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MARTINGALES AND STOCHASTIC INTEGRALS IN THE THEORY OF CONTINUOUS TRADING

J. Michael Harrison, Stanford University

Stanley R. Pliska, Northwestern University

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J. Michael Harrison
Stanford University

Stanley R. Pliska
Northwestern University

Abstract

This paper develops a general stochastic model of a frictionless security market with continuous trading. The vector price process is given by a semimartingale of a certain class, and the general stochastic integral is used to represent capital gains. Within the framework of this model, we discuss the modern theory of contingent claim valuation, including the celebrated option pricing formula of Black and Scholes. It is shown that the security market is complete if and only if its vector price process has a certain martingale representation property. A multidimensional generalization of the Black-Scholes model is examined in some detail, and some other examples are discussed briefly.

Key Words and Phrases

Contingent Claim Valuation, Continuous Trading, Diffusion Processes, Option Pricing, Representation of Martingales, Semimartingales, Stochastic Integrals.

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J. Michael Harrison
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1. Introduction

This paper is intended partly as a tutorial, partly as a survey, and partly as a forum for new results. Its subject is the theory of security markets with continuous trading, a highly specialized but nonetheless important topic in financial economics. We develop a general stochastic model of a frictionless market with continuous trading, hereafter called simply a continuous market, and then discuss the modern theory of contingent claim valuation (option pricing) in the context of that model. The mathematical structure developed here is also potentially useful for study of consumption-investment problems, but that subject will not be dealt with directly.

In mentioning the modern theory of contingent claim valuation, we refer primarily to the option pricing formula of Black and Scholes [2]. It was a desire to better understand their formula that originally motivated our study, so we introduce this paper with a brief account of the Black-Scholes theory and some questions that it naturally suggests. For purposes of introduction, certain terms will be used in a temporary narrow sense, and some of the mathematical definitions will be stated informally or even deleted altogether. Also, to give a more or less concrete motivation for the general theory, excessive emphasis is placed on a single economic issue, involving what we call completeness of the market.

1a. The Option Pricing Formula

Let $W = \{W_t;\ 0 \leq t \leq T\}$ be a standard (zero drift and unit variance) Brownian motion on some probability space $(\Omega, \mathscr{F}, P)$. Let r, μ and σ be real constants with $\sigma > 0$. It will be natural to think in terms of the case $\mu > r > 0$, but this restriction is not necessary. Now define

$$(1.1) \qquad S_t^0 = S_0^0 \exp(rt)\ , \qquad 0 \leq t \leq T\ ,$$

$$(1.2) \qquad S_t^1 = S_0^1 \exp(\sigma W_t + (\mu - \tfrac{1}{2}\sigma^2)t)\ , \qquad 0 \leq t \leq T\ ,$$

where the initial values S_0^0 and S_0^1 are positive constants. (This notational system is used throughout. The time parameter of a process is given by a subscript, and the components of the vector security price process S are indexed by a superscript $k = 0,1,\ldots, K$. The distinction between superscripts and exponents will always be clear from context.) Interpret S_t^0 as the price at time t of a riskless bond, with r being the associated riskless interest rate. Interpret S_t^1 as the price at time t, in dollars per share, of a stock which pays no dividends. In more general terms, we might call S^0 and S^1 the price processes for a riskless security and a risky security respectively. For our purposes, a unit of security k can be viewed simply as a piece of paper which is exchangeable for S_t^k

dollars at any time t $(k = 0,1)$. The market value of the bond grows exponentially at rate r, while that of the stock fluctuates randomly.

Applying Ito's Formula to (1.1) and (1.2), it is seen that our price processes S^0 and S^1 satisfy the stochastic differential equations

$$dS_t^0 = rS_t^0\, dt, \tag{1.3}$$

$$dS_t^1 = \sigma S_t^1\, dW_t + \mu S_t^1\, dt\ . \tag{1.4}$$

One can paraphrase (1.2) and (1.4) by saying that S^1 is a <u>geometric Brownian</u> motion with <u>rate of return</u> $dS_t^1/S_t^1 = \sigma\, dW_t + \mu\, dt$. This terminology is a bit sloppy since W is nondifferentiable, and in the body of the paper we'll simply call $\sigma W_t + \mu t$ the <u>return process</u> for the stock.

Consider an investor, hereafter called <u>you</u>, participating in a securities market where this stock and this bond are traded. Assume that you are allowed to trade continuously, that there are no transaction costs (like brokerage fees) in this market, and that you can sell short without restriction (see below). We summarize these assumptions by saying that this is a <u>frictionless market</u> with continuous trading. Now consider a ticket which entitles its bearer to buy one share of stock at the terminal date T, if he wishes, for a specified price of c dollars. This is a <u>European call option</u> on the stock, with <u>exercise price</u> c and <u>expiration date</u> T. If $S_T^1 < c$ (stock price is below exercise price at expiration date), then the bearer of the ticket will not exercise his option to buy, meaning that the ticket is worthless in the end. But if $S_T^1 \geq c$, the bearer can buy one share of stock for c dollars, then turn around and sell it for S_T^1 dollars, making a profit of $S_T^1 - c$. Thus we see that the call option is completely equivalent to a ticket which entitles the bearer to a payment of $X = (S_T^1 - c)^+$ dollars at time T.

Now how much would you be willing to pay for such a ticket at time zero? Put another way, what is your valuation of the option? On the surface of things, it seems perfectly reasonable that different people might give different answers, depending on their attitudes toward risk bearing, since purchase of the option is unquestionably a risky investment. But Black and Scholes [2] asserted that there is a unique rational value for the option, independent of one's risk attitude. Specifically, defining

$$(1.5) \qquad f(x,t) = x\Phi(g(x,t)) - ce^{-rt}\,\Phi(h(x,t)) \ ,$$

$$\text{where} \qquad g(x,t) = [\ln(x/c) + (r + \tfrac{1}{2}\sigma^2)t]/\sigma\sqrt{t} \ ,$$

$$h(x,t) = g(x,t) - \sigma\sqrt{t} \ ,$$

and $\Phi(\cdot)$ is the standard normal distribution function, this unique rational value is $f(S_0^1,T)$. Observe that the valuation formula (1.5) involves the current stock price x, the expiration date t, the exercise price c, the return variance σ^2 and the riskless interest rate r, but not the mean rate of return μ for the stock.

Before we discuss the reasoning behind this formula, some historical remarks are in order. The first mathematical description of the stochastic process now called Brownian motion was given by Bachelier [1] in a thesis submitted to the Academy of Paris in 1900. Proposing this process as a model of security price fluctuations, his goal was to develop theoretical values for various types of options and compare these against the observed market prices of the options. Thus the problem of option valuation motivated the very first research on what we now call diffusion processes. (Bachelier's work was apparently unknown to Einstein and Wiener when they later developed the mathematical theory of Brownian motion.) From a modern perspective, Bachelier's mathematics and economics were both flawed, so there is no point in describing the valuation theory at which he finally arrived. But he did solve a number of problems correctly, and the paper makes interesting reading.

More than 50 years later, the search for a mathematical theory of option valuation was taken up by Samuelson [34] and others. They replaced Bachelier's ordinary (or arithmetic) Brownian motion with the geometric Brownian motion (1.2), the simplest argument in favor of this change being that stock prices cannot go negative because of limited liability. Using geometric Brownian motion as their model of stock price movement, various authors obtained various valuation theories under various sorts of assumptions. But these theories, developed between 1950 and 1970, all contained ad hoc elements, and they left even their creators feeling vaguely dissatisfied. Then Black and Scholes made the dazzling observation that, in the idealized market described above, investors can actually duplicate the cash flow (or payoff stream) from a call option by adroitly managing a portfolio that contains only stock and bond. Since possession of this portfolio is completely equivalent to possession of the call option, the market value of its constituent securities at time zero is the unique rational value for the option. This argument will be fleshed out and connected with the valuation formula (1.5) shortly.

The mathematical argument given by Black and Scholes in support of their formula is not entirely satisfactory, but there are several alternate explanations and derivations now available in the literature of financial economics. (In fact, explaining the valuation formula has become a minor industry.) The best of these from our perspective, and the one uniquely consistent with the general theory developed here, is the argument by Merton [30] that we now present. For more on the history of option theory, see the surveys by Samuelson [35] and Smith [37].

1b. Portfolio Theory and Option Valuation

It is easy to verify that the function $f(x,t)$ defined by (1.5) satisfies the partial differential equation

$$(1.6)\qquad \frac{\partial}{\partial t} f(x,t) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} f(x,t) + rx\frac{\partial}{\partial x} f(x,t) - rf(x,t)$$

with initial condition

$$(1.7)\qquad f(x,0) = (x-c)^+ .$$

In fact, Black and Scholes originally obtained their valuation formula by solving (1.6)-(1.7). Now define stochastic processes

$$(1.8)\qquad V_t = f(S_t^1,\ T-t)\ , \qquad 0 \le t \le T\ ,$$

$$(1.9)\qquad \phi_t^1 = \frac{\partial}{\partial x} f(S_t^1,\ T-t)\ , \qquad 0 \le t \le T\ ,$$

$$(1.10)\qquad \phi_t^0 = (V_t - \phi_t^1 S_t^1)/S_t^0\ , \qquad 0 \le t \le T\ .$$

Interpret the vector process $\phi_t = (\phi_t^0, \phi_t^1)$ as a <u>trading strategy</u>, with ϕ_t^k specifying the number of units of security k to be held at time t. Simply put, ϕ_t is the <u>portfolio</u> of securities held at time t. From (1.10) we see that the market value of the portfolio held at time t is

$$\phi_t^0 S_t^0 + \phi_t^1 S_t^1 = V_t \,, \qquad 0 \leq t \leq T \,. \tag{1.11}$$

Thus, using (1.8) and (1.7), the initial value of the portfolio is $V_0 = f(S_0^1, T)$ and the terminal value

$$V_T = f(S_T^1, 0) = (S_T^1 - c)^+$$

is precisely equal to the terminal value of the call option. Finally, applying Ito's Formula to (1.8) we obtain

$$dV_t = \frac{\partial}{\partial x} f(S_t^1,\ T-t)\ dS_t^1 + \frac{1}{2}\frac{\partial^2}{\partial x^2} f(S_t^1,\ T-t)(dS_t^1)^2 + \frac{\partial}{\partial t} f(S_t^1,\ T-t)dt \,. \tag{1.12}$$

Using (1.3), (1.4), (1.6) and (1.8)-(1.10), we ultimately reduce (1.12) to

$$dV_t = \phi_t^0\ dS_t^0 + \phi_t^1\ dS_t^1 \,. \tag{1.13}$$

In its precise integral form, (1.13) is

$$V_t - V_0 = \int_0^t \phi_u^0\ dS_u^0 + \int_0^t \phi_u^1\ dS_u^1 \,, \qquad 0 \leq t \leq T \,. \tag{1.14}$$

The right-hand side represents the total earnings, or capital gains, that you realize on your holdings up to time t (see §3). Thus (1.14) says that all changes in the value of your portfolio are due to capital gains, as opposed to withdrawal of cash or infusion of new funds. In the language of Harrison and Kreps [13], this is a self-financing strategy.

The justification of the valuation formula (1.5) is now complete. We have located a trading strategy which requires initial investment $\pi = f(S_0^1, T)$ and thereafter produces exactly the same pattern of cash flows as the call option. In brief, the option is attainable in this market, at a time zero price of π, by dealing only in stock and bond. In the economics literature it is customary to go further, arguing that arbitrage profits could be made if options were sold in a parallel market at any price other than π, and that existence of arbitrage opportunities is inconsistent with equilibrium in the total economic system. See, for example, the original paper of Black and Scholes [2] or the recent article by Cox, Ross and Rubinstein [6]. To reduce verbiage, and to get a self-contained mathematical theory, we shall simply stop with the statement of attainability. Throughout this paper, we focus on an isolated market in which certain securities are traded, assuming that no arbitrage opportunities exist internal to this market (see §2). We seek to characterize the class of contingent claims that investors can attain, and the prices at which they can attain them, by dealing only in the designated securities. In discussing the valuation formula (1.5), for example, we have focused on a market where only the stock and bond are traded, and we've discovered that investors can manufacture call options for themselves in this market, at the price specified in the formula. No comparison is made with the price at which options do sell, or might sell, or should sell outside our market, although it is obviously possible to do so.

Beginning with the statement of the critical balance condition (1.13), our treatment has diverged somewhat from Merton's [30] proof of the valuation formula. In particular, his defense of (1.13) as a zero-net-new-investment condition relies on his own theory of portfolio management with diffusion price processes [26,27].

As a final point, let us return to the assumption of unrestricted short sales. From the standpoint of our formal theory, this means simply that either portfolio component ϕ_t^k can be negative. In the case of the bond, short selling amounts to borrowing (rather than lending) money at the riskless interest rate r. For the particular trading strategy ϕ defined by (1.9) and (1.10), it can be verified that V and ϕ^1 are positive, but ϕ^0 can go negative. Thus, in order to duplicate the cash flow from the call option, you will always hold a positive amount of stock, but it may be necessary to finance some of your stock purchases with riskless borrowing (selling bonds short). In particular, the valuation formula (1.5) for call options does not actually require the assumption that stock can be sold short without restriction, but short sale of stock may be necessary in order to attain other types of options. See Sharpe [36] for an explanation of short sales.

1c. Completeness of the Market

In the preceding section we have defended the valuation formula (1.5) without ever suggesting how it was obtained in the first place. The derivation of the formula, or rather our approach to its derivation, will be explained later in §5, where we also show that the attainability result of the previous section can be greatly generalized. Roughly, the story is as follows. Let

$$\mathscr{F}_T = \mathscr{F}(S_t;\ 0 \leq t \leq T) ,$$

meaning that $\mathscr{F}_T$ consists of all events whose occurrence or nonoccurrence can be determined from the stock price history through time T. Define a contingent claim as a nonnegative random variable X which is measurable with respect to $\mathscr{F}_T$ (hereafter written $X \in \mathscr{F}_T$). This is our formal representation for a ticket which entitles the bearer to a payment, at time T, whose size depends (in an arbitrary way) on the price history up through T. One can of course expand this definition to consider claims payable at other times, but doing so complicates notation, and the added generality is essentially trivial. The European call option discussed above is represented by $X = (S_T - c)^+$. Generalizing the ideas in §1b, a contingent claim X is said to be attainable at price π in our security market if there exists a self-financing trading strategy ϕ, with associated market value process V, such that $V_0 = \pi$ and $V_T = X$ almost surely. To make this precise, one of course needs a general definition of a self-financing strategy (and the associated value process), but we trust that the spirit of the definition is clear. A remarkable property of the diffusion model described in §1a is that every contingent claim is attainable, and one can even write down a general (but rather abstract) valuation formula for the price π associated with a given claim X. The valuation formula is

$$\pi = \exp(-rT)\ E^*(X) , \tag{1.15}$$

where $E^*(\cdot)$ is the expectation operator associated with another (very particular) probability measure P^* on $(\Omega,\mathscr{F})$. This measure P^* is equivalent to P, meaning that $P^*(A) = 0$ if and only if $P(A) = 0$ (the two measures have the same <u>null sets</u>). The Black-Scholes formula (1.5) is a special case of (1.15).

Loosely adopting a standard term in economic theory, we say that a security market model is <u>complete</u> if every contingent claim is attainable. (See §3 for precise definitions.) The completeness of the Black-Scholes model, in a somewhat different sense, and the general valuation formula (1.15) were proved by Harrison and Kreps [13], although the origin of (1.15) lies in an observation by Cox and Ross [5].

1d. <u>An Open Question</u>

It can be argued that the important and interesting feature of the model in §1a is its completeness, not the fact that it yields the explicit valuation formula (1.5) for call options. We shall adopt precisely this point of view throughout most of this paper, investigating the structural features of different models, rather than emphasizing explicit computation. (In the end, however, it is the explicit calculations that give the subject its vitality.) From this viewpoint, the following question is both natural and fundamental.

(1.16) Suppose the vector price process in §1a is replaced by some other positive vector process $S = \{S_t;\ 0 \leq t \leq T\}$ with all other assumptions and definitions unchanged. What processes S yield a complete market?

A significant amount of our attention is directed to this question. A satisfactory general answer will not be obtained, but matters will at least be brought to a point where the question is given a precise mathematical form, and then reduced to an equivalent problem in martingale theory, for which a significant literature exists.

The general question (1.16) probably has a very sharp answer, although much debate is possible over the appropriate criterion of sharpness, and we hope our paper will stimulate interest in this and related mathematical problems. For the moment, we simply wish to make two observations. First, despite the impression one often gets in reading the academic finance literature, it is neither necessary nor sufficient for completeness of the market that the price process S have continuous sample paths. In particular, the attainability of call options in the model of §1a requires much more than continuity of the stock price process, although one can certainly relax the precise distributional assumptions imposed there. See §6c for an example, and compare this against the introductory passage in the survey by Smith [37]. Second, the Markov property is completely irrelevant to the question posed in (1.16). In fact, a much stronger statement can be made. Consider a market model whose securities price process S is defined on some probability space $(\Omega,\mathscr{F},P)$. Now consider a second model identical in all regards except that P is replaced by an equivalent probability measure Q. Then a contingent claim is attainable at price π in the first model if and only if it is attainable at this same price in the second model. Consequently, <u>the first model is complete if and only if the second one is</u>. These statements may not be obvious, since precise definitions have not been given, but we hope they are at least plausible at this point. Putting the assertion another way, only the <u>null sets</u> of the distribution of S are relevant to the question (1.16). In asking whether every contingent claim derived from S is attainable in the market, we are only interested in which sets of sample paths do and do not have positive probability. Thus the parts of probability theory most relevant to the general question (1.16) are those results, usually abstract in appearance and French in origin, that are invariant under substitution of an equivalent measure.

1e. The Probabilistic Setting

Before the completeness question (1.16) can even be stated precisely, one must have a general model of a market with continuous trading. In this section we describe the minimal model structure necessary for a study of completeness, suppressing some features of the theory actually developed later. Our first task is to resolve the following modeling issues.

(1.17) What class of vector processes S might conceivably be used to represent security price fluctuations?

(1.18) How should one define a trading strategy in general, and then what is the proper definition of a self-financing strategy?

To keep things simple, consider only price processes S with $S_t^0 = \exp(rt)$, meaning that the riskless interest rate is both deterministic and constant. Let $\beta_t = \exp(-rt)$ and call β the <u>intrinsic discount process</u> for S. It will be argued that, if we are to obtain an internally consistent theory, we need only consider S such that

(1.19) the discounted vector price process βS is a martingale under some probability measure P^* equivalent to P.

It is this P^*, called the reference measure, that enters in the general valuation formula (1.15) discussed earlier. One implication of (1.19) is that S must be what is called a semimartingale, and we are fortunate to have available a well developed theory dealing with change of measure for semimartingales. This theory, which has evolved from Girsanov's Theorem [12] for Ito processes, is precisely what is needed to verify or refute the condition (1.19) for any given model.

Turning to the modeling issue (1.18), we define a trading strategy ϕ as a predictable vector process, we define the capital gains under strategy ϕ as the stochastic integral of ϕ with respect to the vector price process S, and then we define a self-financing strategy exactly as in (1.14). Because the price process is a semimartingale, the necessary general theory of stochastic integration is readily available. In the end, we find that our model is complete if and only if every process that is a martingale under P^* can be written as a stochastic integral with respect to the process βS in (1.19). In the language of martingale theory, the model is complete if and only if βS has the martingale representation property under our reference measure P^*.

All of this is intended to suggest that the modern theory of martingales and stochastic integrals provides exactly the mathematical framework needed for a theory of continuous trading. As our development unfolds, there will be still more examples of general results in the mathematical theory that look as if they were created for this application. We have begun to feel that all the standard problems studied in martingale theory, and all the major results, must have interpretations and applications in our setting. Be that as it may, the process of searching for such connections has barely even begun.

1f. Outline of the Paper

This paper is aimed at readers with a good command of probability and stochastic processes, but no particular knowledge of economics. On the former dimension, we assume familiarity with the Strasbourg theory of martingales and stochastic integration, as developed in the definitive treatise by Meyer [32]. This assumption is perhaps unrealistic, but we cannot provide a systematic tutorial on stochastic integrals and an adequate treatment of our nominal subject matter in a reasonable amount of space. (Also, the former task is best left to others. We are working dangerously close to the boundaries of our knowledge as things stand.) Most of this paper will be accessible to those who know about stochastic integrals with respect to Brownian motion, and the rest should come into focus after a little study of the relevant foundational material. (On first reading, specialize general results to the case where S is an Ito process.) To facilitate such study, we consistently refer to Meyer [32] by page number for basic definitions and standard results, and his notation and terminology are used wherever possible. For a nice overview of the Strasbourg approach to stochastic integration, plus some new results and illuminating commentary, see the recent survey by Dellacherie [9] in this journal. A comprehensive treatment of stochastic calculus is given by Jacod [18], and it appears that the second volume of Williams [38] will be another good sourcebook on martingales and stochastic integrals in the Strasbourg style. A somewhat different approach to stochastic integrals is developed by

Metivier and Pellaumail [31], and their theory is also discussed briefly by Dellacherie [9]. Some, but not all, of the results used here can be found in the English edition of Liptser and Shiryayev [24].

The heart of his paper is §3, which contains the general theory of continuous markets alluded to earlier. This is preceded by a partial development of the analogous theory for finite markets in §2. (A finite market is one where trading takes place at discrete points in time and the underlying probability space is finite.) Both the formulation and the central results of §2 are taken from the paper by Harrison and Kreps [13], which is in all respects the intellectual progenitor of this work. By treating the finite case first, we are able to ease the exposition in several respects. First, the necessary economic notions are introduced in a simple setting. Having interpreted or defended a definition in the finite case, we typically state its formal analog and proceed without further comment in development of the general theory. Second, we are able to give an adequate treatment in the finite case of certain foundational issues that will be essentially glossed over in development of the general theory. In particular, a key assumption of §3 is defended principally on the basis of its formal similarity to a condition derived from more primitive considerations in §2. Finally, the technical complexity that one encounters with a continuous time parameter obscures the basic structure of the mathematical theory. By treating the finite case first, we hope to establish the natural role of martingale technology and thus motivate the rather intricate developments of §3.

Section 4 serves as a complement to §3, discussing the general relationship between security price processes and their associated return processes. Section 5 analyzes in some detail a multidimensional version of the Black-Scholes model. Section 6 contains further examples relating to completeness of markets, and §7 contains some miscellaneous concluding remarks.

We conclude this section with some general comments on terminology and notation. The term positive is used hereafter in the weak sense, as opposed to strictly positive, and similarly for increasing versus strictly increasing. When we write $X = Y$ for random variables X and Y, this is understood to be an almost sure relationship, and similarly for $X \geq Y$. In the case of processes, $X \geq Y$ means $X_t \geq Y_t$ for all t. As examples of these conventions, we will have frequent occasion to write $X = 0$ or $X \geq 0$, where X may be either a random variable or a process. The symbol $\equiv$ is used to mean equals by definition.

2. The Finite Theory

This section introduces a number of basic concepts by examining the case where time is discrete and the sample space is finite. This presentation is intended not as a comprehensive, systematic study of the finite case, but rather as a device for motivating and facilitating understanding of the continuous trading model that follows in §3. Most of what transpires here can be traced back to the paper by Harrison and Kreps [13].

2a. Formulation of the Market Model

The probability space $(\Omega, \mathscr{F}, P)$ is specified and fixed. The sample space Ω has a finite number of elements, each of which is interpreted as a possible state of the world. We assume $P(\omega) > 0$ for all $\omega \in \Omega$, and this is the only role of the probability measure. We envision a community of investors who agree on which states of the world are possible, but who do not necessarily agree further on their probability assessments. All of our definitions and results remain the same if P is replaced by any equivalent probability measure.

Also specified are a time horizon T, which is a terminal date for all economic activity under consideration, and a filtration $\mathbb{F} = \{\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_T\}$. By this we mean each $\mathscr{F}_t$ is an algebra of subsets of Ω with $\mathscr{F}_0 \subseteq \cdots \subseteq \mathscr{F}_T$. Without any real loss of generality, we assume $\mathscr{F}_0 = \{\phi, \Omega\}$ and $\mathscr{F}_T = \mathscr{F}$ is the set of all subsets.

Securities are traded at times $t = 0, 1, \ldots, T$, and the filtration $\mathbb{F}$ describes how information is revealed to the investors. Each $\mathscr{F}_t$ corresponds to a unique partition $\mathscr{P}_t$ of Ω, and at time t the investors know which cell of this partition contains the true state of the world, but they do not know more than this.

Taken as primitive in our model is a K+1 dimensional stochastic process $S = \{S_t;\ t = 0,1,\ldots,T\}$ with component processes $S^0, S^1, \ldots, S^K$. It is required that each component S^k be strictly positive and adapted to $\mathbb{F}$. The latter means that the function $\omega \to S_t^k(\omega)$ is measurable with respect to $\mathscr{F}_t$ (written $S_t^k \in \mathscr{F}_t$) for each k and t. Interpret S_t^k as the price at time t of security k, so S adapted means that investors know at time t the past and current prices of the K+1 securities.

The zeroth security plays a somewhat special role, because we also assume, and this can be done without loss of generality, that $S_0^0 = 1$. We call this security the bond, even though we make no assumptions that really distinguish it from the other securities. In the continuous theory, the bond will have some special features that set it apart from the other securities. We define a process β by setting $\beta_t = (1/S_t^0)$ and call it the discount process. The reader should think in terms of the special case where $S_t^0 = (1 + r)^t$ with r (the riskless interest rate) constant and positive.

Define a trading strategy to be a predictable vector process $\phi = \{\phi_t;\ t = 1,\ldots,T\}$ with components $\phi^0, \phi^1, \ldots, \phi^K$. Predictable means $\phi_t \in \mathscr{F}_{t-1}$ for $t = 1,\ldots,T$. Interpret ϕ_t^k as the quantity of security k (in physical units like shares) held by the investor between times t-1 and t. The vector ϕ_t will be called the investor's portfolio at time t, and its components may assume negative as well as positive values. In particular, we are permitting unrestricted short sales. By requiring that ϕ be predictable we are allowing the investor to select his time t portfolio after the prices S_{t-1} are observed. However, the portfolio ϕ_t must be established before, and held until after, announcement of the prices S_t.

We pause to introduce some notation. If X and Y are two vector-valued, discrete time stochastic process of the same dimension, then let $X_t Y_s$ denote the inner-product $X_t^1 Y_s^1 + X_t^2 Y_s^2 + \cdots$, and let XY denote the real-valued process whose value at time t is $X_t Y_t$. In addition, let ΔX_t denote the vector $X_t - X_{t-1}$, and let ΔX denote the process whose value at time t is ΔX_t.

Clearly $\phi_t S_{t-1}$ represents the <u>market value</u> of the portfolio ϕ_t just after it has been established at time t-1, whereas $\phi_t S_t$ is its market value just after time t prices are observed, but before any changes are made in the portfolio. Hence $\phi_t \Delta S_t$ is the change in market value due to the changes in security prices that occur between times t-1 and t. If an investor uses trading strategy ϕ, therefore, we see that

$$G_t(\phi) \equiv \sum_{i=1}^{t} \phi_i \Delta S_i , \qquad t = 1, \ldots , T, \tag{2.1}$$

is the cumulative earnings or capital gains that the investor realizes on his holdings up through time t. We set $G_0(\phi) = 0$ and call $G(\phi)$ the <u>gains process</u> associated with ϕ. Note that $G(\phi)$ is an adapted, real-valued stochastic process.

It is important to notice that a general trading strategy ϕ may require the addition of new funds after time zero or allow the withdrawal of funds for consumption. In contrast, we say a trading strategy ϕ is <u>self-financing</u> if

$$\phi_t S_t = \phi_{t+1} S_t, \qquad t = 1, \ldots, T-1 . \tag{2.2}$$

This means that no funds are added to or withdrawn from the value of the portfolio at any of the times $t = 1,\ldots, T-1$. Using (2.1) it is straightforward to check that (2.2) is equivalent to

$$(2.3) \qquad \phi_t S_t = \phi_1 S_0 + G_t(\phi) , \qquad t = 1,\ldots, T .$$

Thus a trading strategy is self-financing if and only if all changes in the value of the portfolio are due to the net gains realized on investments.

We want to add one more restriction. A trading strategy ϕ is called <u>admissible</u> if it is self-financing and $V(\phi)$ is a positive process (hereafter written $V(\phi) \geq 0$), where

$$(2.4) \qquad V_t(\phi) \equiv \begin{cases} \phi_t S_t , & t = 1,\ldots, T \\ \phi_1 S_0, & t = 0 . \end{cases}$$

We call $V(\phi)$ the <u>value process</u> for ϕ, since $V_t(\phi)$ represents the market value of the portfolio held just before time t transactions. By requiring that $V(\phi)$ be positive we are saying not only that the investor must start with positive wealth, but also that his investments must be such that he is never put into a position of debt. This constraint is fairly common in the finance literature. Since security prices are positive, it has the effect of prohibiting certain kinds of short sales. Let Φ denote the set of all admissible trading strategies.

A contingent claim is simply a nonnegative random variable X. It can be thought of as a contract or agreement that pays $X(\omega)$ dollars at time T if state ω pertains. Letting $\mathbf{X}$ denote the set of all such contingent claims, it is easy to see that $\mathbf{X}$ is a convex cone. A contingent claim X is said to be attainable if there exists some $\phi \in \Phi$ such that $V_T(\phi) = X$. In this case we say that ϕ generates X and that $\pi = V_0(\phi)$ is the (time-zero) price associated with this contingent claim. Is this price unique, or can a contingent claim be generated by two different trading strategies with the initial value V_0 being different in each case? This is our next subject.

2b. Viability of the Model

An arbitrage opportunity is some $\phi \in \Phi$ such that $V_0(\phi) = 0$ and yet $E(V_T(\phi)) > 0$. Such a strategy, if one exists, represents a riskless plan for making profit without any investment. It does not require either initial funds or new funds in succeeding periods, but since $V_T(\phi) \geq 0$ it yields, through some combination of buying and selling, a positive gain in some circumstances without a countervailing threat of loss in other circumstances. A security market containing arbitrage opportunities cannot be one in which an economic equilibrium exists.

The purpose of this subsection is to derive two conditions that are equivalent to the assertion that there are no arbitrage opportunities. We begin by defining a price system for contingent claims to be a map $\pi:\mathbf{X} \to [0,\infty)$ satisfying

(2.5a) $\qquad \pi(X) = 0 \quad$ if and only if $\quad X = 0,$

and

(2.5b) $\qquad \pi(aX + bX') = a\pi(X) + b\pi(X')$

$$\text{for all } a, b \geq 0 \text{ and all } X, X' \in \mathbf{X}.$$

Such a price system π is said to be <u>consistent</u> with the market model if $\pi(V_T(\phi)) = V_0(\phi)$ for all $\phi \in \Phi$. Let Π denote the set of all price systems consistent with the model.

Let $\mathbb{P}$ be the set of all probability measures Q that are equivalent to P and are such that the discounted price process βS is a (vector) martingale under Q. The relationship between $\mathbb{P}$ and Π is established in the following (where E_Q is the expectation operator under $Q \in \mathbb{P}$).

(2.6) <u>Proposition</u>. There is one-to-one correspondence between price systems $\pi \in \Pi$ and probability measures $Q \in \mathbb{P}$ via

(i) $\pi(X) = E_Q(\beta_T X)$ and

(ii) $Q(A) = \pi(S_T^0 1_A)$, $A \in \mathscr{F}$.

<u>Proof.</u> Let $Q \in \mathbb{P}$ and define π by (i). Clearly π is a price system. To show it is consistent with the market model, let $\phi \in \Phi$ be arbitrary and notice by (2.2) that

$$\beta_T V_T(\phi) = \beta_T \phi_T S_T + \sum_{i=1}^{T-1} (\phi_i - \phi_{i+1}) \beta_i S_i$$

$$= \sum_{i=2}^{T} \phi_i (\beta_i S_i - \beta_{i-1} S_{i-1}) + \phi_1 \beta_1 S_1.$$

Hence

$$\pi(V_T(\phi)) = E_Q(\beta_T V_T(\phi))$$
$$= E_Q\left(\sum_{i=2}^{T} \phi_i(\beta_i S_i - \beta_{i-1}S_{i-1})\right) + E_Q(\phi_1\beta_1 S_1).$$

Now βS is a martingale under Q and ϕ is predictable, so the first term on the right-hand side equals zero. For the second term we compute $E_Q(\phi_1\beta_1 S_1) = \phi_1 E_Q(\beta_1 S_1) = \phi_1\beta_0 S_0 = V_0(\phi)$, thereby verifying that π is consistent and thus an element of Π.

For the converse, let $\pi \in \Pi$ and define Q by (ii). For each $\omega \in \Omega$ we have $Q(\omega) = \pi(S_T^0 1_\omega) > 0$ since $S_T^0 1_\omega \neq 0$ and π satisfies (2.5a). Now consider the strategy $\phi \in \Phi$ with $\phi^0 = 1$ and $\phi^k = 0$ for $k = 1,\ldots, K$ (hold one bond throughout). Since π is consistent with the model, we have $V_0(\phi) = \pi(V_T(\phi))$, or $1 = \pi(S_T^0)$, or $1 = \pi(S_T^0 1_\Omega)$, or $Q(\Omega) = 1$. Thus Q is a probability measure equivalent to P, and it follows directly from (2.5) that $\pi(X) = E_Q(\beta_T X)$ for any $X \in \mathbf{X}$. Next, let $k \geq 1$ be arbitrary, let $\tau \leq T$ be a stopping time, and consider the strategy $\phi \in \Phi$ defined by

$$\phi_t^k = 1_{\{t \leq \tau\}} \quad , \qquad \phi_t^0 = (S_\tau^k/S_\tau^0)\, 1_{\{t > \tau\}} \quad ,$$

and $\phi^i = 0$ for all other i. This is the strategy which holds one share of stock k up until (through) the stopping time τ, then sells that share of stock and invests all the proceeds in bonds (check that ϕ is predictable). Then $V_0(\phi) = S_0^k$ and $V_T(\phi) = (S_\tau^k/S_\tau^0)S_T^0 = S_T^0\beta_\tau S_\tau^k$, and the consistency of π gives us

$$S_0^k = \pi(S_T^0 \beta_\tau S_\tau^k) = E_Q(\beta_T S_T^0 \beta_\tau S_\tau^k) = E_Q(\beta_\tau S_\tau^k) \ .$$

Since k and τ are arbitrary, this means that βS is a vector martingale under Q, and hence that Q is an element of $\mathbb{P}$.

We now return to the notion of arbitrage opportunities and present the central result of this subsection.

(2.7) <u>Theorem</u>. The market model contains no arbitrage opportunities if and only if $\mathbb{P}$ (or equivalently Π) is nonempty.

<u>Definition</u>. Hereafter we say that the model is <u>viable</u> if the three equivalent conditions of (2.7) hold.

<u>Corollary.</u> If the model is viable, then there is a single price π associated with any attainable contingent claim X, and it satisfies $\pi = E_Q(\beta_T X)$ for each $Q \in \mathbb{P}$.

<u>Remark</u>. This resolves the uniqueness issue raised at the end of §2a. It has also been shown that knowledge of any one $Q \in \mathbb{P}$ allows us to compute (at least in principle) the prices of all attainable claims.

<u>Proof</u>. Suppose $\mathbb{P}$ is nonempty. By (2.6) this is equivalent to Π non-empty. Fix $\pi \in \Pi$ and let $\phi \in \Phi$ be such that $V_0(\phi) = 0$. Then $\pi(V_T(\phi)) = V_0(\phi) = 0$ because π is consistent with the model, and hence

$V_T(\phi) = 0$ by (2.5). Thus no arbitrage opportunities exist. To prove the converse, we need the following preliminary proposition, because we have demanded that admissible strategies have positive value processes.

Lemma. If there exists a self-financing strategy ϕ (not necessarily admissible) with $V_0(\phi) = 0$, $V_T(\phi) \geq 0$, and $E(V_T(\phi)) > 0$, then there exists an arbitrage opportunity.

Proof. If $V(\phi) \geq 0$, then ϕ is admissible and hence is an arbitrage opportunity itself, so we are done. If not, there must exist a $t < T$, $A \in \mathscr{F}_t$ and $a < 0$ such that $\phi_t S_t = a$ on A and $\phi_u S_u \geq 0$ on A for all $u > t$. Define a new trading strategy ψ by setting $\psi_u = 0$ for $u \leq t$, $\psi_u(\omega) = 0$ if $u > t$ and $\omega \notin A$, and, if $u > t$ and $\omega \in A$

$$\psi_u^k(\omega) = \begin{cases} \phi_u^0(\omega) - a/S_t^0(\omega) & \text{for } k = 0 \\ \phi_u^k(\omega) & \text{for } k = 1,2,\ldots, K. \end{cases}$$

Clearly ψ is predictable. For $\omega \in A$ we have

$$\begin{aligned} \psi_{t+1} S_t &= (\phi_{t+1}^0 - a/S_t^0) S_t^0 + \sum_{k=1}^{K} \phi_{t+1}^k S_t^k \\ &= \phi_t S_t - a = 0 \end{aligned}$$

by (2.2) and the definition of a, so it follows that ψ is self-financing. For $u > t$ and $\omega \in A$ we have

$$\psi_u S_u = (\phi_u^0 - a/S_t^0)S_u^0 + \sum_{k=1}^{K} \phi_u^k S_u^k = \phi_u S_u - a(S_u^0/S_t^0) \geq 0 ,$$

so $V(\psi) \geq 0$ and $\psi \in \Phi$. But $S_T^0 > 0$ implies $V_T(\psi) > 0$ on A, so ψ is an arbitrage opportunity. This completes the proof of the lemma.

Back to proving (2.7). Let $\mathbf{X}^+ = \{X \in \mathbf{X}: E(X) \geq 1\}$. Let $\mathbf{X}^0$ be the set of all random variables X on Ω such that $X = V_T(\phi)$ for some self-financing strategy ϕ (not necessarily admissible) with $V_0(\phi) = 0$. Suppose no arbitrage opportunities exist. Then it follows directly from the lemma above that $\mathbf{X}^0$ and $\mathbf{X}^+$ are disjoint (remember that $\mathbf{X}$ contains only positive random variables). Now $\mathbf{X}^+$ is a closed and convex subset of $\mathbb{R}^\Omega$, while $\mathbf{X}^0$ is a linear subspace. Thus by the Separating Hyperplane Theorem there exists a linear functional L on $\mathbb{R}^\Omega$ such that $L(X) = 0$ for all $X \in \mathbf{X}^0$ and $L(X) > 0$ for all $X \in \mathbf{X}^+$. From the latter property (and the linearity) we have $L(1_\omega) > 0$ for all $\omega \in \Omega$. Normalizing, we take $\pi(X) = L(X)/L(S_T^0)$. It is immediate that π satisfies (2.5), so it is a price system. To see that it is consistent with the model ($\pi \in \Pi$), pick $\phi \in \Phi$ and define

$$\psi_t^k = \begin{cases} \phi_t^0 - V_0(\phi) & \text{if } k = 0 \\ \phi_t^k & \text{if } k = 1, \ldots, K. \end{cases}$$

Then ψ is a self-financing strategy (not necessarily admissible) with $V_0(\psi) = 0$ and $V_T(\psi) = V_T(\phi) - V_0(\phi)S_T^0$. Since $V_T(\psi) \in \mathbf{X}^0$, $\pi(X) = 0$ for all $X \in \mathbf{X}^0$, π is linear, and $\pi(S_T^0) = 1$ by normalization, this gives

$$0 = \pi(V_T(\psi)) = \pi(V_T(\phi) - V_0(\phi)\; S_T^0)$$

$$= \pi(V_T(\phi)) - V_0(\phi)\; \pi(S_T^0) = \pi(V_T(\phi)) - V_0(\phi) \quad .$$

So $\pi(V_T(\phi)) = V_0(\phi)$ for all $\phi \in \Phi$, meaning that $\pi \in \Pi$. So no arbitrage opportunities implies Π nonempty, hence $\mathbb{P}$ nonempty by (2.6), and the theorem is proved.

A close look at this proof, and particularly the intermediate lemma, reveals the following. Suppose we had defined admissibility of self-financing strategies by the weaker restriction $V_T(\phi) \geq 0$, meaning that the investor's wealth may go negative at times $t < T$ under plan ϕ, but he must be able to cover all debts in the end. Defining arbitrage opportunities in terms of admissible strategies just as before, Theorem (2.7) would still hold and in the end we would find that $V(\phi) \geq 0$ for all admissible ϕ in a viable model. Thus the weaker definition of admissibility is equivalent to the stronger one if we eventually restrict attention to viable models (as we shall).

Of the three equivalent conditions defining viability, the least abstract and the most meaningful economically is the absence of arbitrage opportunities. Put another way, this condition is the one that justifies our use of the term viable. It is the existence of a martingale measure $Q \in \mathbb{P}$ that is usually easiest to verify in examples, however.

2c. Attainable Claims

We have seen in §2b that for each attainable claim X the associated market price π satisfies $\pi = E_Q(\beta_T X)$ for all $Q \in \mathbb{P}$. But how does one test a given claim X for attainability? First some preliminaries.

(2.8) Proposition. If $\phi \in \Phi$, then the discounted value process $\beta V(\phi)$ is a martingale under each measure $Q \in \mathbb{P}$.

Proof. Since ϕ is self-financing, it is easy to check that $\Delta(\beta V(\phi))_t = \Sigma \phi_t^k \, \Delta(\beta S^k)_t$ with the sum over $k = 1,\ldots, K$ (see the proof of (2.6)). Then (2.8) follows from the predictability of ϕ and the fact that βS is (by definition) a martingale under each $Q \in \mathbb{P}$.

(2.9) Proposition. If $X \in \mathbf{X}$ is attainable, then

$$\beta_t V_t(\phi) = E_Q(\beta_T X | \mathscr{F}_t), \quad t = 0,1,\ldots, T,$$

for any $\phi \in \Phi$ that generates X and each $Q \in \mathbb{P}$.

Proof. Just observe that $V_T(\phi) = X$ for any ϕ that generates X, and then use (2.8).

An immediate implication of (2.9) is the following. If a contingent claim X is attainable, then the value process $V = V(\phi)$ for any $\phi \in \Phi$ that generates X must be

$$V_t = \frac{1}{\beta_t} E_Q(\beta_T X | \mathscr{F}_t), \qquad t = 0,1,\ldots, T, \tag{2.10}$$

where $Q \in \mathbb{P}$ is arbitrary. Furthermore, if V is computed from X by (2.10), and if $\phi \in \Phi$ generates X, then

$$\Delta(\beta V)_t = \sum_{k=1}^{K} \phi_t^k \, \Delta(\beta S^k)_t, \qquad t = 1,\ldots, T, \tag{2.11}$$

as one can easily verify. Note that the bond component ϕ^0 does not enter in (2.11). Finally, one can prove the converse statement as well. The contingent claim X is attainable <u>if and only if</u> there exist predictable processes $\phi^1, \ldots, \phi^K$ such that (2.11) holds, as we'll show in the more general setting of §3. The verification (or refutation) of (2.11) can in principle be done, requiring a separate calculation for each cell of the partition $\mathscr{P}_{t-1}$ and each $t = 1,\ldots, T$. Because this story is quite specific to the finite setting, we'll not continue it, but there is one important qualitative point to understand about the procedure. Its content lies in the fact that V is computed, using (2.10) and any $Q \in \mathbb{P}$, <u>before</u> we know whether or not X is attainable. The question of attainability then comes down to the indicated representation problem.

2d. Complete Markets

The security market model is said to be <u>complete</u> if every contingent claim is attainable. In §3 it will be shown that completeness is equivalent, in the general model, to a certain martingale representation property. Here we wish to state a much sharper characterization of completeness that is entirely specific to the finite case. To eliminate trivial complications, we first impose a non-degeneracy assumption. Recall that $\mathscr{P}_t$ is the partition of Ω underlying $\mathscr{F}_t$. The price process S is said to contain a redundancy if $P(\alpha S_{t+1} = 0|A) = 1$ for some nontrivial vector α, some $t < T$, and some $A \in \mathscr{P}_t$. If such a redundancy exists, then there is an event A possible at time t which makes possession of some one security over the coming period completely equivalent to possession of a linear combination of the other securities over that same period. If no such circumstances exist, then we say that the securities are nonredundant.

For each cell A of $\mathscr{P}_t$ $(t = 0,1,\ldots, T-1)$, let $K_t(A)$ be the number of cells of $\mathscr{P}_{t+1}$ that are contained in A. This might be called the splitting index of A. Assuming that the securities are nonredundant, and (as always) that the model is viable, we must have $K_t(A) \geq K+1$ (the total number of securities) for all t and A. (This fact may not be obvious, but neither is it hard to prove.)

(2.12) Proposition. If the securities are nonredundant, then the model is complete if and only if $K_t(A) = K+1$ for all $A \in \mathscr{P}_t$ and $t = 0, 1,\ldots, T-1$.

A precise proof of this, in its more general form without the nonredundancy assumption, is given by Kreps [20] and we shall not reproduce the argument here. The interested reader should be able to piece together a proof, starting with the single period case $T = 1$. If Ω has n elements, then the space $\mathbf{X}$ of contingent claims is just the positive orthant of $\mathbb{R}^n$, and with $T = 1$ each security k consists of a constant S_0^k and a vector $S_1^k \in \mathbb{R}^n$ whose components specify $S_1^k(\omega)$ for different $\omega \in \Omega$. For completeness it is necessary that each $X \in \mathbf{X}$ be representable as a linear combination of $S_1^0, S_1^1, \ldots, S_1^K$. In the nonredundant case (where $S_1^0, S_1^1, \ldots, S_1^K$ are linearly independent) this comes down precisely to the requirement that $n = K+1$. This argument can then be extended by induction to prove (2.12) for general T.

Thus we see that completeness is a matter of dimension. Speaking very loosely, (2.12) says that in each circumstance A that may prevail at time t, investors must have available enough linearly independent securities to span the space of contingencies that may prevail at time t+1. For a model with many trading dates t and many states ω, completeness depends critically on the way uncertainty resolves itself over time, this being reflected by the splitting indices $K_t(A)$. Again, we refer to Kreps [20] for further discussion.

With continuous trading, no characterization of completeness even remotely similar to (2.12) is known, but a second characterization of completeness for the finite case does have a known general analog. It was observed by Harrison and Kreps [13] that a finite model is complete if and only if $\mathbb{P}$ is a singleton, and a similar result is known to hold in a more general setting, as we'll discuss in §3d.

2e. A Random Walk Model

For a concrete example, consider a finite model with $S_t^0 = (1 + r)^t$, $S_0^1 = \cdots = S_0^K = 1$ and

$$S_t^k = \prod_{s=1}^{t} (1 + a^k \chi_s^k) \qquad \text{for} \quad t = 1, \ldots, T \quad \text{and} \quad k = 1, \ldots, K,$$

where $\{\chi_t^1\}, \ldots, \{\chi_t^K\}$ are independent sequences of IID binary random variables taking values ± 1 with equal probability, and $r, a^1, \ldots, a^K$ are constants satisfying $0 < r < a^k < 1$. The stock price processes are then independent geometric random walks, while security zero is a riskless bond paying interest rate r each period. The reader should have no trouble determining a martingale measure $Q \in \mathbb{P}$ for this model (there are many such Q if $K > 1$, but only one if $K = 1$). Taking $\mathbb{F}$ to be the filtration induced by the price process $\underline{S}$ itself, we see that $K_t(A) = 2^K$ for all A and t. It is easy to verify that these securities are non-redundant, so (2.12) says that <u>this random walk model is complete if and and only if</u> $K = 1$. See Cox, Ross and Rubinstein [6] for an extensive discussion of the $K = 1$ model and its various generalizations. This same paper provides a good introduction to and overview of the modern theory of option pricing, all in the simple setting of a finite model with one stock and one bond.

3. Continuous Trading

This section presents a general model of a frictionless securities market where investors are allowed to trade continuously up to some fixed planning horizon T. The theory closely parallels that developed in §2, so we shall be brief and to the point, pausing only to discuss issues that have no counterparts in the finite case.

We begin now with a probability space $(\Omega, \mathscr{F}, P)$ and a filtration (increasing family of sub-σ-algebras) $\mathbb{F} = \{\mathscr{F}_t;\ 0 \leq t \leq T\}$ satisfying the usual conditions (les conditions habituelles):

(3.1) $\mathscr{F}_0$ contains all the null sets of P, and

(3.2) $\mathbb{F}$ is right-continuous, meaning that

$$\mathscr{F}_t = \bigcap_{s > t} \mathscr{F}_s \quad \text{for} \quad 0 \leq t < T.$$

In fact, without significant loss of generality, it will be assumed that $\mathscr{F}_0$ contains only Ω and the null sets of P, and that $\mathscr{F}_T = \mathscr{F}$. It will ultimately be seen that P plays no role in our theory except to specify the null sets. Hereafter we shall speak of the filtered probability space $(\Omega, \mathbb{F}, P)$.

Let $S = \{S_t; 0 \leq t \leq T\}$ be a vector process whose components $S^0, S^1, \ldots, S^K$ are adapted (meaning $S^k_t \in \mathscr{F}_t$ for $0 \leq t \leq T$), right continuous with left limits (hereafter abbreviated RCLL) and strictly positive. Most of what will be done requires only nonnegative prices, but by assuming strict positivity one avoids various irritating complications.

We assume that S^0 has finite variation and is continuous, interpreting this to mean that security zero (called the bond) is locally riskless. As a convenient normalization, let $S^0_0 = 1$ throughout. If S^0 were absolutely continuous, then we could write

$$S_t^0 = \exp(\int_0^t \gamma_s \, ds) , \qquad 0 \le t \le T ,$$

for some process γ, and then γ_t would be interpreted as the riskless interest rate at time t. We have found that absolute continuity does not significantly simplify any aspect of the theory, however, so we do not assume it. Instead, defining

$$\text{(3.3)} \qquad \alpha_t = \log(S_t^0) , \qquad 0 \le t \le T ,$$

we simply call α the <u>return process</u> for S^0, or the <u>locally riskless return process</u>. Also, let

$$\text{(3.4)} \qquad \beta_t = 1/S_t^0 = \exp(-\alpha_t) , \qquad 0 \le t \le T ,$$

calling β the <u>intrinsic discount process</u> for S. We now interrupt our development of the market model to review some aspects of martingale theory.

3a. Martingales and Stochastic Integrals

A <u>supermartingale</u> is an adapted RCLL process $X = \{X_t; 0 \le t \le T\}$ such that X_t is integrable and $E(X_t|\mathscr{F}_s) \le X_s$ for $0 \le s < t \le T$. The process X is said to be a <u>martingale</u> if both X and -X are supermartingales. All our martingales are uniformly integrable, because they are stopped at time $T < \infty$. This should be kept in mind when comparing our later definitions with those in the general literature. We shall later use the fact that

(3.5) a positive process X is a martingale if and only if it is a supermartingale and $E(X_T) = X_0$,

cf. Lemma (7.10) of Jacod [18]. An adapted RCLL process M is said to be a <u>local martingale</u> [32, p. 291] if there exists an increasing sequence of stopping times $\{T_n\}$ such that

(3.6) $$P\{T_n = T\} \to 1 \quad \text{as} \quad n \to \infty ,$$

and

(3.7) the stopped process $\{M(t \wedge T_n);\ 0 \leq t \leq T\}$ is a martingale for each n,

in which case the sequence $\{T_n\}$ is said to <u>reduce</u> M. As (3.7) illustrates, we shall write the time parameter of a process as a functional argument (rather than a subscript) if this is necessary to avoid clumsy typography.

Clearly, every martingale is a local martingale, and it follows easily from Fatou's Lemma that

(3.8) every positive local martingale is also a supermartingale.

Combining this with (3.5), we see that

(3.9) a positive local martingale M is a martingale if and only if $E(M_T) = M_0$.

A process $A = \{A_t; 0 \leq t \leq T\}$ is said to be in the class VF (for <u>variation finie</u>), or simply a VF process, if it is adapted, RCLL, and has sample paths of finite variation [32, p. 249]. A process X is called a <u>semimartingale</u> [32, p. 298] if it admits a decomposition $X = M + A$, where M is a local martingale and A is a VF process. This <u>canonical decomposition</u> is not generally unique.

We say that $H = \{H_t; 0 \leq t \leq T\}$ is a <u>simple predictable process</u> if there exist times $0 = t_0 < t_1 < \cdots < t_n = T$ and bounded random variables $\xi_0 \in \mathscr{F}_0, \xi_1 \in \mathscr{F}_{t_1}, \ldots, \xi_{n-1} \in \mathscr{F}_{t_{n-1}}$ such that

(3.10) $$H_t = \xi_i \quad \text{if} \quad t_i < t \leq t_{i+1} \quad (i = 0,1,\ldots, n-1).$$

Thus simple predictable processes are bounded, adapted, left-continuous, and piecewise constant. The predictable σ-algebra on $\Omega \times [0,T]$ is defined to be the one generated by the simple predictable processes (a variety of equivalent definitions can be found in the literature). A process $H = \{H_t; 0 \leq t \leq T\}$ is said to be predictable if it is measurable with respect to the predictable σ-algebra. Every predictable process is adapted.

Meyer [32, p. 299] says that a process H is locally bounded if

(3.11) there exist constants $\{C_n\}$ and stopping times $\{T_n\}$ satisfying (3.6) such that $|H_t| \leq C_n$ for $0 \leq t \leq T_n$ and $n = 1,2,\ldots$.

In his discussion of the Lebesgue Stochastic Integral, Dellacherie [9] defines local boundedness by the weaker requirement

$$\sup_{0 \leq t \leq T} |H_t| < \infty , \tag{3.12}$$

but the discrepancy is resolved (for our purposes) by the following result, which Dellacherie [9] cites in a footnote and attributes to Lenglart.

(3.13) Conditions (3.11) and (3.12) are equivalent for predictable processes.

Also, it is well-known that

(3.14) an adapted process that is left-continuous with right limits (LCRL) is both predictable and locally bounded.

Now consider a semimartingale X together with a simple predictable process H satisfying (3.10). The stochastic integral $Z = \int H \, dX$ is then defined path-by-path in the Lebesgue-Stieltjes sense, meaning (remember that H is left-continuous while X is right-continuous) $Z_0 = 0$ and

$$Z_t = \sum_{j=0}^{i-1} \xi_j (X_{t_{j+1}} - X_{t_j}) + \xi_i (X_t - X_{t_i}) \quad \text{if} \quad t_i < t \leq t_{i+1}.$$

If H is now a general locally bounded and predictable process, the stochastic integral $Z = \int H\, dX$ can be defined by continuously extending what we have for simple predictable processes, cf. Dellacherie [9] or Meyer [32, Ch. 4]. Incidentally, when we write $Z = \int H\, dX$ we mean $Z_0 = 0$ and

$$Z_t = \int_0^t H_s\, dX_s = \int_{(0,t]} H_s\, dX_s \,, \qquad 0 < t \leq T \,.$$

Observe that predictability and local boundedness are both preserved under substitution of an equivalent measure, and the semimartingale property is also invariant to such substitutions [32, p. 376]. Finally, the stochastic integral $\int H\, dX$ described above enjoys this same invariance. The fact that all these definitions depend only on the null sets of the underlying probability measure is important in our setting.

The definition of stochastic integrals in terms of predictable integrands is precisely what is needed for economic modeling, and it yields the following key result [32, p. 299].

(3.15) If H is locally bounded and predictable and M is a local martingale, then $\int H\, dM$ is a local martingale as well.

If we further assume that M is a martingale, it may <u>not</u> be true that $\int H\, dM$ is a martingale (there are familiar counter-examples in the Ito theory where M is Brownian motion). It cannot be emphasized too strongly that (3.15) only holds when one restricts attention to <u>predictable</u> integrands H.

If Z is the stochastic integral $\int H\,dX$ as above, then Z is itself a semimartingale (hence RCLL) with

$$\Delta Z_t = H_t\,\Delta X_t\ , \qquad 0 \le t \le T\ , \tag{3.16}$$

where we use the standard notation $\Delta Z_t = Z_t - Z_{t-}$ for the jump of Z at t. We shall write ΔZ and Z_- to denote the processes $\{\Delta Z_t; 0 \le t \le T\}$ and $\{Z_{t-}; 0 \le t \le T\}$ respectively. Incidentally, the definition of the general stochastic integral $\int H\,dX$ agrees with the Ito integral in the case where X is Brownian motion (although we are restricting ourselves to a slightly smaller class of integrands than is customary in developing the Ito theory), and it amounts to a path-by-path Lebesgue-Stieltjes integral when X is a VF process.

Let X and Y be semimartingales. Since X_- and Y_- are LCRL and adapted, (3.14) shows that it is meaningful to define a new process $[X,Y]$ by

$$[X,Y]_t = X_tY_t - \int_0^t X_{s-}\,dY_s - \int_0^t Y_{s-}\,dX_s, \quad 0 \le t \le T\ . \tag{3.17}$$

An equivalent definition is the following [9]. Let $t_i^n = it/2^n$ for $n = 1,2,\ldots$ and $i = 0,1,\ldots, 2^n$. Then

$$[X,Y]_t = X_0Y_0 + \lim_n \sum_i (X(t_{i+1}^n) - X(t_i^n))(Y(t_{i+1}^n) - Y(t_i^n))$$

where the convergence is in probability. This latter definition explains why $[X,Y]$ is called the <u>joint variation</u> of X and Y, with $[X,X]$ called the <u>quadratic variation</u> of X. This is yet another definition which is invariant to substitution of an equivalent probability measure.

Here are a few more properties of the joint variation that will be used later. First, $[X,Y]$ is always a VF process [32, p. 267], and moreover

(3.18) $$[X,Y] = \sum_{s \leq t} \Delta X_s \, \Delta Y_s \quad \text{if either } X \text{ or } Y \text{ is VF.}$$

In particular, if X (say) is continuous and VF, then (3.18) gives $[X,Y] = 0$ for any semimartingale Y. Finally, from (3.17) and the finite variation of $[X,Y]$ it is immediate that

(3.19) the product of two semimartingales is itself a semimartingale.

A process X is said to be integrable (under P) if $E(|X_t|) < \infty$, $0 \leq t \leq T$. It is said to be locally integrable if there exist stopping times $\{T_n\}$ satisfying (3.6) such that $\{X(t \wedge T_n);\ 0 < t \leq T\}$ is integrable for each n.

3b. A Preliminary Market Model

Picking up where we left off before §3a, it will be convenient to define a discounted price process $Z = (Z^1, \ldots, Z^K)$ by setting

$$Z_t^k = \beta_t S_t^k, \qquad 0 \leq t \leq T \ \text{ and } \ k = 1, \ldots, K .$$

Note that Z has only K components. Let $\mathbb{P}$ be the set of probability measures Q on $(\Omega, \mathscr{F})$ that are equivalent to P and such that Z is a (vector) martingale under Q. This is of course the same as requiring that βS be a martingale under Q, since $\beta S^0 = 1$ is a martingale under any measure equivalent to P. Elements of $\mathbb{P}$ are called martingale measures. We shall henceforth impose the following

(3.20) <u>Assumption</u>. $\mathbb{P}$ is nonempty.

The primitive acceptance of (3.20) constitutes a major difference in our treatment of the finite and continuous cases. All of §2b, culminating in Theorem (2.7), was devoted to proving that in a finite setting (3.20) is equivalent to the nonexistence of arbitrage opportunities, which is an economically palatable assumption. For the continuous case, one can in fact prove a general version of Theorem (2.7), but the proper definition of an arbitrage opportunity and the ensuing mathematical development are extremely complex. A proper treatment of viability for continuous models requires a paper in itself, so we just rely here on the formal analogy with the finte theory, referring interested readers to Harrison and Kreps [13] for more on viability in a general setting.

We have that S^0 is a VF process (and thus a semimartingale), that Z^k is a martingale under any $Q \in \mathbb{P}$, and that $S^k = Z^k/\beta = S^0 Z^k$. Then from (3.19) it follows that S^k is a semimartingale under Q and thus also under P (recall that the semimartingale property is invariant under substitution of an equivalent measure). Hence S is a vector semimartingale.

In order to verify (3.20), and later to compute the prices of attainable contingent claims (see §3c), it is necessary to actually determine at least one martingale measure $Q \in \mathbb{P}$. This will be done later for some concrete examples, but it should also be noted that there exists a well developed <u>general</u> theory on change of measure for semimartingales. The general form of <u>Girsanov's Theorem</u> [32, pp. 376-379] shows that to find a $Q \in \mathbb{P}$ one must find a strictly positive martingale M that bears a certain relationship (involving joint variation) to the discounted price process Z. A nice account of this general theory is given in the second volume of Dellacherie and Meyer [10], for which an English translation should soon be available.

A trading strategy is (temporarily) defined as a K+1 dimensional process $\phi = \{\phi_t; 0 \leq t \leq T\}$ whose components $\phi^0, \phi^1, \ldots, \phi^K$ are locally bounded and predictable (see §3a). With each such strategy ϕ we associate a value process $V(\phi)$ and a gains process $G(\phi)$ by

$$(3.21) \qquad V_t(\phi) = \phi_t S_t = \sum_{k=0}^{K} \phi_t^k S_t^k , \qquad 0 \leq t \leq T ,$$

$$(3.22) \qquad G_t(\phi) = \int_0^t \phi_u \, dS_u = \sum_{k=0}^{K} \int_0^t \phi_u^k \, dS_u^k , \quad 0 \leq t \leq T .$$

As in the finite theory, we interpret $V_t(\phi)$ as the market value of the portfolio ϕ_t, and $G_t(\phi)$ as the net capital gains realized under strategy ϕ through time t. But why should trading strategies be predictable, and why does the stochastic integral give the right definition of capital gains? Continuing our practice of ducking foundational issues, we shall say rather little on this important subject. It is obvious that simple predictable strategies (see §3a) should be allowed, and that $G(\cdot)$ gives the right notion of capital gains for such strategies. In fact, the definition of $G(\phi)$ for simple predictable ϕ essentially reduces to that used earlier in the finite theory. The ultimate defense of our set-up must then rely on the fact that each predictable strategy ϕ can be approximated (in a certain sense) by a sequence of simple predictable strategies $\{\phi_n\}$ such that $G(\phi) = \int \phi \, dS$ is the limit (in a certain sense) of $G(\phi_n) = \int \phi_n \, dS$. The restriction to predictable strategies serves to limit in an essential way what investors can do at jumps times of the price process. If S is continuous, one need not worry about predictability at all: using the same forward-looking (or nonanticipating) definition of the stochastic integral, one could allow all trading strategies that are optional (adapted and just a bit more).

We say that a trading strategy ϕ is <u>self-financing</u> if

$$V_t(\phi) = V_0(\phi) + G_t(\phi) , \qquad 0 \leq t \leq T . \tag{3.23}$$

Since the stochastic integral $G(\phi)$ is adapted and RCLL, we see that $V(\phi)$ is adapted and RCLL for any self-financing ϕ. Now let Φ be the class of all self-financing strategies ϕ such that $V(\phi) \geq 0$. This is the precise continuous counterpart to what we had as the set of admissible trading strategies in the finite theory. Unfortunately, Φ will not do as the set of admissible strategies in the continuous theory. Shortly we shall discuss the problems with Φ, and the necessary modifications will be made later. But first a preliminary result is needed. For any trading strategy ϕ, let us agree to write

$$G^*(\phi) = \int \phi \, dZ = \sum_{k=1}^{K} \int \phi^k \, dZ^k ,$$

with the bond component ϕ^0 playing no role. We also introduce the notation

$$V^*(\phi) = \beta V(\phi) = \phi^0 + \sum_{k=1}^{K} \phi^k Z^k ,$$

calling $G^*(\phi)$ and $V^*(\phi)$ the <u>discounted gains process</u> and <u>discounted value process</u> respectively for strategy ϕ.

(3.24) <u>Proposition</u>. Let ϕ be any trading strategy. Then ϕ is self-financing if and only if $V^*(\phi) = V_0^*(\phi) + G^*(\phi)$, and of course $V(\phi) \geq 0$ if and only if $V^*(\phi) \geq 0$.

(3.25) Remark. Thus all our essential definitions can be equivalently recast in terms of discounted quantities. Henceforth we shall deal exclusively in terms of the more convenient discounted formulation. See (3.27) below.

(3.26) Corollary. If $\phi \in \Phi$, then $V^*(\phi)$ is a positive local martingale, and also a supermartingale, under each $Q \in \mathbb{P}$.

Proof. For (3.24), suppose first that ϕ is self-financing, meaning that $V(\phi) = V_0(\phi) + G(\phi)$. Then $\Delta V(\phi) = \Delta G(\phi) = \phi\Delta S$ and hence

$$V_-(\phi) = V(\phi) - \Delta V(\phi) = \phi S - \phi\Delta S = \phi S_- \quad .$$

Since β is a continuous VF process, (3.18) gives $[\beta, V(\phi)] = 0$, and then from the definition (3.17) of the joint variation (and the continuity of β)

$$\begin{aligned} dV^*(\phi) &= d(\beta V(\phi)) = \beta_-\ dV(\phi) + V_-(\phi)\ d\beta \\ &= \beta dV(\phi) + V_-(\phi)\ d\beta = \beta\ dG(\phi) + \phi S_-\ d\beta \\ &= \beta\phi\ dS + \phi S_-\ d\beta = \phi(\beta\ dS + S_-\ d\beta)\ . \end{aligned}$$

But similarly, $dZ = d(\beta S) = \beta\ dS + S_-\ d\beta$, so we have $dV^*(\phi) = \phi\ dZ$, which means precisely that $V^*(\phi) = V_0^*(\phi) + \int \phi\, dZ = V_0^*(\phi) + G^*(\phi)$, the desired conclusion. The proof of the converse is virtually identical, so we delete it. The Corollary (3.26) is immediate from (3.15), the fact that $V^*(\phi) \geq 0$, and (3.8).

(3.27) Remark. Recall that $G^*(\phi)$ does not depend on the bond component ϕ^0. Thus (3.24) shows that a self-financing strategy ϕ is completely determined by its initial value $V_0^*(\phi)$ and its stock components.

More particularly, any set of locally bounded and predictable processes $\phi^1, \ldots, \phi^K$ can be uniquely extended to a self-financing strategy ϕ with specified initial value $V^*(\phi) = v$ by setting

$$\phi_t^0 = v + \sum_{k=1}^{K} \int_0^t \phi_s^k \, dZ_s^k - \sum_{k=1}^{K} \phi_t^k Z_t^k , \qquad 0 \leq t \leq T ,$$

since this is the unique choice of ϕ^0 that will give us $V^*(\phi) = v + G^*(\phi)$. Obviously $\phi \in \Phi$ if and only if $v + G^*(\phi) \geq 0$.

Now what happens if we declare all strategies $\phi \in \Phi$ to be admissible? If one defines an arbitrage opportunity as a strategy $\phi \in \Phi$ for which $V_0(\phi) = 0$ but $V_T(\phi) > 0$ with positive probability, then it follows from (3.26) that none of these exist. Because $V^*(\phi)$ is known to be a positive supermartingale under any $Q \in \mathbb{P}$, it must remain at zero if it starts there. So there are no strategies in Φ that turn nothing into something, but there may be (and generally are) strategies that turn something into nothing. In §6a we'll give an example (for the Black-Scholes model of §1a) of a suicide strategy $\phi \in \Phi$ such that $V_0(\phi) = 1$ but $V_T(\phi) = 0$. If all strategies $\phi \in \Phi$ were allowed, the prices of attainable contingent claims in the Black-Scholes model would therefore never be unique. Having determined that a claim X is attainable at price π using some ϕ, we can always add to ϕ the suicide strategy and thus attain X at price $\pi+1$. (Attainable claims and their associated prices have not been formally defined in this section, but we trust that the spirit of these remarks is clear from all that has gone before.) So the first problem with Φ is that it contains too many strategies, since we want each attainable claim to have a unique associated price. We are going to remedy this by fixing a reference measure $P^* \in \mathbb{P}$ and restricting attention to strategies ϕ for which

$V^*(\phi)$ is a martingale, not just a local martingale, under P^*. This will of course eliminate the suicide strategy alluded to above.

Although Φ is slightly too large in the sense just discussed, it is slightly too small in a different sense. Roughly stated, the space of locally bounded predictable strategies lacks a sort of closure property that we need to get a clean result on completeness. If one wants all contingent claims (or even all bounded claims) to be attainable in the Black-Scholes model, for example, one must allow some strategies that are not locally bounded. We now introduce a set Φ^* of admissible strategies that is just right for our purposes.

3c. The Final Formulation

Let us select and fix a reference measure $P^* \in \mathbb{P}$, denoting by $E^*(\cdot)$ the associated expectation operator. Until further notice, when we speak of martingales and local martingales, the underlying probability measure is understood to be P^*. We define $\mathscr{L}(Z)$ as the set of all predictable processes $H = (H^1, \ldots, H^K)$ such that the increasing process

$$(3.28) \qquad \left(\int_0^t (H_s^k)^2 \; d[Z^k, Z^k]_s\right)^{1/2}, \qquad 0 \leq t \leq T,$$

is locally integrable (see §3a) under P^* for each $k = 1, \ldots, K$. It can be verified that $\mathscr{L}(Z)$ contains all locally bounded and predictable H, and moreover $\int H\,dZ$ is still a local martingale for these integrands [32, p. 341].

We now expand our definition of a trading strategy to include all predictable $\phi = (\phi^0, \phi^1, \ldots, \phi^K)$ such that $(\phi^1, \ldots, \phi^K) \in \mathscr{L}(Z)$. With $V^*(\phi) = \beta\phi S$ and $G^*(\phi) = \int \phi dZ$ as before, a trading strategy ϕ is said to be <u>admissible</u> if $V^*(\phi) \geq 0$, $V^*(\phi) = V_0^*(\phi) + G^*(\phi)$, and

$$V^*(\phi) \text{ is a martingale (under } P^*) \tag{3.29}$$

Let Φ^* be the class of all admissible trading strategies. The last condition (3.29) looks awful, but verifying (or refuting) it is not a problem that ever arises if one is interested only in contingent claim valuation. See (3.33) below. Obviously (3.29) is equivalent to requiring that $G^*(\phi) = \int \phi \, dZ$ be a martingale, and by (3.9) it is also equivalent to the simple condition

$$E^*[V_T^*(\phi)] = V_0^*(\phi) \ . \tag{3.30}$$

A <u>contingent claim</u> is formally defined as a positive random variable X (remember $\mathscr{F} = \mathscr{F}_T$ by convention). Such a claim is said to be <u>attainable</u> if there exists $\phi \in \Phi^*$ such that $V_T^*(\phi) = \beta_T X$, in which case ϕ is said to <u>generate</u> X and $\pi = V_0^*(\phi)$ is called the <u>price associated with</u> X.

(3.31) <u>Proposition.</u> The unique price π associated with an attainable claim X is $\pi = E^*(\beta_T X)$.

This is of course immediate from (3.30). Hereafter we shall say that a claim X is <u>integrable</u> if $E^*(\beta_T X) < \infty$, and similarly <u>bounded</u> means that $\beta_T X$ is bounded. From the definition, it is immediate that only integrable claims can be attainable. We now give a more or less concrete test for attainability.

(3.32) <u>Proposition</u>. Let X be an integrable contingent claim and let V^* be the RCLL modification of

$$V_t^* = E^*(\beta_T X | \mathscr{F}_t), \qquad 0 \leq t \leq T \ .$$

Then X is attainable if and only if V^* can be represented in the form $V^* = V_0^* + \int H\,dZ$ for some $H \in \mathscr{L}(Z)$, in which case $V^*(\phi) = V^*$ for any $\phi \in \Phi^*$ that generates X.

<u>Remark</u>. Note that the candidate value process V^* is computed before we know whether or not X is attainable.

<u>Proof</u>. Suppose X is attainable, generated by some $\phi \in \Phi^*$. Let $H^k = \phi^k$ for $k = 1,\ldots, K$ so that $\int H\,dZ = G^*(\phi)$. Since $\beta_T X = V_T^*(\phi)$ and $V^*(\phi)$ is a martingale by (3.29), we have that

$$V_t^* = E^*(\beta_T X|\mathscr{F}_t) = E^*(V_T^*(\phi)|\mathscr{F}_t) = V_t^*(\phi) \ .$$

But $V^*(\phi) = V_0^*(\phi) + G_t^*(\phi) = V_0^*(\phi) + \int_0^t H\,dZ$ because $\phi \in \Phi^*$, so we have the desired representation.

For the converse, let X be an integrable claim, define V^* as indicated, and suppose that $V^* = V_0^* + \int H\,dZ$ for $H \in \mathscr{L}(Z)$. Set $\phi^1 = H^1,\ldots, \phi^K = H^K$, and then define ϕ^0 as in (3.27), with $v = V_0^*$, thus yielding a trading strategy ϕ with

$$V^*(\phi) = V_0^* + G^*(\phi) = V_0^* + \int H\,dZ = V^* \ .$$

Obviously V^* is a positive martingale by its very definition, so ϕ is an admissible strategy with $V_T^*(\phi) = \beta_T X$. Thus X is attainable, generated by ϕ.

(3.33) <u>Remark</u>. Note that the trading strategy ϕ constructed in the second half of the proof, starting with the integrand H that appears in the representation, <u>automatically</u> satisfies the sticky condition (3.29) because of the way we defined V^* in the first place.

3d. Complete Markets (Representation of Martingales)

We say that the market model of §3c is complete if every integrable claim is attainable. Before proceeding with the analysis of complete markets, let us establish that nothing is added to this definition by considering claims payable before the terminal date T. Suppose we define a (wide sense) contingent claim as a pair (t,X) with $0 \leq t \leq T$ and $X \in \mathscr{F}_t$, making the obvious interpretations. We say that (t,X) is attainable if there exists $\phi \in \Phi^*$ such that $V_t^*(\phi) = \beta_t X$. Defining integrability of (t,X) by the requirement $E^*(\beta_t X) < \infty$, we then say that the model is (wide sense) complete if every integrable (wide sense) claim is attainable. Suppose the model is complete according to our original definition, fix (t,X) and consider the pair (T,X') where $X' = \beta_t X S_T^0$. Obviously $E^*(\beta_T X') = E^*(\beta_t X) < \infty$, so X' is an integrable claim (payable at T). Letting $\phi \in \Phi^*$ be a strategy that attains X' (remember we assumed completeness in the narrow sense), we know that $V^*(\phi)$ is a martingale under P^* with $V_T^*(\phi) = \beta_T X'$. Thus

$$V_t^*(\phi) = E^*(V_T^*(\phi)|\mathscr{F}_t) = E^*(\beta_T X'|\mathscr{F}_t) = E^*(\beta_t X|\mathscr{F}_t) = \beta_t X ,$$

so ϕ also attains (t,X) and we conclude that (wide sense) completeness is equivalent to completeness in the original narrow sense.

All notation and conventions established in the last section remain in force. In particular, the term martingale implicitly refers to the reference measure P^*. Let $\mathscr{M}$ be the set of all martingales, and let $\mathscr{M}(Z)$ consist of all $M \in \mathscr{M}$ repesentable in the form

$$M = M_0 + \int H\, dZ \quad \text{for some } H \in \mathscr{L}(Z) . \tag{3.34}$$

If $\mathscr{M} = \mathscr{M}(Z)$ then we say that Z has the martingale representation property for $(\Omega, \mathbb{F}, P^*)$. This definition of course involves the filtration $\mathbb{F}$ in a fundamental way. Roughly speaking, it says that Z provides a <u>basis</u> for the space $\mathcal{M}$, or that Z <u>spans</u> $\mathcal{M}$, with stochastic integrals playing the role of linear combinations.

(3.35) <u>Theorem</u>. The model is complete if and only if $\mathcal{M} = \mathcal{M}(Z)$.

(3.36) <u>Corollary</u>. If $\mathbb{P}$ is a singleton, then the model is complete.

Theroem (3.35) follows immediately from (3.32), using the fact that any martingale can be expressed as the difference of two positive martingales. Corollary (3.36) comes from the general theory of representation of martingales. Specifically it follows from the results on pp. 337-345 of Jacod [18], using the fact that if P^* is the sole element $\mathbb{P}$, then P^* is an extreme point of the set of all probability measures under which Z is a martingale. Using the general theory in Chapter XI of Jacod [18], (3.36) can actually be strengthened to say that the model is complete <u>if and only if</u> P^* is an extreme point of a certain set. To state this result precisely requires some additional, rather technical definitions, so we shall not pursue the matter further. Jacod's general theorems on representation of martingales have an obvious aesthetic appeal, and they provide a potential means of establishing the completeness of any given market model, but there is nothing comparable to the very explicit characterization of complete finite markets that was given in §2d. That result makes one feel that the ultimate characterization of complete continuous markets should involve the fine structure of filtration $\mathbb{F}$.

Moving on to more concrete issues, suppose that $\mathbb{F} = \mathbb{F}^S$, the minimal filtration (satisfying the usual conditions) with respect to which S is adapted. This is interpreted to mean that investors only have access to (or at least are obliged to base their trading decisions solely on) past and present price information. Let us further assume that S^0 (the bond price process) is deterministic, this giving $\mathbb{F}^S = \mathbb{F}^Z$ because $Z = \beta S$. In the general set-up, completeness is a joint property of $(\Omega, \mathbb{F}, P^*)$ and Z, but now Z actually determines the filtration, so there is no need to mention the underlying space at all. Thus we are led to say that a <u>martingale</u> Z is <u>complete</u> if every other martingale M over $\mathbb{F}^Z$ can be represented as $M = M_0 + \int H\,dZ$ with H predictable.

We shall now discuss some martingales that are known to be complete in at least roughly the sense of the last paragraph. Certainly the oldest known result of this type concerns the completeness of one-dimensional Brownian motion (which implies that every contingent claim is attainable in the Black-Scholes model). Clarke [4] attributes this to Ito [14], and different proofs have been given by Kunita and Watanabe [23], Dellacherie [8], and doubtless many others. Multidimensional Brownian motion is also complete, as we'll discuss in §5, although its natural analog in discrete time is not (see §2e). Jacod [17] says that more general types of diffusion processes are known to be complete, as one can easily deduce from the result for Brownian motion itself, but we do not know a good reference on that subject. The Poisson martingale $cN_t - c\lambda t$, where c is a real constant and N is a Poisson process of intensity λ, is also known to be complete. This result is usually ascribed to Kunita and Watanabe [23], and it has been generalized to arbitrary point processes [15]. Finally, it is well known, although we cannot produce a reference, that the Wiener and Poisson martingales are the only complete one-dimensional martingales having stationary, independent increments.

4. Return Processes and the Semimartingale Exponential

It is customary in financial economics to specify not price processes themselves but rather the corresponding return processes (see §1a). In this section we describe briefly the general mathematical nature of that correspondence.

4a. Exponentiation

Let $X = \{X_t;\ 0 \leq t \leq T\}$ be a semimartingale and consider the equation

$$U_t = U_0 + \int_0^t U_s \, dX_s \ , \qquad 0 \leq t \leq T \ , \tag{4.1}$$

where $U_0 \in \mathscr{F}_0$ is also given. We would like to find a semimartingale U satisfying this equation. It turns out [32, p. 304] that (4.1) always has a (semimartingale) solution, it is unique, and it is given by

$$U_t = U_0 \mathscr{E}_t(X), \qquad 0 \leq t \leq T \ , \tag{4.2}$$

where

$$\mathscr{E}_t(X) = \exp(X_t - X_0 - \frac{1}{2}\,[X,X]_t) \prod_{s \leq t} (1 + \Delta X_s) \exp(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2). \tag{4.3}$$

This process $\mathscr{E}(X)$ is called the <u>exponential of X in the semimartingale sense</u>. Note that $\mathscr{E}_0(X) = 1$. A key property of the semimartingale exponential is [32, p. 306]

$$(4.4) \qquad \mathscr{E}(X)\,\mathscr{E}(Y) = \mathscr{E}(X + Y + [X,Y]) \quad \text{for any two semimartingales } X \text{ and } Y.$$

Since $[X,Y] = 0$ if either X or Y is continuous and VF (see §3a), this means

$$(4.5) \qquad \mathscr{E}(X)\,\mathscr{E}(Y) = \mathscr{E}(X + Y) \quad \text{if } X \text{ is any semimartingale and } Y \text{ is continuous and VF.}$$

Let $\mathscr{R}$ be the set of semimartingales X such that $1 + \Delta X \geq 0$ and let $\mathscr{R}^+$ be those semimartingales X satisfying the stronger condition that $1 + \Delta X > 0$. Then from (4.3) it follows that

$$(4.6) \qquad \begin{aligned} &\mathscr{E}(X) \geq 0 \text{ if and only if } X \in \mathscr{R}, \text{ and} \\ &\mathscr{E}(X) > 0 \text{ if and only if } X \in \mathscr{R}^+. \end{aligned}$$

4b. <u>Return Processes</u>

Our price process S^k and its corresponding return process R^k are related to each other via equation (4.1), with S^k instead of U and R^k instead of X. Rearranging (4.1), we see R^k expressed in terms of S^k by

$$(4.7) \qquad R_t^k = \int_0^t (1/S_{u-}^k)\, dS_u^k, \qquad 0 \leq t \leq T; \quad k = 0,1,\ldots, K .$$

In the case of the continuous VF bond, this simplifies to

$$(4.8) \qquad R_t^0 = \log(S_t^0) = \alpha_t, \qquad 0 \leq t \leq T .$$

We set $R = (R^0, R^1, \ldots, R^K)$ and call R^k the <u>return process</u> for security k.

The following argument shows that (4.7) really does define R^k unambiguously in terms of S^k (remember that we assume $\mathbb{P}$ non-empty throughout). The discounted price process Z^k (see §3b) is a strictly positive martingale under each $Q \in \mathbb{P}$, so Z^k_- is strictly prositive and left continuous, implying that $(1/S^k_-)$ is locally bounded. So the stochastic integral in (4.7) is well defined, the integrand being locally bounded and the integrator being a semimartingale.

Since (4.7) is equivalent to the statement that $dS^k = S^k_- \, dR^k$, we see from §4a that S^k and R^k are also related by the semimartingale exponential. That is,

$$S^k = S^k_0 \, \mathscr{E}(R^k) \;, \qquad k = 0,1,\ldots, K. \tag{4.9}$$

By (4.6) and the strict positivity of S^k we see that $R^k \in \mathscr{R}^+$ for $k = 0,\ldots, K$.

Consider now the discounted price process Z^k. We have $\beta = \exp(-\alpha) = \mathscr{E}(-\alpha)$, so (4.5) gives

$$Z^k = \beta S^k = \mathscr{E}(-\alpha) \; S^k_0 \, \mathscr{E}(R^k) = Z^k_0 \, \mathscr{E}(R^k - \alpha). \tag{4.10}$$

Defining the <u>discounted return process</u> $Y = (Y^1,\ldots, Y^K)$ by

$$Y^k_t = R^k_t - \alpha_t, \qquad 0 \leq t \leq T, \; k = 1,\ldots, K, \tag{4.11}$$

equation (4.10) says

$$Z^k = Z^k_0 \, \mathscr{E}(Y^k) \;. \tag{4.12}$$

Thus Y^k plays the same role for Z^k as R^k does for S^k. We emphasize that the tidy relationship (4.11) depends crucially on our assumption that α is continuous and VF so that $[R^k, -\alpha] = 0$.

5. A Multidimensional Diffusion Model

We consider now a generalization of the Black-Scholes model (see § 1a) that has a bond and K correlated stocks. The bond price process is $S_t^0 = \exp(rt)$, $0 \leq t \leq T$, with r a real constant as before, and each individual stock price process $S^1, \ldots, S^K$ is to be a geometric Brownian motion. To specify the model precisely, it will be convenient to construct first the discounted return process Y (see § 4b), then the discounted stock price process Z (see § 3b), and finally the processes S^k themselves. We continue to denote components of vectors by superscripts, except in a few isolated instances where doing so is hopelessly impractical.

Let $\Lambda = (\lambda_{ij})$ be a non-singular $K \times K$ matrix, and define a covariance matrix (symmetric and positive definite) $A = (a_{ij})$ by setting $A = \Lambda\Lambda'$, meaning that

$$a_{ij} = \sum_{k=1}^{K} \lambda_{ik}\lambda_{jk} \quad \text{for} \quad i,j = 1,\ldots, K \,. \tag{5.1}$$

Let $\mu = (\mu^1, \ldots, \mu^K)$ be a vector of real constants. Next, let $W^1, \ldots, W^K$ be independent standard Brownian motions with $W_0^1 = \cdots = W_0^K = 0$, defined on some probability space $(\Omega, \mathscr{F}, P)$. Then set

$$Y_t = \Lambda W_t + \mu t \quad , \quad 0 \leq t \leq T \,, \quad \text{meaning} \tag{5.2}$$

$$Y_t^k = \sum_{j=1}^{K} \lambda_{kj} W_t^j + \mu^k t, \quad 0 \leq t \leq T \,, \quad k = 1,\ldots, K \,. \tag{5.3}$$

Thus Y is a vector Brownian motion with covariance matrix A and drift vector μ. Now let $Z_0^1, \ldots, Z_0^K$ be strictly positive constants and set

$$(5.4) \qquad Z_t^k = Z_0^k \exp(Y_t^k - \tfrac{1}{2} a_{kk} t) , \qquad 0 \leq t \leq T ,$$

for $k = 1, \ldots, K$. Ito's Formula gives us

$$(5.5) \qquad Z_t^k = Z_0^k + \int_0^t Z_s^k dY_s^k , \qquad 0 \leq t \leq T ,$$

so $Z^k = Z_0 \mathscr{E}(Y^k)$ as in § 4b. Furthermore,

$$(5.6) \qquad d[Z^i, Z^j]_t = Z^i Z^j d[Y^i, Y^j]_t = Z^i Z^j a_{ij} dt .$$

The first equality in (5.6) follows from (5.5) and the basic joint variation property of stochastic integrals [32, p. 271], and the second is a well-known property of Brownian motion. Now define

$$(5.7) \qquad S_t^k = S_t^0 Z_t^k = e^{rt} Z_t^k \quad \text{for} \quad 0 \leq t \leq T , \quad k = 1, \ldots, K ,$$

so that $Z^k = \beta S^k$ as in § 3b. From (5.5)-(5.7) we see that $S^1, \ldots, S^K$ are correlated geometric Brownian motions as promised, the return process for S^k being $R_t^k = Y_t^k + rt$ (a Brownian motion with variance a_{kk} and drift $\mu^k + r$). For the information structure, we take $\mathbb{F} = \mathbb{F}^W = \mathbb{F}^Y = \mathbb{F}^Z = \mathbb{F}^S$ (see § 3d), so that investors are required to base their trading decisions on past and present price information only.

For the explicit calculations of § 5c, the following observation will be helpful. Let $h = (h^1, \ldots, h^K)$ be the function defined by

$$(5.8) \qquad h^k(x,y,t) = x^k \exp(y^k - \tfrac{1}{2} a_{kk} t) , \qquad k = 1, \ldots, K ,$$

for $x, y \in \mathbb{R}^K$ and $t \geq 0$. Then (5.4) says that $Z_t = h(Z_0, Y_t, t)$.

Furthermore, it is easily verified that

$$(5.9) \qquad Z_T = h(Z_t, Y_T - Y_t, T-t) \quad \text{for} \quad 0 \leq t \leq T .$$

5a. The Reference Measure

Because Λ (and hence A) is non-singular by assumption, there exists a unique K-vector γ satisfying

$$(5.10) \qquad \Lambda \gamma = \mu .$$

Now it will be convenient to define a vector process $\xi = (\xi^1, \ldots, \xi^K)$ by

$$(5.11) \qquad \xi_t = W_t + \gamma t , \qquad 0 \leq t \leq T ,$$

so that (5.2) can be restated as

$$(5.12) \qquad Y_t = \Lambda \xi_t , \qquad 0 \leq t \leq T .$$

Now define the martingale (under P)

$$M_t = \exp\{- \sum_{k=1}^{K} \gamma^k W_t^k - \frac{1}{2} \sum_{k=1}^{K} (\gamma^k)^2 t\} , \qquad 0 \leq t \leq T ,$$

and let the reference measure P^* be given by

$$(5.13) \qquad dP^* = M_T dP .$$

Because M is a strictly positive martingale with $M_0 = 1$, we see that P^* is a probability measure equivalent to P. The following proposition, sometimes called the <u>likelihood ratio formula</u> for Brownian motion, is a special case of the original Girsanov Theorem [12].

(5.14) Proposition. The processes $\xi^1, \ldots, \xi^K$ are independent standard Brownian motions under P^*.

From this and (5.12) we have that Y is a Brownian motion with zero drift and covariance matrix A under P^*, and then from (5.4) that Z is a (vector) martingale under P^* as required. From (3.35) and the representation theorem cited in the next subsection, it follows that P^* is in fact the unique element of $\mathbb{P}$, but we'll make no direct use of this fact. We fix P^* as our reference measure and then define admissible trading strategies in terms of it as in § 3c.

5b. Completeness

We now replace P by P^*, so the terms integrable, martingale, and local martingale implicitly refer to P^*. From the definition (5.11) of ξ it is clear that $\mathbb{F} = \mathbb{F}^W = \mathbb{F}^\xi$, meaning that the filtration in our market model is that generated by the standard Brownian motion ξ. Let $\mathcal{M}$ (the space of all martingales) and $\mathcal{M}(Z)$ be defined as in § 3d. We want to show that $\mathcal{M}(Z) = \mathcal{M}$, and hence by (3.35) that the model under discussion is complete.

First suppose $M \in \mathcal{M}$ is square integrable, meaning that $E^*(|M_T|^2) < \infty$. It is well-known [32, pp. 911-913] that M can be represented in the form

$$(5.15) \qquad M_t = M_0 + \int_0^t \theta_s d\xi_s , \qquad 0 \leq t \leq T ,$$

where $\theta = (\theta^1, \ldots, \theta^K)$ is a predictable process satisfying

$$(5.16) \qquad E^*\left(\int_0^T |\theta_t|^2 dt\right) < \infty .$$

Furthermore, every martingale M on the Brownian filtration $\mathbb{F}$ is continuous, hence locally square integrable, and it then follows easily that each $M \in \mathcal{M}$ can be represented in the form (5.15) with θ satisfying (5.16) locally, which means simply that

$$(5.17) \qquad P^*\left(\int_0^T |\theta_t|^2 dt < \infty\right) = 1 .$$

From (5.11) and the non-singularity of Λ, this is obviously equivalent to the following. Each $M \in \mathcal{M}$ can be represented in the form

$$(5.18) \qquad M_t = M_0 + \int_0^t \eta_s dY_s , \qquad 0 \leq s \leq T ,$$

where $\eta = (\eta^1, \ldots, \eta^K)$ is predictable and satisfies

$$(5.19) \qquad P^*\left(\int_0^T |\eta_t|^2 dt < \infty\right) = 1 .$$

Now let us define $H = (H^1, \ldots, H^K)$ by

$$(5.20) \qquad H_t^k = \eta_t^k / Z_t^k \quad \text{for} \quad 0 \leq t \leq T , \quad k = 1, \ldots, K .$$

Using (5.5), we can then rewrite (5.18) as

$$(5.21) \qquad M_t = M_0 + \int_0^t H_s dZ_s , \qquad 0 \leq t \leq T .$$

Furthermore, the increasing process (3.28) occurring in the definition of

$\mathscr{L}(Z)$ is, by (5.6),

$$(5.22) \qquad \left(\int_0^t (H_s^k)^2 d[Z^k, Z^k]_s\right)^{1/2} = \left(\int_0^t (\eta_s^k)^2 a_{kk} ds\right)^{1/2} .$$

This process is continuous, so it is locally integrable (under P^*) by (5.19), and we conclude that $H \in \mathscr{L}(Z)$. To repeat, every $M \in \mathscr{M}$ can be represented as $M = M_0 + \int HdZ$ for some $H \in \mathscr{L}(Z)$, so $\mathscr{M} = \mathscr{M}(Z)$ and hence the model is complete by (3.35).

One can greatly generalize the diffusion model discussed in this section and still have completeness. The diffusion coefficients a_{ij} can be made to depend on past and present prices in a more or less arbitrary way, and the drift coefficients μ^k may depend on even more than that. (We'll not attempt to even make these statements precise, let alone justify them.) But it appears that the riskless interest rate must be deterministic if one is to get completeness, although it may vary with time, and that the diffusion coefficients may not depend on more than past and present prices. We're not sure how one even states this latter property precisely, but see the example of § 6c.

5c. Explicit Computations

We now consider a class of contingent claims X for which one can calculate quite explicitly the associated price $\pi = E^*(\beta_T X)$ <u>and</u> a trading strategy φ that generates X. Specifically, we assume in this section that

$$(5.23) \qquad X = e^{rt}\psi(Z_T) \quad \text{for some} \quad \psi : \mathbb{R}_+^K \to \mathbb{R}_+ .$$

Since $Z_T^k = \exp(-rT)S_T^k$ for $k = 1,\ldots, K$, this means simply that X is a function of the final stock prices only. As usual, though, it is more convenient to speak in terms of the discounted price process Z throughout. It is easy to verify that the European call option discussed in § 1a corresponds to the function

$$\psi(x) = [x^1 - ce^{-rt}]^+ ,$$

assuming that we're talking about a call option on stock $k = 1$ (with exercise price c and expiration date T).

Let X be given by (5.23) and assume hereafter that it is integrable, meaning that

$$\pi = E^*(\beta_T X) = E^*(e^{-rT}X) = E^*[\psi(Z_T)] < \infty .$$

Then we know from the completeness result of § 5b that X is attainable at price π. Moreover, we know from § 3c that the discounted value process $V^* = V^*(\varphi)$ for any φ generating X is given by

$$(5.24) \qquad V_t^* = E^*(\beta_T X \mid \mathscr{F}_t) = E^*[\psi(Z_T) \mid \mathscr{F}_t] , \qquad 0 \leq t \leq T .$$

Our objective now is to calculate V^* and hence π (since $\pi = V_0^*$). First let's define the normal density function

$$(5.25) \qquad \Gamma_t(z) = (2\pi t)^{-K/2} \exp(-\tfrac{1}{2}|z|^2)$$

for $t > 0$ and $z \in \mathbb{R}^K$. Observe that

$$(5.26) \qquad P^*\{(\xi_T - \xi_t) \in dz \mid \mathscr{F}_t\} = \Gamma_{T-t}(z)dz ,$$

meaning that $\xi_T - \xi_t$ is independent of $\mathscr{F}_t$ and has density $\Gamma_{T-t}(\cdot)$ under P^*. This, of course, follows from (5.14) and the fact that $\mathbb{F} = \mathbb{F}^{\xi}$. Now (5.9) and (5.12) give us

$$(5.27) \qquad Z_T = h(Z_t, \Lambda(\xi_T - \xi_t), T-t), \qquad 0 \le t \le T,$$

so combining (5.24)-(5.27) we have

$$(5.28) \qquad V_t^* = E^*[\psi(h(Z_t, \Lambda(\xi_T - \xi_t), T-t)) \mid \mathscr{F}_t]$$

$$= \int \psi(h(Z_t, \Lambda z, T-t))\Gamma_{T-t}(z)dz,$$

where the integral is over $\mathbb{R}^K$. Defining

$$(5.29) \qquad f^*(x,t) = \int \psi(h(x, \Lambda z, t))\Gamma_t(z)dz$$

for $x \in \mathbb{R}_+^K$ and $t \ge 0$, (5.28) is more compactly stated as

$$(5.30) \qquad V_t^* = f^*(Z_t, T-t), \qquad 0 \le t \le T.$$

In particular, our final valuation formula for X is

$$(5.31) \qquad \pi = V_0^* = f^*(Z_0, T).$$

Obviously (5.29) and (5.31) give the most explicit valuation formula possible without further information on the payoff function ψ.

To determine the trading strategy φ that generates X, we compute the differential of V^* from (5.30) and Ito's Formula, observing that f^* has the necessary regularity by its definition (5.29). Letting $\frac{\partial}{\partial u} f^*$ denote the partial derivative of f^* with respect to its second argument, and using (5.6), we have

$$(5.32) \qquad dV_t^* = \sum_{k=1}^{K} \frac{\partial}{\partial x^k} f^*(Z_t, T-t)dZ_t^k + (L^* - \frac{\partial}{\partial u})f^*(Z_t, T-t)dt ,$$

where L^* is the linear partial differential operator

$$L^* = \frac{1}{2} \sum_{i=1}^{K} \sum_{j=1}^{K} a_{ij} x^i x^j \frac{\partial^2}{\partial x^i \partial x^j} .$$

Starting from the fact that $\Gamma_t(z)$ satisfies the heat equation

$$\frac{\partial}{\partial t} \Gamma_t(z) = \frac{1}{2} \sum_{k=1}^{K} \frac{\partial^2}{dz_k^2} \Gamma_t(z) ,$$

and fighting through all the transformations that define f^* in (5.29), it can be verified that $\frac{\partial}{\partial u} f^* = L^* f^*$. Thus, taking

$$(5.33) \qquad \varphi_t^k = \frac{\partial}{\partial x^k} f^*(Z_t, T-t) \quad \text{for} \quad 0 \le t \le T , \quad k = 1, \ldots, K ,$$

we see that (5.32) gives

$$(5.34) \qquad V_t^* = V_0^* + \sum_{k=1}^{K} \int_0^t \varphi_t^k dZ_t^k , \qquad 0 \le t \le T .$$

Then (3.32) shows that strategy $\varphi = (\varphi^0, \varphi^1, \ldots, \varphi^K)$ generates X, where the bond component φ^0 is given by

$$\varphi_t^0 = V_t^* - \sum_{k=1}^{K} \varphi_t^k Z_t^k = f^*(Z_t, T-t) - \sum_{k=1}^{K} \varphi_t^k Z_t^k .$$

From the general representation result (3.32) and the completeness result of § 5b, we knew that our process $V_t^* = f^*(Z_t, T-t)$ was going to be representable in the form (5.34), and from (5.32) we see that this is the case if and only if f^* satisfies the differential equation $\frac{\partial}{\partial u} f^* = L^* f^*$. Thus the differential equation has arisen here as a logical consequence of various general propositions. In contrast, it was by solving an analogous differential equation that Black and Scholes [2] originally obtained their option pricing formula.

Because all the calculations of this section had been done in discounted terms, they do not mesh precisely with the earlier discussion of option pricing in § 1. The interested reader should have no trouble making the linkage, however, by recasting the earlier discussion in discounted terms. In particular, the function $f(x,t)$ defined by (1.5) can be gotten by evaluating (5.29) for $\psi(x) = [x^1 - c \exp(-rT)]^+$, as we've indicated earlier.

6. Further Examples

We collect in this section four concrete examples that illustrate the diversity, and some of the intricacy, that one encounters in models with continuous trading. The first example is of a trading strategy that turns something into nothing. The remaining three are chosen to shed light on the important subject of completeness. We make no attempt to connect these examples with any realistic problems, and the analyses are neither systematic nor rigorous.

6a. A Bad Strategy

Consider the Black-Scholes model of § 1a, specialized to the case $r = 0$ (so that $S^0 = 1$), $T = 1$, and $S_0^1 = 1$. As before, we call S^0 and S^1 the bond price process and stock price process respectively. As a first step in constructing the suicide strategy alluded to in § 3b, suppose $b > 0$ and consider the strategy

$$\varphi_t^k = \begin{cases} 1 + b & \text{if } k = 0 \text{ and } 0 \leq t \leq \tau(b) \\ -b & \text{if } k = 1 \text{ and } 0 \leq t \leq \tau(b) \\ 0 & \text{otherwise} \end{cases},$$

where

$$\tau(b) = \inf\{t : S_t^1 = 1 + 1/b\} = \inf\{t : V_t(\varphi) = 0\} .$$

The investor starts with one dollar of wealth, he sells b shares of stock short and buys $1 + b$ bonds, holding this portfolio up until $t = 1$ or he is ruined, whichever comes first. The probability of ruin under this strategy is $p(b) = P(\tau(b) < 1)$, and it's clear that $p(b)$ increases from zero to one as b increases from zero to infinity. By selling short a very large amount of stock, the investor makes his own ruin almost certain, but he will probably make a great deal of money if he survives.

The chance of survival can be completely eliminated, however, by escalating the amount of stock sold short in the following way. On the time interval $[0, 1/2]$ we follow the strategy of the last paragraph with parameter $b = 1$. The probability of ruin during $[0, 1/2]$ is then $p \equiv P(\tau(1) \leq 1/2)$. If $\tau(1) > 1/2$, we adjust the amount of stock sold short to a new level b_1 at time $1/2$, simultaneously changing the amount of bond held in a self-financing fashion. Specifically, the number b_1 is chosen so as to make the conditional probability of ruin during the interval $(\frac{1}{2}, \frac{3}{4}]$ equal to p again. In general, if at any time $t_n = 1 - (1/2)^n$ we still have positive wealth, then we readjust (typically increase) the amount of stock sold short so that the conditional probability of ruin during $(t_n, t_{n+1}]$ is again p. To keep the strategy self- financing, the amount of bond held must be adjusted at each time t_n as well, of course. The probability of survival through time t_n is then $(1-p)^n$, which vanishes as $n \to \infty$ $(t_n \to 1)$. Thus we obtain a piecewise constant, self-financing strategy φ with $V_0(\varphi) = 1$, $V(\varphi) \geq 0$, and $V_1(\varphi) = 0$. This is closely related to an example presented by Kreps [20].

6b. A Point Process Model

Consider the model with $K = 1$, $S^0 = 1$, and

$$S_t^1 = S_0^1 \exp(bN_t - \mu t) , \tag{6.1}$$

where $N = \{N_t;\ 0 \leq t \leq T\}$ is a Poisson process with intensity $\lambda > 0$, and b and μ are positive constants. This is the model of Cox and Ross [5], specialized to the case of zero riskless interest rate. Corresponding to S^1 is the return process (see § 4b)

$$(6.2) \qquad R_t^1 = (\exp(b) - 1)N_t - \mu t \ .$$

For the filtration $\mathbb{F}$ we take the one generated by S^1 itself.

Let

$$(6.3) \qquad \lambda^* = \mu/(\exp(b) - 1) \quad \text{and}$$

$$M_t = (\lambda^*/\lambda)^{N_t} \exp((\lambda-\lambda^*)t) \ , \qquad 0 \leq t \leq T \ .$$

Observing that M is a strictly positive martingale with $M_0 = 1$, we define an equivalent probability measure P^* by $dP^* = M_T dP$. From the change of measure theorem for point processes [3, pp. 377-379] we have that N is a Poisson process with intensity λ^* under P^*. It follows from (6.2) and (6.3) that R^1 is a martingale under P^*, and then from (6.1) that S^1 is too. Hence we can (and do) adopt P^* as our reference measure.

It is well-known, cf. Jacod [18, p. 347], that R^1 has the martingale representation property for $(\Omega, \mathbb{F}, P^*)$, and it is straightforward to verify that the same must be true of S^1. Thus this model is complete (see §3d), and the price associated with any integrable contingent claim X is

$$\pi = E^*(X)$$

because $\beta = 1$. In particular, consider the call option

$$X = [S^1_T - c]^+ .$$

Using the fact that N has intensity λ^* under P^*, we have the valuation formula

$$\begin{aligned}
\pi &= E^*([S^1_T - c]^+) \\
&= E^*([S^1_0 \exp(bN_T - \mu T) - c]^+) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda^* T)^n [S^1_0 \exp(bn - \mu T) - c]^+ .
\end{aligned}$$

This is a special case (the riskless interest rate is zero) of the formula obtained by Cox and Ross [5]. The precise trading strategy that generates this contingent claim X can be computed much as in §5c.

6c. A Model That Is Not Complete

Let $(\Omega, \mathscr{F}, P)$ be a probability space on which is defined a standard Brownian motion $W = \{W_t;\ 0 \leq t \leq T\}$ and an independent process $\sigma = \{\sigma_t;\ 0 \leq t \leq T\}$ such that

$$\sigma_t = \begin{cases} 2 & \text{for} \quad 0 \leq t < T/2 \quad \text{with probability } 1 \\ 1 & \text{for} \quad T/2 \leq t \leq T \quad \text{with probability } 1/2 \\ 3 & \text{for} \quad T/2 \leq t \leq T \quad \text{with probability } 1/2 . \end{cases}$$

Let $K = 1$, assume $S^0 \equiv 1$ (the riskless interest rate is zero throughout), and define

$$R^1_t = \int_0^t \sigma_s dW_s , \qquad 0 \leq t \leq T .$$

Thus the return process R^1 for the stock evolves as a driftless Brownian motion with variance parameter $\sigma_t^2 = 4$ over the interval $[0, T/2)$ and then a coin is flipped. If head is observed, then the variance parameter increases to $\sigma_t^2 = 9$, but if a tail is observed it decreases to $\sigma_t^2 = 1$. Observe that

$$(6.4) \qquad [R^1, R^1]_t = \int_0^t \sigma_s^2 ds , \qquad 0 \leq t \leq T .$$

Let the filtration $\mathbb{F}$ be that generated by R^1, or equivalently by S^1, so investors have access only to past and present price information. The filtration $\mathbb{F}$ is the same as that generated by W except that, by (6.4) and the right continuity of $\mathbb{F}$, $\mathscr{F}_t$ is augmented by the outcome of the coin flip for $T/2 \leq t \leq T$. Obviously R^1 is a martingale, and it is easy to check that the same is true for $S^1 = \mathscr{E}(R^1) = \exp(R^1 - \frac{1}{2}[R^1, R^1])$. Of course $Z^1 = S^1$, because $\beta = 1$. Thus we can (and do) adopt P itself as our reference measure.

It is easy to prove, using the fact that W has the martingale representation property for its own filtration, that every martingale on $(\Omega, \mathbb{F}, P)$ has the form

$$(6.5) \qquad dM = \psi^1 dS^1 + \psi^2 d\sigma , \qquad 0 \leq t \leq T ,$$

where ψ^1 and ψ^2 are predictable. Since $Z^1 = S^1$ is continuous, only continuous martingales M can be represented as stochastic integrals with respect to Z^1, so by (3.35) this model is not complete. The investors do not have available enough financial instruments to span all sources of uncertainty.

This model can be made complete, however, by the introduction of another security. Let

$$S_t^2 = \begin{cases} 1 \, , & 0 \leq t < T/2 \\ 0 \, , & T/2 \leq t \leq T \quad \text{and} \quad \sigma_T = 1 \\ 2 \, , & T/2 \leq t \leq T \quad \text{and} \quad \sigma_T = 3 \, . \end{cases}$$

This is the price process for a ticket which can be bought (or sold) at a price of one dollar at any time before $T/2$. If a head (a variance increase) is then observed, then the ticket becomes worth two dollars, but the ticket becomes worthless if a tail is observed. The tickets represent an institutionalized means of betting on the outcome of the coin flip, and we impose the strong assumption that the price of the tickets is certain to remain constant up until the time of the coin toss (this assumption is not essential, but it eliminates a lot of complexity). Clearly $S^2 = Z^2$ is a martingale, so P remains a valid reference measure.

Now from (6.5) and the definitions of σ and S^2 we have that every martingale M satisfies

$$dM = \psi^1 dS^1 + \psi^2 dS^2 = \psi^1 dZ^1 + \psi^2 dZ^2$$

for some predictable integrands ψ^1 and ψ^2, so the model is now complete by (3.35).

This example suggests another natural sort of question that one might ask about security markets with continuous trading. Given only a filtered probability space $(\Omega, \mathbb{F}, P^*)$, what is the minimal number of securities adapted to $\mathbb{F}$ with which one can create a complete market, and what is their form? See Davis and Varaiya [7] for a discussion of this question (cast in purely mathematical terms).

6d. A Model of Mixed Type

This subsection is devoted to yet another example with a bond and one stock $(K = 1)$. We believe, but cannot prove, that this model is complete. Be that as it may, this example provides a vehicle for discussion of several important points. The bond price process is $S^0 = 1$, so the riskless interest rate is zero. To simplify notation, the stock price process will be denoted by S rather than S^1, and the corresponding return process by R rather than R^1. Because $\beta = 1$, there is no distinction between S and $Z = \beta S$, or between R and $Y = R - \alpha$, so we're free to (and shall) reuse the letters Y and Z with completely new meanings. The time parameters of various processes will appear as subscripts at some points and as functional arguments at others, depending on which is more convenient.

Begin with a probability space $(\Omega, \mathscr{F}, P)$ on which is defined a standard (zero drift and unit variance) Brownian motion $W = \{W_t;\ t \geq 0\}$, a Poisson process $N = \{N(t);\ t \geq 0\}$ with intensity $\lambda > 0$, and an IID sequence of binary random variables $\{X_1, X_2, \ldots\}$ such that $X_n = \pm 1$ with equal probability. We assume that W, N and $\{X_n\}$ are also independent of one another, with $W_0 = N(0) = 0$.

Let $\ell = \{\ell_t;\ t \geq 0\}$ be the <u>local time</u> of W at the origin, meaning that

$$(6.6) \qquad \ell_t = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|W_s| \leq \epsilon\}} ds , \qquad t \geq 0 .$$

From this definition, it is apparent that

$$(6.7) \qquad \ell \text{ increases only at times } t \text{ where } W_t = 0 ,$$

and it is well-known that ℓ is continuous but not absolutely continuous. In fact, because the set $\{t:W_t = 0\}$ has zero Lebesgue measure (almost surely), we have from (6.6) that ℓ is flat except on a set of measure zero. Next, let $\tau_0 = 0$,

$$\tau_n = \inf\{t \geq 0 : W_t = n\} \quad \text{for} \quad n = 1,2,\ldots, \quad \text{and} \tag{6.8}$$

$$M(t) = \sup\{n \geq 0 : \tau_n \leq t\} \ , \qquad t \geq 0 \ .$$

Finally, let γ be a constant ($0 < \gamma < 1$) and define

$$R_t = W_t + X_t + Y_t \ , \quad \text{where} \tag{6.9}$$

$$X_t = N(\ell_t) - \lambda \ell_t \quad , \quad \text{and} \tag{6.10}$$

$$Y_t = \gamma[X_1 + \cdots + X_{M(t)}] \ .$$

Note that each of the jump times $T_1, T_2, \ldots$ of X must be a point of increase for ℓ, and thus $W(T_n) = 0$ for all n by (6.7). In contrast, Y jumps by $\gamma X_1, \gamma X_2, \ldots$ at the hitting times $\tau_1, \tau_2, \ldots$ respectively, so the two sequences of jump times are disjoint. Also, ℓ is a continuous VF process. Thus (see § 3a) we have

$$[W,W]_t = t \ ,$$

$$[X,X]_t = \sum_{s \leq t} (\Delta X_s)^2 = N(\ell_t) \ ,$$

$$[Y,Y]_t = \sum_{s \leq t} (\Delta Y_s)^2 = \gamma^2 M(t) \ ,$$

$$[W,X] = [W,Y] = [X,Y] = 0 \ .$$

We now set $S = \mathscr{E}(R)$, taking $S_0 = 1$ for convenience. From the preceding equations and §4a we see that $S = \mathscr{E}(R) = \mathscr{E}(W)\,\mathscr{E}(X)\,\mathscr{E}(Y)$. The general formula (4.3) for the semimartingale exponential then gives us

$$(6.11) \qquad S_t = \exp(W_t - \tfrac{1}{2}t)2^{N(\ell_t)} \exp(-\lambda \ell_t) \prod_{n=1}^{M(t)} (1 + \gamma X_n) \, .$$

Observe that our stock price process S satisfies $dS = SdW$ when the underlying Brownian motion W is not at an integer level. At each of the times τ_n where W hits a positive integer level n for the first time, S either jumps to $(1 + \gamma)$ times its previous value or else drops to $(1 - \gamma)$ times its previous value (with equal probability). Also, there are times $T_1, T_2, \ldots$ at which S jumps to double its previous value, but these only occur when W is in state zero, and it is at just such times that the factor $\exp(-\lambda \ell_t)$ is pulling the stock price down (in a continuous fashion).

We take the filtration for our example to be $\mathbb{F} = \mathbb{F}^R = \mathbb{F}^S$ (see §3d), meaning that investors have access only to past and present stock price information. It is apparent that W, X, Y and hence R are martingales over $\mathbb{F}$, so $S = \mathscr{E}(R)$ is at least a local martingale. Direct calculation shows that S is moreover a martingale, so we can (and shall) take P itself as our reference measure.

Readers familiar with martingale theory will recognize (6.9) as the decomposition of R into its continuous martingale part (W), the sum of its predictable jumps (Y), and the compensated sum of its totally inaccessible jumps (X). Meyer [32, pp. 261-267] explains how an arbitrary martingale can be so decomposed, and we shall review here just the two essential definitions. A stopping time τ is said to be <u>predictable</u> if there exists an increasing sequence of stopping times $\{\tau^k\}$ such that

$\tau^k \uparrow \tau$ almost surely as $k \uparrow \infty$, in which case the sequence $\{\tau^k\}$ is said to announce τ. Each of the hitting times τ_n in (6.8) is predictable, because we can construct a sequence $\{\tau^k\}$ announcing τ_1 (for example) by taking

$$\tau^k = \inf\{t \geq 0 : W_t = 1 - \tfrac{1}{k}\}, \qquad k = 1, 2, \ldots .$$

At the other extreme, a stopping time τ is said to be totally inaccessible if $P(\tau = \tau') = 0$ for every predictable stopping time τ'. The jump times of a Poisson process are the canonical examples of totally inaccessible stopping times, and from this one can quite easily show that the jump times $T_1, T_2, \ldots$ above are totally inaccessible. This categorization of stopping times is of fundamental importance in martingale theory, and the definitions also seem natural and useful for purposes of economic modeling.

The return process R (or equivalently S) in this example was devised so as to exhibit both predictable and totally inaccessible jumps, plus a nontrivial continuous martingale part, and in this sense it is representative of the most general martingale possible. Our example also has the feature that R (or S) can have only finitely many jumps in a finite amount of time, however, and in this regard it is quite special. A general martingale may have a countably infinite number of jumps in a finite amount of time, and it is this feature that generates most of the difficulties in the general theory of stochastic integration.

Now what is the general form of a predictable trading strategy in this model? That is a very long story, which we'll not go into here. The reader with a serious interest in the general theory of continuous trading will find further analysis of this example an educational exercise, however, and we'll say just a few more words to facilitate such study.

If f, g and h are any three predictable processes, then the process φ defined by

$$\varphi_t(\omega) = \begin{cases} f_t(\omega) & \text{if } W_t(\omega) = 0 \\ g_t(\omega) & \text{if } \tau_n(\omega) = t \text{ for some } n \\ h_t(\omega) & \text{otherwise} \end{cases}$$

is also predictable because the sets $\{(t,\omega) : W_t(\omega) = 0\}$ and $\{(t,\omega) : \tau_n(\omega) = t \text{ for some } n\}$ are elements of the predictable σ-algebra. Furthermore, with φ defined in this way we have

$$\int \varphi dS = \int \varphi S_- dR = \int \varphi S_- (dW + dX + dY)$$

$$= \int h S_- dW + \int f S_- dX + \int g S_- dY ,$$

using the fact that $\varphi = h$ except on a set of time points having zero Lebesgue measure. What this ultimately means is that investors are able to use completely different trading strategies relative to the three components (W, X and Y) of the return process R. From the known completeness of Brownian motion, the Poisson martingale $N(t) - \lambda t$, and the one-dimensional random walk in discrete time, we then conjecture that this model is complete.

7. Concluding Remarks

This section presents a list of unresolved questions that we think merit further study by probabilists and/or economists. It may be that some of the answers are already known, or that they can be gotten by straightforward application of existing theory. At the end, we discuss briefly the questions of why one ought to study continuous trading at all.

In §3 we sidestepped the whole question of viability with continuous trading. How does one justify the critical assumption (3.20) from more primitive economic considerations, or is (3.20) even the right expression of viability? Should we replace (3.20) by the weaker requirement that Z be just a local martingale under some equivalent measure Q, or perhaps by the stronger requirement that Z be a square integrable martingale under some such Q? Again we refer the interested reader to Harrison and Kreps [20] for more on this very complex subject.

The definition of an attainable contingent claim depends directly on the definition of a self-financing strategy, which in turn depends on how one defines the gains operator G. In §3 we have not defended our restriction to predictable trading strategies, nor our definition of G as a stochastic integral. We have no doubt that these <u>are</u> the right definitions, but a careful study of this issue is certainly needed. It should be possible to show, for example, that a claim is attainable according to our definition if and only if it is the limit (in some appropriate sense) of claims generated by simple (see §3a) self-financing strategies.

In §3b we temporarily restricted attention to locally bounded predictable strategies. For any integrand ϕ of this class, and any semi-martingale S, the stochastic integral $\int \phi \, dS$ is well defined, this definition depending on the underlying probability measure only through its null sets. There is furthermore a well developed stochastic calculus for locally bounded integrands [32, Ch. IV], and we used parts of this calculus to show that all our essential definitions could be recast in terms of discounted quantities. Then we fixed a reference measure P^* and used it to define a new class of strategies Φ^*, some of whose members are not locally bounded. Can the undiscounted gains process $G(\phi) = \int \phi \, dS$ be meaningfully defined for each $\phi \in \Phi^*$? If so, can the final formulation of §3c, which was expressed entirely in terms of discounted quantities, be equivalently recast in undiscounted terms?

Another important question concerns the extent to which our choice of reference measure (when there is a choice) affects the set of contingent claims that are ultimately found to be attainable and the prices associated with these claims. There is of course some effect, but we believe it is relatively small. More particularly, we conjecture (but cannot prove) that the following two statements are true. First, a bounded claim is attainable with one choice of reference measure if and only if it is attainable with any choice of reference measure. Second, if a claim is attainable under two different choices of reference measure, then it has the same associated price under each. Resolution of these issues is a matter of highest priority.

The definition of Φ^* in §3c retains only those self-financing strategies ϕ for which $V^*(\phi)$ is a martingale, this ensuring that the price associated with each attainable claim is unique. One would like to know that in making this definition, we have discarded only logically dominated strategies. This requires a result of the following type. Let X be a contingent claim, and let $\Phi(X)$ be the set of all self-financing strategies ϕ such that $V^*(\phi) \geq 0$ and $V_T^*(\phi) = \beta_T X$.

(7.1) <u>Conjecture</u>. If $\Phi(X)$ is nonempty, then $\Phi(X) \cap \Phi^*$ is nonempty.

We know from (3.26) that $V^*(\phi)$ is a local martingale, and hence a supermartingale, under P^* for each $\phi \in \Phi(X)$, so a proof of (7.1) would show that we have retained only that strategy (or perhaps those strategies) which attain X at the lowest possible price.

In the first paragraph of §1 we said that the mathematical structure developed here is potentially useful for study of consumption-investment problems. Consider first the pure investment problem where one starts with wealth π at $t = 0$ and wishes to find a self-financing strategy ϕ such that $V_0(\phi) = \pi$ and $V_T(\phi)$ has maximal expected utility. In this problem the choice set is essentially the set of all contingent claims attainable at price π , so our conceptual framework is precisely appropriate. For a true consumption-investment problem, however, one must allow investors to withdraw wealth for consumption over the interval $[0,T]$. Roughly speaking, this requires that the formulation of §3 be generalized in the following way.

The set of admissible trading strategies would be enlarged to include those ϕ for which $V(\phi) \geq 0$ and $I(\phi) \equiv V_0(\phi) + G(\phi) - V(\phi)$ is an <u>increasing process</u> (rather than just identically zero), where $V(\phi) = \phi S$ and $G(\phi) = \int \phi \, dS$ as before. We would interpret $I_t(\phi)$ as the cumulative amount of wealth withdrawn from the portfolio over the interval $[0,t]$ for consumption, calling $I(\phi)$ the <u>consumption stream</u> or <u>cash flow</u> generated by strategy ϕ . An investor starting with wealth π would then choose among those admissible strategies ϕ with $V_0(\phi) = \pi$, making his selection in such a way that $I(\phi)$ and $V_T(\phi)$ jointly maximize some measure of felicity. Here we are thinking in terms of the case where there is utility associated both with consumption during $[0,T]$ and with terminal wealth. For a treatment of consumption-investment problems with diffusion price processes see Merton [26,27].

We have observed in §3d that existing general results on the martingale representation property do not give much insight as to the conditions that yield complete markets. More specifically, the result cited in §2d for discrete models suggests that the ultimate characterization of completeness with continuous trading ought to involve the fine structure of the filtration $\mathbb{F}$. Perhaps the relationships between completeness and the martingale representation property (see §3d) will suggest new lines of attack on the mathematical problem itself. Be that as it may, one of our central conclusions is that there exists potential benefit for financial economics in continued study of the martingale representation property.

Finally, let's consider the question of why bother with continuous trading, focusing solely on the problem of contingent claim valuation. Recall from §2e that a finite market with a deterministic bond and two independent stocks following geometric random walks is _not_ complete. In contrast, we've seen in §5 that the continuous limit of this model, having a deterministic bond and two independent stocks following geometric Brownian motions, _is_ complete. It should then be possible to demonstrate that, under the usual conditions justifying a diffusion approximation, the finite market is in some sense _nearly complete_, or that each contingent claim is in some sense _nearly attainable_. This point of view has been discussed by Kreps [20], who quite rightly observes that making these statements precise is a mathematical task of imposing proportions. Still we feel confident that a satisfactory convergence theory can be developed, and the notion of asymptotic completeness, if accompanied by a reasonable understanding of how and when it occurs, is of great potential importance. The Black-Scholes model and its various generalizations are important precisely because they may approximate so many other types of models that are not themselves complete.

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