Asset Allocation in a Downside Risk Framework

Executive Summary. *The traditional Markowitz portfolio optimization has two serious drawbacks. First, mean-variance portfolio optimization is inadequate when asset returns are skewed. Second, investor risk aversion is ignored. A more efficient measure of risk that focuses only on the deviation below a pre-specified target rate of return is defined in a generalized lower partial moment (LPM) framework. The concepts of LPM and co-LPM, a downside measure of the covariance of return, are extended to Markowitz's model to provide a more efficient and robust optimization process. This article demonstrates that downside risk models can be easily implemented using spreadsheet programs and illustrates how investor risk aversion can be incorporated into a downside risk asset optimization model.*

by Tien Foo Sing*
Seow Eng Ong**

*National University of Singapore, Singapore 117566 or rststf@nus.edu.sg.

**National University of Singapore, Singapore 117566 or rstongse@nus.edu.sg.

Introduction

Optimizing asset allocation is simply defined as the process of mixing asset weights of a portfolio within the constraints of an investor's capital resources to yield the most favorable risk-return trade-off. For typical risk-averse investors, an optimal combination of investment assets that gives a lower risk and a higher return is always preferred (Markowitz, 1959). In a complete market without riskless lending and borrowing, a whole range of efficient asset portfolios having the stochastic dominance features could be determined, which collectively delineates a convex mean-variance frontier.

In practice, a wide range of portfolio "optimizers," from a simple rule of thumb to a full-scale quadratic programming technique, have been proposed to delineate the boundary of the mean-variance efficient frontier. Most, if not all, of the optimization algorithms are developed strictly based on the standard mean-variance concepts. The mean-variance approach has two important limitations nonetheless. First, bounded by the strict assumption that asset returns follow a symmetric bell-shaped distribution, the application of the mean-variance model is limited when asset returns are skewed. It tends to include only an insignificant proportion of stochastically dominant assets into the efficient frontier on one hand, and prematurely precludes assets having negatively skewed return on another hand, because a high threshold return is imposed. Second, investor risk aversion is ignored.

Markowitz (1959) recognized the "asymmetrical" inefficiencies inherited in the traditional mean-variance models, and suggested a semi-variance measure of asset risk that focuses only on the risks below a target rate of return, an intuitively more appealing alternative. However, the alternative measure of risk focusing on the downside of asset returns has not gained much attention. The relative unfamiliarity and the intractability of the semi-variance measure are believed to be the main causes for resistance amongst portfolio managers against the downside risk concept.

The semi-variance measure is adopted in a proposed downside risk asset optimization framework of Harlow and Rao (1989) and Harlow (1991). In their optimization model, the objective is to minimize the lower partial moment (LPM), a proxy for semi-variance, of the portfolio subject to the constraints on the portfolio return and the weight composition. The proposed model does not take into account the correlation between the LPMs of individual assets, which is the most important condition for diversification in the Markowitz's concept. Therefore, the optimization of the asset allocation based on the Harlow and Rao (1989) and Harlow (1991) framework is restrictive in cases where the LPMs of individual assets are highly correlated. Bearing in mind the limitations in the earlier downside risk optimization model, we generalize the model by incorporating the downside covariance of correlated asset returns into the minimization algorithm via two different co-LPM measures. Further, we extend the downside risk asset allocation model to include real estate as an asset class in addition to stocks and bonds.

This study demonstrates the application of the proposed downside risk models using spreadsheet programs with stocks, bonds and real estate data from Singapore as a case in point. It also illustrates how investor risk aversion can be incorporated into the downside risk asset optimization model. The results of the analysis showed that a downside optimal portfolio is always preferred over a mean-variance portfolio because of the higher downside risk protection for a same level of return. At a given portfolio return of 1.00% on the efficient frontiers, the downside risks are estimated at 1.26% and 1.50%, compared to 2.74% in the traditional mean-variance estimation. By asset type allocation, real estate assets appeared to be the most important composition in the portfolio when the portfolio return exceeds 1.00%. In terms of risk aversion, the results are consistent with the expected utility hypothesis that a comparatively more risk-averse investor prefers a lower downside risk compared to one with higher risk preference, given that both investors are indifferent with the levels of portfolio return.

This article is organized into five sections. The next section presents the underlying concepts of the downside risk. The following section first discusses the standard Markowitz's quadratic-programming algorithm and then extends the algorithm to incorporate downside risk measures. Next, the computations of the standard and the downside optimization algorithms are illustrated in a three-asset portfolio using empirical inputs from Singapore capital markets. The final section is the conclusion.

Concepts of Downside Risk

The concept of downside risk is not new, its existence dates back to 1952 (Roy, 1952). The earlier concern of the downside deviation was addressed by Roy (1952) in the form of a "safety first" rule that measures the probability of outcomes falling below a target return. Kataoka (1963) and Telser (1956) extended the rule in a single period setting, which was further developed by Tse, Uppal and White (1993) in a dynamic framework.

The safety first rule, together with other measures of risk, namely the expected value of loss, the expected absolute deviation, the maximum expected loss, the semi-variance and the variance, were evaluated when Markowitz (1959) formalized his seminal portfolio theory. The semi-variance, defined as the squared deviation of return below a target return, was found to be a theoretically more robust measure of risk, though the variance was subsequently chosen for technical reasons. Markowitz argued for the importance of the tailed-end return distribution over the upside potential of the investment because he believed that investors' risk perception should be heuristically asymmetric. Mao (1970) also found support for the semi-variance measure among business executives who were more sensitive to losses below some benchmark returns compared with the likelihood of the project return going above the benchmarks.

Bawa (1975) generalized the semi-variance measure of risk to reflect a less restrictive class of decreasing absolute risk-averse (DARA) utility function.[1] The generalized concept of downside risk is called lower partial variance or LPM. The measure of the co-variance of the below target returns dispersion as proposed by Bawa and Linderberg (1977) is another important downside risk measure required for the downside portfolio optimization algorithms.

Lower Partial Moment

Bawa (1975) shows that the second order mean-LPM ($MLPM_2$), for a class of DARA utility functions, is a preferred approximation for the optimal third order stochastic dominance selection rule[2] compared to the mean-variance criteria. The definition of the second order LPM function by Bawa could be generalized into n-order LPMs to cover a range of risk measures as:

$$LPM_n(\tau, R_i) = \int_{-\infty}^{\tau} (\tau - R_i)^n dF(R_i), \qquad (1)$$

where τ is "target return," R_i is the return of asset i, $dF(R_i)$ is the probability density function of return on asset i and n is the order of moment that characterizes an investor's preference of return dispersion below the target rate. The common classes of LPM are the probability of loss ($n = 0$), the target shortfall ($n = 1$), the target semi-variance ($n = 2$) and the target skewness ($n = 3$). The variable n can also be viewed as a measure of risk aversion where risk aversion increases with n. Risk as measured by the n-LPM reflects explicitly the asymmetry and skewness of the probability distribution of asset returns. For computational reasons, if we assume that there are T number of return observations for asset i, then the n-LPM can be described as a discrete distribution:

$$LPM_n(\tau, R_i) = \frac{1}{T-1} \sum_{t=1}^{T} [Max(0, (\tau - R_{it}))]^n. \qquad (2)$$

Co-Lower Partial Moment

In extending the semi-variance measure of risk to the capital asset pricing model (CAPM), Hogan and Warren (1974) introduced the co-semivariance concept, an asymmetric measure of the relative risk between a risky asset and an efficient market portfolio. Bawa and Lindenberg (1977) generalized the co-semivariance measure into an n-degree LPM structure, which is called a generalized or asymmetric co-LPM (GCLPM) and defined as:

$$GCLPM_n(\tau, R_i, R_j) = \int_{-\infty}^{\tau} \int_{-\infty}^{+\infty} (\tau - R_i)^{n-1} (\tau - R_j) dF(R_i, R_j), \qquad (3)$$

$$GCLPM_n(\tau, R_i, R_j) \neq GCLPM_n(\tau, R_j, R_i), \qquad (4)$$

$$GCLPM_n(\tau, R_i, R_j) = LPM_n(\tau, R_i), \quad when\ R_i = R_j, \qquad (5)$$

where $dF(R_i, R_j)$ in Equation (3) is the joint probability density function of the returns of assets i and j. A discrete form of the GCLPM can be written as:

$$GCLPM_n(\tau, R_i, R_j) = \frac{1}{T-1} \sum_{t=1}^{T} [Max(0, (\tau - R_{it}))]^{n-1} (\tau - R_{jt}). \qquad (6)$$

Nantell and Price (1979) and Harlow and Rao (1989) proposed an unrestricted version of the Bawa-Lindenberg's generalized CLPM, where the target rate (τ) is not equal to the risk-free interest rate (R_f), *i.e.*, $\tau \neq R_f$. Another variation of the CLPM measure assumes symmetry between the returns of assets i and j, given as:

$$GCLPM_n(\tau, R_i, R_j) = GCLPM_n(\tau, R_j, R_i). \qquad (7)$$

Nawrocki (1991) found that the symmetric co-LPM

(SCLPM) provides a more consistent estimation for a shorter period. The SCLPM is defined as:

$$SCLPM_n(\tau, R_i, R_j) = SCLPM_n(\tau, R_j, R_i)$$
$$= [LPM_n(\tau, R_i)^{1/n}\ [LPM_n(\tau, R_j)]^{1/n}(\rho_{i,j}), \qquad (8)$$

where $\rho_{i,j}$ is the correlation coefficient between the returns of assets i and j.

Portfolio Optimization Algorithms

Portfolio optimization as defined is to determine the most favorable combination of assets such that the portfolio is stochastically dominant with a minimum risk at all levels of expected return. There are many optimization algorithms that are capable of ascertaining the most efficient mix of asset portfolio. Based on the assumptions of no short selling and no riskless lending and borrowing, Markowitz's quadratic programming model, which is the most commonly used optimization models, is illustrated for a N-asset case. The N-asset Markowitz's model is then generalized in the LPM framework.

Markowitz's Quadratic Programming Optimization Model

In the Markowitzian's risk-return trade-off conditions, the efficient portfolio is constructed by a convex combination of risky assets that give the risk-averse investor the lowest risk for an expected return. The optimization of asset portfolio can be modeled as a quadratic programming function consisting of a risk minimization objective (G) and three constraints (C):

$$Minimize \quad G(x) = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij}$$

Subject to

$$C_1(x_i) = \sum_{i=1}^{N} x_i \overline{R}_i - \overline{R}_P \qquad (9)$$

$$C_2(x_i) = \sum_{i=1}^{n} x_i - 1$$

$$x_i \geq 0, \quad i = 1, 2, \ldots, N,$$

where:

x_i = The proportion of portfolio allocated to asset i;
$\overline{R}_P$ = The expected portfolio return;
$\overline{R}_i$ = The expected return on asset i; and
σ_{ij} = The covariance between asset i returns and asset j returns, where σ_{ii} is the variance of asset i.

The model implies that an investor aims to obtain an efficient combination of assets at the lowest level of risk ($G(x)$), subject to three conditions. $C_1(x)$ requires the weighted returns of the assets to be higher than the expected portfolio return. $C_2(x)$ eliminates idle investment capitals by equating the total portfolio weight to one. The third constraint restricts any short selling of assets. Equation (9) can be solved mathematically as a standard constrained optimization problem.

Downside Risk Optimization Model

Asset allocation in a downside risk framework determines an optimal investment opportunity set for downside risk-averse investors. In the downside risk asset allocation framework proposed by Harlow and Rao (1989) and Harlow (1991), the optimization problem involves essentially the selection of an optimal asset mix such that the probability of the portfolio return (R_p) falling below the target rate of return (τ) is minimized. The three constraints as in Markowitz's model (Equation (9)) are retained in the downside risk minimization process.

Formally, the downside risk asset allocation model of Harlow and Rao (1989) and Harlow (1991) can be represented as:

$$Minimize\ x_i\ in \quad LPM_n(\tau, x_i)$$
$$= \frac{1}{T-1}\sum_{t=1}^{T} Max\left[0, \left(\tau - \sum_{t=1}^{N} x_{it}R_{it}\right)^n\right]$$

Subject to

$$n = 1\ or\ 2$$
$$C_1(x_i) = \sum_{i=1}^{N} x_i \overline{R}_i - R_P \qquad (13)$$
$$C_2(x_i) = \sum_{i=1}^{N} x_i - 1$$
$$x_i \geq 0, \quad i = 1, 2, \ldots, N.$$

The minimization of the target deviation of the portfolio return as in Equation (13) implicitly assumes that the aggregate downside effects are ex-ante efficient. There are two major limitations in the above downside asset allocation algorithm. First, the downside distributions of the ex-post returns of individual assets are taken to be irrelevant as long as the return of the portfolio composing individual assets is a downside efficient portfolio with minimum target shortfalls. Second, by only evaluating the downside deviation of the portfolio return in aggregate terms, the model also neglects the co-movement between individual asset returns that fall below the target return. It may violate the fundamental rule of diversification if two assets with perfectly correlated downside shortfalls are included in the portfolio composition.

Taking into account the two critical limitations in the Harlow and Rao (1989) and Harlow (1991) optimization model, we further generalize the downside risk measure in CLPM framework. In this extended optimization model, the ex-post downside risks and the co-variances of individual assets with respect to the target return are explicitly reflected in the minimization function. Based on the same set of constraints adopted in Markowitz's quadratic model, the risk minimization algorithm of the extended n-degree Mean-Co-LPM (M-CLPM$_n$) is defined as:

$$\textit{Minimize} \quad G(x) = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \, CLPM_n(\tau, x_i, x_j)$$

Subject to

$$C_1(x_i) = \sum_{i=1}^{N} x_i \overline{R}_i - \overline{R}_P \tag{14}$$

$$C_2(x_i) = \sum_{i=1}^{N} x_i - 1$$

$$x_i \geq 0, \quad i = 1, 2, \ldots, N.$$

Empirical Analysis

We apply the extended downside risk asset allocation algorithm (Equation (14)) to the capital markets of Singapore and test the performance of the model with a three-asset case. A comparison of the optimization outcomes is made with respect to the classical mean-variance models. All models are computed using Microsoft Excel spreadsheets.[3]

Data

The returns of the three assets, namely real estate, stocks and bonds, are collected and estimated on a quarterly basis for the period from 1983:2 to 1997:2 for the country under study. In Singapore, the Urban Redevelopment Authority (URA) all-property index is a reliable transaction based indicator of the real estate market performance.[4] The stock and bond data are respectively represented by the Stock Exchange of Singapore (SES) all-share and the short-term government bond indices, both of which are available from the TREND database.

The historical statistics of the asset markets are summarized in Exhibit 1. The results show that real estate assets are the most attractive investment generating the highest quarterly return of 2.01%. Short-term government bonds as a proxy of a safe investment yield a negative return of 0.02% on average from 1983 to 1999. The fluctuation of the rate of change of Singapore's bond yields is relatively small and within a standard deviation of 2.19% per quarter. Singapore stocks with an ex-post standard deviation of returns of 9.03% per quarter appear to be the most risky investment over the sample period.

Exhibit 1
Historical Statistics of Singapore Asset Markets

Asset Type	Real Estate	Stock	Bond
Abbreviation	SProp	SStk	SBond
Mean (%)	2.01	1.24	−0.02
Median (%)	2.00	0.29	0.02
Std. Dev. (%)	4.73	9.03	2.19
Sample Variance (%)	0.22	0.82	0.05
Kurtosis	−0.23	0.71	1.90
Skewness	0.02	−0.43	−0.44
Range (%)	21.03	45.91	12.02
Min. (%)	−8.03	−27.97	−6.65
Max. (%)	13.01	17.94	5.37
Number	57	57	57
Correlation Matrix			
Real Estate	1.000		
Stocks	0.513	1.000	
Bonds	0.121	0.046	1.000

In term of the third moment of expected returns (skewness), stocks and bonds have considerable negative skews. The real estate returns, however, are approximately normal in the distribution. For the fourth moment of expected returns, the fat-tailed distribution is not significantly observed. With a kurtosis of less than 3, the return distributions of the three assets contain a large proportion of medium-sized deviations. They have a flat-topped distribution that is known technically as platykurtic distribution.

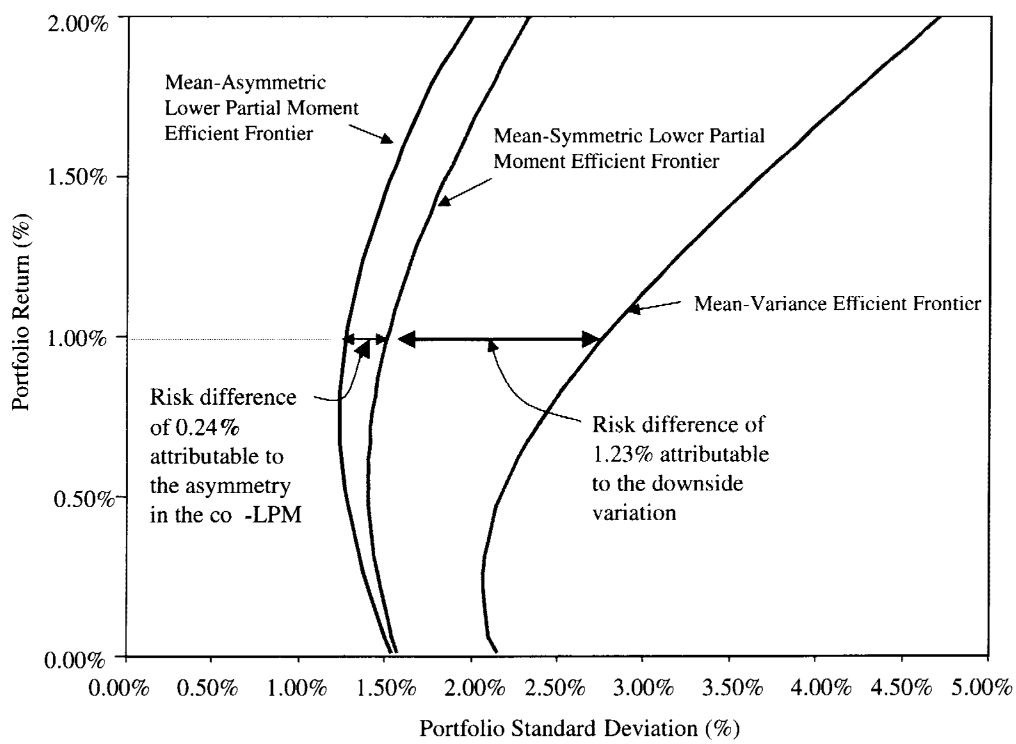
Analysis of Results

Downside-risk vs. mean variance efficient frontiers. The convex efficient frontiers computed based on the classical mean-variance optimization and the downside risk optimization algorithms are shown in Exhibit 2. For the downside risk optimization models, the order of moment is set at $n = 2$ and the target rate or return is fixed at zero (*i.e.*, $\tau = 0.00\%$). For an efficient portfolio with a return ranging from 0.00% to 2.00%, the standard deviation of the classical Markowitz's portfolio is a convex function varying within a range of 2.63%. The downside standard deviations, on the other hand, have a smaller range of 0.80% for the non-symmetric downside risk model and 0.92% for the symmetric version of the mean-LPM model.

It is also apparent from the graphical comparison in Exhibit 2 that the downside risk models push the efficient frontiers outward to the left of the classical Markowitz's mean-variance curve to produce efficient portfolios that are stochastically dominant. At a quarterly portfolio return of 1%, the optimal portfolios of real estate, stocks and bonds determined by the mean-LPM models provide a larger downside risk protection with standard deviations of 1.26% (asymmetric model) and 1.50% (symmetric model), compared to 2.73% for the Markowitz's portfolio mix. The results imply that for the same level of portfolio return, it is possible to further reduce the threshold of the risk frontier by as much as 1.23% by focusing only on the shortfalls below the target rate of returns. The portfolio target semi-deviation is further reduced by 0.24% if the co-LPM matrix is assumed to be asymmetric across its diagonal.

Exhibit 2

Efficient Frontiers of Downside Risk and Mean-Variance Portfolios



In terms of asset weight allocation, there are some interesting observations to be made by comparing the results from different optimization algorithms. Based on selective portfolio returns, the standard deviations and asset weights for the three assets included in the portfolio are summarized in Exhibit 3. It is shown that with the exception of the portfolio composition that yields a low quarterly return of 0.1%, the three asset allocation models—the classical Markowitz's mean-variance, the mean-SCLPM and the mean-LPM (Equation (13))—allocate the same asset weights to the three asset classes. In terms of the minimum downside standard deviations of portfolio returns, the estimated values for the three models with same asset weight allocation vary with the changes in the portfolio returns. The results, however, indicate that the downside return deviations for portfolio optimized by the mean-GCLPM model are the lowest vis-à-vis other models.

For the type of assets included in the risk-minimizing portfolio, real estate appears to be the most important composition in the portfolio when the portfolio return in excess of 1.00% is to be attained (Exhibit 3). It is also interesting to note that except for the asymmetric co-LPM model, all three other models actually drop the most risky asset, *i.e.*, stocks that have a historical average standard deviation of 9.03% (refer to Exhibit 1), from the portfolio. In contrast, the optimization rules proposed by the asymmetric co-LPM (mean-GCLPM) framework favor more stocks when the expected portfolio return increases. For a range of portfolio returns between 0.10% and 1.80%, the weight of stock in the mean-GCLPM portfolio increases from 5.11% to 17.47%. The negative downside covariances between real estate and stock that were asymmetrically different over the sample period are deemed to be the factor that explains the diversification strategy adopted by the mean-GCLPM model.

Effects of investor's downside risk preference (n). The risk aversion of investor increases when n is adjusted from 1 to 3 correspondingly. Exhibit 4

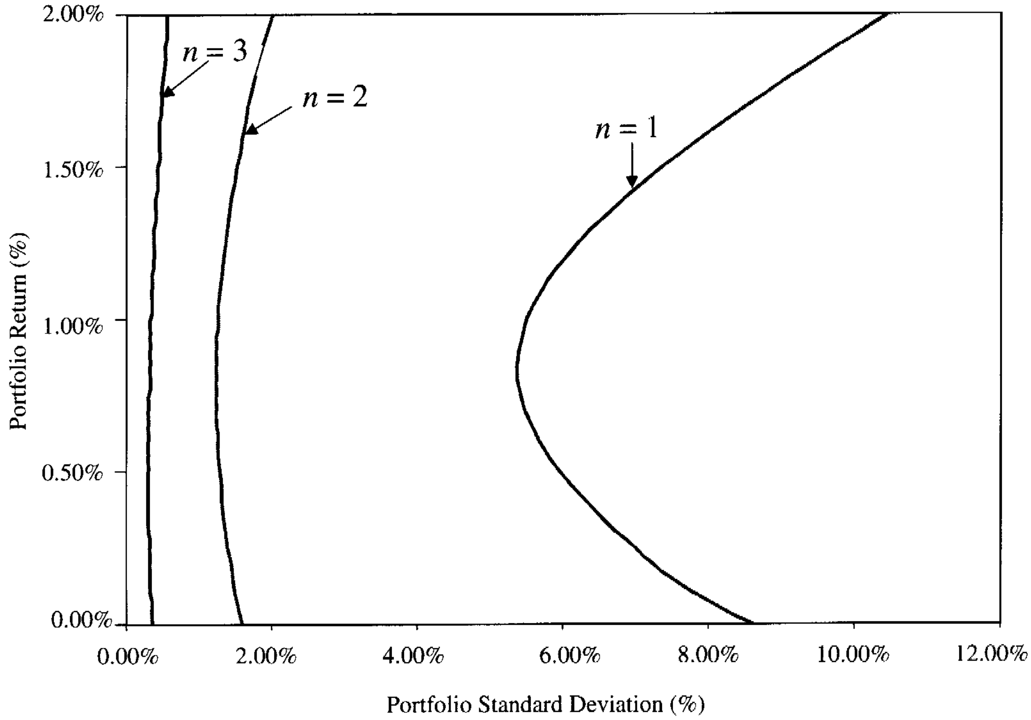
Exhibit 3

Asset Weights for Optimal Portfolios from Different Optimization Models

Optimization Models	Portfolio Return (%)	Portfolio Std. Dev. (%)	Real Estate Weight (%)	Stock Weight (%)	Bond Weight (%)
I	0.10	2.10	5.04	1.76	93.20
II	0.10	1.50	2.97	5.11	91.93
III	0.10	1.54	4.88	2.03	93.09
IV	0.10	1.49	6.14	0.00	93.86
I	0.50	2.15	25.79	0.00	74.21
II	0.50	1.29	20.80	8.01	71.18
III	0.50	1.41	25.79	0.00	74.21
IV	0.50	1.29	25.79	0.00	74.21
I	1.00	2.74	50.36	0.00	49.64
II	1.00	1.26	43.11	11.65	45.24
III	1.00	1.50	50.36	0.00	49.64
IV	1.00	1.44	50.36	0.00	49.64
I	1.50	3.65	74.93	0.00	25.07
II	1.50	1.52	65.42	15.29	19.30
III	1.50	1.84	74.93	0.00	25.07
IV	1.50	1.83	74.93	0.00	25.07
I	1.80	4.28	89.67	0.00	10.33
II	1.80	1.77	78.80	17.47	3.74
III	1.80	2.12	89.67	0.00	10.33
IV	1.80	2.12	89.67	0.00	10.33

Notes:
I = Classical Markowitz Quadratic Programming model.
II = Generalized (Asymmetric) Mean-Co-Lower Partial Moment model.
III = Symmetric Mean-Co-Lower Partial Moment model.
IV = Mean-Lower Partial Moment model by Harlow and Rao (1989) and Harlow (1991) (Equation (13)).

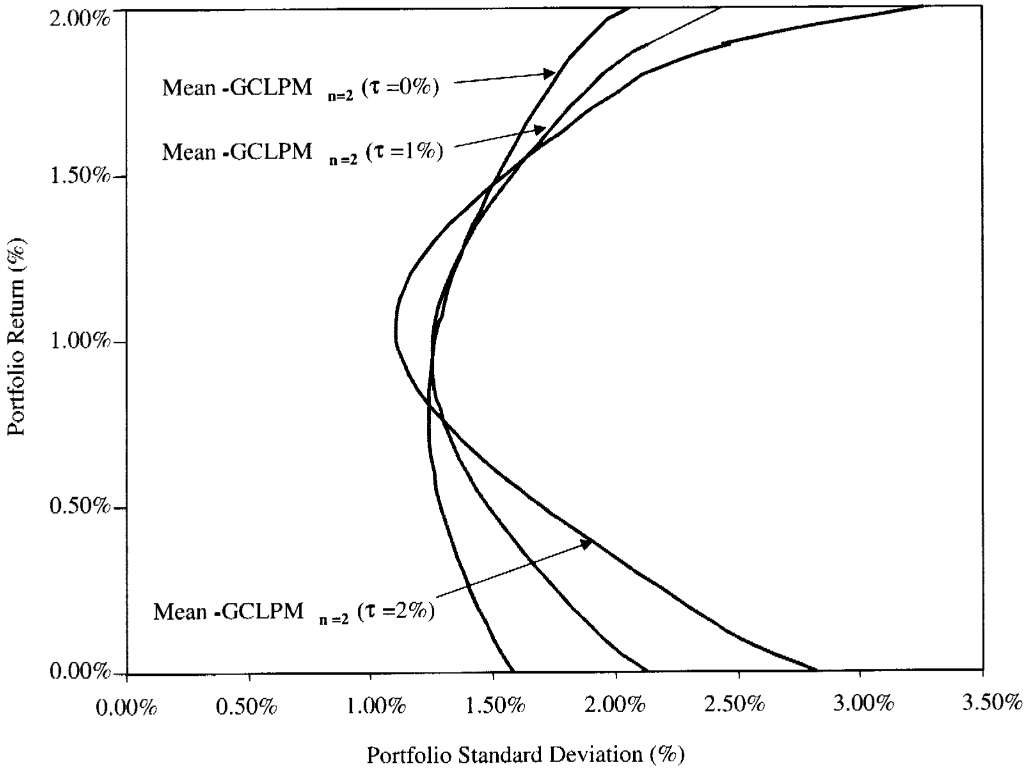
Exhibit 4
Effects of Risk Aversion in a Mean-GCLPM Optimization Framework



shows that the efficient frontiers lie further to the left when the risk aversion of investors declines. In other words, a risk-averse investor who prefers less downside risk would choose the right-most efficient portfolio compared to those on the left. At a given portfolio return of 0.5%, the downside risks on the efficient mean-GCLPM curve are estimated at 5.96%, 1.28% and 0.31% for models with $n = 1$, 2 and 3, respectively. The results are consistent with the expected utility hypothesis that a more risk-averse investor ($n = 3$), prefers less risk than a less risk-averse investor ($n = 1$) for a same level of return. The less risk-averse investor has a higher risk preference threshold in the optimization process.

Effects of target rate of return (τ). For the effects of the target rate of return, we vary τ from 0.0% to 2.0% and plot the efficient convex combinations of the asset portfolio based on the asymmetric CLPM optimization model (Exhibit 5). There is no strict second order stochastic dominance[5] among the three efficient frontiers with different target rates of return (τ). At both extreme ranges of the portfolio returns, approximately for $R_P < 0.85\%$ and $R_P > 1.45\%$, the optimal portfolio of the mean-GCLPM model with $\tau = 0$ dominates those with higher target rates of return ($\tau = 1$ and 2). The downside standard deviations of the mean-$GCLPM_{\tau=0}$ portfolio are the smallest at every level of portfolio return below 0.85% or exceeding 1.45%. However, for the mid-range portfolio returns (R_P) between approximately 0.85% and 1.45%, the mean-$GCLPM_{\tau=2\%}$ efficient portfolio is stochastically dominant over the mean-$GCLPM_{\tau=0\%}$ and the mean-$GCLPM_{\tau=1\%}$ portfolios (*i.e.*, the mean-GCLPM portfolio is preferred for its lowest downside risks at a given range of returns approximately $0.85\% < R_P < 1.45\%$). The change in the target rate of returns also has a significant effect on the convexity of the efficient curves. When τ increases, the spread of the portfolio returns becomes narrower at any given level of portfolio downside risk. From another perspective, the convexity of the lower and upper tails of the curves are stretched rightward along the downside risk axis.

Exhibit 5
Effects of Target Rate of Return in a Mean-GCLPM Optimization Framework



Conclusion

In practice, Markowitz's mean-variance optimization is one of the most commonly used algorithms for estimating optimal portfolio weights. However, built on the strict assumption that asset returns are normally and independently distributed, Markowitz's algorithm is ineffective in optimizing portfolios that comprise assets with skewed returns. The traditional mean-variance model, which treats both the above and the below target returns equally, tends to over-estimate the risks and imposes unnecessary conditions that rule out portfolios that are downside efficient.

Intuitively, investors are more concerned about the probability of investment returns that fall below the target return. Bawa (1975) conceptualized an alternative risk measure in a generalized LPM framework that focuses only on deviations below a pre-specified target rate of return. Harlow and Rao (1989) and Harlow (1989) then incorporated the downside risk concept to a downside risk portfolio optimization model. Their model is designed to directly minimize the target semi-deviation of portfolio returns, as measured by the LPM. However, the model is subject to two limitations. First, the model ignores the downside distributions of individual asset ex-post returns. Second, the model neglects the co-movement between individual asset returns that fall below the target return, which is an important criterion for risk diversification. We extend their model by applying the co-LPM concept (Hogan and Warren, 1974; and Bawa and Lindenberg, 1977), a downside measure of the covariance of asset returns, to the classical Markowitz quadratic algorithm to improve the robustness of the optimization process.

The proposed downside risk versions of Markowitz's optimization model were applied to determine the optimal weights for a three-asset portfolio in Singapore using Microsoft Excel. For a range of efficient portfolio returns between 0.00% and 2.00%, the results show that the portfolio risks estimated in the downside risk models vary by a smaller range of 0.80% for the mean-GCLPM model and 0.92% for the mean-SCLPM vis-à-vis the 2.63% range for the classical mean-variance model. In absolute terms, at a given portfolio return of 1.00% on the efficient frontiers, the downside risks for the asymmetric and the symmetric CLPM models are estimated at 1.26% and 1.50% respectively, compared to 2.74% for the classical Markowitz model. The results imply that a risk-averse investor would always prefer the downside optimal portfolio compared to the mean-variance portfolio because the CLPM models offer a larger buffer below the threshold risk preference for the same level of return.

In terms of asset weight allocation, the optimization rules of the asymmetric CLPM framework favor a higher allocation to stock in comparison with the Markowitz mean-variance and the mean-symmetric CLPM models. It can thus be expected that a higher proportion of stocks that are relatively more risky would be included in a downside efficient portfolio to trade-off for a higher expected return.

The effects of the risk aversion of an investor as captured by the variable n in the CLPM formulae are also examined. The results suggest that a comparatively more risk-averse investor as indicated by a higher n obtains a lower downside risk compared to one with higher risk preference, i.e., for $n = 1$, given that both investors are indifferent to the levels of portfolio return. It is also found that by varying the target rate of return from 0.00% to 2.00%, the convexity of the efficient frontiers is adjusted such that both the lower and upper tails of the curves are stretched rightward along the downside risk axis. However, there is no strict stochastic dominance among the three efficient frontiers with different target rates of return (τ).

Endnotes

1. In a decreasing absolute risk aversion (DARA) function, a risky asset is a normal good that implies that the demand for such risky asset increases with an increase in individual wealth (Arrow, 1970).
2. The Third Order Stochastic Dominance is a complex optimal selection rule that compares the means and the lower partial variance function of the probability distributions of alternative asset returns (refer to Bawa (1975) for detailed derivations of the rules).
3. These programs are available from the authors on request.
4. The URA property price index is computed from information obtained in caveats lodged with the Singapore Land Registry. It is a quarterly index that is derived by dividing the current median price per square meter with the median price in the base year 1990. The composition of the index includes five major types of private properties—residential, office, shop, flatted factory and warehouse. It differs from the Russell-NCREIF Index, the benchmark institutional real estate index in the United States, which is based on the appraised values of the properties held in portfolios of the member firms of the National Council of Real Estate Investment Fiduciaries. Given the size of the fund that includes over 1500 properties with an appraised value of above $22 billion, a policy that staggers the period of re-assessment of the properties was adopted. In contrast, the URA index is updated quarterly based on all the caveats that have been lodged for that particular quarter, which may comprise cases of repeated sale caveats during the quarter.
5. In the absence of information on the entire distribution functions for the three optimal portfolio frontiers, the comparison of their performances under the second order stochastic dominance condition, in our context, is a more restrictive representation based solely on the mean and variance of the portfolio returns that are assumed to have the same distribution functions.

References

Arrow, K., *Essays in the Theory of Risk-Bearing*, North-Holland: Amsterdam, 1970.

Bawa, V. S., Optimal Rules for Ordering Uncertain Prospects, *Journal of Financial Economics*, 1975, 2, 95–121.

Bawa, V. S. and E. B. Linderberg, Capital Market Equilibrium in a Mean-Lower Partial Moment Framework, *Journal of Financial Economics*, 1977, 5, 189–200.

Harlow, W. V., Asset Allocation in a Downside Risk Framework, *Financial Analysts Journal*, 1991, 28–40.

Harlow, W. V. and R. K. S. Rao, Asset Pricing in a Generalized Mean-Lower Partial Moment Framework: Theory and Evidence, *Journal of Financial And Quantitative Analysis*, 1989, 24:3, 285–309.

Hogan, W. W. and J. M. Warren, Computation of the Efficient Boundary in the E-S Portfolio Selection Model, *Journal of Financial and Quantitative Analysis*, 1974, 7, 1881–96.

Kataoka, S., A Stochastic Programming Model, *Econometrica*, 1963, 21, 181–96.

Mao, J. C. T., Survey of Capital Budgeting: Theory and Practice, *Journal of Finance*, 1970, 25, 349–60.

Markowitz, H., *Portfolio Selection*, John Wiley & Sons: New York, NY, 1959.

Nantell, T. J. and B. Price, An Analytical Comparison of Variance and Semivariance Capital Market Theories, *Journal of Financial and Quantitative Analysis*, 1979, 14:2, 221–42.

Nawrocki, D. N., Optimal Algorithms and Lower Partial Moment: Ex-post Results, *Applied Economics*, 1991, 23, 465–70.

Roy, A. D., Safety First and the Holding of Assets, *Econometrica*, 1952, 20, 431–39.

Telser, L. G., Safety First and Hedging, *Review of Economic Studies*, 1956, 23, 1–16.

Tse, M. K. S., J. Uppal and M. A. White, Downside Risk and Investment Choice, *The Financial Review*, 1993, 28:4, 585–605.

The authors thank Gerald Brown, Kanak Patel, David Ho Kim Hin and Chua Chong Jin and anonymous referee for their comments.