



# Optimal portfolios for logarithmic utility

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## Abstract

We consider the problem of maximizing the expected logarithmic utility from consumption or terminal wealth in a general semimartingale market model. The solution is given explicitly in terms of the semimartingale characteristics of the securities price process. © 2000 Elsevier Science B.V. All rights reserved.

*Keywords:* Portfolio optimization; Logarithmic utility; Semimartingale characteristics; Martingale method

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## 1. Introduction

A classical problem in mathematical finance is the computation of optimal portfolios, where *optimal* here refers to maximization of expected utility from terminal wealth or consumption (cf. Korn, 1997 for a well-written introduction). Merton (1969, 1971) determined optimal strategies in a Markovian Itô-process setting using a *dynamic programming* approach. The Hamilton–Jacobi–Bellman equation from stochastic control theory leads to a non-linear partial differential equation (PDE) for the optimal expected utility as a function of time and current wealth. If one can solve this PDE, the optimal portfolio is immediately obtained. In multiperiod discrete-time models, a similar approach leads to a recursive equation instead of a PDE (cf. Mossin, 1968; Samuelson, 1969; Hakansson, 1970, 1971).

Harrison and Kreps (1979) and Harrison and Pliska (1981) introduced the martingale methodology to finance. They relate absence of arbitrage and completeness of securities markets to the existence resp. uniqueness of *equivalent martingale measures*. Their results were applied by Pliska (1986), Karatzas et al. (1987) and Cox and Huang (1989) to portfolio optimization in complete models. With the help of the pricing measure, they can determine the optimal terminal wealth basically as in a simple one-period model. The corresponding generating trading strategy is computed in a second step. Using a different terminology, this alternative path to portfolio optimization had already been discovered by Bismut (1975).

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Applying *duality* methods, the martingale approach could be generalized to incomplete models by He and Pearson (1991a,b) Karatzas et al. (1991), Cvitanić and Karatzas (1992), Kramkov and Schachermayer (1999), Schachermayer (1999). Roughly speaking, the optimal portfolio and wealth process in the given incomplete model and in a fictitious completed market coincide if the completion is performed in the least favourable manner (cf. also Kallsen, 1998).

It is usually quite hard to compute optimal strategies explicitly unless the market is of a certain simple structure (e.g. time-homogeneous, cf. Samuelson, 1969; Merton, 1969; Framstad et al., 1999; Benth et al., 1999; Kallsen, 2000) or the logarithm is chosen as utility function (cf. Hakansson, 1971 in discrete time; Merton, 1971 for continuous Markov processes; Aase, 1984 for a class of processes with jumps; Karatzas et al., 1991; Cvitanić and Karatzas, 1992 in an Itô-process setting). Since we want to restrict the class of market models as little as possible, we consider logarithmic utility in this paper. Optimal trading and consumption strategies are determined in a general semimartingale setting, yielding earlier results as special cases (cf. Section 4).

Intuitively speaking, the optimal portfolio depends only on the local behaviour of the price process in the case of logarithmic utility. Since the *characteristics* in the sense of Jacod (1979), Jacod and Shiryaev (1987) describe exactly this local behaviour of a semimartingale, they turn out to be the appropriate tool at hand. Moreover, they provide a framework in which very diverse models can be expressed.

The paper is organized as follows. In Section 2 we state the problem and our version of the above-mentioned duality link to martingale measures. The explicit solution in terms of the characteristics of the price process can be found in the subsequent section. Various examples are given in Section 4. Finally, the appendix contains results from stochastic calculus that are needed in Sections 3 and 4.

We generally use the notation of Jacod and Shiryaev (1987) and Jacod (1979, 1980) The transposed of a vector or matrix  $x$  is denoted as  $x^T$  and its components by superscripts. Stochastic and Stieltjes integrals are written as  $\int_0^t H_s dX_s = H \cdot X_t$ . Increasing processes are identified with their corresponding Lebesgue–Stieltjes measure.

## 2. Optimal portfolios and martingale measures

Our mathematical framework for a frictionless market model is as follows. We work with a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$  in the sense of Jacod and Shiryaev (1987), Definition I.1.2. *Securities*  $0, \dots, d$  are modelled by their *price process*  $S := (S^0, \dots, S^d)$ . Security 0 is assumed to be positive and plays a special role. It serves as a *numeraire* by which all other securities are discounted. More specifically, we denote the *discounted price process* as  $\hat{S} := (1/S^0)S := (1, (1/S^0)S^1, \dots, (1/S^0)S^d)$ . We assume that  $\hat{S}$  is a  $\mathbb{R}^{d+1}$ -valued semimartingale. Occasionally, we will identify  $\hat{S}$  with the  $\mathbb{R}^d$ -valued process  $(\hat{S}^1, \dots, \hat{S}^d)$ .

We consider an investor (hereafter called “you”), who disposes of an *initial endowment*  $\varepsilon S_0^0 \in (0, \infty)$ . *Trading strategies* are modelled by  $\mathbb{R}^{d+1}$ -valued, predictable stochastic processes  $\varphi = (\varphi^0, \dots, \varphi^d)$ , where  $\varphi_t^i$  denotes the number of shares of security  $i$  in your portfolio at time  $t$ .

**Proposition 2.1.** Assume that  $S^0$  is a semimartingale such that  $S^0, S^0_-$  are positive. Then we have equivalence between

1.  $\varphi \in L(S)$  and  $\varphi_t^\top S_t = \varphi_0^\top S_0 + \int_0^t \varphi_s^\top dS_s$  for any  $t \in \mathbb{R}_+$ ,
2.  $\varphi \in L(\hat{S})$  and  $\varphi_t^\top \hat{S}_t = \varphi_0^\top \hat{S}_0 + \int_0^t \varphi_s^\top d\hat{S}_s$  for any  $t \in \mathbb{R}_+$ .

(Note that it is not necessary to assume that  $S^0$  is predictable as is – for simplicity – often done in the literature. For the definition of multidimensional integrals cf. Jacod (1980).)

**Proof.**  $2 \Rightarrow 1$ : Partial integration of  $\varphi^\top S = (\varphi^\top \hat{S})S^0$  and Propositions A.1 and A.2 yield

$$\begin{aligned} \varphi^\top S &= \varphi_0^\top S_0 + (\varphi^\top \hat{S})_- \cdot S^0 + S^0_- \cdot (\varphi^\top \cdot \hat{S}) + [\varphi^\top \cdot \hat{S}, S^0] \\ &= \varphi_0^\top S_0 + (\varphi^\top \hat{S})_- \cdot S^0 + (\varphi S^0_-)^\top \cdot \hat{S} + \varphi^\top \cdot [\hat{S}, S^0]. \end{aligned}$$

Note that  $\Delta(\varphi^\top \hat{S}) = \Delta(\varphi^\top \cdot \hat{S}) = \varphi^\top \Delta \hat{S}$  and hence  $(\varphi^\top \hat{S})_- = \varphi^\top \hat{S}_-$ . Again using Proposition A.1, we obtain  $\varphi^\top S = \varphi_0^\top S_0 + \varphi^\top \cdot (\hat{S}_- \cdot S^0 + S^0_- \cdot \hat{S} + [\hat{S}, S^0]) = \varphi_0^\top S_0 + \varphi^\top \cdot (\hat{S} S^0_-)$ . In particular,  $\varphi \in L(S)$ .

$1 \Rightarrow 2$ : This is shown as above, but with exchanged roles of  $S, \hat{S}$  and with  $1/S_0$  instead of  $S_0$ .  $\square$

We call a trading strategy  $\varphi \in L(\hat{S})$  with  $\varphi_0 = 0$  *self-financing* if  $\varphi_t^\top \hat{S}_t = \int_0^t \varphi_s^\top d\hat{S}_s$  for any  $t \in \mathbb{R}_+$ . A self-financing strategy  $\varphi$  belongs to the set  $\mathfrak{S}$  of all *admissible* strategies if its *discounted gain process*  $\int_0^\cdot \varphi_t^\top d\hat{S}_t$  is bounded from below by  $-\varepsilon$  (no debts allowed).

Fix a terminal time  $T \in \mathbb{R}_+$ . We assume that your *discounted consumption* up to time  $t$  is of the form  $\int_0^t \kappa_s dK_s$ , where  $\kappa$  denotes your *discounted consumption rate* according to the “clock”  $K$ . We assume that  $K$  is an increasing function with  $K_0 = 0$ . Typical choices are  $K_t := 1_{[T, \infty)}$  (consumption only at time  $T$ ),  $K_t := t$  (consumption uniformly in time),  $K_t := \sum_{s \leq t} 1_{\mathbb{N}}(s)$  (consumption only at integer times).  $\kappa$  is supposed to be an element of the set  $\mathfrak{K}$  of all non-negative, optional processes satisfying  $\int_0^T \kappa_s dK_s < \infty$   $P$ -almost surely. For  $\kappa \in \mathfrak{K}$ , the corresponding *undiscounted consumption rate* at time  $t$  is  $\kappa_t S_t^0$ . Your *discounted wealth* at time  $t$  is given by  $V_t(\varphi, \kappa) := \varepsilon + \int_0^t \varphi_s^\top d\hat{S}_s - \int_0^t \kappa_s dK_s$ . A pair  $(\varphi, \kappa) \in \mathfrak{S} \times \mathfrak{K}$  belongs to the set  $\mathfrak{P}$  of *admissible portfolio/consumption pairs* if the discounted wealth process  $V_t(\varphi, \kappa)$  is non-negative.

**Definition 2.2.** 1. We say that  $(\varphi, \kappa) \in \mathfrak{P}$  is an *optimal portfolio/consumption pair* if it maximizes  $(\tilde{\varphi}, \tilde{\kappa}) \mapsto E(\int_0^T \log(\tilde{\kappa}_t) dK_t)$  over all  $(\tilde{\varphi}, \tilde{\kappa}) \in \mathfrak{P}$ .

2. We say that  $\varphi \in \mathfrak{S}$  is an *optimal portfolio for terminal wealth* if it maximizes  $\tilde{\varphi} \mapsto E(\log(\varepsilon + \int_0^T \tilde{\varphi}_t^\top d\hat{S}_t))$  over all  $\tilde{\varphi} \in \mathfrak{S}$ .

**Remark.** 1. Suppose that  $E(\int_0^T |\log(S_t^0)| dK_t) < \infty$ . If  $(\varphi, \kappa)$  is an optimal portfolio/consumption pair, then it maximizes also  $(\tilde{\varphi}, \tilde{\kappa}) \mapsto E(\int_0^T \log(\tilde{\kappa}_t S_t^0) dK_t)$  (i.e., the expected logarithm of *undiscounted consumption*) over all  $(\tilde{\varphi}, \tilde{\kappa}) \in \mathfrak{P}$ .

2. Suppose that  $S^0$  is a semimartingale and  $E(|\log(S_T^0)|) < \infty$ . If  $\varphi$  is an optimal portfolio for terminal wealth, then it maximizes also  $\tilde{\varphi} \mapsto E(\log(\varepsilon S_T^0 + \int_0^T \tilde{\varphi}_t^\top d\hat{S}_t))$  (i.e., the expected logarithm of *undiscounted* terminal wealth).

3. If we set  $K := 1_{[T, \infty)}$ , then  $\varphi \in \mathfrak{S}$  is an optimal portfolio for terminal wealth if and only if  $(\varphi, \kappa) \in \mathfrak{P}$  is an optimal portfolio/consumption pair, where  $\kappa_T := \varepsilon + \int_0^T \varphi_t^\top d\hat{S}_t$  and  $\kappa_t$  can be chosen arbitrarily for  $t < T$ . Therefore the terminal wealth problem can be treated as a special case of maximization of utility from consumption.

**Lemma 2.3.** *Let  $Z$  be a positive local martingale with  $Z_0 = K_T/\varepsilon$  and such that  $(Z\hat{S})^T$  is a local martingale. Then  $E(\int_0^T \log(\kappa_t) dK_t) \leq E(-\int_0^T \log(Z_t) dK_t)$  for any  $(\varphi, \kappa) \in \mathfrak{P}$ . In particular,  $E(\log(\varepsilon + \int_0^T \varphi_t^\top d\hat{S}_t)) \leq E(-\log(Z_T))$  if  $Z_0 = 1/\varepsilon$ .*

**Proof.** Let  $(T_n)_{n \in \mathbb{N}}$  be a sequence of stopping times with  $T_n \uparrow \infty$   $P$ -almost surely and such that  $Z^{T_n}$  is a martingale for any  $n$ . Fix  $n \in \mathbb{N}$ . Then  $(\varepsilon/K_T)Z_{T_n \wedge T}$  is the density of a probability measure  $P^\star \sim P$ . Since  $Z^{T_n \wedge T} \hat{S}^{T_n \wedge T} = (Z\hat{S})^{T_n \wedge T}$  is a local martingale,  $\hat{S}^{T_n \wedge T}$  is a  $P^\star$ -local martingale (cf. Jacod and Shiryaev 1987, III.3.8b).

Let  $(\varphi, \kappa) \in \mathfrak{P}$  such that  $E(\int_0^T \log(\kappa_t) dK_t)$  is defined. We have that

$$\begin{aligned} E\left(\int_0^{T_n \wedge T} Z_t \kappa_t dK_t\right) &= \int_0^T E(1_{[0, T_n \wedge T]}(t) \kappa_t E(Z_{T_n \wedge T} | \mathcal{F}_t)) dK_t \\ &= \frac{K_T}{\varepsilon} E_{P^\star} \left( \int_0^{T_n \wedge T} \kappa_t dK_t \right) \\ &\leq \frac{K_T}{\varepsilon} E_{P^\star} (\varepsilon + \varphi^\top \cdot \hat{S}_{T_n \wedge T}). \end{aligned}$$

By Ansel and Stricker (1994), Corollary 3.5,  $\varphi^\top \cdot \hat{S}^{T_n \wedge T}$  is a  $P^\star$ -local martingale and hence a  $P^\star$ -supermartingale. Thus,  $E(\int_0^{T_n \wedge T} Z_t \kappa_t dK_t) \leq K_T$  and therefore  $E(\int_0^T Z_t \kappa_t dK_t) \leq K_T$  by monotone convergence. This implies  $E(\int_0^T \log(\kappa_t) dK_t) \leq E(\int_0^T (\log(\kappa_t) - Z_t \kappa_t) dK_t) + K_T$ . Since the logarithm is concave, we have  $\log(\kappa_t) - Z_t \kappa_t \leq \log(1/Z_t) - 1$ . Together, it follows that  $E(\int_0^T \log(\kappa_t) dK_t) \leq E(-\int_0^T \log(Z_t) dK_t)$ .  $\square$

**Remark.** (1) If  $Z$  is a martingale, then  $Z_T/Z_0$  is the density of an equivalent local martingale measure.

(2) The above lemma implies that  $(\varphi, \kappa) \in \mathfrak{P}$  is optimal if  $\kappa = Z^{-1}$  for a process  $Z$  as above. Similarly,  $\varphi \in \mathfrak{S}$  is optimal for terminal wealth if  $\varepsilon + \int_0^T \varphi_t^\top d\hat{S}_t = Z_T^{-1}$  for such a process  $Z$ . Kramkov and Schachermayer (1999) showed that an optimal portfolio for terminal wealth necessarily solves an equation of the form  $\varepsilon + \int_0^T \varphi_t^\top d\hat{S}_t = Z_T^{-1}$ , where  $Z$  is some non-negative process such that  $(Z\hat{S})^T$  is a supermartingale.

(3) Using a different language, a version of the previous lemma can be found in Karatzas et al. (1991), Theorem 9.3. The proof of Lemma 2.3 is essentially classical (cf., e.g. the proof of Theorem 2.0 in Kramkov and Schachermayer, 1999).

(4) An inspection of the proof reveals that the assumptions in Lemma 2.3 can be slightly relaxed. If  $(Z\hat{S})^T$  is not a local martingale, then the resulting inequality still holds for any  $(\varphi, \kappa) \in \mathfrak{P}$  such that  $E_{P^\star}(\varphi^\top \cdot \hat{S}_{T_n \wedge T}) \leq 0$  for any  $n \in \mathbb{N}$ , where

$P^\star$  is defined as in that proof. This fact will allow us to consider some constrained optimization problems as well (cf. Corollary 3.2).

The following lemma addresses the uniqueness of optimal portfolio/consumption pairs.

**Lemma 2.4.** *Let  $(\varphi, \kappa)$  and  $(\tilde{\varphi}, \tilde{\kappa})$  be optimal portfolio/consumption pairs with finite expected utility  $E(\int_0^T \log(\kappa_t) dK_t)$ . Then  $\kappa = \tilde{\kappa}$  holds  $(P \otimes K)$ -almost everywhere on  $\Omega \times [0, T]$ . Moreover,  $\int_0^t \varphi_s^\top d\hat{S}_s = \int_0^t \tilde{\varphi}_s^\top d\hat{S}_s$  and hence  $V_t(\varphi, \kappa) = V_t(\tilde{\varphi}, \tilde{\kappa})$  for any  $t$  with  $K_{t-} < K_T$ . An analogous statement holds for optimal portfolios for terminal wealth.*

**Proof.** Let  $(\varphi, \kappa), (\tilde{\varphi}, \tilde{\kappa}) \in \mathfrak{P}$  be optimal portfolio/consumption pairs.

*Step 1:* Define  $\hat{\varphi} := \frac{1}{2}(\varphi + \tilde{\varphi})$ ,  $\hat{\kappa} := \frac{1}{2}(\kappa + \tilde{\kappa})$ . Obviously,  $(\hat{\varphi}, \hat{\kappa}) \in \mathfrak{P}$ . By optimality of  $(\varphi, \kappa), (\tilde{\varphi}, \tilde{\kappa})$ , we have  $\int_{\Omega \times [0, T]} (\log(\hat{\kappa}_t) - \frac{1}{2}(\log(\kappa_t) + \log(\tilde{\kappa}_t))) d(P \otimes K) = \int_{\Omega \times [0, T]} \log(\hat{\kappa}_t) d(P \otimes K) - \int_{\Omega \times [0, T]} \log(\kappa_t) d(P \otimes K) \leq 0$ . Since the logarithm is concave, the integrand  $\log(\hat{\kappa}_t) - \frac{1}{2}(\log(\kappa_t) + \log(\tilde{\kappa}_t))$  is non-negative, which implies that it is 0  $(P \otimes K)$ -almost everywhere. Therefore  $\tilde{\kappa} = \kappa$   $(P \otimes K)$ -almost everywhere because the logarithm is strictly concave.

*Step 2:* Let  $t_0 \in [0, T]$  with  $K_{t_0-} < K_T$ , moreover  $A := \{V_{t_0}(\varphi, \kappa) < V_{t_0}(\tilde{\varphi}, \tilde{\kappa})\} \in \mathcal{F}_{t_0}$  and  $D := 1_A(V_{t_0}(\tilde{\varphi}, \tilde{\kappa}) - V_{t_0}(\varphi, \kappa)) \geq 0$ . Define a new portfolio/consumption pair  $(\bar{\varphi}, \bar{\kappa})$  by

$$\bar{\varphi}_t(\omega) := \begin{cases} \tilde{\varphi}_t(\omega) & \text{if } t \leq t_0 \text{ or } \omega \in A^c, \\ \varphi_t(\omega) & \text{if } t > t_0 \text{ and } \omega \in A, \end{cases}$$

$$\bar{\kappa}_t := \begin{cases} \kappa_t & \text{for } t < t_0, \\ \kappa_t + \frac{D}{K_T - K_{t_0-}} & \text{for } t \geq t_0. \end{cases}$$

More precisely, let  $\bar{\varphi}_t^0 := \varphi_t^0 + D$  for  $t > t_0$  so that  $\bar{\varphi}$  is a self-financing strategy. Since  $\kappa = \tilde{\kappa}$ , we have  $\varphi^\top \cdot S_{t_0} < \tilde{\varphi}^\top \cdot S_{t_0}$  on  $A$ . This implies that  $\bar{\varphi}$  is admissible. Moreover, we have  $V_t(\bar{\varphi}, \bar{\kappa}) = V_t(\varphi, \kappa) + D - D((K_t - K_{t_0-}) / (K_T - K_{t_0-})) \geq V_t(\varphi, \kappa)$  for  $t \geq t_0$ , which implies that  $(\bar{\varphi}, \bar{\kappa}) \in \mathfrak{P}$ . Obviously,  $\bar{\kappa} > \kappa$  on  $A \times [t_0, T]$ . In view of the first step, this is only possible if  $P(A) = 0$ .  $\square$

**Remark.** If  $K_\infty := \lim_{t \rightarrow \infty} K_t < \infty$ , then Definition 2.2 makes sense for  $T = \infty$  as well. In this case, Lemmas 2.3 and 2.4 still hold. We do not want to consider terminal wealth for  $T = \infty$ , since the limit  $\int_0^\infty \varphi_t^\top d\hat{S}_t$  is usually non-existent.

### 3. Solution in terms of characteristics

In this section, we turn to the explicit solution of the logarithmic utility maximization problem. Fix a truncation function  $h: \mathbb{R}^d \rightarrow \mathbb{R}^d$ , i.e. a bounded function with compact support that satisfies  $h(x) = x$  in a neighbourhood of 0. We assume that the characteristics  $(B, C, \nu)$  of the  $\mathbb{R}^d$ -valued semimartingale  $(\hat{S}^1, \dots, \hat{S}^d)$  relative to  $h$  are given in

the form

$$B = \int_0^\cdot b_t \, dA_t, \quad C = \int_0^\cdot c_t \, dA_t, \quad v = A \otimes F, \tag{3.1}$$

where  $A \in \mathcal{A}_{\text{loc}}^+$  is a predictable process,  $b$  is a predictable  $\mathbb{R}^d$ -valued process,  $c$  is a predictable  $\mathbb{R}^{d \times d}$ -valued process whose values are non-negative, symmetric matrices, and  $F$  is a transition kernel from  $(\Omega \times \mathbb{R}_+, \mathcal{P})$  into  $(\mathbb{R}^d, \mathcal{B}^d)$ . By Jacod and Shiryaev (1987, Proposition II.2.9) such a representation always exists. Corollary A.7 shows how to obtain  $(b, c, F)$  if the characteristics of the undiscounted price process  $S$  are known.

We are now ready to establish the main result of this paper.

**Theorem 3.1.** *Assume that there exists a  $\mathbb{R}^d$ -valued process  $H \in L(\hat{S})$  with the following properties:*

1.  $1 + H_t^\top x > 0$  for  $(A \otimes F)$ -almost all  $(t, x) \in [0, T] \times \mathbb{R}^d$ ,
2.  $\int |x/(1 + H_t^\top x) - h(x)| F_t(dx) < \infty$  ( $P \otimes A$ )-almost everywhere on  $\Omega \times [0, T]$ ,
- 3.

$$b_t - c_t H_t + \int \left( \frac{x}{1 + H_t^\top x} - h(x) \right) F_t(dx) = 0 \tag{3.2}$$

( $P \otimes A$ )-almost everywhere on  $\Omega \times [0, T]$ .

Let

$$\begin{aligned} \kappa_t &:= \frac{\varepsilon}{K_T} \mathcal{E} \left( \int_0^\cdot H_s^\top \, d\hat{S}_s \right)_t, \\ V_t &:= \kappa_t (K_T - K_t), \\ \varphi_t^i &:= H_t^i V_{t-} \text{ for } i = 1, \dots, d, \quad \varphi_t^0 := \int_0^t \varphi_s^\top \, d\hat{S}_s - \sum_{i=1}^d \varphi_t^i \hat{S}_t^i \end{aligned}$$

for  $t \in [0, T]$ , where we set  $V_{0-} := 0$ . Then  $(\varphi, \kappa) \in \mathfrak{B}$  is an optimal portfolio/consumption pair with discounted wealth process  $V$ .

**Proof.** *Step 1:* We have  $E(\sum_{t \leq T} 1_{(-\infty, 0]}(1 + \Delta(H^\top \cdot \hat{S})_t)) = E(1_{(-\infty, 0]}(1 + H^\top x) * \mu_T^{\hat{S}}) = E(1_{(-\infty, 0]}(1 + H^\top x) * \nu_T) = 0$  by Condition 1. Therefore,  $P(\text{Ex. } t \in [0, T] \text{ with } \Delta(H^\top \cdot \hat{S})_t \leq -1) = 0$ . By Jacod and Shiryaev (1987, I.4.64 and I.4.61c), this implies that  $\kappa = \varepsilon/K_T \mathcal{E}(H^\top \cdot \hat{S})$  is positive on  $[0, T]$ .

*Step 2:* Define  $Z_t := 1/\kappa_t$  and  $N_t := -H^\top \cdot \hat{S}_t^c + (1/(1 + H^\top x) - 1) * (\mu^{\hat{S}} - \nu)_t$  for  $t \in [0, T]$ . We will show that  $Z = K_T/\varepsilon \mathcal{E}(N)$ , which implies that  $Z$  is a positive local martingale.

Note that  $\langle \kappa^c, \kappa^c \rangle_t = (\kappa_-^2 H^\top cH) \cdot A_t$  and  $\mu^\kappa([0, t] \times G) = 1_G (\kappa_- H^\top x) * \mu_t^{\hat{S}}$  for  $t \in [0, T]$ ,  $G \in \mathcal{B}$ . An application of Itô's formula (cf. Lemma A.5) yields that  $Z = Z_0 + 1/\kappa_- \cdot (-H^\top \cdot \hat{S} + (H^\top cH) \cdot A + (1/(1 + H^\top x) - 1 + H^\top x) * \mu^{\hat{S}})$ . It remains to show that

$$\begin{aligned} & -H^\top \cdot \hat{S}_t + (H^\top cH) \cdot A_t + \left( \frac{1}{1 + H^\top x} - 1 + H^\top x \right) * \mu_t^{\hat{S}} \\ &= -H^\top \cdot \hat{S}_t^c + \left( \frac{1}{1 + H^\top x} - 1 \right) * (\mu^{\hat{S}} - \nu)_t \end{aligned} \tag{3.3}$$

for  $t \in [0, T]$ . Define the set  $\Delta := \{(\omega, t, x) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^d : |x| > 1 \text{ or } |H_t^T(\omega)x| > 1\} \in \mathcal{P} \otimes \mathcal{B}^d$ . By Proposition A.2 we have  $\hat{S} = \hat{S}_0 + x1_{\Delta}(x) * \mu^{\hat{S}} + \hat{S}^c + x1_{\Delta^c}(x) * (\mu^{\hat{S}} - \nu) + \tilde{B}$  and

$$H^T \hat{S} = H^T x1_{\Delta}(x) * \mu^{\hat{S}} + H^T \cdot \hat{S}^c + H^T x1_{\Delta^c}(x) * (\mu^{\hat{S}} - \nu) + H^T \cdot \tilde{B}$$

for some predictable process  $\tilde{B}$  whose components are in  $\mathcal{V}$ . The semimartingale  $\hat{S}$  can also be written in its canonical representation  $\hat{S} = \hat{S}_0 + (x - h(x)) * \mu^{\hat{S}} + \hat{S}^c + h(x) * (\mu^{\hat{S}} - \nu) + B$  (cf. Jacod and Shiryaev (1987, II.2.34)). By Proposition A.3 we have that  $\tilde{B} = B - (h(x) - x1_{\Delta^c}(x)) * \nu$ . This implies that  $\tilde{B} = \tilde{b} \cdot A$ , where

$$\begin{aligned} \tilde{b}_t &= b_t - \int (h(x) - x1_{\Delta^c}(x))F_t(dx) \\ &= c_t H_t - \int \left( \frac{x}{1 + H_t^T x} - x1_{\Delta^c}(x) \right) F_t(dx) \end{aligned}$$

(cf. Eq. (3.2)). Summing up the terms on the left-hand side of Eq. (3.3), we obtain  $-H^T \cdot \hat{S}_t^c + (1/(1 + H^T x) - 1 + H^T x1_{\Delta^c}(x)) * \mu_t^{\hat{S}} - H^T x1_{\Delta^c}(x) * (\mu^{\hat{S}} - \nu)_t - (1/(1 + H_t^T x) - 1 + H^T x1_{\Delta^c}(x)) * \nu_t$ , which equals the right-hand side of Eq. (3.3).

Step 3: Fix  $i \in \{1, \dots, d\}$ . For fixed  $t$ , Jacod and Shiryaev (1987, II.2.14) implies that  $\int x/(1 + H_t^T x)\nu(\{t\} \times dx) = (b_t + \int (x/(1 + H_t^T x) - h(x))F_t(dx))\Delta A_t = 0$ . Moreover, Eq. (3.2) yields  $(c^i \cdot H)^T \cdot A = B^i + (x^i/(1 + H_t^T \cdot x) - h^i(x)) * \nu$ . Therefore,

$$\begin{aligned} [\hat{S}^i, N]_t &= \langle \hat{S}^{i,c}, N^c \rangle_t + \sum_{s \leq t} \Delta \hat{S}_s^i \Delta N_s \\ &= - \sum_{j=1}^d H^j \cdot \langle \hat{S}^{i,c}, \hat{S}^{j,c} \rangle_t \\ &\quad + \sum_{s \leq t} \Delta \hat{S}_s^i \left( \frac{1}{1 + H_s^T \Delta \hat{S}_s} - 1 - \int \frac{-H_s^T x}{1 + H_s^T x} \nu(\{s\} \times dx) \right) \\ &= -B_t^i - \left( \frac{x^i}{1 + H^T x} - h^i(x) \right) * \nu_t + x^i \left( \frac{1}{1 + H^T x} - 1 \right) * \mu_t^{\hat{S}} \\ &= -B_t^i + \left( \frac{x^i}{1 + H^T x} - h^i(x) \right) * (\mu^{\hat{S}} - \nu)_t + (h^i(x) - x^i) * \mu_t. \end{aligned}$$

In view of the canonical representation of the semimartingale  $\hat{S}^i$ , this implies that  $\hat{S}^i + [\hat{S}^i, N]$  is a local martingale. Hence,  $\hat{S}^i Z = \hat{S}_0^i Z_0 + Z_- \cdot \hat{S}^i + \hat{S}_-^i \cdot Z + Z_- \cdot [\hat{S}^i, N]$  is a local martingale as well.

Step 4: Partial integration in the sense of Jacod and Shiryaev (1987, I.4.49a) yields that  $V_t = V_0 + (K_T - K_-) \cdot \kappa + \kappa \cdot (K_T - K) = \varepsilon + \varphi^T \cdot \hat{S} - \kappa \cdot K$ . It follows that  $(\varphi, \kappa)$  is an admissible portfolio/consumption pair with discounted wealth process  $V$ . Note that  $\varphi^0$  is well defined, since  $\varphi^T \cdot \hat{S} = (\varphi^1, \dots, \varphi^d)^T \cdot (\hat{S}^1, \dots, \hat{S}^d)$ . In view of Lemma 2.3, we are done.  $\square$

**Remark.** (1) If  $K_\infty := \lim_{t \rightarrow \infty} K_t < \infty$ , then Theorem 3.1 holds for  $T = \infty$  as well (cf. the remark at the end of Section 2).

(2) If Conditions 1–3 in Theorem 3.1 are met and  $\varphi \in \mathfrak{G}$  is defined by

$$\varphi_t^i := H_t^i \varepsilon \mathcal{E} \left( \int_0^t H_s^T d\hat{S}_s \right)_{t-} \quad \text{for } i = 1, \dots, d, \quad \varphi_t^0 := \int_0^t \varphi_s^T d\hat{S}_s - \sum_{i=1}^d \varphi_t^i \hat{S}_t^i$$

for  $t \in (0, T]$ , then  $\varphi$  is an optimal portfolio for terminal wealth and its discounted wealth process equals  $\varepsilon \mathcal{E}(\int_0^t H_t^T d\hat{S}_t)$ .

(3) Since we have  $\varphi_t^i = H_t^i V_{t-}$ , the optimal portfolio is proportionate to the current discounted wealth. The factor  $H_t$  depends only on the local behaviour of the price process. This reflects the well-known fact that the logarithmic utility is *myopic* (cf. Mossin, 1968, Hakansson, 1971).

(4) In the terminal wealth case, the crucial condition (3.2) allows a nice interpretation in terms of the *portfolio return process*  $R := \log(V) = \log(\varepsilon + \int_0^t \varphi_s^T d\hat{S}_s)$ . Let  $\varphi \in \mathfrak{G}$  be any trading strategy such that  $R$  is a special semimartingale, i.e. such that  $R$  can be decomposed into a predictable process of finite variation  $D$  (*drift process*) and a local martingale. By Jacod and Shiryaev (1987, II.2.29) we have  $D = \tilde{B} + (x - h_1(x)) * \tilde{\nu}$ , where  $(\tilde{B}, \tilde{C}, \tilde{\nu})$  denotes the characteristics of  $R$  relative to a truncation function  $h_1 : \mathbb{R} \rightarrow \mathbb{R}$ . Application of Corollary A.6 and Propositions A.2, A.3 yields after straightforward calculations that  $D = \int_0^t d_t dA_t$  with

$$d_t = H_t^T b_t - \frac{1}{2} H_t^T c_t H_t + \int (\log(1 + H_t^T x) - H_t^T h(x)) F_t(dx), \tag{3.4}$$

where we set  $H_t := (1/V_{t-})(\varphi_t^1, \dots, \varphi_t^d)$ . Observe that the *drift rate*  $d$  is a concave function of  $H_t(\omega)$  for fixed  $(\omega, t)$ . If we may differentiate under the integral sign, we obtain exactly Eq. (3.2) as a condition for maximal points of this function. In this sense, optimal trading means pointwise maximization of the drift rate of the portfolio return process.

In view of Eq. (3.4), this connection is not surprising. The drift rate  $d_t$  depends only on  $H_t$  and not on past values of  $H$ . Therefore, pointwise optimization leads to a global maximum of  $D_T$  as a function of  $H$ . One may now loosely reason that the martingale part of  $R$  does not contribute to the expected value  $E(R_T)$  so that our candidate  $H$  indeed maximizes the expected logarithmic terminal wealth. However, since local martingales need not be martingales and because of other technical obstacles, it would require some efforts to make this intuitive argument precise (cf. Aase, 1984 in this context).

(5) Note that solving the reduced maximization problem (18) in Aase (1984) leads to Eq. (3.2) if  $\hat{S}$  is of the particular form in that paper. He considers semimartingales driven by Brownian motion and marked point processes.

The preceding theorem can be generalized to allow for cone constraints, in particular short-sale restrictions. To this end, let  $\Gamma \subset \mathbb{R}^d$  be any closed convex cone and denote by  $\Gamma^\circ := \{y \in \mathbb{R}^d : x^T y \leq 0 \text{ for any } x \in \Gamma\}$  the polar cone of  $\Gamma$ . Define the *constrained* sets of trading strategies  $\mathfrak{G}(\Gamma)$  and of portfolio/consumption pairs  $\mathfrak{P}(\Gamma)$  as in Section 2, but with the additional requirement that  $(\varphi^1, \dots, \varphi^d) \in \Gamma$  pointwise on  $\Omega \times [0, T]$ . The most important example is  $\Gamma := (\mathbb{R}_+)^d$  (no short sales), in which case  $\Gamma^\circ = (-\infty, 0]^d$ . We define optimal portfolio/consumption pairs and optimal portfolios for terminal wealth



relative to the constraint set  $\Gamma$  analogous to Definition 2.2 by substituting  $\mathfrak{P}(\Gamma)$  and  $\mathfrak{G}(\Gamma)$  for  $\mathfrak{P}$  and  $\mathfrak{G}$ . The following corollary extends Theorem 3.1 to this slightly more general setting.

**Corollary 3.2** (Cone constraints). *Assume that there exists a  $\Gamma$ -valued process  $H \in L(\hat{S})$  and a  $\Gamma^\circ$ -valued process  $A \in L(A)$  such that*

1.  $1 + H_t^\top x > 0$  for  $(A \otimes F)$ -almost all  $(t, x) \in [0, T] \times \mathbb{R}^d$ ,
2.  $\int |x/(1 + H_t^\top x) - h(x)| F_t(dx) < \infty$   $(P \otimes A)$ -almost everywhere on  $\Omega \times [0, T]$ ,
- 3.

$$b_t - c_t H_t + \int \left( \frac{x}{1 + H_t^\top x} - h(x) \right) F_t(dx) = A_t$$

$(P \otimes A)$ -almost everywhere on  $\Omega \times [0, T]$ ,

4.  $H_t^\top A_t = 0$   $(P \otimes A)$ -almost everywhere on  $\Omega \times [0, T]$ .

Define  $\kappa, V, \varphi$  as in Theorem 3.1. Then  $(\varphi, \kappa) \in \mathfrak{P}(\Gamma)$  is an optimal portfolio/consumption pair relative to the constraint set  $\Gamma$ . Its discounted wealth process is  $V$ .

**Proof.** *Step 1:* Define the  $\mathbb{R}^d$ -valued semimartingale  $\tilde{S} = (\tilde{S}^1, \dots, \tilde{S}^d)$  by  $\tilde{S} := \hat{S} - A \cdot A$ . Obviously, its semimartingale characteristics are of the form (3.1) with  $\tilde{b} := b - A$  instead of  $b$ . Since  $H \in L(\hat{S})$ , Condition 4 implies  $H \in L(\tilde{S})$  and  $H^\top \cdot \tilde{S} = H^\top \cdot \hat{S}$ . From Theorem 3.1 it follows that  $(\varphi, \kappa) \in \mathfrak{P}(\Gamma)$  is an optimal portfolio/consumption pair for the discounted price process  $\tilde{S}$  instead of  $\hat{S}$ . On an intuitive level, it is now tempting to reason that  $\hat{\varphi}^\top \cdot \hat{S} = \hat{\varphi}^\top \cdot \tilde{S} + (\hat{\varphi}^\top A) \cdot A \leq \hat{\varphi}^\top \cdot \tilde{S}$  for any  $(\hat{\varphi}, \hat{\kappa}) \in \mathfrak{P}(\Gamma)$ . This would imply that such a portfolio/consumption pair is admissible for the market  $\tilde{S}$  as well, which in turn yields the assertion. However, in general  $\hat{\varphi}$  may not be integrable with respect to  $\tilde{S}$  and so we have to argue more carefully.

*Step 2:* Let  $(\hat{\varphi}, \hat{\kappa}) \in \mathfrak{P}(\Gamma)$  be an admissible portfolio/consumption pair relative to the original price process  $\hat{S}$ . Define the local martingale  $Z$  as in the proof of Theorem 3.1 and choose a localizing sequence  $(T_n)_{n \in \mathbb{N}}$  for  $Z$ . Fix  $n \in \mathbb{N}$  and define  $P^\star$  as in the proof of Lemma 2.3. Note that  $\hat{S}^{T_n \wedge T} = \tilde{S}^{T_n \wedge T} + A \cdot A^{T_n \wedge T}$ , where  $\tilde{S}^{T_n \wedge T}$  is a  $P^\star$ -local martingale. Proposition A.4 yields that  $\hat{\varphi}^\top \cdot \hat{S}^{T_n \wedge T} = \hat{\varphi}^\top \cdot U + \hat{\varphi}^\top \cdot V$  for some  $\mathbb{R}^d$ -valued  $P^\star$ -local martingale  $U$  and a process  $V$  such that  $\hat{\varphi}^\top \cdot V$  is decreasing. Since  $\hat{\varphi}^\top \cdot U$  is bounded from below, Ansel and Stricker (1994, Corollary 3.5) yields as in the proof of Lemma 2.3 that  $E_{P^\star}(\hat{\varphi}^\top \cdot U_T) \leq 0$ . Moreover,  $E_{P^\star}(\hat{\varphi}^\top \cdot V_T) \leq 0$  (possibly  $-\infty$ ). In view of Remark 4 following Lemma 2.3, we are done.  $\square$

**Remark.** (1) The remarks following Theorem 3.1 hold accordingly. Moreover, the uniqueness result Lemma 2.4 applies to the constrained case as well.

(2) In the proof of Corollary 3.2 the solution to the constrained optimization problem is obtained by solving a related perturbed unconstrained problem. This is a standard approach in convex optimization (cf., e.g. Rockafellar, 1970). In this sense, one may interpret the process  $A$  entering the drift of the perturbed price process  $\tilde{S}$  as a *Lagrange multiplier* or *Kuhn–Tucker vector*. The same idea was applied in an Itô-process setting by Cvitanić and Karatzas (1992).

(3) Under additional regularity conditions, one could go one step further and extend Corollary 3.2 to arbitrary (even random) closed convex constraint sets as in Cvitanić and Karatzas (1992). Note, however, that the constraints would have to be imposed on  $H := 1/V_-(\varphi^1, \dots, \varphi^d)$  rather than  $(\varphi^1, \dots, \varphi^d)$ .

Unfortunately, such a generalization is only of limited use for hedging problems where  $\varphi^i$  itself is fixed for the security  $i$  that is to be hedged. In this case, the optimal solution may no longer depend only on the local behaviour of the securities price process  $\hat{S}$ .

#### 4. Examples

In this section we consider several particular settings where the conditions in Theorem 3.1 can be reformulated in different terms. The resulting optimal consumption/portfolio pairs are essentially well known for Itô processes and discrete-time models. As far as we know, the general exponential Lévy process setting has not been treated yet for logarithmic utility (but cf. Benth et al. 1999; Kallsen, 2000).

**Example 4.1** (Discrete-time models). We consider a discrete-time market, i.e.  $\hat{S}$  is piecewise constant on the open intervals between integer times. Conditions 1–3 in Theorem 3.1 are satisfied if

1.  $P(1 + H_t^T \Delta \hat{S}_t \leq 0 | \mathcal{F}_{t-}) = 0$  for  $t = 1, \dots, T$ ,
2.  $E(|\Delta \hat{S}_t / (1 + H_t^T \Delta \hat{S}_t)| | \mathcal{F}_{t-}) < \infty$  for  $t = 1, \dots, T$ ,
3.  $E(\Delta \hat{S}_t / (1 + H_t^T \Delta \hat{S}_t) | \mathcal{F}_{t-}) = 0$  for  $t = 1, \dots, T$ .

The process  $\kappa$  in Theorem 3.1 is of the form  $\kappa_t = (\varepsilon/K_T) \prod_{s=1}^t (1 + H_s^T \Delta \hat{S}_s)$ .

Suppose that, in addition,  $\hat{S}_t^i = \hat{S}^0 \prod_{s=1}^t X_s^i$ , where  $X_1, \dots, X_T$  are identically distributed  $(0, \infty)$ -valued random variables such that  $X_t$  is independent of  $\mathcal{F}_{t-}$ . Then Conditions 1–3 in Theorem 3.1 can be replaced with the following assumption: There exists some  $\gamma \in \mathbb{R}^d$  such that

1.  $1 + \gamma^T (X_1 - 1^d) > 0$   $P$ -almost surely, where  $1^d := (1, \dots, 1)$ ,
2.  $E(|(X_1 - 1^d) / (1 + \gamma^T (X_1 - 1^d))|) < \infty$ ,
3.  $E((X_1 - 1^d) / (1 + \gamma^T (X_1 - 1^d))) = 0$ ,
4.  $H_t^i = \gamma^i / \hat{S}_{t-1}^i$  for  $i = 1, \dots, d$ .

In this case, the process  $\kappa$  is of the form  $\kappa_t = (\varepsilon/K_T) \prod_{s=1}^t (1 + \gamma^T (X_s - 1^d))$ . The constant  $\gamma^i$  can be interpreted as the fraction of current wealth that is invested in security  $i$ . Note that it does not depend on  $t$  resp.  $T$ . This surprising fact has already been pointed out by Mossin (1968) and Samuelson (1969).

**Proof.** Choose  $A_t := \sum_{s \leq t} 1_{\mathbb{N}}(s)$ . Note that  $c_t = 0$ ,  $b_t = \int h(x) F_t(dx)$ ,  $F_t(G) = E(1_G(\Delta \hat{S}_t) | \mathcal{F}_{t-})$  for  $t = 1, \dots, T$ ,  $G \in \mathcal{B}^d$  (cf. Jacod and Shiryaev, 1987, II.1.26, II.2.14). The results follow from simple calculations.  $\square$

**Example 4.2** (Itô processes). Assume that  $\hat{S}$  is a solution to

$$d\hat{S}_t^i = \hat{S}_t^i \left( b_t^i dt + \sum_{j=1}^n \sigma_t^{ij} dW_t^j \right),$$

where  $b^i, \sigma^{ij}$  are predictable processes for  $i=1, \dots, d, j=1, \dots, n$  and  $W$  is a  $\mathbb{R}^n$ -valued standard Wiener process. In this case, Conditions 1–3 in Theorem 3.1 can be replaced with the following assumption: There exists some predictable,  $\mathbb{R}^d$ -valued process  $\gamma$  such that

1.  $b_t - \sigma_t \sigma_t^\top \gamma_t = 0$  ( $P \otimes \lambda$ )-almost everywhere on  $\Omega \times [0, T]$ ,
2.  $H_t^i = \gamma_t^i / \hat{S}_t^i$  for  $i = 1, \dots, d$ .

The process  $\kappa$  in Theorem 3.1 is of the form  $\kappa = (\varepsilon/K_T)\mathcal{E}(\int_0^\cdot \gamma_t^\top b_t dt + \int_0^\cdot \gamma_t^\top \sigma_t dW_t)$ . These results coincide with those derived by Merton (1969, 1971), Karatzas et al. (1991), Cvitanic and Karatzas (1992).

**Example 4.3** (Exponential Lévy processes). Suppose that  $\hat{S}^1, \dots, \hat{S}^d$  are positive processes of the form  $\hat{S}^i := \hat{S}_0^i \mathcal{E}(L^i)$ , where  $L$  is a  $\mathbb{R}^d$ -valued Lévy process with characteristic triplet  $(b, c, F)$ . By Lemma A.8, these processes coincide with those of the form  $\hat{S}^i = \hat{S}_0^i e^{\tilde{L}^i}$  for  $\mathbb{R}^d$ -valued Lévy processes  $\tilde{L}$ . In the last couple of years, processes of this type have become popular for securities models, since they are mathematically tractable and provide a good fit to real data (cf. Eberlein and Keller, 1995; Eberlein et al., 1998; Madan and Senata, 1990; Barndorff-Nielsen, 1998). In this setting, Conditions 1–3 in Theorem 3.1 can be replaced with the following assumption: There exists some  $\gamma \in \mathbb{R}^d$  such that

1.  $F(\{x \in \mathbb{R}^d : 1 + \gamma^\top x \leq 0\}) = 0$ ,
2.  $\int |x/(1 + \gamma^\top x) - h(x)|F(dx) < \infty$ ,
3.  $b - c\gamma + \int (x/(1 + \gamma^\top x) - h(x))F(dx) = 0$ ,
4.  $H_t^i = \gamma^i / \hat{S}_{t-}^i$  for  $i = 1, \dots, d$ .

The process  $\kappa$  equals  $(\varepsilon/K_T)\mathcal{E}(\gamma^\top L)$ , where  $\gamma^\top L$  is a Lévy process whose triplet  $(\tilde{b}, \tilde{c}, \tilde{F})$  relative to some truncation function  $\tilde{h} : \mathbb{R} \rightarrow \mathbb{R}$  is given by  $\tilde{b} = \gamma^\top b + \int (\tilde{h}(\gamma^\top x) - \gamma^\top h(x))F(dx)$ ,  $\tilde{c} = \gamma^\top c\gamma$ ,  $\tilde{F}(G) = \int 1_G(\gamma^\top x)F(dx)$  for  $G \in \mathcal{B}$ . Again,  $\gamma^i$  can be interpreted as the fraction of wealth that is invested in security  $i$ . As in Example 4.1, it does not depend on the time horizon.

**Proof.** Choose  $A_t := t$ . From Lemma A.8 and Corollary A.6, it follows that the characteristics  $(\hat{b} \cdot A, \hat{c} \cdot A, A \otimes \hat{F})$  of  $\hat{S}$  are given by

$$\hat{b}_t^i = \hat{S}_{t-}^i b^i + \int (h^i(\hat{S}_{t-}^1 x^1, \dots, \hat{S}_{t-}^d x^d) - \hat{S}_{t-}^i h^i(x^1, \dots, x^d))F(d(x^1, \dots, x^d)),$$

$$\hat{c}_t^{ij} = \hat{S}_{t-}^i c^{ij} \hat{S}_{t-}^j,$$

$$\hat{F}_t(G) = \int 1_G(\hat{S}_{t-}^1 x^1, \dots, \hat{S}_{t-}^d x^d)F_t(d(x^1, \dots, x^d))$$

for  $t \in [0, T]$ ,  $G \in \mathcal{B}^d$ . The triplet of  $\gamma^T L$  can be obtained from Corollary A.6 applied to the mapping  $\mathbb{R}^d \rightarrow \mathbb{R}$ ,  $x \mapsto \gamma^T x$ . The assertion follows from straightforward calculations.  $\square$

**Remark.** In spite of its generality, Theorem 3.1 does not provide a necessary condition. Kramkov and Schachermayer (1999) give an example (Example 5.1 bis in that paper) where Eq. (3.2) has no solution but an optimal portfolio for terminal wealth still exists.

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**Appendix A**

In this section, we state results from stochastic calculus which are needed in the previous sections. Partially, they are of interest in their own right. Some of them are slight generalizations of properties that can be found in Jacod (1979), Jacod and Shiryaev (1987), or other textbooks. Truncation functions  $h, h_d, h_{d+1}, h_n$  on  $\mathbb{R}, \mathbb{R}^d, \mathbb{R}^{d+1}, \mathbb{R}^n$ , respectively, are supposed to be fixed in the appendix. We start with three technical propositions.

**Proposition A.1.** *Let  $H, K$  be predictable processes with values in  $\mathbb{R}^d$  (resp.  $\mathbb{R}$ ).*

1. *Suppose that  $M$  is a  $\mathbb{R}^d$ -valued local martingale and  $K \in L_{\text{loc}}^1(M^i)$  for  $i = 1, \dots, d$ . Then  $H \in L_{\text{loc}}^1(K \cdot M)$  if and only if  $HK \in L_{\text{loc}}^1(M)$ , where  $K \cdot M := (K \cdot M^1, \dots, K \cdot M^d)$ . In this case  $H^T \cdot (K \cdot M) = (HK)^T \cdot M$ .*
2. *Suppose that  $B$  is a  $\mathbb{R}^d$ -valued semimartingale whose components are in  $\mathcal{V}$  and let  $K \in L_s(B^i)$  for  $i = 1, \dots, d$ . Then  $H \in L_s(K \cdot B)$  if and only if  $HK \in L_s(B)$ , where  $K \cdot B := (K \cdot B^1, \dots, K \cdot B^d)$ . In this case  $H^T \cdot (K \cdot B) = (HK)^T \cdot B$ .*
3. *Suppose that  $X$  is a  $\mathbb{R}^d$ -valued semimartingale and let  $K \in L(X^i)$  for  $i = 1, \dots, d$ . Then  $H \in L(K \cdot X)$  if and only if  $HK \in L(X)$ , where  $K \cdot X := (K \cdot X^1, \dots, K \cdot X^d)$ . In this case  $H^T \cdot (K \cdot X) = (HK)^T \cdot X$ .*
4. *Statements 1–3 hold accordingly if  $H$  is  $\mathbb{R}$ -valued and  $K$   $\mathbb{R}^d$ -valued. In this case  $H \cdot (K^T \cdot X) = (KH)^T \cdot X$ .*
5. *Let  $\mu$  be an integer-valued random measure with compensator  $\nu$  and  $W = (W^1, \dots, W^d)$  with  $W^i \in G_{\text{loc}}(\mu)$  for  $i = 1, \dots, d$ . Then  $H^T W \in G_{\text{loc}}(\mu)$  if and only if  $H \in L_{\text{loc}}^1(W * (\mu - \nu))$ . In this case  $(H^T W) * (\mu - \nu) = H^T \cdot (W * (\mu - \nu))$ .*

**Proof.** (1) Let  $[M^i, M^j] = a^{ij} \cdot A$ , where  $A \in \mathcal{V}^+$  and  $a$  is a  $\mathbb{R}^{d \times d}$ -valued process. Then  $[K \cdot M^i, K \cdot M^j] = (Ka^{ij}K) \cdot A$  and  $(\sum_{i,j=1}^d H^i (Ka^{ij}K) H^j) \cdot A = (\sum_{i,j=1}^d (H^i K) a^{ij} (H^j K)) \cdot A$ . The statement follows from the definition of stochastic integration in Jacod (1980).

(2) Let  $B^i = a^i \cdot A$ , where  $A \in \mathcal{V}^+$  and  $a$  is a  $\mathbb{R}^d$ -valued process. Then  $K \cdot B^i = (Ka^i) \cdot A$  and  $(\sum_{i=1}^d H^i(Ka^i)) \cdot A = (\sum_{i=1}^d (H^i K)a^i) \cdot A$ . The statement follows again immediately from the definition in Jacod (1980).

(3) Define  $D := \{|\Delta X| > 1\} \cup \bigcup_{i=1}^d \{|K \Delta X^i| > 1\} \cup \{|(HK)^\top \Delta X| > 1\}$ . If  $HK \in L(X)$  or  $H \in L(K \cdot X)$ , then  $D$  is discrete by Jacod (1980, Proposition 3). Let  $\tilde{X}^D := X_0 + \sum_{s \leq \cdot} \Delta X_s 1_D(s)$  and  $X^D := X - \tilde{X}^D$ . Denote the canonical decomposition of the special semimartingale  $X^D$  as  $X^D = N + \tilde{B}$ . Let  $B := \tilde{X}^D + \tilde{B}$ . Then  $K \in \bigcap_{i=1}^d (L_{\text{loc}}^1(N^i) \cap L_s(B^i))$  by Jacod (1980, Proposition 3). If  $H \in L(K \cdot X)$ , the same proposition implies that  $H \in L_{\text{loc}}^1(K \cdot N) \cap L_s(K \cdot B)$ . Hence,  $HK \in L_{\text{loc}}^1(N) \cap L_s(B)$  and  $H^\top \cdot (K \cdot X) = (HK)^\top \cdot X$  by parts 1 and 2. If, on the other hand,  $HK \in L(X)$ , then  $HK \in L_{\text{loc}}^1(N) \cap L_s(B)$  again by Jacod (1980, Proposition 3). As before, the statement follows from parts 1 and 2.

(4) This is proved along the same lines as statements 1–3.

(5) Note that  $\hat{W}' = H^\top \hat{W}$  and  $\tilde{W}' = H^\top \tilde{W}$  for  $W' := H^\top W$  and  $\hat{W} := (\hat{W}^1, \dots, \hat{W}^d)$  and  $\tilde{W} := (\tilde{W}^1, \dots, \tilde{W}^d)$ . Let  $[W^i * (\mu - \nu), W^j * (\mu - \nu)] = a^{ij} \cdot A$ , where  $A \in \mathcal{V}^+$  and  $a$  is a  $\mathbb{R}^{d \times d}$ -valued process. Observe that  $\sum_{s \leq \cdot} (H^\top \tilde{W})^2 = \sum_{s \leq \cdot} \sum_{i,j=1}^d H_s^i H_s^j \Delta(W^i * (\mu - \nu))_s \Delta(W^j * (\mu - \nu))_s = (\sum_{i,j=1}^d H^i a^{ij} H^j) \cdot A$ . In view of the definitions of  $G_{\text{loc}}(\mu)$  and  $L_{\text{loc}}^1(W * (\mu - \nu))$ , this implies the first claim. The second statement follows from the fact that both sides of the equation are purely discontinuous local martingales with the same jumps.  $\square$

**Proposition A.2.** *Let  $X$  be a  $\mathbb{R}^d$ -valued semimartingale and  $H \in L(X)$ .*

1. *Then  $H \in L(X^c)$ ,  $(H^\top \cdot X)^c = H^\top \cdot X^c$ , and  $\Delta(H^\top \cdot X) = H^\top \Delta X$ , where  $X^c := (X^{1,c}, \dots, X^{d,c})$ .*
2. *If  $Y$  is a real-valued semimartingale, then  $[H^\top \cdot X, Y] = H^\top \cdot [X, Y]$ . In particular,  $H \in L([X, Y])$ .*
3. *For  $\Delta := \{(\omega, t, x) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^d : |x| > 1 \text{ or } |H_t^\top(\omega)x| > 1\} \in \mathcal{P} \otimes \mathcal{B}^d$  we have*

$$X = X_0 + X^c + x1_{\Delta^c}(x) * (\mu^X - \nu^X) + x1_{\Delta}(x) * \mu^X + B \tag{A.1}$$

and

$$H^\top \cdot X = H^\top \cdot X^c + H^\top x1_{\Delta^c}(x) * (\mu^X - \nu^X) + H^\top x1_{\Delta}(x) * \mu^X + H^\top \cdot B, \tag{A.2}$$

where  $B$  is some predictable process whose components are in  $\mathcal{V}$ .

**Proof.** (1) For  $d = 1$ , this follows from the definition in Jacod (1979, Definitions 2.67, and 2.68). In the  $d$ -dimensional case, let  $X = M + B$  be a decomposition of  $X$  with  $H \in L_{\text{loc}}^1(M)$  and  $X \in L_s(B)$ . Since  $[M^i, M^j] = [X^{i,c} + M^{i,d}, X^{j,c} + M^{j,d}] = [X^{i,c}, X^{j,c}] + [M^{i,d}, M^{j,d}]$ , one easily concludes that  $H \in L_{\text{loc}}^1(X^c) \cap L_{\text{loc}}^1(M^d)$ . From  $H^\top \cdot B \in \mathcal{V}$  it follows that  $(H^\top \cdot X)^c = (H^\top \cdot M)^c$ . The statements  $(H^\top \cdot M)^c = H^\top \cdot X^c$  and  $\Delta(H^\top \cdot M) = H^\top \Delta M$  can be shown as in Jacod (1979, 2.48). Since  $\Delta(H^\top \cdot B) = H^\top \Delta B$  holds for Stieltjes integrals, the claim follows.

(2) We have  $[H^\top \cdot X, Y] = \langle (H^\top \cdot X)^c, Y^c \rangle + \sum_{s \leq \cdot} \Delta(H^\top \cdot X) \Delta Y = H^\top \cdot \langle X^c, Y^c \rangle + \sum_{s \leq \cdot} H^\top \Delta X \Delta Y$  by Statement 1 and the definition of  $H^\top \cdot X^c$ . Note that the second term equals  $H^\top \cdot (\sum_{s \leq \cdot} \Delta X \Delta Y)$  in the Stieltjes sense. Together, the claim follows.

(3) Let  $D := \{|\Delta X| > 1\} \cup \{|H^T \Delta X| > 1\}$ . Eq. (A.1) follows from the canonical representation of the special semimartingale  $X^D := X - X_0 - x1_{\Delta}(x) * \mu^X$  (cf. Jacod and Shiryaev, 1987, II.2.38). By Jacod (1980, Proposition 3) we have  $H \in L_s(B)$  and  $H \in L_s(x1_{\Delta}(x) * \mu^X)$ . Moreover,  $H \in L(X^c)$  has been shown in the first part of the proof.  $H \in L(x1_{\Delta^c}(x) * (\mu^X - \nu^X))$  and Eq. (A.2) now follows from properties of stochastic integration (cf. Jacod, 1980, (11) and Proposition 5.1).  $\square$

**Proposition A.3.** *Assume that*

$$M + W_1 * (\mu - \nu) + W_2 * \mu + B = \tilde{M} + W_3 * (\mu - \nu) + W_4 * \mu + \tilde{B},$$

where  $\mu$  is an integer-valued random measure with compensator  $\nu$ ,  $M, \tilde{M}$  are continuous local martingales,  $B, \tilde{B} \in \mathcal{V}$  are predictable,  $W_1, W_2 \in G_{\text{loc}}(\mu)$ ,  $|W_2| * \mu \in \mathcal{V}$ ,  $|W_4| * \mu \in \mathcal{V}$ , and  $W_1 + W_2 = W_3 + W_4$ . Then we have  $M = \tilde{M}$  and  $\tilde{B} = B + (W_2 - W_4) * \nu$  up to indistinguishability.

**Proof.**  $M = \tilde{M}$  follows from Jacod and Shiryaev (1987, I.4.27). This implies that  $(W_1 - W_3) * (\mu - \nu)$  is the sum of processes in  $\mathcal{V}$ . Since it is a local martingale, it must be in  $\mathcal{A}_{\text{loc}}$  (cf. Jacod and Shiryaev, 1987, I.3.11). Hence,  $(W_2 - W_4) * \mu$  is the sum of processes in  $\mathcal{A}_{\text{loc}}$ . Therefore its variation process  $|W_2 - W_4| * \mu$  is in  $\mathcal{A}_{\text{loc}}^+$ , which implies that  $|W_2 - W_4| * \nu \in \mathcal{A}_{\text{loc}}^+$  and  $(W_2 - W_4) * (\mu - \nu) = (W_2 - W_4) * \mu - (W_2 - W_4) * \nu$  (cf. Jacod and Shiryaev, 1987, II.1.28). The claim follows from a simple calculation.  $\square$

**Proposition A.4.** *Let  $X = M + A \cdot A$ , where  $M$  denotes a  $\mathbb{R}^d$ -valued local martingale,  $A \in \mathcal{A}_{\text{loc}}^+$  a predictable process, and  $A \in L(A)$  a  $\mathbb{R}^d$ -valued process. Moreover, suppose that  $H \in L(X)$  with  $H^T A \leq 0$  ( $P \otimes A$ )-almost everywhere. Then there exists a  $\mathbb{R}^d$ -valued local martingale  $U$  and a  $\mathbb{R}^d$ -valued semimartingale  $V$  such that  $X = U + V$ ,  $H \in L(U) \cap L(V)$ , and  $H^T \cdot V$  is decreasing.*

**Proof.** W.l.o.g., the compensator  $\nu$  of the measure of jumps  $\mu^X$  is of the form  $\nu = A \otimes F$  for some transition kernel  $F$  from  $(\Omega \times \mathbb{R}_+, \mathcal{P})$  into  $(\mathbb{R}^d, \mathcal{B}^d)$ . Otherwise, replace  $A$  with some predictable process  $\tilde{A} \in \mathcal{A}_{\text{loc}}^+$  such that  $A \ll \tilde{A}$  (cf. Jacod and Shiryaev, 1987, II.2.9, I.3.13, Proposition 5.1).

Define the set  $\Delta := \{(\omega, t, x) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^d : |x| > 1 \text{ or } |H_t^T(\omega)x| > 1\} \in \mathcal{P} \otimes \mathcal{B}^d$ . From  $M = X_0 + X^c + x * (\mu^X - \nu)$  and Statement 3 of Proposition A.2 it follows that  $B := A \cdot A + x1_{\Delta}(x) * (\mu^X - \nu) - x1_{\Delta}(x) * \mu^X$  is a predictable process whose components are in  $\mathcal{V}$ . Moreover, the proof of this statement shows that  $H \in L_s(B) \cap L_s(x1_{\Delta}(x) * \mu^X)$ . Note that  $|x1_{\Delta}(x) * \nu \in \mathcal{A}_{\text{loc}}^+$  and  $B = A \cdot A - x1_{\Delta}(x) * \nu = (A - \int x1_{\Delta}(\cdot, x)F(\cdot, dx)) \cdot A$  by Proposition A.3.

Now, let  $\Delta' := \Delta \cap \{(\omega, t, x) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^d : H_t^T(\omega)x \leq 0\}$  and define a predictable mapping  $Y : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,

$$Y(t, x) := \begin{cases} x1_{\Delta'}(t, x) & \text{if } \int (-H_t^T \tilde{x})1_{\Delta'}(t, \tilde{x})F_t(d\tilde{x}) \leq -H_t^T A_t, \\ x1_{\Delta'}(t, x) \frac{H_t^T A_t}{\int H_t^T \tilde{x}1_{\Delta'}(t, \tilde{x})F_t(d\tilde{x})} & \text{else.} \end{cases}$$

$|Y(t, x)| \leq |x1_{\Delta}(t, x)|$  implies that  $|Y| * \nu \in \mathcal{A}_{\text{loc}}^+$  as well. Since  $H_t^T(A_t - \int Y(t, x)F_t(dx)) = 0 \wedge H_t^T(A_t - \int x1_{\Delta}(t, x)F_t(dx))$  and  $0 \wedge H_t^T(A_t - \int x1_{\Delta}(t, x)F_t(dx)) \leq 0 \wedge H_t^T(A_t - \int x1_{\Delta'}(t, x)F_t(dx)) \leq 0$ , we have  $H \in L_s(A \cdot A - Y * \nu)$  and  $H^T \cdot (A \cdot A - Y * \nu)$  is a decreasing process. Moreover,  $|H_t^T Y(t, x)| \leq |H_t^T x1_{\Delta}(t, x)|$  implies that  $H \in L_s(Y * \mu^X)$  and  $H^T \cdot (Y * \mu^X) = (H^T Y) * \mu^X$  is a decreasing process as well. Together, it follows that  $\mathcal{V} := A \cdot A + Y * (\mu^X - \nu) = (A \cdot A - Y * \nu) + Y * \mu^X$  is a well-defined semimartingale such that  $H \in L(\mathcal{V})$  and  $H^T \cdot \mathcal{V}$  is decreasing. Finally,  $U := M - Y * (\mu^X - \nu) = X - V$  is a  $\mathbb{R}^d$ -valued local martingale with  $H \in L(X) \cap L(V) \subset L(U)$ .  $\square$

The following lemma is a simple reformulation of Itô’s formula.

**Lemma A.5** (Itô’s formula). *Let  $U$  be an open subset of  $\mathbb{R}^d$  and  $X$  a  $U$ -valued semimartingale such that  $X_-$  is  $U$ -valued as well. Moreover, let  $f : U \rightarrow \mathbb{R}$  be a function of class  $C^2$ . Then  $f(X)$  is a semimartingale, and we have*

$$f(X_t) = f(X_0) + \int_0^t Df(X_{s-})^T dX_s + \frac{1}{2} \sum_{i,j=1}^d \int_0^t D_{ij}^2 f(X_{s-}) d\langle X^{i,c}, X^{j,c} \rangle_s + \int_{[0,t] \times \mathbb{R}^d} (f(X_{s-} + x) - f(X_{s-}) - Df(X_{s-})^T x) \mu^X(ds, dx) \tag{A.3}$$

for any  $t \in \mathbb{R}_+$ . Here,  $Df = (D_1 f, \dots, D_d f)$  and  $(D_{ij}^2 f)_{i,j=1, \dots, d}$  denote the first and second derivatives of  $f$ , respectively.

**Proof.** This follows immediately from Jacod (1979, (2.54)). Note that  $\bigcup_{n \in \mathbb{N}} [0, R_n] = \mathbb{R}_+$  if  $X_-$  is  $U$ -valued.  $\square$

We now consider the effect of  $C^2$ -mappings on the characteristics.

**Corollary A.6.** *Let  $X, U$  be as in the previous lemma. Moreover, let  $f : U \rightarrow \mathbb{R}^n$  be a function of class  $C^2$ . If  $(B, C, \nu)$  denote the characteristics of  $X$ , then the characteristics  $(\tilde{B}, \tilde{C}, \tilde{\nu})$  of the  $\mathbb{R}^n$ -valued semimartingale  $f(X)$  are given by*

$$\begin{aligned} \tilde{B}_t^k &= \int_0^t Df^k(X_{s-})^T dB_s + \frac{1}{2} \sum_{i,j=1}^d \int_0^t D_{ij}^2 f^k(X_{s-}) dC_s^{ij} \\ &+ \int_{[0,t] \times \mathbb{R}^d} (h_n^k(f(X_{s-} + x) - f(X_{s-})) - Df^k(X_{s-})^T h_d(x)) \nu(ds, dx), \\ \tilde{C}_t^{kl} &= \sum_{i,j=1}^d \int_0^t D_i f^k(X_{s-}) D_j f^l(X_{s-}) dC_s^{ij}, \end{aligned} \tag{A.4}$$

$$\tilde{\nu}([0, t] \times G) = \int_{[0,t] \times \mathbb{R}^n} 1_G(f(X_{s-} + x) - f(X_{s-})) \nu(ds, dx) \tag{A.5}$$

for  $k, l \in \{1, \dots, d\}$ ,  $t \in \mathbb{R}_+$ ,  $G \in \mathcal{B}^n$ .

**Proof.**  $\Delta f(X)_t = f(X_{t-} + \Delta X_t) - f(X_{t-})$  implies  $\mu^{f(X)}([0, t] \times G) = 1_G(f(X_- + x) - f(X_-)) * \mu_t$  and hence Eq. (A.5). From the previous lemma and the canonical representation of  $X$  (cf. Jacod and Shiryaev (1987, II.2.34)), we have that  $f^k(X) = f^k(X_0) + Df^k(X_-)^T \cdot X^c + Df^k(X_-)^T h_d(x) * (\mu^X - \nu) + Df^k(X_-)^T \cdot B + \frac{1}{2} \sum_{i,j=1}^d D_{ij}^2 f^k(X_-) \cdot C^{ij} + (f^k(X_- + x) - f^k(X_-) - Df^k(X_{s-})^T h_d(x)) * \mu^X$ . Moreover, the canonical semimartingale representation of  $f(X)$  equals  $f(X) = f(X_0) + (f(X))^c + h_n(f(X_- + x) - f(X_-)) * (\mu^X - \nu) + (f(X_- + x) - f(X_-) - h_n(f(X_- + x) - f(X_-))) * \mu^X + \tilde{B}$ . By Proposition A.3, we obtain the first equation and  $(f^k(X))^c = Df^k(X_-)^T \cdot X^c$ . Hence, Eq. (A.4) also follows.  $\square$

The effect of discounting on the characteristics is considered below. Note that many terms vanish if  $S^0$  is very simple (e.g.  $S_t^0 = e^{rt}$  for  $r \in \mathbb{R}$ ).

**Corollary A.7** (Discounting). *If, in Section 3, the price process  $S$  is a  $\mathbb{R}^{d+1}$ -valued semimartingale whose characteristics  $(\tilde{B}, \tilde{C}, \tilde{\nu})$  are of the form (3.1) (for some  $(\tilde{b}, \tilde{c}, \tilde{F})$ ) instead of  $(b, c, F)$  and if  $S^0, S_-^0$  are positive, then the discounted process  $(\hat{S}^1, \dots, \hat{S}^d)$  is a semimartingale whose characteristics are of the form (3.1) as well, where*

$$\begin{aligned}
 b_t^i &= \frac{1}{S_{t-}^0} \tilde{b}_t^i - \frac{\hat{S}_{t-}^i}{S_{t-}^0} \tilde{b}_t^0 - \frac{1}{(S_{t-}^0)^2} \tilde{c}_t^{0i} + \frac{\hat{S}_{t-}^i}{(S_{t-}^0)^2} \tilde{c}_t^{00} \\
 &+ \int_{\mathbb{R}^{d+1}} \left( h_d^i \left( \frac{S_{t-}^1 + x^1}{S_{t-}^0 + x^0} - \hat{S}_{t-}^1, \dots, \frac{S_{t-}^d + x^d}{S_{t-}^0 + x^0} - \hat{S}_{t-}^d \right) \right. \\
 &\left. - \frac{1}{S_{t-}^0} h_{d+1}^i(x^0, \dots, x^d) + \frac{\hat{S}_{t-}^i}{S_{t-}^0} h_{d+1}^0(x^0, \dots, x^d) \right) \tilde{F}_t(dx^0, \dots, x^d), \\
 c_t^{ij} &= \frac{1}{(S_{t-}^0)^2} \tilde{c}_t^{ij} - \frac{\hat{S}_{t-}^i}{(S_{t-}^0)^2} \tilde{c}_t^{0j} - \frac{\hat{S}_{t-}^j}{(S_{t-}^0)^2} \tilde{c}_t^{0i} + \frac{\hat{S}_{t-}^i \hat{S}_{t-}^j}{(S_{t-}^0)^2} \tilde{c}_t^{00},
 \end{aligned}$$

$$F_t(G) = \int 1_G \left( \frac{S_{t-}^1 + x^1}{S_{t-}^0 + x^0} - \hat{S}_{t-}^1, \dots, \frac{S_{t-}^d + x^d}{S_{t-}^0 + x^0} - \hat{S}_{t-}^d \right) \tilde{F}_t(dx)$$

for  $i, j \in \{1, \dots, d\}$ ,  $t \in \mathbb{R}_+$ ,  $G \in \mathcal{B}^d$  with  $0 \notin G$ .

**Proof.** The claim follows from the previous corollary applied to the mapping  $f : (0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $(x^0, \dots, x^d) \mapsto (x^1/x^0, \dots, x^d/x^0)$ .  $\square$

The following lemma shows how stochastic and usual exponentials of Lévy processes relate to each other.

**Lemma A.8** (Exponential Lévy processes). 1. *Let  $\tilde{L}$  be a real-valued Lévy process (i.e., a PIIS in the sense of Jacod and Shiryaev (1987, Definition II.4.1), with characteristic triplet  $(\tilde{b}, \tilde{c}, \tilde{F})$ . Then the process  $Z := e^{\tilde{L}}$  is of the form  $\mathcal{E}(L)$  for some Lévy process  $L$  whose triplet  $(b, c, F)$  is given by*

$$b = \tilde{b} + \frac{\tilde{c}}{2} + \int (h(e^x - 1) - h(x)) \tilde{F}(dx),$$



$$c = \tilde{c},$$

$$F(G) = \int 1_G(e^x - 1) \tilde{F}(dx) \quad \text{for } G \in \mathcal{B}.$$

2. Let  $L$  be a real-valued Lévy process with characteristic triplet  $(b, c, F)$ . Suppose that  $Z := \mathcal{E}(L)$  is positive. Then  $Z = e^{\tilde{L}}$  for some Lévy process  $\tilde{L}$  whose triplet  $(\tilde{b}, \tilde{c}, \tilde{F})$  is given by

$$\tilde{b} = b - \frac{c}{2} + \int (h(\log(1+x)) - h(x)) F(dx),$$

$$\tilde{c} = c,$$

$$\tilde{F}(G) = \int 1_G(\log(1+x)) F(dx) \quad \text{for } G \in \mathcal{B}.$$

**Proof.** 1. By Itô's formula, we have  $e^{\tilde{L}} = 1 + e^{\tilde{L}-} \cdot L$ , where

$$\begin{aligned} L_t &= \tilde{L}_t + \frac{1}{2} \langle \tilde{L}^c, \tilde{L}^c \rangle_t + (e^x - 1 - x) * \mu_t^{\tilde{L}} \\ &= \tilde{L}_t^c + h(x) * (\mu_t^{\tilde{L}} - \nu_t^{\tilde{L}}) + \tilde{b}t + \frac{1}{2} \langle \tilde{L}^c, \tilde{L}^c \rangle_t + (e^x - 1 - h(x)) * \mu_t^{\tilde{L}}. \end{aligned}$$

Since  $\Delta L = e^{-\tilde{L}-} \Delta e^{\tilde{L}} = e^{\Delta \tilde{L}} - 1$ , we have  $\mu^L([0, t] \times G) = 1_G(e^x - 1) * \mu^{\tilde{L}}$  and likewise for  $\nu^L, \nu^{\tilde{L}}$ . From the canonical representation of  $L$  we get  $L = L^c + h(e^x - 1) * (\mu^{\tilde{L}} - \nu^{\tilde{L}}) + B + (e^x - 1 - h(e^x - 1)) * \mu^{\tilde{L}}$ , where  $B$  is the first semimartingale characteristic of  $L$ . Hence  $L^c = \tilde{L}^c$  and  $B_t = \tilde{b}t + \frac{1}{2} \langle \tilde{L}^c, \tilde{L}^c \rangle_t + (h(e^x - 1) - h(x)) * \nu_t^{\tilde{L}}$  by Proposition A.3. It follows that  $L$  is a Lévy process whose triplet is as claimed.

2. By Itô's formula, we have  $\log(Z) = \tilde{L}$ , where

$$\begin{aligned} \tilde{L}_t &= L_t - \frac{1}{2} \langle L^c, L^c \rangle_t + (\log(1+x) - x) * \mu_t^L \\ &= L_t^c + h(x) * (\mu_t^L - \nu_t^L) + bt - \frac{1}{2} \langle L^c, L^c \rangle_t + (\log(1+x) - h(x)) * \mu_t^L. \end{aligned}$$

Obviously,  $\tilde{L}$  is a Lévy process. Its triplet is immediately obtained from Statement 1.  $\square$

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