

RELATIVE ROBUST PORTFOLIO OPTIMIZATION

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Abstract. Considering mean-variance portfolio problems with uncertain model parameters, we contrast the classical absolute robust optimization approach with the relative robust approach based on a maximum regret function. Although the latter problems are NP-hard in general, we show that tractable inner and outer approximations exist in several cases that are of central interest in asset management.

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1. Introduction. We consider decision-making tools for problems that are defined by uncertain or unknown parameters. Uncertainty and the risk of undesirable outcomes are inevitable features of most productive activities. From engineers to economists, from health care providers to investment managers, many professionals must take decisions under considerable uncertainty on a daily basis. Our objective is to improve the quality of decisions made in these environments by understanding, modeling, quantifying and managing uncertainty. To achieve this goal we propose a robust optimization modeling methodology based on extending the idea of relative robust optimization of Kouvelis and Yu [13] in the context of discrete optimization.

Many decision problems with uncertainty can be formulated as optimization problems. In recent years, robust optimization (RO) has emerged as a powerful tool for managing uncertainty in such optimization problems [4, 5]. An excellent overview can be gained from the recent survey paper [2]. Robust optimization is a generic term that is used to describe a class of modeling strategies as well as solution methods for optimization problems that are defined by uncertain inputs. Decisions made with incomplete information may result in undesirable outcomes when the realized values of the uncertain inputs are unfavorable. Robust optimization models and algorithms aim to mitigate the effects of uncertainty and obtain a solution that is guaranteed to perform reasonably well for all, or at least most, possible realizations of the uncertain input parameters.

There is growing evidence, both empirical and theoretical, that robust-optimized solutions have better characteristics than their non-robust counterparts. For example, a theoretical study by Schöttle and Werner demonstrates that the map between the model parameters of an optimization problem and its set of optimal solutions can become much smoother if one uses robust optimization with ellipsoidal uncertainty sets [19], [20]. In a portfolio optimization setting with uncertain expected return estimates, Ceria and Stubbs report simulated results where the ex-post performance of

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robust-optimal portfolios outperform those of standard mean-variance optimal portfolios with high frequency [6].

While the stated goals of robust optimization are intuitive, it is not always clear what metrics one should use to achieve these goals. Most robust optimization approaches discussed in the existing literature use the “worst-case objective value” as the comparison metric among alternative sets of decisions. Despite the advantages we mentioned in the previous paragraph, the focus on the worst-case objective value in robust optimization is a source of frequent criticism. Modelers, often with good reason, worry that the extreme scenarios in the uncertainty set have an undue influence on the final decisions in such robust formulations. The “worst-case objective value” is an *absolute* metric. While there are many situations where it is the appropriate metric for evaluating robustness, it is inadequate for measuring robustness in a *relative* sense.

A typical example arises in the investment management context where managers are frequently evaluated and compensated based on their performance relative to the competition. For robust decision-making in an uncertain decision environment, rather than protecting themselves against worst-case scenarios, investment managers may thus prefer to choose decisions that avoid falling severely behind their competitors under a range of scenarios. This view of robustness was formalized by Kouvelis and Yu [13]. For each choice of the decision variables and each scenario one compares the attained objective value with the optimal objective value attainable under the model parameter values described by the scenario. The difference between these two values, or alternatively their ratio, can be seen as measures of *regret* based on hindsight after the true values of the uncertain parameters are revealed. With the objective of limiting such regret measures Kouvelis and Yu arrive at the *robust deviation* and *relative robust* decision problem formulations. We will provide these formulations in Section 2 as they form the focus of our study. We will refer to the robust deviation and relative robust decision problems collectively as relative robust problems.

While Kouvelis and Yu explore robust deviation and relative robust decisions in several classes of discrete optimization problems, similar studies for continuous optimization problems are mostly missing in the literature. A rare exception is Taguchi’s master’s thesis [25] and the subsequent paper [26]. As these authors also observed, relative robust formulations are typically more difficult than the corresponding absolute robust formulations. Since they involve the optimal value function whose argument is the vector of uncertain parameters inside a min-max optimization problem, relative robust problems are three-level optimization problems. This is in contrast to the two-level absolute robust formulations. Since the optimal value function is rarely available in closed form, tractability is an important concern for these models. Our study shows that for many uncertainty structures on quadratic programming and other optimization problems, the resulting relative robust formulation can be reduced to one or a series of single-level deterministic optimization problems that can be solved using conic optimization methods.

The simplest uncertainty sets are finite sets, corresponding to the intuitive notion of a collection of scenarios. Both the absolute and relative robust formulations with finite uncertainty sets are relatively easy as they can be solved as a finite sequence of deterministic problems. Using simple convexity arguments, we show that robust problems with polytopic uncertainty structures (uncertainty sets defined as convex hull of a finite number of points) can be reduced to the finite case and are therefore of the same complexity. Relative robust optimization problems with polytopic uncer-

tainty sets were also considered by Taguchi et. al. [26], but our discussion is based on the second author’s MSc thesis [14] which predated the work of the aforementioned authors and formed the basis of the original draft of this paper. Gregory et. al. [10] also investigated the polyhedral case.

In the second part of the paper we move beyond polytopic uncertainty sets and consider relative robust models with ellipsoidal uncertainty. In the application on which we concentrate, mean-variance portfolio optimization, we assume ellipsoidal uncertainty only in the vector of expected returns and assume that the covariance matrix is fixed. This assumption is justified, as practitioners typically use matrix shrinkage and factor models to ensure that the covariance is estimated robustly. Ellipsoidal uncertainty sets for the vector of expected returns appear quite naturally as confidence regions in their statistical estimation. Taguchi et. al. [26] propose to approximate ellipsoidal uncertainty sets by a polytope obtained as the convex hull of a random sample of points from the ellipsoid. The resulting relative robust problems are relaxations of the original relative robust problem with ellipsoidal uncertainty, and since these relaxations are based on polyhedral uncertainty, they can be solved via the approach discussed earlier. Adom [1] investigated similar relaxations but based on polyhedra generated by pseudo-randomly chosen points, which leads to faster convergence.

Our approach to relative robustness under ellipsoidal uncertainty is very different. By developing inner approximations to the relative robust problem in the form of a symmetric cone programming problem, we obtain solutions that are guaranteed to be feasible. In several cases of interest our inner approximations are provably tight, as we show using a theory developed by Sturm and Zhang [24]. Thus, while our models approximate the feasible set of the relative robust problem from the inside, Taguchi et. al.’s approach [26] approximates it from the outside, which yield solutions that are not guaranteed to be feasible but give a bound on the optimal objective value. The two approaches can be combined to obtain approximation guarantees for both – a distance to optimality in the case of our model, and a distance to feasibility in the case of the model of Taguchi et. al..

Since we have in mind the mean-variance portfolio optimization framework of Markowitz [15] as a particular application of the methods investigated in this paper, most of the notation we use will be inspired by this framework which is explained in some detail in Section 3. Much of the other notation is of standard use in the optimization literature, such as \succeq and \succ to denote positive semidefiniteness and definiteness of a matrix, for example, $\text{cone}(\cdot)$, $\text{conv}(\cdot)$ and $\text{aff}(\cdot)$ for conic, convex and affine hulls of a subset of a vector space respectively, \bullet for the trace inner-product of two matrices of equal size, \cdot^* for duals of functionals and cones, e and I for the vector of all ones and the identity matrix of appropriate size, and \cdot^T for the transpose of a matrix.

2. Absolute Robust versus Relative Robust Optimization. We consider a generic optimization problem whose input parameters are denoted by the vector p .

$$\begin{aligned} \max_{x \in \mathbb{R}^n} f(x, p) \\ \text{s.t. } x \in \mathcal{X}_p, \end{aligned} \tag{2.1}$$

where $f(x, p)$ and \mathcal{X}_p represent the objective function and the feasible set of the problem. For a given p , let $z^*(p)$ and $\Omega^*(p)$ denote, respectively, the optimal value

and the set of optimal solutions of the above given problem, provided that they exist. We will also use the notation $x^*(p)$ to denote a generic element of $\Omega^*(p)$.

2.1. The Absolute Robust Optimization Framework. When p is known (2.1) is a standard optimization problem. Robust optimization (RO) is concerned with the case where p is not known with certainty. In recent years, RO has emerged as an alternative to traditional approaches to optimization under uncertainty such as sensitivity analysis and stochastic programming. As mentioned in the introduction, its primary objective is to find solutions that will have a good performance under a variety of scenarios for the uncertain input parameters. RO models are especially well-suited in situations where there are constraints with uncertain parameters that must be satisfied regardless of the values of these parameters or when the optimal solutions are particularly sensitive to perturbations. Additionally, RO is an attractive modeling option when the decision-maker cannot afford low-probability high-magnitude risks.

One of the essential elements of a RO model is the *uncertainty set*. The uncertainty set, say \mathcal{U} , represents the set of possible scenarios/realizations for the parameters p . When p is uncertain and must be estimated, uncertainty sets can represent or be formed by difference of opinions, alternative estimates, confidence regions of statistical estimators, or based on Bayesian or Kalman filtering methods for tracking the evolution of an assumed probability distribution for p . While the current literature does not provide clear guidelines on their construction, their shape often reflects the sources of uncertainty while their size depends on the desired level of robustness. Common types of uncertainty sets include: (i) $\mathcal{U} = \{p_1, p_2, \dots, p_k\}$ (a finite set of scenarios), (ii) $\mathcal{U} = \text{conv}(p_1, p_2, \dots, p_k)$ (a polytopic set), (iii) $\mathcal{U} = \{p : l \leq p \leq u\}$ (intervals), and (iv) $\mathcal{U} = \{p : p = p_0 + Mu, \|u\| \leq 1\}$ (an ellipsoidal set).

RO formulations optimize some variation of a worst-case performance metric, where the “worst-case” is computed over the uncertainty set. In most cases [6, 8, 9, 11], the objective is to optimize the worst-case realization of the objective function. For the optimization problem (2.1), this leads to the following formulation:

$$\max_{x \in \bigcap_{p \in \mathcal{U}} \mathcal{X}_p} \left(\min_{p \in \mathcal{U}} f(x, p) \right). \quad (2.2)$$

Kouvelis and Yu [13] classify (2.2) as the *absolute robust decision problem*. This name reflects the fact that the worst-case objective value is an absolute metric. One potential consequence of this emphasis on the worst-case is that the decisions are disproportionately affected by extreme scenarios in the uncertainty set. As this is not always desirable, Bertsimas and Sim [3] study this cost of robustness as a function of the level of conservatism. An alternative we consider in this paper is to seek robustness in a relative sense.

2.2. The Relative Robust Optimization Framework. For this purpose, we consider a *regret function* that measures the difference between the performance of the solution with and without the benefit of hindsight. If we choose x as decision vector when p is the vector of realized parameter values, then the *regret* associated with having chosen x rather than $x^*(p)$ as decision vector is defined as follows,

$$r(x, p) := z^*(p) - f(x, p) = f(x^*(p), p) - f(x, p). \quad (2.3)$$

Note that since $x^*(p)$ is an optimal decision vector for the parameter values p , the regret $r(x, p)$ is always nonnegative.

The regret function is not useful at the decision-making stage since we cannot measure the regret before we observe the realized value of the parameters. Furthermore, in many a context p cannot be observed even *after* its realization. For example, financial data typically yield a single sample of a random return vector R while the parameters $p = (\mathbb{E}[R], \text{Cov}(R))$ that serve as input parameters to the optimal investment problem are neither directly observable nor inferable from this single sample. For this reason we consider the maximum regret function instead, which provides an upper bound on the true regret,

$$R(x) := \max_{p \in \mathcal{U}} r(x, p) = \max_{p \in \mathcal{U}} (z^*(p) - f(x, p)). \quad (2.4)$$

If the function $z^*(p)$ is positive everywhere, one can also consider a scaled version of the regret function,

$$\tilde{r}(x, p) = \frac{z^*(p) - f(x, p)}{z^*(p)}, \quad (2.5)$$

$$\tilde{R}(x) = \max_{p \in \mathcal{U}} \tilde{r}(x, p) = \max_{p \in \mathcal{U}} \frac{z^*(p) - f(x, p)}{z^*(p)}. \quad (2.6)$$

In Kouvelis and Yu [13], vectors x that minimize the maximum regret functions $R(x)$ and $\tilde{R}(x)$ are called *robust deviation decisions* and *relative robust decisions* respectively. We collectively refer to problems seeking such decisions as relative robust problems and focus on the function $R(x)$ for most of the rest of our discussion.

Let us consider the simpler case where the uncertain parameters are only in the objective function and the feasible set $\mathcal{X}_p \equiv \mathcal{X}$ is independent of p . In most models, the dependence of the objective function on the uncertain parameters is linear. When this is the case, it is easy to see that the optimal value function $z^*(p)$ is a convex function. In fact, it is sufficient that f be convex in p to guarantee the convexity of $z^*(p)$:

LEMMA 2.1. *Let \mathcal{U} be a convex set. For all $p \in \mathcal{U}$, define*

$$z^*(p) = \sup_{x \in \mathcal{X}} f(x, p)$$

where f is convex in p . Then z^* is a convex function on \mathcal{U} .

Lemma 2.1 is part of the folklore on convex analysis. For the sake of completeness we include a proof in Appendix A. As we will see in the next section, this simple convexity result is responsible for reducing relative robust optimization models with polytopic uncertainty sets to problems with finite uncertainty sets.

3. Application to Mean-Variance Portfolio Optimization. Portfolio theory deals with the problem of deciding what proportion of investable wealth to allocate to each of several risky investment opportunities so as to achieve a chosen goal, which is usually to maximize the expected return while limiting risk. Under the mean-variance optimization (MVO) approach of Markowitz [15], all n investments are assumed to be held during the same fixed investment period over which they generate random returns R_i . Assembled in a random vector R , the expectation $\mu = \mathbb{E}[R]$ and the positive definite covariance matrix $Q = \text{Cov}(R)$ of the asset returns serve as input parameters to one of the quadratic programming problems (3.1)–(3.3) described

below. Their solutions yield optimal portfolio weights. Though these problems are convex and computationally tractable and can therefore be solved to global optimality, MVO models can produce portfolios that are highly sensitive to the values of the model parameters (μ, Q) and show unsatisfactory diversification. Since (μ, Q) have to be estimated statistically, output sensitivity to these parameters is an important practical issue. Robust optimization models have therefore emerged as favourable alternatives to plain MVO models [9, 28, 6].

3.1. Classical Mean-Variance Portfolio Models. Although the mathematical methods investigated in this paper are applicable more widely, the MVO framework constitutes their main motivation and application. We shall therefore briefly describe some of the models that arise in this context. Let x_i be the proportion of wealth invested in the i -th investment opportunity (or asset), and let these weights be collected in a vector x of size n . The portfolio corresponding to the weights x then has the overall return $R^T x$ with expectation $\mu^T x$ and variance $x^T Q x$. Apart from the *budget constraint* $e^T x = 1$ (where $e := [1 \dots 1]^T$), fund managers usually restrict the set \mathcal{X} of *feasible* portfolios (investment decisions they are willing to consider) by introducing further constraints that impose limits on short-selling, diversification, rebalancing costs and other criteria. Typically, all constraints are linear, leading to a polyhedral feasible set $\mathcal{X} = \{x \in \mathbb{R}^n : Fx = f, Gx \leq g\}$. Here we make the minimal assumption that \mathcal{X} be a convex tractable set, that is, a set for which it can be decided in polynomial time whether or not a given point is a member.

3.1.1. Convex MVO Models. Taking the variance of the portfolio return as a risk measure, MVO formulations are obtained by either choosing to minimize the variance subject to a lower bound target return ρ , to maximize the return subject to an upper bound target risk σ^2 or to maximize the risk-adjusted expected return $\mu^T x - \lambda x^T Q x$ defined by a specific choice of a risk-aversion parameter $\lambda > 0$,

$$\begin{aligned} \min_{x \in \mathbb{R}^n} f_Q(x) &:= x^T Q x & (3.1) \\ \text{s.t. } \mu^T x &\geq \rho, \\ x &\in \mathcal{X}, \end{aligned}$$

$$\begin{aligned} \max_{x \in \mathbb{R}^n} f_\mu(x) &:= \mu^T x & (3.2) \\ \text{s.t. } x^T Q x &\leq \sigma^2, \\ x &\in \mathcal{X}, \end{aligned}$$

$$\begin{aligned} \max_{x \in \mathbb{R}^n} f_{\mu, Q}(x) &:= \mu^T x - \lambda x^T Q x & (3.3) \\ \text{s.t. } x &\in \mathcal{X}. \end{aligned}$$

It is well-known and can easily be established using KKT conditions that the three formulations presented above are equivalent in the sense that they produce identical solutions for appropriately chosen values of ρ , σ^2 and λ . For example, there exists a function $\sigma^2(\mu, Q, \rho)$ such that the solutions of (3.1) and (3.2) coincide when $\sigma^2 = \sigma^2(\mu, Q, \rho)$. We note however that this functional dependence also depends on the model parameters. A portfolio x is said to be *efficient* if it optimizes (3.1)–(3.3) for

some choice of ρ , σ^2 and λ respectively. Because of the above-mentioned equivalence, it does not matter which problem we use for this definition. The set $\{((x^T Q x)^{1/2}, \mu^T x) : x \text{ efficient}\}$ is the efficient frontier.

3.1.2. Sharpe Ratio Maximization. Another variant of MVO is the Sharpe Ratio maximization problem. Let r be the return of a risk-free investment held over the same period as the above considered assets, e.g., a short-term government bond or the money market. If this risk-free asset is included among the considered assets – we call it asset 0 and refer to it as cash – and if the feasible set \mathcal{X}_0 of portfolios containing a cash position is of affine form,

$$\mathcal{X}_0 = \text{aff} \left(\left\{ \begin{bmatrix} 0 \\ x \end{bmatrix} : x \in \mathcal{X} \right\} \cup \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \right), \quad (3.4)$$

where $\text{aff}(\cdot)$ denotes the affine hull of a set and \mathcal{X} is the set of feasible portfolios containing only positions in the risky assets, then cash can freely be borrowed to invest in the risky assets. In this case the efficient frontier is a straight line going through the point $(0, r)$ and with a gradient given by

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \frac{\mu^T x - r}{\sqrt{x^T Q x}} \\ \text{s.t. } x \in \mathcal{X}. \end{aligned} \quad (3.5)$$

The *Sharpe ratio* [22] of a portfolio x is defined as the ratio of the excess expected return of the portfolio over the risk-free asset and the standard deviation of the portfolio return. Correspondingly, Model (3.5) is called the *Maximum Sharpe Ratio problem* (MSR). In the case where this problem has a unique optimal solution, this solution is called the *market portfolio*. It can be easily seen that any efficient portfolio is then an affine combination of cash and the market portfolio. This observation forms the basis of Sharpe’s capital asset pricing theory [21].

While (3.5) is a nonlinear and nonconvex problem, it can be solved by the tractable convex programming problem

$$\begin{aligned} \max_{y \in \mathbb{R}^n} g(y) &= (\mu - re)^T y \\ \text{s.t. } y &\in \mathbb{R}_+ \mathcal{X}, \\ y^T Q y &\leq 1. \end{aligned} \quad (3.6)$$

See Appendix B for detailed explanations, and also [9] and [28] for similar techniques.

3.2. Absolute Robust Portfolio Models. Motivated by the sensitivity of the solutions of Problems (3.1), (3.2), (3.3) and (3.5) as functions of the model parameters (μ, Q) , we next consider robust counterparts of these models. Depending on how the uncertainty set \mathcal{U} for the model parameters (μ, Q) is chosen and which of the models one chooses to robustify, one arrives at different robust formulations. We note that the ensuing models are no longer all equivalent in the sense in which the nonrobust versions (3.1)–(3.3) were. The main reason for this difference is that the feasible sets of Problems (3.1) and (3.2) depend on the model parameters (μ, Q) while those of Problems (3.3) and (3.5) do not, with the consequence that in the context of the former two problems joint uncertainty structures in μ and Q cannot be exploited, while in the context of the latter two they can, at least conceptually.

To be more specific, let us write $\mathcal{U}_\mu := \{\mu : \exists (\mu, Q) \in \mathcal{U}\}$ for the projection of \mathcal{U} onto the μ -component, and $\mathcal{U}_Q := \{Q : \exists (\mu, Q) \in \mathcal{U}\}$ for its projection onto the Q -component, and note that when μ and Q have joint uncertainty structure, then \mathcal{U} is usually strictly contained in the Cartesian product $\mathcal{U}_\mu \times \mathcal{U}_Q$. Yet the absolute robust counterpart of (3.1) has the following equivalent formulations,

$$\begin{aligned} & \min_{\{x \in \mathcal{X} : \mu^\top x \geq \rho \forall (\mu, Q) \in \mathcal{U}\}} \left(\max_{(\mu, Q) \in \mathcal{U}} x^\top Q x \right) \\ \Leftrightarrow & \min_{\{x \in \mathcal{X} : \min_{\mu \in \mathcal{U}_\mu} \mu^\top x \geq \rho\}} \left(\max_{Q \in \mathcal{U}_Q} x^\top Q x \right), \quad (3.7) \\ \Leftrightarrow & \min_{\{x \in \mathcal{X} : \mu^\top x \geq \rho \forall (\mu, Q) \in \mathcal{U}_\mu \times \mathcal{U}_Q\}} \left(\max_{(\mu, Q) \in \mathcal{U}_\mu \times \mathcal{U}_Q} x^\top Q x \right). \end{aligned}$$

Thus, the joint uncertainty structure of μ and Q cannot be exploited in the context of the robust problem (3.7), and neither can it be in the context of the absolute robust counterpart of (3.2),

$$\max_{\{x \in \mathcal{X} : \max_{Q \in \mathcal{U}_Q} x^\top Q x \leq \sigma^2\}} \left(\min_{\mu \in \mathcal{U}_\mu} \mu^\top x \right). \quad (3.8)$$

In contrast, joint uncertainty in μ and Q can be exploited, at least conceptually, in the framework of the absolute robust counterpart of (3.3),

$$\max_{x \in \mathcal{X}} \left(\min_{(\mu, Q) \in \mathcal{U}} \mu^\top x - \lambda x^\top Q x \right), \quad (3.9)$$

as well as in the absolute robust counterpart of (3.5),

$$\max_{x \in \mathcal{X}} \left(\min_{(\mu, Q) \in \mathcal{U}} \frac{\mu^\top x - r}{\sqrt{x^\top Q x}} \right). \quad (3.10)$$

Special cases of the above-described models appear in the literature as follows: Goldfarb and Iyengar [9] discussed the problems (3.7), (3.8) in the case where \mathcal{U} is an uncertainty set of Cartesian type $\mathcal{U} = \mathcal{U}_\mu \times \mathcal{U}_Q$ corresponding to a confidence region for the statistical estimators arising in the context of the load-factor model for (μ, Q) . Halldórsson and Tütüncü [11] discussed the problem (3.9) in the case where $\mathcal{U} = \mathcal{U}_\mu \times \mathcal{U}_Q$, and where \mathcal{U}_μ is a box of confidence intervals for the individual components of μ and \mathcal{U}_Q is a box of confidence intervals intersected with the cone of positive semidefinite symmetric matrices.

Note that the absolute robust problems (3.1)–(3.3) are all two-level optimization problems and thus a priori harder to solve than the classical nonrobust models (3.1)–(3.3). However, when \mathcal{U} is tractable, then the robust problems are tractable too. See the above cited papers and the other literature on robust optimization for details.

3.3. Relative Robust Portfolio Models. To contrast the relative robust framework with the absolute robust setting, we next formulate relative robust counterparts of problems (3.1)–(3.3).

Let us first consider problem (3.1), which has two meaningful relative robust analogues: In the first version,

$$z^*(\mu, Q) := \min_{\{y \in \mathcal{X} : \mu^\top y \geq \rho\}} y^\top Q y,$$

is defined as the maximum objective value achievable by an *omniscient* adversary (one who knows the parameters (μ, Q) with certitude), leading to the maximum regret

$$R(x) := \max_{(\mu, Q) \in \mathcal{W}} (x^\top Q x - \min_{\{y \in \mathcal{X} : \mu^\top y \geq \rho\}} y^\top Q y)$$

and the relative robust problem

$$\min_{\{x \in \mathcal{X} : \min_{\mu \in \mathcal{U}_\mu} \mu^\top x \geq \rho\}} \left(\max_{(\mu, Q) \in \mathcal{W}} (x^\top Q x - \min_{\{y \in \mathcal{X} : \mu^\top y \geq \rho\}} y^\top Q y) \right). \quad (3.11)$$

In the second version regrets are computed merely relative to the solution of a *fortuitous* adversary who is bound to choosing a portfolio that is feasible for all parameter values in \mathcal{W} but happens to choose the one that is optimal among these for the true parameter values. Thus, one would have to define

$$z^*(\mu, Q) := \min_{\{y \in \mathcal{X} : \min_{\nu \in \mathcal{U}_\mu} \nu^\top y \geq \rho\}} y^\top Q y,$$

$$R(x) := \max_{Q \in \mathcal{U}_Q} (x^\top Q x - \min_{\{y \in \mathcal{X} : \min_{\nu \in \mathcal{U}_\mu} \nu^\top y \geq \rho\}} y^\top Q y),$$

which leads to the relative robust problem

$$\min_{\{x \in \mathcal{X} : \min_{\mu \in \mathcal{U}_\mu} \mu^\top x \geq \rho\}} \left(\max_{Q \in \mathcal{U}_Q} (x^\top Q x - \min_{\{y \in \mathcal{X} : \min_{\nu \in \mathcal{U}_\mu} \nu^\top y \geq \rho\}} y^\top Q y) \right). \quad (3.12)$$

Note that, similarly to what we observed in the context of Problem (3.7), the dependence of the feasible set of Problem (3.1) on the model parameter μ introduces limitations on the exploitation of joint uncertainty in μ and Q . The emergence of two conceptually different relative robust analogues is also due to this dependence.

In complete similarity, Problem (3.2) has the following two relative robust analogues with similar limitations on the exploitation of structured uncertainty sets,

$$\min_{\{x \in \mathcal{X} : \max_{C \in \mathcal{U}_Q} x^\top C x \leq \sigma^2\}} \left(\max_{(\mu, Q) \in \mathcal{W}} \left(\max_{\{y \in \mathcal{X} : y^\top Q y \geq \sigma^2\}} \mu^\top (y - x) \right) \right), \quad (3.13)$$

$$\min_{\{x \in \mathcal{X} : \max_{C \in \mathcal{U}_Q} x^\top C x \leq \sigma^2\}} \left(\max_{\mu \in \mathcal{U}_\mu} \left(\max_{\{y \in \mathcal{X} : \max_{Q \in \mathcal{U}_Q} y^\top Q y \geq \sigma^2\}} \mu^\top (y - x) \right) \right). \quad (3.14)$$

Let us now turn our attention to the relative robust counterparts of (3.3) and (3.5). In these cases, the feasible set is independent of the model parameters, and for any given uncertainty structure there exists only one relative robust counterpart model.

3.3.1. The Relative Robust Counterpart of Problem (3.3).

We define

$$z^*(\mu, Q) := \max_{y \in \mathcal{X}} (\mu^\top y - \lambda y^\top Q y),$$

$$R(x) := \max_{(\mu, Q) \in \mathcal{W}} \left(\max_{y \in \mathcal{X}} (\mu^\top y - \lambda y^\top Q y) - (\mu^\top x - x^\top Q x) \right).$$

Problem (3.3) then has the following relative robust counterpart,

$$\min_{x \in \mathcal{X}} \left(\max_{(\mu, Q) \in \mathcal{W}} \left(\max_{y \in \mathcal{X}} (\mu^\top y - \lambda y^\top Q y) - (\mu^\top x - x^\top Q x) \right) \right). \quad (3.15)$$

Introducing an artificial variable γ that expresses an upper bound on the regret, we can equivalently reformulate this problem as follows,

$$\begin{aligned} \min_{(x,\gamma) \in \mathbb{R}^{n+1}} \quad & \gamma & (3.16) \\ \text{s.t.} \quad & x \in \mathcal{X}, \\ & \gamma \geq z^*(\mu, Q) - \mu^\top x + x^\top Q x, \quad \forall (\mu, Q) \in \mathcal{U}. \end{aligned}$$

Alternatively, using the definition of $z^*(\mu, Q)$, we obtain another equivalent formulation,

$$\begin{aligned} \min_{(x,y,\gamma) \in \mathbb{R}^{2n+1}} \quad & \gamma & (3.17) \\ \text{s.t.} \quad & x \in \mathcal{X}, \\ & \gamma \geq \mu^\top y - y^\top Q y - \mu^\top x + x^\top Q x, \quad \forall (\mu, Q) \in \mathcal{U}, y \in \mathcal{X}. \end{aligned}$$

We will use the formulation (3.16) in Section 4, while the formulation (3.17) will be preferable in Section 5.

3.3.2. The Relative Robust Counterpart of Problem (3.5). We now describe the relative robust maximum Sharpe-ratio problem and reformulate it using the convexification approach described in Appendix B.

We will use the notation introduced in Section 3.1.2 and Appendix B, and we assume that the feasible set \mathcal{X}_0 of portfolios containing a cash position takes the affine form (3.4).

For any given $(\mu, Q) \in \mathcal{U}$ the maximum Sharpe Ratio achievable in \mathcal{X} is given by

$$z^*(\mu, Q) := \max_{x \in \mathcal{X}} \frac{(\mu - re)^\top x}{\sqrt{x^\top Q x}} \quad (3.18)$$

and can be computed by solving a convex problem of the form (3.6).

Furthermore, introducing an artificial variable γ , the relative robust counterpart of Problem (3.5) can be formulated as follows,

$$\begin{aligned} \min_{\gamma \in \mathbb{R}, x \in \mathcal{X}} \quad & \gamma \\ \text{s.t.} \quad & \frac{(\mu - re)^\top x}{\sqrt{x^\top Q x}} \geq z^*(\mu, Q) - \gamma, \quad \forall (\mu, Q) \in \mathcal{U}. \end{aligned}$$

We see that, in effect, this model corresponds to comparing the Sharpe ratio achieved by the optimal decisions x^* with the Sharpe Ratio achieved by an omniscient adversary. Although the absolute robust counterpart problem is tractable when the uncertainty set is of cartesian form $\mathcal{U} = \mathcal{U}_\mu \times \mathcal{U}_Q$, the relative robust counterpart problem is not. We therefore restrict ourselves to the case where only μ is uncertain, and Q is known with certainty, that is, $\mathcal{U} = \mathcal{U}_\mu \times \{Q\}$. This is a realistic assumption, as in practical applications it is typically more interesting to model uncertainty in μ only and guard against uncertainty in Q via matrix shrinkage techniques.

Assuming an uncertainty structure of the form $\mathcal{U} = \mathcal{U}_\mu \times \{Q\}$ and using the convexification approach described in Appendix B, the above relative robust model is

equivalent to

$$\begin{aligned}
& \min_{\gamma \in \mathbb{R}, y \in \mathbb{R}^n} \gamma & (3.19) \\
& \text{s.t. } (\mu - re)^T y \geq z^*(\mu) - \gamma, \quad \forall \mu \in \mathcal{U}_\mu, \\
& \quad y \in \mathbb{R}_+ \mathcal{X}, \\
& \quad y^T Q y \leq 1.
\end{aligned}$$

Note that since Q is certain, z^* can be considered to be a function of μ only. We also remark that if we had considered the relative robust counterpart of Problem (3.6), we would have arrived at the same model under the chosen uncertainty structure.

3.3.3. Complexity of Relative Robust Optimization. All relative robust problems introduced above are three-level optimization problems. In contrast to the two-level absolute robust models (3.7)–(3.9), relative robust problems are generally intractable, even for tractable uncertainty sets \mathcal{U} . This is further illustrated in Section 5 in the case where \mathcal{U}_μ is an ellipsoid and \mathcal{U}_Q a singleton. The best we can hope to achieve in this case is to identify tractable approximations. Most of Section 5 is therefore spent on deriving good polynomial-time solvable inner approximations to this problem. Outer approximations – that is, relaxations – that rely on the tractability results for polytopic uncertainty sets derived in Section 4 were discussed by Taguchi et. al. [26] and Adom [1].

4. Finite and Polytopic Uncertainty Sets. In this section we will show that if the uncertainty set \mathcal{U} is chosen as a polytope – that is, the convex hull of k points – or a set of k points, then the relative robust optimization problems (3.11), (3.12), (3.13), (3.14), (3.15) and (3.19) are polynomial-time solvable as a function of k and the problem dimension. The complexity is also polynomial in the logarithm of a condition number [7], as any conic programming problem, but we will not discuss details here.

4.1. Solving Problem (3.15). We start by considering Problem (3.15) in the form (3.16) and by assuming that the uncertainty set

$$\mathcal{U} = \{(\mu^{[i]}, Q^{[i]}) : (i = 1, \dots, k)\}$$

consists of finitely many scenarios. For any given μ and positive definite Q , $z^*(\mu, Q)$ is easily computed by solving a convex quadratic optimization problem. Therefore, each instance of the last inequality in the formulation (3.16) is a convex quadratic constraint that can be efficiently handled using, for example, conic optimization methods. Model (3.16) can thus be rewritten by enumerating the possibilities,

$$\begin{aligned}
& \min_{x, \gamma} \gamma & (4.1) \\
& \quad x \in \mathcal{X}, \\
& \quad \mu^{[i]T} x - \lambda x^T Q^{[i]} x \geq z^*(\mu^{[i]}, Q^{[i]}) - \gamma, \quad (i = 1, \dots, k)
\end{aligned}$$

This shows that the relative robust problem (3.15) can be solved by first obtaining the optimal values $z^*(\mu^{[i]}, Q^{[i]})$ and then solving problem (4.1) as a second-order cone programming problem (SOCP) with k convex quadratic constraints. While this process may be tedious and time consuming, the resulting formulation is a single level

deterministic optimization problem that can be solved efficiently for realistic problem sizes both in terms of the dimension and the number of parameter scenarios.

Next, we consider polytopic uncertainty sets, namely those defined as the convex hull of a finite number of extreme scenarios,

$$\mathcal{U} = \text{conv}(\{(\mu^{[i]}, Q^{[i]}) : i = 1, \dots, k\}). \quad (4.2)$$

Using Lemma 2.1 we can now observe that the relative robust model (3.15) that corresponds to this uncertainty set is also solved by (4.1). This follows immediately from the following corollary:

COROLLARY 4.1. *For \mathcal{U} given in (4.2) and $x \in \mathbb{R}^n$, the following are equivalent,*

- i) $\mu^T x - \lambda x^T Q x \geq z^*(\mu, Q) - \gamma$ for all $(\mu, Q) \in \mathcal{U}$,
- ii) $\mu^{[i]T} x - \lambda x^T Q^{[i]} x \geq z^*(\mu^{[i]}, Q^{[i]}) - \gamma$ for $(i = 1, \dots, k)$.

Proof. We only need to show that ii) implies i), as the reverse implication is trivial. For each $(\mu, Q) \in \mathcal{U}$ there exist weights $\alpha^{[i]} \geq 0$ such that $\sum_{i=1}^k \alpha^{[i]} = 1$ and $(\mu, Q) = \sum_{i=1}^k \alpha^{[i]} (\mu^{[i]}, Q^{[i]})$. Multiplying each inequality in ii) by $\alpha^{[i]}$ and taking the sum, one obtains the required inequality

$$\mu^T x - \lambda x^T Q x \geq \sum_{i=1}^k \alpha^{[i]} z^*(\mu^{[i]}, Q^{[i]}) - \gamma \geq z^*(\mu, Q) - \gamma,$$

where the second inequality follows from the linearity of the function $(\mu, Q) \mapsto \mu^T x - \lambda x^T Q x$ and application of Lemma 2.1. \square

Thus, when the uncertainty set is given as a convex hull the relative robust problem (3.15) is tractable. Similar results hold for Models (3.11), (3.12), (3.13) and (3.14). The reader will find it easy to work out the details.

4.2. Solving Problem (3.19). Recall that in the case of Model (3.19), we assumed the uncertainty set to be of the form $\mathcal{U} = \mathcal{U}_\mu \times \{Q\}$, that is, we assumed the covariance matrix Q to be known with certainty. In the case where \mathcal{U}_μ is once again given by a finite number of extreme scenarios

$$\mathcal{U}_\mu = \{\mu^{[i]} : i = 1, \dots, k\},$$

the k values $z(\mu^{[i]})$ need to be computed by solving convex quadratic programming problems of the form (3.6), and then (3.19) turns into the convex quadratic programming problem

$$\begin{aligned} \min_{\gamma \in \mathbb{R}, y \in \mathbb{R}^n} \quad & \gamma \\ \text{s.t.} \quad & (\mu^{[i]} - re)^T y \geq z^*(\mu^{[i]}) - \gamma, \quad (i = 1, \dots, k) \\ & y \in \mathbb{R}_+ \mathcal{X}, \\ & y^T Q y \leq 1. \end{aligned} \quad (4.3)$$

In the case where \mathcal{U}_μ is a convex hull of extreme scenarios

$$\mathcal{U}_\mu = \text{conv}(\{\mu^{[i]} : i = 1, \dots, k\}),$$

one can once again exploit the fact that the function $\mu \mapsto z^*(\mu)$ is convex, so that Lemma 2.1 implies that (4.3) solves Problem (3.19).

5. Ellipsoidal Uncertainty Sets. In the remaining sections we assume the covariance matrix Q to be known with certainty. The vector of expected returns μ is assumed to be uncertain and lie in an ellipsoidal uncertainty set

$$\mathcal{U}_\mu = \{\bar{\mu} + Mu : \|u\| \leq 1\},$$

where M is a $n \times k$ matrix with $k \leq n$. Although the ellipsoid \mathcal{U} need not be full-dimensional, in applications it is often natural to choose $k = n$. Throughout this section we treat u or $\mu = \mu(u) = \bar{\mu} + Mu$ as the vector of uncertain model parameters interchangeably. Further, we assume that the set of feasible decision vectors is of the form

$$\mathcal{X} = \{x \in \mathbb{R}^n : Fx = f, Gx \leq g\},$$

where $F \in \mathbb{R}^{m_f \times n}$ has full row rank, $f \in \mathbb{R}^{m_f}$, $G \in \mathbb{R}^{m_g \times n}$ and $g \in \mathbb{R}^{m_g}$. We write F_i and G_i for the i -th rows of F and G respectively, and

$$\begin{aligned} f_\mu(x) &:= \mu^\top x - \lambda x^\top Q x, \\ z^*(\mu) &:= \max_{y \in \mathcal{X}} f_\mu(y), \\ R(x) &:= \max_{\mu \in \mathcal{U}_\mu} z^*(\mu) - f_\mu(x). \end{aligned}$$

The relative robust problem we wish to solve is then given by

$$\text{(RRP)} \quad \min_{x \in \mathcal{X}} R(x).$$

5.1. Copositivity Cones. We begin by introducing the technical tools that make an analysis of (RRP) possible. Most of the notation is adopted from the elucidating paper of Sturm and Zhang [24].

If D is subset of \mathbb{R}^n , let

$$\mathcal{H}(D) := \text{clo} \left\{ z = \begin{bmatrix} x \\ \tau \end{bmatrix} \in \mathbb{R}^{n+1} : \tau > 0, \tau^{-1}x \in D \right\}$$

be its homogenization, where $\text{clo}(\cdot)$ denotes the topological closure. Let $q : x \mapsto x^\top A x + 2b^\top x + c$ be an arbitrary quadratic polynomial on \mathbb{R}^n , where $A \in \mathcal{S}^n$ is a symmetric $n \times n$ matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$, and let

$$\mathcal{M}(q) := \begin{bmatrix} A & b \\ b^\top & c \end{bmatrix}.$$

Then

$$q(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^\top \mathcal{M}(q) \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}^\top \bullet \mathcal{M}(q),$$

where we write $X \bullet Y = \text{trace}(X^\top Y)$ for the trace inner product of two matrices of equal size (this inner product is the polarization of the Frobenius norm). In what follows we will refer to $\mathcal{M}(q)$ as the *matrix representation* of $q(\cdot)$. This defines a 1-1 correspondence between the set of quadratic functions on \mathbb{R}^n and the set \mathcal{S}^{n+1} of symmetric matrices of size $(n+1)$. In the sequel we will write $X \succeq 0$ if X is a symmetric positive semidefinite matrix and

$$\mathcal{S}_+^{(n+1)} := \{X \in \mathcal{S}^{n+1} : X \succeq 0\}.$$

Let

$$\begin{aligned}\mathcal{F}\mathcal{C}_+(D) &:= \{\mathcal{A} \in \mathcal{S}^{n+1} : \begin{bmatrix} x \\ 1 \end{bmatrix}^\top \mathcal{A} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \quad \forall x \in D\} \\ &= \{\mathcal{A} \in \mathcal{S}^{n+1} : z^\top \mathcal{A} z \geq 0 \quad \forall z \in \mathcal{H}(D)\}\end{aligned}$$

be the set of quadratic functions that are nonnegative on D . In the case where $D = \{x : q(x) \geq 0\}$ for some quadratic polynomial q , $\mathcal{F}\mathcal{C}_+(D)$ is called the set of quadratic functions copositive with q . Here we use an abuse of language and speak of $\mathcal{F}\mathcal{C}_+(D)$ as the *copositivity cone* associated with D when D is a more general set.

LEMMA 5.1 (Corollary 1, [24]). $\mathcal{F}\mathcal{C}_+(D) = \text{conv}\{zz^\top : z \in \mathcal{H}(D)\}^*$.

Proof. Lemma 5.1 is the same as Corollary 1 in Sturm & Zhang [24]. Here we give an alternative proof for completeness. We have

$$\begin{aligned}\mathcal{F}\mathcal{C}_+(D) &= \{X \in \mathcal{S}^{n+1} : z^\top X z \geq 0, \quad \forall z \in \mathcal{H}(D)\} \\ &= \bigcap_{z \in \mathcal{H}(D)} \{X \in \mathcal{S}^{n+1} : \langle X, zz^\top \rangle \geq 0\} \\ &= (\text{conv}\{zz^\top : z \in \mathcal{H}(D)\})^*.\end{aligned}$$

□

Unfortunately, for general D , the cone $\text{conv}\{zz^\top : z \in \mathcal{H}(D)\}^*$ does not have a tractable characterization. For example, when $D = \mathbb{R}_+^n$ then

$$\mathcal{H}(D) = \text{clo}\left(\left\{\begin{bmatrix} x \\ \tau \end{bmatrix} : \tau > 0, \tau^{-1}x \in \mathbb{R}_+^n\right\}\right) = \mathbb{R}_+^{n+1},$$

and

$$\mathcal{F}\mathcal{C}_+(D) = \text{conv}\{zz^\top : z \in \mathbb{R}_+^{n+1}\}^*$$

is the *co-positive cone*. Testing whether a given matrix belongs to this cone is co-NP-hard [16]. We will see in the sequel that (RRP) is equivalent to solving a conic optimization problem with a conic constraint of type $\mathcal{F}\mathcal{C}_+(D)$ for a convex set D defined by multiple linear and one quadratic inequality. If the cone $\mathcal{F}\mathcal{C}_+(D)$ is intractable, then the conic formulation of (RRP) is intractable too. To render the relative robust approach computationally viable in this situation, we will identify a tractable convex cone

$$K \subseteq \mathcal{F}\mathcal{C}_+(D)$$

which can be used in an inner approximation of (RRP). Most of the technical details regarding the construction of K are deferred to Section 6

5.2. A Conic Formulation of (RRP). Introducing an artificial variable γ , (RRP) is easily seen to be equivalent to

$$\begin{aligned}(\text{RRP.i}) \quad & \min_{x, \gamma} \gamma \\ \text{s.t.} \quad & x \in \mathcal{X} \\ & \gamma \geq z^*(\mu) - f_\mu(x) \quad \forall \mu \in \mathcal{U}_\mu.\end{aligned}$$

Since $z^*(\mu) := \max_{y \in \mathcal{X}} f_\mu(y)$, this problem can be further rewritten as

$$\begin{aligned}
(\text{RRP.ii}) \quad & \min_{x, \gamma} \gamma \\
& \text{s.t. } x \in \mathcal{X} \\
& \gamma \geq f_\mu(y) - f_\mu(x) \quad \forall \mu \in \mathcal{U}_\mu, y \in \mathcal{X}.
\end{aligned} \tag{5.1}$$

This is a semi-infinite optimization problem, that is, a finite-dimensional problem with infinitely many constraints. To render this problem amenable to numerical computations, we have to replace these infinitely many constraints by a finitely many.

Parameterizing μ by u , the set of values of $(\mu(u), y)$ that appear in the right-hand side of (5.1) corresponds to

$$D := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} : u^\top u \leq 1, Fy = f, Gy \leq g \right\}. \tag{5.2}$$

It follows from Lemma 4 of Sturm-Zhang [24] that the homogenization of this set is characterized by

$$\begin{aligned}
\mathcal{H}(D) = \left\{ \begin{bmatrix} u \\ y \\ \tau \end{bmatrix} \in \mathbb{R}^{k+n+1} : \tau \geq 0, \begin{bmatrix} u \\ y \\ \tau \end{bmatrix}^\top \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ y \\ \tau \end{bmatrix} \geq 0, \right. \\
\left. \begin{bmatrix} 0 \\ -F_i \\ f_i \end{bmatrix}^\top \begin{bmatrix} u \\ y \\ \tau \end{bmatrix} = 0, (i = 1, \dots, m_f), \right. \\
\left. \begin{bmatrix} 0 \\ -G_i \\ g_i \end{bmatrix}^\top \begin{bmatrix} u \\ y \\ \tau \end{bmatrix} \geq 0, (i = 1, \dots, m_g) \right\}.
\end{aligned}$$

Further, for fixed (x, γ) , the expression

$$\begin{aligned}
q_{x, \gamma}(u, y) &:= \gamma - f_\mu(y) + f_\mu(x) \\
&= \begin{bmatrix} u \\ y \end{bmatrix}^\top \begin{bmatrix} 0 & -\frac{1}{2}M^\top \\ -\frac{1}{2}M & \lambda Q \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} + \begin{bmatrix} M^\top x \\ -\bar{\mu} \end{bmatrix}^\top \begin{bmatrix} u \\ y \end{bmatrix} + (\gamma - \lambda x^\top Q x + \bar{\mu}^\top x)
\end{aligned}$$

is a quadratic function of (u, y) whose matrix representation in the homogenized space is given by

$$\mathcal{M}_{x, \gamma} := \mathcal{M}(q_{x, \gamma}) = \begin{bmatrix} 0 & -\frac{1}{2}M^\top & \frac{1}{2}M^\top x \\ -\frac{1}{2}M & \lambda Q & -\frac{1}{2}\bar{\mu} \\ \frac{1}{2}x^\top M & -\frac{1}{2}\bar{\mu}^\top & (\gamma - \lambda x^\top Q x + \bar{\mu}^\top x) \end{bmatrix}.$$

Using Lemma 5.1, Condition (5.1) is seen to be the same as

$$\mathcal{M}_{x, \gamma} \in \text{conv} \{ z z^\top : z \in \mathcal{H}(D) \}^*. \tag{5.3}$$

Therefore, (RRP.ii) can be written in conic form,

$$\begin{aligned}
(\text{RRP.iii}) \quad & \min_{x, \gamma} \gamma \\
& \text{s.t. } x \in \mathcal{X}, \\
& \mathcal{M}_{x, \gamma} \in \text{conv} \{ z z^\top : z \in \mathcal{H}(D) \}^*.
\end{aligned}$$

5.3. A Tractable Inner Approximation. Let H be the trailing $n \times (n - m_f)$ block of the orthogonal factor in the QR-decomposition $[\star H]R$ of F^\top , so that the

columns of H form a basis of $\ker(F)$. Further, let $x_p \in \mathbb{R}^n$ be a particular solution of the system $Fy = f$, and let us write $r := n - m_f$, so that

$$\{x \in \mathbb{R}^n : Fx = f\} = \{x_p + Hw : w \in \mathbb{R}^r\}.$$

Let $p_0^\top = [0 \ 1]$, where 0 is a zero row vector of size $k + r$, and let p_i^\top ($i = 1, \dots, m_g$) be the row vectors of the matrix $[0 \ -GH \ g - Gx_p]$, where 0 is now a zero matrix of size $m_g \times k$. And finally, let $Q = U^\top U$ be the Cholesky factorization of Q . With this notation, Corollary 6.11 of Section 6 shows that the following problem is an inner approximation of (RRP), where the minimisation is over the decision variables $w \in \mathbb{R}^r$, $\gamma, s, \eta, \xi_{ij} \in \mathbb{R}$, ($i \neq j = 0, \dots, m_g$), and $\tau_i \in \mathbb{R}$, $u_i \in \mathbb{R}^k$, ($i = 0, \dots, m_g$):

$$\begin{aligned} (\text{ARRP}) \quad & \min_{w, \gamma, s, \eta, \xi, \tau, u} \gamma \\ & \text{s.t. } g - Gx_p - GHw \in \mathbb{R}_+^{m_g}, \\ & \eta, \xi_{ij} \in \mathbb{R}_+, \quad (i \neq j = 0, \dots, m_g), \\ & \begin{bmatrix} \tau_i \\ u_i \end{bmatrix} \in L_{k+1}, \quad (i = 0, \dots, m_g), \\ & \begin{bmatrix} 0 \\ 0 \\ Ux_p \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ & & UH \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ s \\ w \end{bmatrix} \in L_{n+2}, \\ & \begin{bmatrix} 0 & -\frac{1}{2}M^\top H & \frac{1}{2}M^\top Hw \\ -\frac{1}{2}H^\top M & \lambda \bar{H}^\top QH & H^\top (\lambda Qx_p - \frac{1}{2}\bar{\mu}) \\ \frac{1}{2}w^\top H^\top M & (\lambda Qx_p - \frac{1}{2}\bar{\mu})^\top H & \gamma - \lambda s + (\bar{\mu} - 2\lambda Qx_p)^\top Hw \end{bmatrix} \\ & \quad - \eta \begin{bmatrix} -1 & & \\ & 0_r & \\ & & 1 \end{bmatrix} - \sum_{i \neq j=0}^{m_g} \xi_{ij} (p_i p_j^\top + p_j p_i^\top) \\ & \quad + \sum_{i=0}^{m_g} \left(p_i \begin{bmatrix} u_i \\ 0 \\ \tau_i \end{bmatrix}^\top + \begin{bmatrix} u_i \\ 0 \\ \tau_i \end{bmatrix} p_i^\top \right) \in \mathcal{S}_+^{k+r+1}. \end{aligned}$$

By construction, every solution $(w, \gamma, s, \eta, \xi, \tau, u)$ to (ARRP) provides a feasible solution $(x_p + Hw, \gamma)$ to (RRP.iii). Since the feasible set of (ARRP) is thus smaller than the feasible set of (RRP.iii), a (ARRP)-optimal solution $(w^*, \gamma^*, s^*, \eta^*, \xi^*, \tau^*, u^*)$ does not necessarily correspond to a (RRP)-optimal solution, this does not necessarily imply that $(x_p + Hw^*, \gamma^*)$ is optimal for (RRP.iii). However, since (ARRP) is equivalent to (RRP.iii) in the case $m_g \in \{0, 1\}$ (see Corollary 6.11), it is reasonable to expect that $(x_p + Hw^*, \gamma^*)$ is a feasible solution to (RRP.iii) which is quite close to optimal even in the case $m_g \geq 2$.

Note that each of the constraints of (ARRP) is formulated as a conic inequality of an expression that is linear in the decision variables. Thus, the great advantage of working with the model (ARRP) rather than (RRP) is the fact that, while (RRP) may be NP-hard, (ARRP) is readily solvable via standard polynomial-time conic programming implementations such as SDTP3 [27] or Sedumi [23].

6. Tightness of Inner Approximations. In this section we discuss some of the technical details and tightness results surrounding the inner approximation of the cones $\mathcal{FC}_+(D)$ used in Section 5.

6.1. The Case of General D . We begin with the discussion of inner approximations of $\mathcal{F}\mathcal{C}_+(D)$ where $D \subseteq \mathbb{R}^n$ is an arbitrary set.

LEMMA 6.1. $\mathcal{F}\mathcal{C}_+(D)^* \subseteq \{X \in \mathcal{S}_+^{n+1} : Xw \in \mathcal{H}(D) \forall w \in \mathcal{H}(D)^*\}$.

Proof. By Lemma 5.1, any $X \in \mathcal{F}\mathcal{C}_+(D)^*$ can be written as a limit $X = \lim_{j \rightarrow \infty} X_j$, where $X_j = \sum_{i=1}^{k_j} \xi_{ij} z_{ij} z_{ij}^T$ for some $z_{ij} \in \mathcal{H}(D)$ and $\xi_{ij} \geq 0$ ($i = 1, \dots, k_j$). Clearly this implies that $X \succeq 0$, and since for any $w \in \mathcal{H}(D)^*$ we have

$$X_j w = \sum_{i=1}^{k_j} \xi_{ij} (z_{ij}^T w) z_{ij} \in \mathcal{H}(D)$$

and $\mathcal{H}(D)$ is closed, $Xw \in \mathcal{H}(D)$. \square

Taking duals in the inclusion of Lemma 6.1, we obtain the following inner approximation of $\mathcal{F}\mathcal{C}_+(D)$,

$$\mathcal{F}D_+(D) \supseteq \mathcal{S}^{n+1} + K^*, \quad (6.1)$$

where

$$K = \{X \in \mathcal{S}^{n+1} : Xw \in \mathcal{H}(D) \forall w \in \mathcal{H}(D)^*\}. \quad (6.2)$$

To make this result useful, we need to characterize K^* .

LEMMA 6.2. *Let $C \subseteq \mathbb{R}^n$ be a closed convex cone and $w \in \mathbb{R}^n$. Then*

$$\{X \in \mathcal{S}^n : Xw \in C\}^* = \{vw^T + wv^T : v \in C^*\}.$$

Proof. Consider the linear map $\varphi_w(X) = Xw$ from \mathcal{S}^n to \mathbb{R}^n . Endowing these spaces with their canonical inner products $\langle X, Y \rangle := X \bullet Y$ and $\langle x, y \rangle := x^T y$, the adjoint map $\varphi_w^* : \mathbb{R}^{n+1} \rightarrow \mathcal{S}^{n+1}$ is defined by the relation

$$\langle X, \varphi_w^*(v) \rangle = \langle \varphi_w(X), v \rangle, \quad (X \in \mathcal{S}^{n+1}, v \in \mathbb{R}^{n+1}).$$

The right-hand side in this equation equals $(Xw)^T v = \frac{1}{2} \langle X, (wv^T + vw^T) \rangle$, showing that

$$\varphi_w^*(v) = \frac{1}{2} (wv^T + vw^T). \quad (6.3)$$

Now we have $Xw \in C$ if and only if $\langle \varphi_w(X), v \rangle \geq 0$ for all $v \in C^*$ (using biduality and the assumption that C is a closed convex cone). Taking adjoints, this is further equivalent to $\langle X, \varphi_w^*(v) \rangle \geq 0$ for all $v \in C^*$, and finally to

$$X \in (\varphi_w^*(C^*))^*. \quad (6.4)$$

Since C^* is a closed convex cone and φ_w^* is a linear map between finite-dimensional vector spaces, $\varphi_w^*(C^*)$ is a closed convex cone, so that taking duals in (6.4) yields

$$\{X \in \mathcal{S}^n : Xw \in C\}^* = \varphi_w^*(C^*).$$

Using (6.3), this is seen to be equivalent to the claim of the lemma. \square

LEMMA 6.3. *Let K be the cone defined in (6.2). Then*

$$K^* = \text{clo}(\text{cone}\{wv^T + vw^T : v, w \in \mathcal{H}(D)^*\}).$$

Proof. For each $w \in \mathcal{H}(D)^*$, let $K_w := \{X \in \mathcal{S}^n : Xw \in \mathcal{H}(D)\}$. Since $\mathcal{H}(D)$ is a closed convex cone, Lemma 6.2 shows that $K_w^* = \{vw^T + wv^T : v \in \mathcal{H}(D)^*\}$. Therefore, we have

$$\begin{aligned} K^* &= \left(\bigcap_{w \in \mathcal{H}(D)^*} K_w \right)^* \\ &= \text{clo} \left(\text{cone} \left(\bigcup_{w \in \mathcal{H}(D)^*} K_w^* \right) \right) \\ &= \text{clo}(\text{cone}\{wv^T + vw^T : v, w \in \mathcal{H}(D)^*\}), \end{aligned}$$

as claimed. \square

EXAMPLE 6.4. Let $D = \{x \in \mathbb{R}^n : b^T x \geq c\}$. Then $\mathcal{H}(D) = \{z \in \mathbb{R}^{(n+1)} : a^T z \geq 0\}$, where $a = \begin{bmatrix} b \\ -c \end{bmatrix}$ and $\mathcal{H}(D)^* = \text{cone}\{a\}$. Furthermore, we have

$$\begin{aligned} K &= \{X \in \mathcal{S}^{n+1} : Xa \in \mathcal{H}(D)\} \\ &= \{X \in \mathcal{S}^{n+1} : a^T Xa \geq 0\} \\ &= \{X \in \mathcal{S}^{n+1} : X \bullet aa^T \geq 0\}, \end{aligned}$$

so that $K^* = \text{cone}\{aa^T\}$. This is confirmed by Lemma 6.3, which says that $K^* = \text{cone}\{wv^T + vw^T : v, w \in \text{cone}\{a\}\} = \text{cone}\{aa^T\}$.

EXAMPLE 6.5. Let $D = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Then

$$\mathcal{H}(D) = \left\{ z \in \mathbb{R}^{n+1} : z^T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} z \geq 0, \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T z \geq 0 \right\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} L_{n+1},$$

where L_{n+1} is the $n+1$ -dimensional Lorenz cone or second-order cone and the operator $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ permutes the first component of a vector into last place. Since L_{n+1} is self-dual, we have

$$K = \{X \in \mathcal{S}^{n+1} : Xw \in \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} L_{n+1} \forall w \in \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} L_{n+1}\}. \quad (6.5)$$

Lemma 6.3 thus shows that $K^* = \text{cone}\{wv^T + vw^T : v, w \in \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} L_{n+1}\}$.

Combining the inclusion (6.1) with Lemma 6.3, we arrive at the following result.

THEOREM 6.6. For any set $D \subseteq \mathbb{R}^n$ it is true that

$$\mathcal{FC}_+(D) \supseteq \mathcal{S}^{n+1} + \text{clo}(\text{cone}\{wv^T + vw^T : v, w \in \mathcal{H}(D)^*\}).$$

6.2. Approximation Tightness in the General Case. The inner approximation of Theorem 6.6 is valid for arbitrary $D \subset \mathbb{R}^n$, and when $\mathcal{H}(D)$ can be explicitly characterized it becomes a useful computational tool in conjunction with Carathéodory's theorem. It is therefore natural to ask if the inclusion given by the theorem is in fact an equality. Unfortunately, this is not true in general, as we shall now see.

EXAMPLE 6.7. Consider Example 6.5 again, and note that in this case any $z \in \mathcal{H}(D) = L_{n+1}$ satisfies

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \bullet zz^T = z^T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} z \geq 0,$$

and the same holds true for convex combinations of matrices zz^T with $z \in \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} L_{n+1}$. Thus, using this extra information, we could have used the tighter approximation

$$\mathcal{F}\mathcal{C}_+(D)^* \subseteq \mathcal{S}_+^{n+1} \cap K \cap C,$$

where K is defined as in (6.5) and

$$C := \{X \in \mathcal{S}^{n+1} : \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \bullet X \geq 0\}.$$

This yields the inner approximation

$$\begin{aligned} \mathcal{F}\mathcal{C}_+(D) &\supseteq \mathcal{S}_+^{n+1} + K^* + C^* \\ &= \mathcal{S}_+^{n+1} + \text{cone}\{vw^T + vv^T : v, w \in \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} L_{n+1}\} + \text{cone}\left\{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \end{aligned} \quad (6.6)$$

which is strictly larger than the inner approximation

$$\mathcal{F}\mathcal{C}_+(D) \supseteq \mathcal{S}_+^{n+1} + \text{cone}\{vw^T + vv^T : v, w \in \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} L_{n+1}\} \quad (6.7)$$

provided by Theorem 6.6, as the right-hand side of (6.7) does not contain the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Further, applying the s -Lemma (see Lemma 6.8 below) in the context of this Example, one finds

$$\mathcal{F}D_+(D) = \mathcal{S}_+^{n+1} + \text{cone}\left\{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}, \quad (6.8)$$

an identity which was first discovered by Rendl-Wolkowicz [18]. In other words, the approximation (6.6) is not only an improvement over that of Theorem 6.6, but it is in fact tight, that is, the inclusion becomes an equality. Note that this also shows that

$$\{vw^T + vv^T : v, w \in \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} L_{n+1}\} \subset \mathcal{S}_+^{n+1} + \text{cone}\left\{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}. \quad (6.9)$$

The following classical result from the theory of robust control theory was used in the analysis of the above example:

LEMMA 6.8 (s -Lemma, Yakubovich [29]). *If $D = \{x \in \mathbb{R}^n : q(x) \geq 0\}$, where $q(\cdot)$ is a quadratic function that takes a strictly positive value somewhere, then*

$$\mathcal{F}D_+(D) = \mathcal{S}_+^{n+1} + \text{cone}\{\mathcal{M}(q)\}.$$

For proofs see e.g. [29], [17] and [12].

6.3. Improved Approximation for Use in Section 5. Next we will generalize the improved inner approximation (6.6) to copositivity cones associated with the convex set D we found in (5.2), (5.1),

$$D := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbb{R}^{k+n} : u^T u \leq 1, Fy = f, Gy \leq g \right\}.$$

Recall that $F \in \mathbb{R}^{m_f \times n}$ has full row rank, and $G \in \mathbb{R}^{m_g \times n}$. Let H be the trailing $n \times (n - m_f)$ block of the Q-factor of the QR-decomposition $[\star H]R$ of F^T , so that the columns of H form a basis of $\ker(F)$. Further, let $x_p \in \mathbb{R}^n$ be a particular solution of the system $Fy = f$, and let us write $r := n - m_f$, so that

$$\{y \in \mathbb{R}^n : Fy = f\} = \{x_p + Hw : w \in \mathbb{R}^r\}. \quad (6.10)$$

Consider the linear map

$$\Lambda : \mathcal{S}^{k+n+1} \rightarrow \mathcal{S}^{k+r+1},$$

$$\begin{bmatrix} A_{11} & A_{12} & b_1 \\ A_{12}^T & A_{22} & b_2 \\ b_1^T & b_2^T & c \end{bmatrix} \mapsto \begin{bmatrix} A_{11} & A_{12}H & b_1 + A_{12}x_p \\ H^T A_{12}^T & H^T A_{22}H & H^T(b_2 + A_{22}x_p) \\ b_1^T + x_p^T A_{12}^T & (b_2 + A_{22}x_p)^T H & x_p^T A_{22}x_p + 2b_2^T x_p + c \end{bmatrix}.$$

For any quadratic function h on \mathbb{R}^{k+n} with matrix representation $\mathcal{A} \in \mathcal{S}^{k+n+1}$ let h_Λ be the corresponding quadratic function on \mathbb{R}^{k+r} with matrix representation $\Lambda(\mathcal{A})$. If $y = x_p + Hw$, then by construction of Λ we have

$$h(u, y) \geq 0 \Leftrightarrow h_\Lambda(u, w) \geq 0. \quad (6.11)$$

Let $q(u, y) = 1 - u^T u$, so that

$$D = \left\{ \begin{bmatrix} u \\ y \end{bmatrix} : q(u, y) \geq 0, [0 \ F] \begin{bmatrix} u \\ y \end{bmatrix} = f, [0 \ G] \begin{bmatrix} u \\ y \end{bmatrix} \leq g \right\},$$

and note that

$$\mathcal{M}(q) = \begin{bmatrix} -I & \\ & 0_n & \\ & & 1 \end{bmatrix}, \quad \mathcal{M}(q_\Lambda) = \begin{bmatrix} -I & \\ & 0_r & \\ & & 1 \end{bmatrix},$$

where I is an identity matrix of size k and $0_n, 0_r$ are zero matrices of size n and r respectively. Let

$$D_\Lambda := \left\{ \begin{bmatrix} u \\ w \end{bmatrix} \in \mathbb{R}^{k+r} : q_\Lambda(u, w) \geq 0, [0 \ GH] \begin{bmatrix} u \\ w \end{bmatrix} \leq g - Gx_p \right\}.$$

We now obtain the following result, which shows that we can work directly in the reduced space \mathbb{R}^{k+r} instead of \mathbb{R}^{k+n} :

PROPOSITION 6.9.

- i) $D = \left\{ \begin{bmatrix} u \\ x_p + Hw \end{bmatrix}^T : \begin{bmatrix} u \\ w \end{bmatrix}^T \in D_\Lambda \right\}$,
- ii) $\mathcal{F}\mathcal{C}_+(D) = \Lambda^{-1}(\mathcal{F}\mathcal{C}_+(D_\Lambda))$.

Proof. i) follows from (6.10) and (6.11), while ii) follows from part i) and (6.11).

□

Next, let $p_0^T = [0 \ 1]$, where 0 is a zero row vector of size $k+r$, and let p_i^T ($i = 1, \dots, m_g$) be the row vectors of the matrix $[0 \ -GH \ g - Gx_p]$, where 0 is now a zero matrix of size $m_g \times k$. Then it follows from Lemma 4 in Sturm-Zhang [24] that

$$\mathcal{H}(D_\Lambda) = \left\{ z \in \mathbb{R}^{k+r+1} : z^T \begin{bmatrix} -I & \\ & 0_r & \\ & & 1 \end{bmatrix} z \geq 0, p_i^T z \geq 0, (i = 0, \dots, m_g) \right\}. \quad (6.12)$$

THEOREM 6.10. *An inner approximation of the cone $\mathcal{F}\mathcal{C}_+(D_\Lambda)$ is given by*

$$\begin{aligned} \mathcal{F}\mathcal{C}_+(D_\Lambda) \supseteq & \mathcal{S}_+^{k+r+1} + \text{cone} \left\{ \begin{bmatrix} -I & \\ & 0_r & \\ & & 1 \end{bmatrix} \right\} \\ & + \text{cone} \{ p_i p_j^T + p_j p_i^T : i \neq j \in \{0, \dots, m_g\} \} \\ & + \sum_{i=0}^{m_g} \left\{ p_i \begin{bmatrix} u \\ 0 \\ \tau \end{bmatrix}^T + \begin{bmatrix} u \\ 0 \\ \tau \end{bmatrix} p_i^T : \begin{bmatrix} u \\ \tau \end{bmatrix} \in L_{k+1} \right\}. \end{aligned} \quad (6.13)$$

Furthermore, if $m_g \in \{0, 1\}$ then the inclusion in (6.13) is an equality.

Proof. Lemma 5.1 and (6.12) establish

$$\begin{aligned} \mathcal{F}C_+(D_\Lambda)^* &= \text{conv} \left\{ zz^\text{T} : z \in \mathbb{R}^{k+r+1}, z^\text{T} \begin{bmatrix} -\text{I} & \\ & 0_r & \\ & & 1 \end{bmatrix} z \geq 0, p_i^\text{T} z \geq 0, (i = 0, \dots, m_g) \right\} \\ &\subseteq \mathcal{S}_+^{k+r+1} \cap K^q \cap K_0^l \cap \dots \cap K_{m_g}^l \cap K_{\mathcal{H}}, \end{aligned} \quad (6.14)$$

where

$$\begin{aligned} K^q &:= \left\{ X \in \mathcal{S}^{k+r+1} : X \bullet \begin{bmatrix} -\text{I} & \\ & 0_r & \\ & & 1 \end{bmatrix} \geq 0 \right\}, \\ K_i^l &:= \left\{ X \in \mathcal{S}^{k+r+1} : X p_i \in \mathcal{H}(D_\Lambda) \right\}, \quad (i = 0, \dots, m_g), \\ K_{\mathcal{H}} &:= \left\{ X \in \mathcal{S}^{k+r+1} : X v \in \mathcal{H}(D_\Lambda) \forall v \in \mathcal{H}(D_\Lambda)^* \right\}. \end{aligned}$$

Using the self-duality of L_{k+1} and (6.12), we get

$$\mathcal{H}(D_\Lambda)^* = \text{cone} \{ p_i : i = 0, \dots, m_g \} + \left\{ \begin{bmatrix} u \\ 0 \\ \tau \end{bmatrix} : [\tau] \in L_{k+1} \right\},$$

so that Lemma 6.3 implies

$$\begin{aligned} K_{\mathcal{H}}^* &= \text{cone} \left\{ p_i p_j^\text{T} + p_j p_i^\text{T} : i, j \in \{0, \dots, m_g\} \right\} \\ &\quad + \left\{ \begin{bmatrix} u \\ 0 \\ \tau \end{bmatrix} \begin{bmatrix} v \\ 0 \\ \sigma \end{bmatrix}^\text{T} + \begin{bmatrix} v \\ 0 \\ \sigma \end{bmatrix} \begin{bmatrix} u \\ 0 \\ \tau \end{bmatrix}^\text{T} : [\tau], [\sigma] \in L_{k+1} \right\} \\ &\quad + \sum_{i=0}^{m_g} \left\{ p_i \begin{bmatrix} u \\ 0 \\ \tau \end{bmatrix}^\text{T} + \begin{bmatrix} u \\ 0 \\ \tau \end{bmatrix} p_i^\text{T} : [\tau] \in L_{k+1} \right\}. \end{aligned} \quad (6.15)$$

Next, using Lemma 6.2 to take duals in (6.14), we find

$$\begin{aligned} \mathcal{F}C_+(D_\Lambda) &\supseteq \mathcal{S}_+^{k+r+1} + \text{cone} \left\{ \begin{bmatrix} -\text{I} & \\ & 0_r & \\ & & 1 \end{bmatrix} \right\} \\ &\quad + \sum_{i=0}^{m_g} \left\{ p_i z^\text{T} + z p_i^\text{T} : z \in \mathcal{H}(D_\Lambda)^* \right\} + K_{\mathcal{H}}^*. \end{aligned} \quad (6.16)$$

Substituting (6.15) into (6.16), exploiting the fact that

$$\left\{ \begin{bmatrix} u \\ 0 \\ \tau \end{bmatrix} \begin{bmatrix} v \\ 0 \\ \sigma \end{bmatrix}^\text{T} + \begin{bmatrix} v \\ 0 \\ \sigma \end{bmatrix} \begin{bmatrix} u \\ 0 \\ \tau \end{bmatrix}^\text{T} : [\tau], [\sigma] \in L_{k+1} \right\} \subset \mathcal{S}_+^{k+r+1} + \text{cone} \left\{ \begin{bmatrix} -\text{I} & \\ & 0_r & \\ & & 1 \end{bmatrix} \right\},$$

which follows from (6.9), and using

$$\text{cone} \{ p_i p_i^\text{T} + p_i p_i^\text{T} : i = 1, \dots, m_g \} \subset \mathcal{S}_+^{k+r+1},$$

the inclusion claimed in the theorem is seen to hold true. Furthermore, it follows from (6.8) that the inclusion is an equality when $m_g = 0$. The fact that this is also true for $m_g = 1$ follows from Sturm-Zhang [24], Theorem 3. \square

COROLLARY 6.11. *Let $Q = U^\text{T}U$ be the Cholesky factorization of Q . Then the following are sufficient conditions for (5.3) to hold: $\exists, \eta, \xi_{ij} \geq 0, (i \neq j = 0, \dots, m_g), s \in \mathbb{R}$ and $[\tau_i^i] \in L_{k+1}, (i = 0, \dots, m_g)$ such that*

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & & & \\ & \frac{1}{\sqrt{2}} & & \\ & & -\frac{1}{\sqrt{2}} & \\ & & & 1 \end{bmatrix} U \begin{bmatrix} \frac{1}{2} \\ s \\ x \end{bmatrix} \in L_{n+2} \quad (6.17)$$

and

$$\begin{aligned}
& \begin{bmatrix} 0 & -\frac{1}{2}M^T H & \frac{1}{2}M^T(x - x_p) \\ -\frac{1}{2}H^T M & \lambda \bar{H}^T Q H & \bar{H}^T(Qx_p - \frac{1}{2}\bar{\mu}) \\ \frac{1}{2}(x - x_p)^T M & (Qx_p - \bar{\mu})^T H & \gamma - \lambda s + \bar{\mu}^T(x - x_p) + \lambda x_p^T Q x_p \end{bmatrix} \\
& - \eta \begin{bmatrix} -1 & & \\ & 0_r & \\ & & 1 \end{bmatrix} - \sum_{i \neq j=0}^{m_g} \xi_{ij} (p_i p_j^T + p_j p_i^T) \\
& + \sum_{i=0}^{m_g} \left(p_i \begin{bmatrix} u_i \\ 0 \\ \tau_i \end{bmatrix}^T + \begin{bmatrix} u_i \\ 0 \\ \tau_i \end{bmatrix} p_i^T \right) \succeq 0. \quad (6.18)
\end{aligned}$$

Furthermore, for $m_g \in \{0, 1\}$ the above conditions are both necessary and sufficient for (5.3) to hold.

Proof. With $\mathcal{M}_{x,\gamma}$ defined as in Section 5, condition (5.3) is of course the same as $\mathcal{M}_{x,\gamma} \in \mathcal{FC}_+(D)$, see Lemma 5.1. Proposition 6.9 shows that this is further equivalent to $\Lambda(\mathcal{M}_{x,\gamma}) \in \mathcal{FC}_+(D_\Lambda)$. By Theorem 6.10, for this latter condition to hold it is sufficient to demand that

$$\begin{aligned}
\Lambda(\mathcal{M}_{x,\gamma}) \in & \mathcal{S}_+^{k+r+1} + \text{cone} \left\{ \begin{bmatrix} -1 & & \\ & 0_r & \\ & & 1 \end{bmatrix} \right\} \\
& + \text{cone} \{ p_i p_j^T + p_j p_i^T : i \neq j \in \{0, \dots, m_g\} \} \\
& + \sum_{i=0}^{m_g} \left\{ p_i \begin{bmatrix} u \\ 0 \\ \tau \end{bmatrix}^T + \begin{bmatrix} u \\ 0 \\ \tau \end{bmatrix} p_i^T : \begin{bmatrix} u \\ \tau \end{bmatrix} \in L_{k+1} \right\}. \quad (6.19)
\end{aligned}$$

Further, since $\lambda \geq 0$, (6.19) is equivalent to the existence of a $s \geq x^T Q x$ such that

$$\begin{aligned}
\Lambda(\mathcal{M}_{x,\gamma}) + \lambda \begin{bmatrix} 0 & & \\ & 0 & x^T Q x - s \end{bmatrix} \in & \mathcal{S}_+^{k+r+1} + \text{cone} \left\{ \begin{bmatrix} -1 & & \\ & 0_r & \\ & & 1 \end{bmatrix} \right\} \\
& + \text{cone} \{ p_i p_j^T + p_j p_i^T : i \neq j \in \{0, \dots, m_g\} \} \\
& + \sum_{i=0}^{m_g} \left\{ p_i \begin{bmatrix} u \\ 0 \\ \tau \end{bmatrix}^T + \begin{bmatrix} u \\ 0 \\ \tau \end{bmatrix} p_i^T : \begin{bmatrix} u \\ \tau \end{bmatrix} \in L_{k+1} \right\}
\end{aligned}$$

which is the same as (6.18) for some $\eta, \xi_{ij} \geq 0$, ($i \neq j = 0, \dots, m_g$), and $\begin{bmatrix} \tau_i \\ u_i \end{bmatrix} \in L_{1+k}$, ($i = 0, \dots, m_g$). Finally, the condition $s \geq x^T Q x$ is easily seen to be equivalent to (6.17). The last claim holds because by Theorem 6.10, (6.19) is equivalent to $\Lambda(\mathcal{M}_{x,\gamma}) \in \mathcal{FC}_+(D_\Lambda)$ when $m_g \in \{0, 1\}$. \square

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7. Appendix A: Proof of Lemma 2.1. *Proof.* Let $p_1, p_2 \in \mathcal{U}$ and $\alpha \in [0, 1]$. Let us first assume that $z^*(\alpha p_1 + (1 - \alpha)p_2) < +\infty$. For $\varepsilon > 0$ there exists $x \in \mathcal{X}$ such that

$$\begin{aligned} z^*(\alpha p_1 + (1 - \alpha)p_2) - \varepsilon &\leq f(x, \alpha p_1 + (1 - \alpha)p_2) \\ &\leq \alpha f(x, p_1) + (1 - \alpha)f(x, p_2) \\ &\leq \alpha z^*(p_1) + (1 - \alpha)z^*(p_2). \end{aligned}$$

Since this is true for all ε , we find

$$z^*(\alpha p_1 + (1 - \alpha)p_2) \leq \alpha z^*(p_1) + (1 - \alpha)z^*(p_2). \quad (7.1)$$

Now assume $z^*(\alpha p_1 + (1 - \alpha)p_2) = +\infty$. Then there exists a sequence $(x_n)_{\mathbb{N}} \subset \mathcal{X}$

such that

$$f(x_n, \alpha p_1 + (1 - \alpha)p_2) \xrightarrow{n \rightarrow \infty} +\infty. \quad (7.2)$$

By the same argument as above, we have

$$f(x_n, \alpha p_1 + (1 - \alpha)p_2) \leq \alpha z^*(p_1) + (1 - \alpha)z^*(p_2),$$

whence (7.2) establishes that at least one of $z^*(p_i)$ ($i = 1, 2$) equals $+\infty$, so that (7.1) holds once again. \square

8. Appendix B: Convexification of Problem (3.5). We use the notation introduced in Section 3.1.2. Using the fact that any $x \in \mathcal{X}$ was assumed to satisfy the budget constraint $e^\top x$, Problem (3.5) has an equivalent formulation

$$\begin{aligned} \max_{x \in \mathbb{R}^n} f(x) &= \frac{(\mu - re)^\top x}{\sqrt{x^\top Q x}} \\ \text{s.t. } &x \in \mathcal{X}. \end{aligned} \quad (8.1)$$

The objective function $f(x)$ of this formulation is homogeneous of degree 0 in x . Consider also the normalized problem,

$$\begin{aligned} \max_{y \in \mathbb{R}^n} g(y) &= (\mu - re)^\top y \\ \text{s.t. } &y \in \mathbb{R}_+ \mathcal{X}, \\ &y^\top Q y = 1, \end{aligned} \quad (8.2)$$

where $\mathbb{R}_+ \mathcal{X} = \{\tau x : \tau \geq 0, x \in \mathcal{X}\}$ is a tractable cone. For example, if \mathcal{X} is a polyhedron, as it is in most applications, $\mathbb{R}_+ \mathcal{X}$ is a polyhedral cone.

Since $e^\top x = 1$ for all $x \in \mathcal{X}$ and $Q \succ 0$, we have $x^\top Q x > 0$ for all $x \in \mathcal{X}$. Therefore, any feasible solution x of (8.1) provides $y = (x^\top Q x)^{-1/2} x$ as a feasible solution of (8.2), and furthermore, $g(y) = f(x)$. Conversely, since any feasible y of (8.2) satisfies $y \neq 0$, the vector $x = (e^\top y)^{-1} y \in \mathcal{X}$ is feasible for (8.1) and satisfies $f(x) = g(y)$. This shows that (8.1) and (8.2) are equivalent: Instead of solving (8.1), we may solve (8.2) and then construct an optimal solution $x^* = (e^\top y^*)^{-1} y^*$ of the first problem from an optimal solution y^* of the second.

Formulation (8.2) is furthermore equivalent to its relaxation

$$\begin{aligned} \max_{y \in \mathbb{R}^n} g(y) &= (\mu - re)^\top y \\ \text{s.t. } &y \in \mathbb{R}_+ \mathcal{X}, \\ &y^\top Q y \leq 1, \end{aligned} \quad (8.3)$$

since a feasible solution of (8.3) cannot be optimal unless $y^\top Q y = 1$, assuming that there exist feasible y for which $g(y) > 0$ (if this is not the case, it is not rational to invest at all). Thus, the optimal solution y^* of (8.2) may be found by solving the tractable convex problem (8.3).