A Stochastic Feedback Model for Volatility

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Financial time series exhibit a number of interesting properties that are difficult to explain with simple models. These properties include fat-tails in the distribution of price fluctuations (or *returns*) that are slowly removed at longer timescales, strong autocorrelations in absolute returns but zero autocorrelation in returns themselves, and multifractal scaling. Although the underlying cause of these features is unknown, there is growing evidence they originate in the behavior of *volatility*, i.e., in the behavior of the magnitude of price fluctuations. In this paper, we posit a feedback mechanism for volatility that reproduces many of the non-trivial properties of empirical prices. The model is parsimonious, requires only two parameters to fit a specific financial time series, and can be grounded in a straightforward framework where volatility fluctuations are driven by the estimation error of an exogenous Poisson rate.

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Financial time series exhibit a number of interesting regularities (or “stylized facts”, as economists call them) that are well-documented in the literature[1–7]. They include fat tails in the distribution of price fluctuations (known as *returns* in finance) that are slowly removed at longer timescales, strong autocorrelations in absolute returns but zero autocorrelation in returns themselves, multifractal scaling, and a negative correlation between past returns and the future magnitude of returns (known as the leverage effect).

Although there is currently no consensus on the underlying cause of these properties, there is growing evidence they are all rooted in the behavior of *volatility*, i.e., in the behavior of the magnitude of returns[3]. In addition, there is evidence that returns – like earthquakes, turbulent flow, and Barkhausen noise – are driven by strong endogenous, or internal, feedback effects[8].

In this paper, we present a stochastic feedback model for volatility that generates many of the stylized facts of financial time series. The model is motivated by several recent studies that have found the variance of price fluctuations, i.e., the squared volatility, is slowly varying and inverse gamma distributed so that returns are well-fit by a Student’s t-distribution over intraday timescales[9, 10]. Here we extend these results by modelling the properties of returns over timescales longer than one day. We propose a simple mechanism that generates inverse gamma distributed variances and introduce a feedback parameter that allows the variance to change slowly over time as it does in real price series. As a result, returns are t-distributed at daily intervals but slowly approach a Gaussian as timescales are increased to weekly, monthly, and yearly intervals.

The model is based on the following insight: when the rate of a Poisson process is estimated using a fixed number of events, the resulting estimate is inverse gamma distributed. Assuming that price fluctuations are driven by market participants who act according to this estimated rate, squared volatility will be inverse gamma distributed and returns will be t-distributed.

Although the results of the model match empirical data very well, we make no strong claim that we have uncovered *the* mechanism driving real-world volatility fluctuations. Instead, we offer the proposed mechanism as a novel explanation for these fluctuations and leave any conclusions regarding the true mechanism for later analysis. Alternative models that also produce inverse gamma distributed variances (and therefore Student t-distributed returns) include the well-known GARCH model[3, 11, 12] which can be motivated by the position of stop-loss orders in markets[13] and the Minimal Market Model[14, 15] which describes the dynamics of a growth optimal portfolio with deterministic drift.

To begin our analysis, we define the t^{th} return as the difference in logarithmic price from time t to time $t+\Delta t$ where t is measured in days,

$$r_t(\Delta t) = \ln(p_{t+\Delta t}) - \ln(p_t). \tag{1}$$

We model daily returns, $r_t(\Delta t = 1)$, as a discrete time stochastic process with a fluctuating variance,

$$r_t = \mu + \sigma_t \xi_t, \tag{2}$$

where μ is the daily average return, ξ_t is an IID Gaussian $N(0,1)$ random variable, and σ_t^2 is the local variance of returns (the square of the daily volatility, σ_t).

We assume that the daily variance of returns, σ_t^2, is determined by a stochastic feedback process so that,

$$\sigma_t^2 \sim \frac{\sigma_0^2}{Gamma(1 + B\sigma_0^2/\sigma_{t-1}^2, 1 + B)}, \tag{3}$$

where $Gamma(a,b)$ is the gamma distribution, $f(x|a,b) = (b^a/\Gamma[a])x^{a-1}e^{-bx}$.

Only two parameters are used in Eq. 3: $1/\sigma_0^2$ is the equilibrium inverse variance of the return process and $B > 1$ is a feedback parameter. Notice that when $\sigma_{t-1}^2 = \sigma_0^2$, the expected value of $1/\sigma_t^2$ is $1/\sigma_0^2$. Deviations from this equilibrium value are removed over a length of time determined by B. If B is large, the variance requires many iterations to relax back towards σ_0^2, but for small B, the relaxation is quick.

The model can be motivated by the following mechanism. Suppose that detrended intraday returns are uncorrelated, are all of size $\pm\delta$, and occur at times determined by a Poisson process with changing rate parameter $\lambda(t)$. $\lambda(t)$ is determined by market participants at the beginning of the day as follows: (1) they observe a separate exogenous Poisson process with rate parameter λ_e (this process could describe, for example, the arrival of new orders to the market or the arrival of economic news), (2) they estimate the rate of this exogenous process using the past M events, and (3) they set $\lambda(t)$ such that a target number N price fluctuations occur per M exogenous events. Denoting their estimate of the exogenous rate by $\widehat{\lambda}_e(t)$,

$$\lambda(t) = (N/M)\widehat{\lambda}_e(t). \tag{4}$$

M exogenous events will occur in an amount of time, $\tau \sim Gamma(M, \lambda_e)$, making the estimated rate inverse gamma distributed, $\widehat{\lambda}_e(t) \sim M/Gamma(M, \lambda_e)$. The local variance of daily returns is therefore,

$$\sigma_t^2 = \delta^2\lambda(t) \sim \frac{\delta^2 N}{Gamma(M, \lambda_e)}. \tag{5}$$

Notice that this mechanism produces inverse gamma distributed variances, but these variances are not autocorrelated.

To introduce feedback effects we assume that M varies through time so that the sensitivity of the market to exogenous events (measured by N/M) is high when the local variance is high and low when the local variance is low. The simplest form of this relationship (taking into account that $M \geq 1$) is,

$$M(t) = 1 + A/\sigma_{t-1}^2. \tag{6}$$

The final result is,

$$\sigma_t^2 \sim \frac{\delta^2 N}{Gamma(1 + A/\sigma_{t-1}^2, \lambda_e)}. \tag{7}$$

The equation can be simplified by introducing the equilibrium inverse variance $1/\sigma_0^2$, defined by the relation $E[1/\sigma_t^2|\sigma_{t-1}^2 = \sigma_0^2] = 1/\sigma_0^2$. Therefore, $\sigma_0^2 = \delta^2 N\lambda_e - A$. Using this relation and the simplifying assumption that at equilibrium, $E[\tau|\sigma_{t-1}^2 = \sigma_0^2] = 1$, so that $A = \sigma_0^2(\lambda_e - 1)$, we have,

$$\sigma_t^2 \sim \frac{\sigma_0^2}{Gamma(1 + (\lambda_e - 1)\sigma_0^2/\sigma_{t-1}^2, \lambda_e)}. \tag{8}$$

Notice this equation is identical to Eq. 3 with the change of variable $\lambda_e = 1 + B$.

To determine how the parameters of the model affect returns, we simulate the model using different choices of σ_0^2 and B (each run includes 50 million time steps). σ_0^2 only influences the scale of the process and leaves the statistical properties of returns unchanged (a result confirmed in our simulations), so we do not show the results of varying σ_0^2 here.

In Fig. 1, we show the properties of the model with different B. We set $\sigma_0^2 = 10^{-4}$ and let $B = 10, B = 100, B = 1000$ respectively. To facilitate the presentation of results, we define the following normalized variables:

$$\begin{aligned} r_t'(\Delta t) &\equiv (r_t(\Delta t) - \mu)/(\sigma_0\sqrt{\Delta t}), &(9)\\ \sigma_t' &\equiv \sigma_t/\sigma_0, &(10)\\ \beta_t' &\equiv \sigma_0^2/\sigma_t^2. &(11)\end{aligned}$$

As seen in Figs. 1(a and b), the parameter B sets the strength of volatility autocorrelations with larger values of B producing stronger autocorrelations. Notice that the autocorrelation function (Fig. 1(b)) drops rapidly at the point where the lag approximately equals B. As seen in Fig. 1(c), the distribution of the normalized inverse squared volatility, β', is gamma distributed and largely unaffected by B (although for $B = 10$, the peak of the distribution is slightly below the others). A gamma distributed β' should produce t-distributed returns for $\Delta t = 1$, which is observed in the topmost curves in Fig. 1(d). As Δt increases (moving to the lower curves in Fig. 1(d)), the return distribution adjusts from a Student's t-distribution to a Gaussian, with the speed of adjustment determined by B. Higher values of B correspond to a slowly varying volatility and therefore to a return distribution that retains its non-Gaussian shape at larger Δt.

In Fig. 2 we compare the results of the model ($\sigma_0^2 = 1$, $B = 175$) with three empirical financial time series: the daily price series of (a) the Dow Jones Industrial Average (DJIA) from January 4, 1960 to December 31, 1984, (b) the DJIA from January 2, 1985 to December 31, 2009, and (c) the FTSE 100 from January 2, 1985 to December 31, 2009. All data is from finance.yahoo.com. From the price series, daily returns are calculated as in Eq. 1. Squared volatilities are estimated using a rolling window of two months of daily returns (42 trading days),

$$\hat{\sigma}_t^2 = \frac{\sum_{i=-21}^{20}(r_{t+i} - \hat{\mu})^2}{42}, \tag{12}$$

where $\hat{\mu}$ is the estimated mean of the daily returns.

We have found that a good estimate of σ_0^2, is the inverse of the mean inverse variance,

$$\hat{\sigma}_0^2 = \frac{1}{\langle 1/\sigma_t^2\rangle}. \tag{13}$$

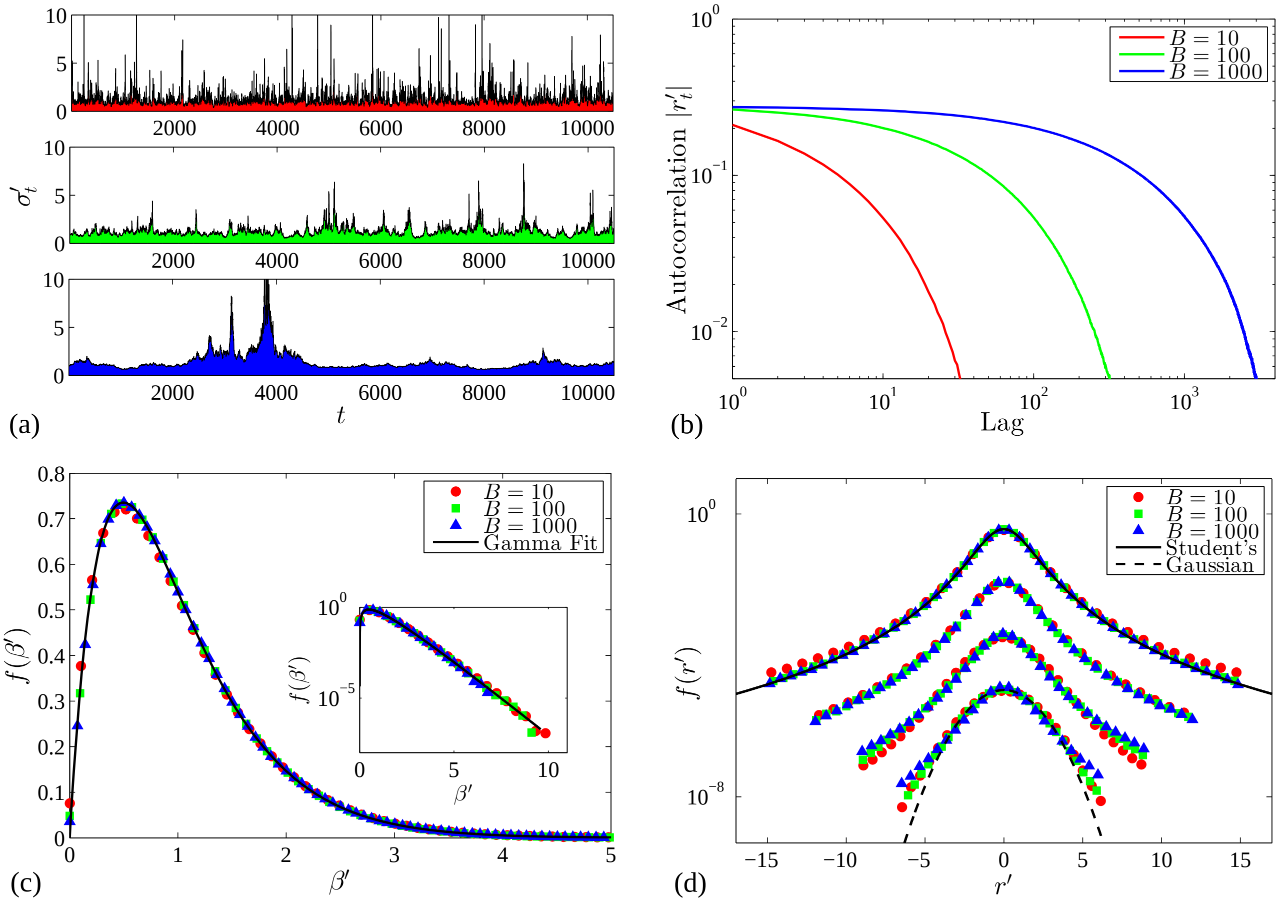

FIG. 1: Properties of the model with various choices of the parameter B. (a) The volatility series for $B = 10$ (top), $B = 100$ (middle), and $B = 1000$ (bottom). (b) The autocorrelation function of absolute scaled returns. (c) The probability density function of the scaled inverse variance (the inset plot is in semilog coordinates). A fit to the Gamma distribution is shown for B=100. (d) The probability density function of scaled returns compared to a Student's t-distribution and a Gaussian. The curves at different Δt are arbitrarily offset vertically and from top to bottom are for $\Delta t = 1$, $\Delta t = 10$, $\Delta t = 100$, and $\Delta t = 1000$.

Using this equation, we find $\hat{\sigma}_0^2 = 4.2 \times 10^{-5}$ for the early DJIA data, $\hat{\sigma}_0^2 = 6.0 \times 10^{-5}$ for the late DJIA data, and $\hat{\sigma}_0^2 = 6.3 \times 10^{-5}$ for the FTSE data.

We estimate B for each series by minimizing the sum of the squared difference between $1/\sigma_t^2$ and its expected value, i.e., we find the B that minimizes $\sum e_t^2$, where $e_t = 1/\sigma_t^2 - (1 + B\sigma_0^2/\sigma_{t-1}^2)/((1+B)\sigma_0^2)$. Using this method, we find $\widehat{B} = 202$ for the early DJIA data, $\widehat{B} = 156$ for the late DJIA data, and $\widehat{B} = 167$ for the FTSE data.

Although we obtain a different $\widehat{B}$ for the each of the empirical price series, the values are sufficiently similar that we choose to compare all three series to the model using only one set of parameters, $\sigma_0^2 = 1$ and $B = 175 = (202 + 156 + 167)/3$ (see Fig. 2). As in Fig. 1, the plots in Fig. 2 use the renormalizations of Eqs. 9-11 but with the estimated values of the parameters. As seen in Figs. 2(a-d), the statistical properties of the empirical prices are well-reproduced by the model.

Financial time series have been studied by mathematicians and physicists over many years[1, 6, 16, 17]. Although the dynamics of prices are now well-characterized and understood, it is still unclear why prices exhibit the interesting properties that they do. This lack of understanding is especially troublesome because prices fluctuate in a universal, regular way, i.e., the returns of many different traded items all possess the same nontrival properties. It is therefore quite likely that some simple, robust mechanism underlies price dynamics, even if we have not yet discovered it[18].

We have presented a simple feedback model for volatility that matches empirical data very well and that can be motivated by the estimation error of an exogenous Poisson rate. The model reproduces several important features of empirical prices: returns are t-distributed at daily intervals but slowly become Gaussian when measured over longer timescales, and the absolute value of returns is strongly autocorrelated. Although not shown here, we have found that the model also replicates the

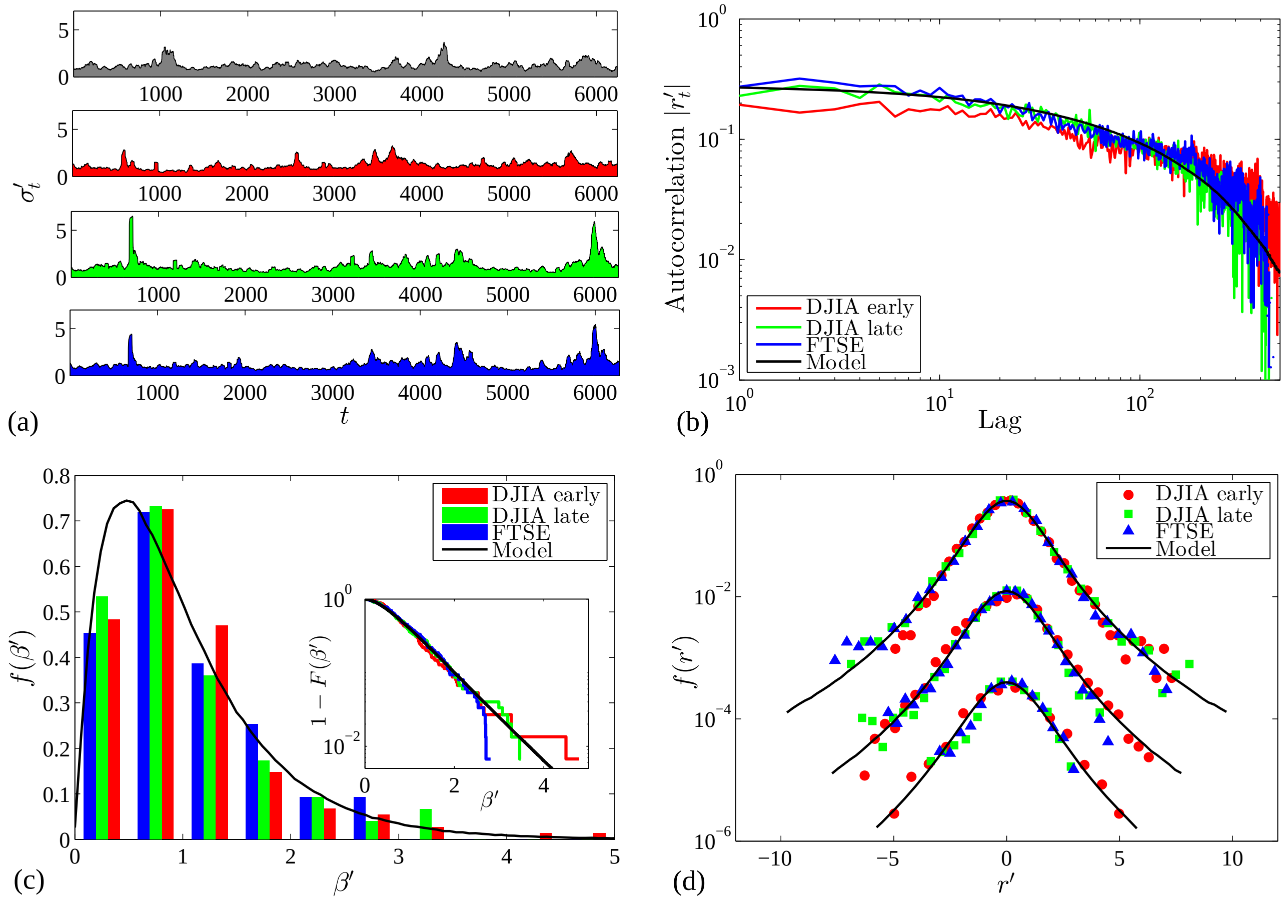

FIG. 2: Comparison of the model ($\sigma_0^2 = 1$, $B = 175$) with data. (a) The volatility series for the model (top), the early DJIA data (top middle), the late DJIA data (bottom middle), and the FTSE data (bottom). (b) The autocorrelation function of absolute scaled returns. (c) The probability density function of the scaled inverse variance. The inset plot shows 1 minus the cumulative distribution function. (d) The probability density function of scaled returns. The curves at different Δt are arbitrarily offset vertically and from top to bottom are for $\Delta t = 1$, $\Delta t = 10$, and $\Delta t = 100$.

multifractal structure of returns. The model does not produce the well-known correlation between negative returns and volatility (known as the leverage effect) nor does it explicitly include feedback effects on multiple timescales[13, 19–21], but these features could be added to the model without difficulty.

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