also straightforward to show directly, by extracting the data of an adjoint inverse from the contractions on the fibres.

The converses of these implications are a little less immediate, but still not too difficult: for instance, (2) ⇒ (4) may be shown essentially by the classical proof that any equivalence of categories may be improved to an adjoint equivalence. □

Note that we did *not* say, as one might expect, "the following are equivalent", since we wish to emphasise the difference between mere interderivability and actual equivalence of types. The stronger condition, however, does hold for three of the four notions:

Proposition B.0.9. *For any types* X, Y *and function* $f \colon X \longrightarrow Y$*, the following types are equivalent*

(1) $\mathsf{isWEq}(f)$*;*
(2) $\mathsf{LInv}(f) \times \mathsf{RInv}(f)$*;*
(3) $\mathsf{AdjInv}(f)$*.*

To prove this, it is convenient to introduce a general notion of being "just a property".

Definition B.0.10. X is a *homotopy-proposition* (briefly, an h-proposition) if one can give, for any two elements of X, a path between them:

$$\mathsf{isHProp}(X) := \Pi_{x,y:X} \mathsf{Id}(x, y).$$

Some easy observations:

Proposition B.0.11.

- *The product of any pair or family of h-propositions is again an h-proposition.*
- *For any type* X*, the type* $\mathsf{isContr}(X)$ *is an h-proposition.*
- *To show* $\mathsf{isHProp}(X)$*, it is sufficient to do so under the assumption of an inhabitant of* X*.*
- *If two h-propositions are interderivable, then they are moreover equivalent.*

Proposition B.0.9 thus follows from B.0.8, together with:

Proposition B.0.12. *For any* X*,* Y*,* f*, the types* $\mathsf{isWEq}(f)$*,* $\mathsf{LInv}(f) \times \mathsf{RInv}(f)$*, and* $\mathsf{AdjInv}(f)$ *are all h-propositions.*

Partial proof. $\mathsf{isWEq}(f)$ is a product of h-propositions, so is one itself.

For $\mathsf{LInv} \times \mathsf{RInv}$, we may assume an inhabitant of it, and hence of $\mathsf{isWEq}(f)$; given this, it is straightforward to show that both $\mathsf{LInv}(f)$ and $\mathsf{RInv}(f)$ are h-propositions, and hence their product is. (Note, however, that we do need the assumption of an inhabitant: for general f, $\mathsf{LInv}(f)$ and $\mathsf{RInv}(f)$ may fail to be h-props.)

The case of $\mathsf{AdjInv}(f)$ is less straightforward than the others, so we omit it here. □

This leaves the question of comparing HEq with the other notions. For this, first note:

Proposition B.0.13. *For any type X, two-sided homotopy inverses for 1_X are equivalent to homotopies from 1_X to itself.*

Sketch. If (g, η, ϵ) is a homotopy inverse for 1_X, then $\eta \cdot \epsilon^{-1}$ gives a homotopy from 1_X to itself. One may show directly that this construction is an equivalence. □

Beyond this, the theory is agnostic: it is consistent either that HInv is always an h-proposition, or that it may fail to be one. In the standard model in **Sets**, with trivial identity types, HInv is always an h-proposition, and hence HEq is equivalent to the other notions. However, in various homotopically non-trivial models, the circle (or some analogous object) gives a type on which it is clear that there are non-trivial homotopies from the identity map to itself, and hence HInv is not an h-proposition. This is most easily verified in the groupoid model, using the object $\mathbf{B}\mathbb{Z}$, the free groupoid on a single endo-arrow. With a little more work, the same counterexample may be constructed in the simplicial sets model; or taking a different approach, one can also show that the Univalence Axiom implies that HInv of the identity map on the free loop space of the universe is not an h-prop.

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