POLYCATEGORIES

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This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text.

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Chapter 1

Introduction

The concept of *polycategory*, as introduced in [Sza75], has for a long time been something of a poor relation in the family of generalised categorical structures. Intuitively, a polycategory is a category in which maps can have many 'inputs' and many 'outputs'; composition proceeds by plugging precisely one output of one map into precisely one input of another.

Despite this seemingly simple description, the concept of polycategory has for a long time lacked an urbane mathematical formulation: indeed, there has been little development on Szabo's original hands-on definition, whose slew of data and axioms make it very hard to develop a coherent 'theory of polycategories'.

The polycategory has a simpler cousin, the *multicategory*, whose maps can have many inputs but only one output. The multicategory admits an elegant formalism, originally developed by Burroni [Bur71] (under the name 'T-categories'), and later independently rediscovered and popularised by Leinster [Lei04b] and Hermida [Her00] under the name 'T-multicategories'. However, this approach does not generalise easily to the case of polycategories; indeed, the paper [Kos03] is, to date, the only such attempt. Though it does yield a more abstract framework within which to consider polycategories, there is still a sense that this framework is being bent somewhat in order to obtain the desired results.

There is another approach to multicategories in terms of the *free strict monoidal category pseudomonad* $\hat{S}$ on **Mod**, the bicategory of categories, profunctors and transformations. We can form the 'Kleisli bicategory' $Kl(\hat{S})$ of this pseudomonad – which is a higher-dimensional analogue of an ordinary Kleisli category – and recover multicategories as monads in this bicategory.

This is the approach adopted by [BD98] and [CT03], and has certain advantages not possessed by the 'T-multicategory' approach: it generalises straightforwardly to what one might term '$\mathcal{V}$-enriched multicategories' upon replacing **Mod** with $\mathcal{V}$-**Mod**; and it allows one to consider *symmetric* multicategories, whose inputs and outputs may be freely re-ordered, upon replacing the strict monoidal category pseudomonad with the symmetric strict monoidal category pseudomonad.

Furthermore, this approach has an extension from the case of multicategories to that of polycategories, an extension which is the primary concern of this thesis. In it, we consider the free symmetric strict monoidal category as a pseudo*co*monad $\hat{T}$ as well as a pseudomonad $\hat{S}$ on **Mod**, and look for a *pseudo-distributive law* of $\hat{T}$ over $\hat{S}$. Pseudo-distributive laws generalise distributive laws in the sense of Beck [Bec69], and have been studied by [Mar99] and more comprehensively by [Tan04].

Now, given an honest distributive law $\delta \colon TS \Rightarrow ST$ of a comonad T over a monad S on a category $\mathbf{C}$, we can form the 'two-sided Kleisli category' $Kl(\delta)$ of δ, whose objects are those of $\mathbf{C}$, whose maps from X to Y are maps $TX \to SY$ of $\mathbf{C}$, and whose composition proceeds using the distributive law:

$$(TY \xrightarrow{g} SZ) \circ (TX \xrightarrow{f} SY) = TX \xrightarrow{\Delta_X} TTX \xrightarrow{Tf} TSY \xrightarrow{\delta_Y} STY \xrightarrow{Sg} SSZ \xrightarrow{\mu_Z} SZ.$$

Similarly, given a pseudo-distributive law δ of a pseudocomonad over a pseudomonad, we can produce the higher-dimensional analogue of the above, namely the 'two-sided Kleisli bicategory' $Kl(\delta)$; we would like to view polycategories as monads in a suitable such $Kl(\delta)$.

This is our theoretical framework; however, its practical implementation is potentially somewhat wrought. In order to give a pseudo-distributive law, we must specify five pieces of bicategorical data, each itself consisting of non-trivial data and axioms, and check ten equalities of pastings. Clearly, a brute force approach is hopeless, and therefore we seek a subtler way to derive this data.

For this, we turn to the theory of *clubs*, introduced and later reformulated abstractly by Kelly [Kel72a, Kel72b, Kel74b, Kel92]. Clubs capture the intuition of adding structure to categories in a 'generic way': given a description of this added structure at the terminal category 1, we should be able to derive it at an arbitrary category **C** by 'labelling with objects and maps of **C**'.

We should like to use this theory to reduce the problem of giving our pseudo-distributive law on **Mod** to that of giving a pseudo-distributive law 'at the terminal category 1', a statement that we will make precise in the course of this thesis. If we can perform this reduction, then much of the coherence and data which should be obvious comes for free, data which we would otherwise be required to provide. This frees us to concentrate on providing the (non-trivial) combinatorial core of the pseudo-distributive law.

However, the theory of clubs as it stands is inadequate; it deals with categories with pullbacks, and we need to work with **Mod**, which neither is a category nor has pullbacks. Therefore we must first look for a suitable generalisation of the theory of clubs which is amenable to application in **Mod**. Now, taking pullbacks is fundamental to the theory of clubs, so we are led to question whether or not **Mod** is the correct place to work; ideally, we should like to replace it with something where we can take lots of pullbacks.

Now, observe that **Mod** has certain peculiar properties: it has all lax colimits, but these lax colimits have a universal property up to isomorphism rather than up to equivalence; unfortunately, the language of bicategories cannot express what this universal property is. Similarly, the operation given on objects by cartesian product of categories induces a structure of monoidal bicategory on **Mod**; again, this structure ought to be associative up to isomorphism rather than equivalence, and again, the language of bicategories is simply unable to express this.

Inspired by this, we are led to consider the *pseudo double categories* of [GP99] and [GP04] (and also considered briefly by [Lei04a]). These are a weakening of Ehresmann's notion of *double category* [Ehr63, Ehr65], and have two directions, one 'category-like' and the other 'bicategory-like'. The presence of a 'category-like' direction allows us to express 'up-to-isomorphism' as well as 'up-to-equivalence' notions, and more saliently, to take lots of pullbacks. Indeed, in our case, we can generalise **Mod** to the pseudo double category $\mathbb{C}at$ of 'categories, functors, profunctors and transformations' which in an appropriate sense, has all pullbacks.

Thus our first task is to develop a suitable generalisation of the theory of clubs from plain categories to pseudo double categories; our second is to exhibit a suitable such 'double club' on the pseudo double category $\mathbb{C}at$, and our third is to apply this theory to the construction of the pseudo-distributive law for polycategories on **Mod**. Corresponding to these three tasks are the three Parts of this thesis.

A subsidiary theme running throughout is the relationship between higher-dimensional monoidal structures, such as monoidal bicategories and the *monoidal double categories* of Chapter 4, and corresponding higher-dimensional structures equipped with a notion of left and right 'whiskerings'. A structure equipped with the latter structure always possesses the former, and the converse is almost true. Likewise, a map which preserves whiskerings will preserve monoidal structure as well, and again, the converse is almost true. Although many of the results of this thesis are phrased in terms of monoidal structures, it is the whiskerings which we shall be more concerned with in practice.

Part I begins by summarising some of the basic concepts and definitions of pseudo double categories (Chapter 1), before recapping the theory of plain clubs (Chapter 2). We then explore some further aspects of the theory of pseudo double categories which will be necessary in order to generalise the theory of clubs: we consider comma double categories, equivalences of double categories and cartesian maps in double categories (Chapter 3), and aspects of the theory of 'monoidal double categories' (Chapter 4). With this in place, we are ready to give our definition of 'double club', and to prove important results mirroring those for the theory of plain clubs (Chapter 5).

Part II moves from the general to the specific by examining the pseudo double category $\mathbb{C}at$ and some of its more pertinent properties (Chapter 6) and showing that we can extend the club S for symmetric strict monoidal categories on **Cat** to a double club on $\mathbb{C}at$ (Chapter 7).

Part III then applies the preceding theory to the description of polycategories. We start by laying out in more detail the theoretical framework for polycategories outlined above (Chapter 8); the remainder of the thesis we devote to its practical implementation, namely the establishment of a suitable pseudo-distributive law on **Mod**. After a few necessary technical results about dual monads and pseudomonads (Chapter 9), we use the theory of double clubs to reduce the construction of a pseudo-distributive law on **Mod** to that of a pseudo-distributive law 'at 1' (Chapter 10), a construction which we then carry out (Chapter 11).

Part I

Clubs and double clubs

Chapter 2

Pseudo double categories I

In this chapter, we provide definitions of the notion of *pseudo double category*, together with the apposite notions of functor, transformation and modification. Pseudo double categories are a weakening of the well-known notion of (plain) double category, as introduced by Ehresmann [Ehr65, Ehr63], and have been studied by Grandis and Paré [GP99, GP04] and Leinster [Lei04a] (under the name 'weak double category').

This chapter summarises material from [GP99], though the definitions given here emphasise more strongly the fact that a pseudo double category is something like a cross between a bicategory and a monoidal category. One may complain that this statement is nonsensical, that a monoidal category is just a special case of bicategory: but there is more to it than that. For example, monoidal categories naturally form a 2-category whilst bicategories do not; and indeed, we shall see that pseudo double categories, despite their bicategory-like aspects, also naturally form a 2-category.

2.1 Pseudo double categories

Let K_0 be a category, and consider the strict slice 2-category $\mathbf{Cat}/(K_0 \times K_0)$. The underlying ordinary category of this 2-category has a monoidal structure given by pullback; that is

$$\begin{tikzcd}
& & & & & & & & & & G \otimes F \arrow[dl] \arrow[dr] & & \\
& F \arrow[dl, "s_F"'] \arrow[dr, "t_F"] & & & & G \arrow[dl, "s_G"'] \arrow[dr, "t_G"] & & & & F \arrow[dl, "s_F"'] \arrow[dr, dotted, "t_F"] & & G \arrow[dl, dotted, "s_G"'] \arrow[dr, "t_G"] & \\
K_0 & & K_0 & \otimes & K_0 & & K_0 & = & K_0 & & K_0 & & K_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
& K_0 \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\
\mathbf{I} = \quad K_0 & & K_0.
\end{tikzcd}$$

Now, by Section 3 of [Kel89], these pullbacks are in fact **Cat**-pullbacks, and so this enriches to make $\mathbf{Cat}/(K_0 \times K_0)$ into a monoidal **Cat**-category; in particular, it is a *monoidal bicategory* [GPS95], and hence we can consider *pseudomonoids* [DS97, McC99] in it.

Definition 1. A **pseudo double category** $\mathbb{K}$ consists of:

- A category K_0;
- A pseudomonoid $K_1 \overset{s}{\underset{t}{\rightrightarrows}} K_0$ in $\mathbf{Cat}/(K_0 \times K_0)$.

We now give an elementary description of this structure. First we need a little notation: let us write a typical object of K_1 as $\mathbf{X}$, and write X_s and X_t for $s(\mathbf{X})$ and $t(\mathbf{X})$; similarly, let us write a typical map of K_1 as $\mathbf{f}$ and write f_s and f_t for $s(\mathbf{f})$ and $t(\mathbf{f})$. We may also write $\mathbf{X}$ as $\mathbf{X}\colon X_s \relbar\joinrel\mapstochar\joinrel\rightarrow X_t$ and $\mathbf{f}$ as

$$\begin{tikzcd}
X_s \arrow[r, "\mathbf{X}", "\shortmid"{marking}] \arrow[d, "f_s"'] \arrow[dr, phantom, "\Downarrow \mathbf{f}"] & X_t \arrow[d, "f_t"] \\
Y_s \arrow[r, "\mathbf{Y}"', "\shortmid"{marking}] & Y_t.
\end{tikzcd}$$

Now let us expand the above definition; a pseudo double category consists of the following data:

(DD1) A category K_0 of ‘objects and vertical maps’;

(DD2) A category K_1 of ‘horizontal maps and cells’;

(DD3) ‘Source’ and ‘target’ functors $s, t\colon K_1 \to K_0$;

(DD4) A ‘horizontal units’ functor $\mathbf{I}\colon K_0 \to K_1$;

(DD5) A ‘horizontal composition’ functor $\otimes\colon K_1 {}_s\times_t K_1 \to K_1$;

(DD6) Special natural isomorphisms

$$\begin{tikzcd}
K_1 \arrow[r, "{[\mathbf{I}t,\mathrm{id}]}"] \arrow[dr, "\mathrm{id}"'] \arrow[r, phantom, shift right=4ex, "\overset{\mathfrak{l}}{\Rightarrow}"] & K_1 {}_s\times_t K_1 \arrow[d, "\otimes"] & K_1 \arrow[l, "{[\mathrm{id},\mathbf{I}s]}"'] \arrow[dl, "\mathrm{id}"] \arrow[l, phantom, shift left=4ex, "\overset{\mathfrak{r}}{\Leftarrow}"] \\
 & K_1 &
\end{tikzcd}$$

with components

$$\mathfrak{l}_{\mathbf{X}}\colon \mathbf{X} \to \mathbf{I}_{X_t} \otimes \mathbf{X}$$
$$\text{and } \mathfrak{r}_{\mathbf{X}}\colon \mathbf{X} \to \mathbf{X} \otimes \mathbf{I}_{X_s}$$

in K_1;

(DD7) A special natural isomorphism

$$\begin{tikzcd}
K_1 {}_s\times_t K_1 {}_s\times_t K_1 \arrow[r, "\mathrm{id}_s\times_t\otimes"] \arrow[d, "\otimes_s\times_t\mathrm{id}"'] \arrow[dr, phantom, "\Downarrow\mathfrak{a}"] & K_1 {}_s\times_t K_1 \arrow[d, "\otimes"] \\
K_1 {}_s\times_t K_1 \arrow[r, "\otimes"'] & K_1
\end{tikzcd}$$

with components

$$\mathfrak{a}_{\mathbf{XYZ}}\colon \mathbf{X} \otimes (\mathbf{Y} \otimes \mathbf{Z}) \to (\mathbf{X} \otimes \mathbf{Y}) \otimes \mathbf{Z}$$

in K_1.

Here, *special* means the following:

Definition 2. Given a diagram $K_1 \overset{s}{\underset{t}{\rightrightarrows}} K_0$ of categories, we say that a natural transformation

$$\begin{tikzcd}
J \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & K_1
\end{tikzcd}$$

is **special** if $sF = sG$, $tF = tG$, $s\alpha = \mathrm{id}_{sG}$ and $t\alpha = \mathrm{id}_{tG}$. In particular, setting $J = 1$, we say that a map $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ in K_1 is **special** if $X_s = Y_s$, $X_t = Y_t$, $f_s = \mathrm{id}_{X_s}$ and $f_t = \mathrm{id}_{X_t}$. Clearly, a natural transformation is special if and only if all its components are special maps.

Now, this data is required to satisfy the following axioms:

(DA1) The following diagram commutes:

$$\begin{tikzcd}
& K_0 \arrow[dl, "\mathrm{id}"'] \arrow[d, "\mathbf{I}"] \arrow[dr, "\mathrm{id}"] & \\
K_0 & K_1 \arrow[l, "s"] \arrow[r, "t"'] & K_0;
\end{tikzcd}$$

(DA2) The following diagram commutes:

$$\begin{tikzcd}
& K_1 {}_s\times_t K_1 \arrow[dl, "s\circ\pi_2"'] \arrow[d, "\otimes"] \arrow[dr, "t\circ\pi_1"] & \\
K_0 & K_1 \arrow[l, "s"] \arrow[r, "t"'] & K_0;
\end{tikzcd}$$

(DA3) For all $(\mathbf{X}, \mathbf{Y})$ in $K_1 {}_s\times_t K_1$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{X} \otimes \mathbf{Y} \arrow[r, "\mathbf{X}\otimes\mathfrak{l}_{\mathbf{Y}}"] \arrow[d, "\mathfrak{r}_{\mathbf{X}}\otimes\mathbf{Y}"'] & \mathbf{X} \otimes (\mathbf{I}_{Y_t} \otimes \mathbf{Y}) \arrow[d, "\mathfrak{a}_{\mathbf{X},\mathbf{I}_{Y_t},\mathbf{Y}}"] \\
(\mathbf{X} \otimes \mathbf{I}_{X_s}) \otimes \mathbf{Y} \arrow[r, equal] & (\mathbf{X} \otimes \mathbf{I}_{Y_t}) \otimes \mathbf{Y};
\end{tikzcd}$$

(DA4) For all $(\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in $K_1 {}_s\times_t K_1 {}_s\times_t K_1 {}_s\times_t K_1$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{W} \otimes \big(\mathbf{X} \otimes (\mathbf{Y} \otimes \mathbf{Z})\big) \arrow[r, "\mathfrak{a}_{\mathbf{W},\mathbf{X},(\mathbf{Y}\otimes\mathbf{Z})}"] \arrow[d, "\mathbf{W}\otimes\mathfrak{a}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}"'] & (\mathbf{W} \otimes \mathbf{X}) \otimes (\mathbf{Y} \otimes \mathbf{Z}) \arrow[dd, "\mathfrak{a}_{(\mathbf{W}\otimes\mathbf{X}),\mathbf{Y},\mathbf{Z}}"] \\
\mathbf{W} \otimes \big((\mathbf{X} \otimes \mathbf{Y}) \otimes \mathbf{Z}\big) \arrow[d, "\mathfrak{a}_{\mathbf{W},(\mathbf{X}\otimes\mathbf{Y}),\mathbf{Z}}"'] & \\
\big(\mathbf{W} \otimes (\mathbf{X} \otimes \mathbf{Y})\big) \otimes \mathbf{Z} \arrow[r, "\mathfrak{a}_{\mathbf{W},\mathbf{X},\mathbf{Y}}\otimes\mathbf{Z}"'] & \big((\mathbf{W} \otimes \mathbf{X}) \otimes \mathbf{Y}\big) \otimes \mathbf{Z}.
\end{tikzcd}$$

Now, given a bicategory $\mathcal{K}$, we can form a pseudo double category $\mathbb{D}\mathcal{K}$ from it, by taking $(\mathbb{D}\mathcal{K})_0$ to be the set of objects of $\mathcal{K}$ (viewed as a discrete category), and $(\mathbb{D}\mathcal{K})_1$ to be the disjoint union of the hom-categories of $\mathcal{K}$; horizontal composition, identities, associativity and unitality are now derived from those of $\mathcal{K}$ in the evident way.

Conversely, any pseudo double category $\mathbb{K}$ contains a bicategory $\mathcal{B}\mathbb{K}$, with objects the objects of K_0, 1-cells the objects of K_1 and 2-cells the special maps in K_1. In light of this, the following notation will be useful: given $\mathbf{X}\colon A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ and $\mathbf{Y}\colon B \relbar\joinrel\mapstochar\joinrel\rightarrow C$ in K_1, we draw $\mathbf{Y} \otimes \mathbf{X}$ as

$$\mathbf{Y} \otimes \mathbf{X}\colon A \xrightarrow{\mathbf{X}} B \xrightarrow{\mathbf{Y}} C.$$

Since horizontal composition is not associative, we cannot extend this notation unambiguously to chains of three or more such composites; any such chain will need a choice of 'bracketing' in order to specify a composite horizontal arrow of $\mathbb{K}$.

We can extend this notation by using bicategorical pasting diagrams to specify composites of special maps in K_1. It follows from the pasting theorem for bicategories (see [Pow90, Ver92]) that such pasting diagrams uniquely specify a special map in K_1 once a bracketing for the start and end edge has been chosen.

2.2 Morphisms of pseudo double categories

Note that given a functor $F_0\colon K_0 \to L_0$, we induce a 2-functor

$$(F_0)_*\colon \mathbf{Cat}/(L_0 \times L_0) \to \mathbf{Cat}/(K_0 \times K_0)$$

by pulling back along $F_0 \times F_0$. Moreover, with respect to the monoidal structure outlined above, this 2-functor becomes a monoidal 2-functor, and so in particular, sends pseudomonoids to pseudomonoids. Thus we have:

Definition 3. A **morphism of pseudo double categories** (or **double morphism** for short) $F\colon \mathbb{K} \to \mathbb{L}$ consists of

- A functor $F_0\colon K_0 \to L_0$;
- A (lax) pseudomonoid morphism $F_1\colon K_1 \to (F_0)_*(L_1)$ in $\mathbf{Cat}/(K_0 \times K_0)$.

Again, let us spell this out more explicitly. A double morphism consists of data:

(DMD1) A functor $F_0 \colon K_0 \to L_0$;

(DMD2) A functor $F_1 \colon K_1 \to L_1$;

(DMD3) A special natural transformation

$$\begin{tikzcd}
K_0 \arrow[r, "F_0"] \arrow[d, "\mathbf{I}"'] \arrow[dr, phantom, "\Downarrow \mathfrak{e}_F"] & L_0 \arrow[d, "\mathbf{I}"] \\
K_1 \arrow[r, "F_1"'] & L_1;
\end{tikzcd}$$

(DMD4) A special natural transformation

$$\begin{tikzcd}
K_1 \, {}_s\!\times_t K_1 \arrow[r, "F_1 {}_s\!\times_t F_1"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Downarrow \mathfrak{m}_F"] & L_1 \, {}_s\!\times_t L_1 \arrow[d, "\otimes"] \\
K_1 \arrow[r, "F_1"'] & L_1.
\end{tikzcd}$$

In order to keep the notation under control, we shall usually write 'F' for both 'F_0' and 'F_1', and write '$\mathfrak{e}$' and '$\mathfrak{m}$' for '$\mathfrak{e}_F$' and '$\mathfrak{m}_F$'. Thus we write the components of our coherence natural transformations in L_1 as

$$\begin{aligned} \mathfrak{m}_{\mathbf{X},\mathbf{Y}} \colon F\mathbf{X} \otimes F\mathbf{Y} &\to F(\mathbf{X} \otimes \mathbf{Y}) \\ \text{and } \mathfrak{e}_X \colon \mathbf{I}_{FX} &\to F\mathbf{I}_X. \end{aligned}$$

This data is required to satisfy the following axioms:

(DMA1) The following squares commute:

$$\begin{tikzcd}
& K_1 \arrow[dl, "s"'] \arrow[d, "F_1"] \arrow[dr, "t"] & \\
K_0 \arrow[d, "F_0"'] & L_1 \arrow[dl, "s"] \arrow[dr, "t"'] & K_0 \arrow[d, "F_0"] \\
L_0 & & L_0;
\end{tikzcd}$$

(DMA2) For all $\mathbf{X} \in K_1$, the following diagrams commute:

$$\begin{tikzcd}
F\mathbf{X} \arrow[r, "\mathfrak{r}_{F\mathbf{X}}"] \arrow[d, "F\mathfrak{r}_{\mathbf{X}}"'] & F\mathbf{X} \otimes \mathbf{I}_{FX_s} \arrow[d, "F\mathbf{X}\otimes\mathfrak{e}_{X_s}"] \\
F(\mathbf{X} \otimes \mathbf{I}_{X_s}) & F\mathbf{X} \otimes F\mathbf{I}_{X_s} \arrow[l, "\mathfrak{m}_{\mathbf{X},\mathbf{I}_{X_s}}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
F\mathbf{X} \arrow[r, "\mathfrak{l}_{F\mathbf{X}}"] \arrow[d, "F\mathfrak{l}_{\mathbf{X}}"'] & \mathbf{I}_{FX_t} \otimes F\mathbf{X} \arrow[d, "\mathfrak{e}_{X_t}\otimes F\mathbf{X}"] \\
F(\mathbf{I}_{X_t} \otimes \mathbf{X}) & F\mathbf{I}_{X_t} \otimes F\mathbf{X}; \arrow[l, "\mathfrak{m}_{\mathbf{I}_{X_t},\mathbf{X}}"]
\end{tikzcd}$$

(DMA3) For all $(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \in K_1 {}_s\times_t K_1 {}_s\times_t K_1$, the following diagram commutes:

$$\begin{tikzcd}
F\mathbf{X} \otimes (F\mathbf{Y} \otimes F\mathbf{Z}) \arrow[r, "\mathfrak{a}_{F\mathbf{X},F\mathbf{Y},F\mathbf{Z}}"] \arrow[d, "F\mathbf{X}\otimes\mathfrak{m}_{\mathbf{Y},\mathbf{Z}}"'] & (F\mathbf{X} \otimes F\mathbf{Y}) \otimes F\mathbf{Z} \arrow[d, "\mathfrak{m}_{\mathbf{X},\mathbf{Y}}\otimes F\mathbf{Z}"] \\
F\mathbf{X} \otimes F(\mathbf{Y} \otimes \mathbf{Z}) \arrow[d, "\mathfrak{m}_{\mathbf{X},(\mathbf{Y}\otimes\mathbf{Z})}"'] & F(\mathbf{X} \otimes \mathbf{Y}) \otimes F\mathbf{Z} \arrow[d, "\mathfrak{m}_{(\mathbf{X}\otimes\mathbf{Y}),\mathbf{Z}}"] \\
F\big(\mathbf{X} \otimes (\mathbf{Y} \otimes \mathbf{Z})\big) \arrow[r, "F\mathfrak{a}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}"'] & F\big((\mathbf{X} \otimes \mathbf{Y}) \otimes \mathbf{Z}\big).
\end{tikzcd}$$

Now, pseudo double categories and the morphisms between them form themselves into a category **DblCat**, whose identity maps and composition are given as follows:

- **Identity** at $\mathbb{K}$ is given by $(\mathrm{id}_{\mathbb{K}})_0 = \mathrm{id}_{K_0}$, $(\mathrm{id}_{\mathbb{K}})_1 = \mathrm{id}_{K_1}$, $\mathfrak{m}_{\mathrm{id}_{\mathbb{K}}} = \mathrm{id}_{\otimes}$ and $\mathfrak{e}_{\mathrm{id}_{\mathbb{K}}} = \mathrm{id}_{\mathbf{I}}$;

- **Composition** of $F\colon \mathbb{K} \to \mathbb{L}$ and $G\colon \mathbb{L} \to \mathbb{M}$ is given by GF with $(GF)_0 = G_0F_0$, $(GF)_1 = G_1F_1$ and comparison transformations given by the pastings

$$\begin{tikzcd}
K_1 {}_s\times_t K_1 \arrow[r, "F_1{}_s\times_t F_1"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Downarrow\mathfrak{m}_F"] & L_1 {}_s\times_t L_1 \arrow[r, "G_1{}_s\times_t G_1"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Downarrow\mathfrak{m}_G"] & M_1 {}_s\times_t M_1 \arrow[d, "\otimes"] \\
K_1 \arrow[r, "F_1"'] & L_1 \arrow[r, "G_1"'] & M_1
\end{tikzcd}$$

and

$$\begin{tikzcd}
K_0 \arrow[r, "F_0"] \arrow[d, "\mathbf{I}"'] \arrow[dr, phantom, "\Downarrow \mathfrak{e}_F"] & L_0 \arrow[r, "G_0"] \arrow[d, "\mathbf{I}"'] \arrow[dr, phantom, "\Downarrow \mathfrak{e}_G"] & M_0 \arrow[d, "\mathbf{I}"] \\
K_1 \arrow[r, "F_1"'] & L_1 \arrow[r, "G_1"'] & M_1.
\end{tikzcd}$$

Likewise, we may define categories $\mathbf{DblCat}_o$ and $\mathbf{DblCat}_\psi$ whose maps are respectively double opmorphisms and double homomorphisms; a double opmorphism F has $\mathfrak{m}_F$ and $\mathfrak{e}_F$ oriented the other way, whilst a double homomorphism F has $\mathfrak{m}_F$ and $\mathfrak{e}_F$ invertible.

The data given above is very reminiscent of that for a (homo, op)morphism of bicategories, and in fact, given a (homo, op)morphism of pseudo double categories $F\colon \mathbb{K} \to \mathbb{L}$, we induce a (homo, op)morphism of bicategories $\mathcal{B}F\colon \mathcal{B}\mathbb{K} \to \mathcal{B}\mathbb{L}$. Indeed, since the components of $\mathfrak{m}_F$ and $\mathfrak{e}_F$ are special cells of $\mathbb{L}$, they lie in $\mathcal{B}\mathbb{L}$ and hence provide the required coherence data for a (homo, op)morphism of bicategories.

In fact, we see that $\mathcal{B}$ becomes a functor $\mathbf{DblCat} \to \mathbf{Bicat}$ where $\mathbf{Bicat}$ is the category of bicategories and morphisms between them. We have evident 'op' and 'homo' variants for this last statement.

2.3 Vertical transformations

There are two types of transformation we may consider between double morphisms. The simpler is the vertical transformation, which is akin to a monoidal natural transformation between monoidal functors.

Definition 4. Given morphisms $F, G\colon \mathbb{K} \to \mathbb{L}$ of pseudo double categories, a **vertical transformation** $\alpha\colon F \Rightarrow G$ consists of data:

(VTD1) A natural transformation $\alpha_0\colon F_0 \Rightarrow G_0$;

(VTD2) A natural transformation $\alpha_1\colon F_1 \Rightarrow G_1$,

(and again, we shall use 'α' indifferently for α_0 and α_1), subject to the following axioms:

(VTA1) The following pasting equalities hold:

$$\begin{tikzcd} K_1 \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow \alpha_1"] \arrow[d, "s"'] & L_1 \arrow[d, "s"] \\ K_0 \arrow[r, bend right, "G_0"'] & L_0 \end{tikzcd} = \begin{tikzcd} K_1 \arrow[r, bend left, "F_1"] \arrow[d, "s"'] & L_1 \arrow[d, "s"] \\ K_0 \arrow[r, bend left, "F_0"] \arrow[r, bend right, "G_0"'] \arrow[r, phantom, "\Downarrow \alpha_0"] & L_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} K_1 \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow \alpha_1"] \arrow[d, "t"'] & L_1 \arrow[d, "t"] \\ K_0 \arrow[r, bend right, "G_0"'] & L_0 \end{tikzcd} = \begin{tikzcd} K_1 \arrow[r, bend left, "F_1"] \arrow[d, "t"'] & L_1 \arrow[d, "t"] \\ K_0 \arrow[r, bend left, "F_0"] \arrow[r, bend right, "G_0"'] \arrow[r, phantom, "\Downarrow \alpha_0"] & L_0; \end{tikzcd}$$

(VTA2) For all $X \in K_0$, the following diagram commutes in L_1:

$$\begin{tikzcd} \mathbf{I}_{FX} \arrow[r, "\mathfrak{e}_X"] \arrow[d, "\mathbf{I}_{\alpha_X}"'] & F\mathbf{I}_X \arrow[d, "\alpha_{\mathbf{I}_X}"] \\ \mathbf{I}_{GX} \arrow[r, "\mathfrak{e}_X"'] & G\mathbf{I}_X; \end{tikzcd}$$

(VTA3) For all $(\mathbf{X}, \mathbf{Y}) \in K_1 {}_s\times_t K_1$, the following diagram commutes in L_1:

$$\begin{tikzcd} F\mathbf{X} \otimes F\mathbf{Y} \arrow[r, "\mathfrak{m}_{\mathbf{X},\mathbf{Y}}"] \arrow[d, "\alpha_{\mathbf{X}} \otimes \alpha_{\mathbf{Y}}"'] & F(\mathbf{X} \otimes \mathbf{Y}) \arrow[d, "\alpha_{\mathbf{X} \otimes \mathbf{Y}}"] \\ G\mathbf{X} \otimes G\mathbf{Y} \arrow[r, "\mathfrak{m}_{\mathbf{X},\mathbf{Y}}"'] & G(\mathbf{X} \otimes \mathbf{Y}). \end{tikzcd}$$

Now, given pseudo double categories $\mathbb{K}$ and $\mathbb{L}$, the double morphisms $\mathbb{K} \to \mathbb{L}$ and vertical transformations between them form a category $[\mathbb{K}, \mathbb{L}]_v$:

- The identity at F has $(\mathrm{id}_F)_0 = \mathrm{id}_{F_0}$ and $(\mathrm{id}_F)_1 = \mathrm{id}_{F_1}$;
- The composition $\beta\alpha$ has $(\beta\alpha)_0 = \beta_0\alpha_0$ and $(\beta\alpha)_1 = \beta_1\alpha_1$,

and it's straightforward to check that these satisfy (VTA1)–(VTA3) as required.

In fact, these categories $[\mathbb{K}, \mathbb{L}]_v$ provide us with hom-categories enriching the category **DblCat** above to a 2-category. Horizontal composition of 2-cells is given by the usual horizontal composition in **Cat** of the underlying natural transformations, and it's easy to check that such composites satisfy (VTA1)–(VTA3).

We can single out the full subcategory $[\mathbb{K}, \mathbb{L}]_{v\psi}$ of $[\mathbb{K}, \mathbb{L}]$ given by the double *homo*morphisms and vertical transformations. Further, as the horizontal composite of two homomorphisms is another homomorphism, we can form a locally full sub-2-category $\mathbf{DblCat}_\psi$ of $\mathbf{DblCat}$, consisting of the pseudo double categories, double homomorphisms and vertical transformations.

2.4 Horizontal transformations

The second, and slightly more involved type of transformation is the horizontal transformation. This acts more like a pseudo-natural transformation between morphisms of bicategories:

Definition 5. Given double morphisms $A_s, A_t \colon \mathbb{K} \to \mathbb{L}$, a **horizontal transformation** $\mathbf{A}\colon A_s \nRightarrow A_t$ consists of the following data:

(HTD1) A 'components functor' $A_c\colon K_0 \to L_1$. To simplify notation, we shall write $\mathbf{A}X$ for A_cX and $\mathbf{A}f$ for A_cf;

(HTD2) A 'pseudonaturality' special invertible transformation

$$\begin{tikzcd}
K_1 \arrow[r, "{[(A_t)_1, A_c s]}"] \arrow[d, "{[A_c t, (A_s)_1]}"'] \arrow[dr, phantom, "\Downarrow A"] & L_1 {}_s\!\times_t L_1 \arrow[d, "\otimes"] \\
L_1 {}_s\!\times_t L_1 \arrow[r, "\otimes"'] & L_1,
\end{tikzcd}$$

(with components

$$A_{\mathbf{X}} \colon A_t\mathbf{X} \otimes \mathbf{A}X_s \to \mathbf{A}X_t \otimes A_s\mathbf{X}$$

in L_1, or, in pasting notation

$$\begin{tikzcd}
A_sX_s \arrow[r, "A_s\mathbf{X}", "\shortmid" marking] \arrow[d, "\mathbf{A}X_s"', "\shortmid" marking] \arrow[dr, phantom, "\Uparrow A_{\mathbf{X}}"] & A_sX_t \arrow[d, "\mathbf{A}X_t", "\shortmid" marking] \\
A_tX_s \arrow[r, "A_t\mathbf{X}"', "\shortmid" marking] & A_tX_t.
\end{tikzcd} \quad)$$

subject to the following axioms:

(HTA1) The following triangles commute:

$$\begin{tikzcd}
& K_0 \arrow[dl, "(A_s)_0"'] \arrow[d, "A_c"] \arrow[dr, "(A_t)_0"] & \\
L_0 & L_1 \arrow[l, "s"] \arrow[r, "t"'] & L_0;
\end{tikzcd}$$

(HTA2) Given $X \in K_0$, the following pastings agree:

$$\begin{tikzcd}
A_sX \arrow[r, "\mathbf{I}_{A_sX}"', "|" marking] \arrow[r, bend left=50, "A_s\mathbf{I}_X", "|" marking] \arrow[r, phantom, bend left=25, "\Uparrow\mathfrak{e}_{A_sX}"] \arrow[d, "\mathbf{A}X"', "|" marking] \arrow[dr, "\mathbf{A}X"', "|" marking] \arrow[dr, phantom, shift left=4ex, "\Uparrow\mathfrak{r}_{\mathbf{A}X}"] \arrow[dr, phantom, shift right=4ex, "\Uparrow\mathfrak{l}^{-1}_{\mathbf{A}X}"] & A_sX \arrow[d, "\mathbf{A}X", "|" marking] \\
A_tX \arrow[r, "\mathbf{I}_{A_tX}"', "|" marking] & A_tX
\end{tikzcd}
=
\begin{tikzcd}
A_sX \arrow[r, bend left=50, "A_s\mathbf{I}_X", "|" marking] \arrow[d, "\mathbf{A}X"', "|" marking] \arrow[dr, phantom, "\Uparrow A_{\mathbf{I}_X}"] & A_sX \arrow[d, "\mathbf{A}X", "|" marking] \\
A_tX \arrow[r, "\mathbf{I}_{A_tX}"', "|" marking] \arrow[r, bend left=50, "A_t\mathbf{I}_X", "|" marking] \arrow[r, phantom, bend left=25, "\Uparrow\mathfrak{e}_{A_tX}"] & A_tX;
\end{tikzcd}$$

(HTA3) Given $(\mathbf{Y},\mathbf{X}) \in K_1 \; {}_s\!\times_t \; K_1$, the following pastings agree:

$$\begin{tikzcd}
A_sX \arrow[r, "A_s\mathbf{X}", "|" marking] \arrow[rr, bend left=45, "A_s(\mathbf{Y}\otimes\mathbf{X})", "|" marking] \arrow[rr, phantom, bend left=20, "\Uparrow\mathfrak{m}_{\mathbf{Y},\mathbf{X}}"] \arrow[d, "\mathbf{A}X"', "|" marking] \arrow[dr, phantom, "\Uparrow A_{\mathbf{X}}"] & A_sY \arrow[r, "A_s\mathbf{Y}", "|" marking] \arrow[d, "\mathbf{A}Y", "|" marking] \arrow[dr, phantom, "\Uparrow A_{\mathbf{Y}}"] & A_sZ \arrow[d, "\mathbf{A}Z", "|" marking] \\
A_tX \arrow[r, "A_t\mathbf{X}"', "|" marking] & A_tY \arrow[r, "A_t\mathbf{Y}"', "|" marking] & A_tZ
\end{tikzcd}
=
\begin{tikzcd}
A_sX \arrow[rr, bend left=45, "A_s(\mathbf{Y}\otimes\mathbf{X})", "|" marking] \arrow[d, "\mathbf{A}X"', "|" marking] \arrow[drr, phantom, "\Uparrow A_{\mathbf{Y}\otimes\mathbf{X}}"] & & A_sZ \arrow[d, "\mathbf{A}Z", "|" marking] \\
A_tX \arrow[r, "A_t\mathbf{X}"', "|" marking] \arrow[rr, bend left=45, "A_t(\mathbf{Y}\otimes\mathbf{X})", "|" marking] \arrow[rr, phantom, bend left=20, "\Uparrow\mathfrak{m}_{\mathbf{Y},\mathbf{X}}"] & A_tY \arrow[r, "A_t\mathbf{Y}"', "|" marking] & A_tZ,
\end{tikzcd}$$

Although we have asked for the 'pseudonaturality' special transformation A to be invertible, we could just as easily have dropped this requirement, thereby arriving at a notion of *lax horizontal transformation*. However, we shall only need the 'pseudo' version in this thesis, and hence shall use 'horizontal transformation' to refer to this notion without further comment.

Again, the above data is rather reminiscent of that for a pseudo-natural transformation between morphisms of bicategories; and indeed, given a horizontal transformation $\mathbf{A}\colon A_s \Rrightarrow A_t$, we induce a pseudo-natural transformation $\mathcal{B}\mathbf{A}$ between the morphisms of bicategories $\mathcal{B}A_s$ and $\mathcal{B}A_t$.

2.5 Modifications

There are evident candidates for 'identity' and 'composition' of horizontal transformations, but this structure will be neither unital nor associative on the nose. To specify what it is unital and associative 'up to', we shall need the notion of a modification.

Definition 6. Given horizontal transformations $\mathbf{A}\colon A_s \Rrightarrow A_t$ and $\mathbf{B}\colon B_s \Rrightarrow B_t$, a **modification** $\boldsymbol{\gamma}\colon \mathbf{A} \Rrightarrow \mathbf{B}$ consists of the following data:

(MD1) A pair of vertical transformations $\gamma_s\colon A_s \Rightarrow B_s$ (the 'vertical source') and $\gamma_t\colon A_t \Rightarrow B_t$ (the 'vertical target');

(MD2) A natural transformation $\gamma_c\colon A_c \Rightarrow B_c$ (the 'central natural transformation'). To simplify notation, we shall refer to the components of γ_c as 'the components of $\boldsymbol{\gamma}$', and write a typical such component as $\boldsymbol{\gamma}_X$,

subject to the following axioms:

(MA1) The following pastings agree:

$$\begin{tikzcd} K_0 \arrow[r, bend left, "A_c", ""{name=U, below}] \arrow[r, bend right, "B_c"', ""{name=D}] \arrow[Rightarrow, from=U, to=D, "\gamma_c"] & L_1 \arrow[r, "s"] & L_0 \end{tikzcd} = \begin{tikzcd} K_0 \arrow[rr, bend left, "(A_s)_0", ""{name=U, below}] \arrow[rr, bend right, "(B_s)_0"', ""{name=D}] \arrow[Rightarrow, from=U, to=D, "(\gamma_s)_0"] & & L_0 \end{tikzcd}$$

and

$$\begin{tikzcd} K_0 \arrow[r, bend left, "A_c", ""{name=U, below}] \arrow[r, bend right, "B_c"', ""{name=D}] \arrow[Rightarrow, from=U, to=D, "\gamma_c"] & L_1 \arrow[r, "t"] & L_0 \end{tikzcd} = \begin{tikzcd} K_0 \arrow[rr, bend left, "(A_t)_0", ""{name=U, below}] \arrow[rr, bend right, "(B_t)_0"', ""{name=D}] \arrow[Rightarrow, from=U, to=D, "(\gamma_t)_0"] & & L_0; \end{tikzcd}$$

(MA2) For all $\mathbf{X} \in K_1$, the following diagram commutes:

$$\begin{tikzcd}
A_t\mathbf{X} \otimes \mathbf{A}X_s \arrow[r, "A_{\mathbf{X}}"] \arrow[d, "(\gamma_t)_{\mathbf{X}} \otimes \boldsymbol{\gamma}_{X_s}"'] & \mathbf{A}X_t \otimes A_s\mathbf{X} \arrow[d, "\boldsymbol{\gamma}_{X_t} \otimes (\gamma_s)_{\mathbf{X}}"] \\
B_t\mathbf{X} \otimes \mathbf{B}X_s \arrow[r, "B_{\mathbf{X}}"'] & \mathbf{B}X_t \otimes B_s\mathbf{X}.
\end{tikzcd}$$

We shall notate such a modification as:

$$\begin{tikzcd}
A_s \arrow[r, Rightarrow, "\mathbf{A}", "\shortmid" marking] \arrow[d, Rightarrow, "\gamma_s"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\gamma}"] & A_t \arrow[d, Rightarrow, "\gamma_t"] \\
B_s \arrow[r, Rightarrow, "\mathbf{B}"', "\shortmid" marking] & B_t.
\end{tikzcd}$$

We observe that given a *special* modification, $\boldsymbol{\alpha}\colon \mathbf{A} \Rrightarrow \mathbf{B}$, i.e., a modification $\boldsymbol{\alpha}$ for which α_s and α_t are identity vertical transformations, we induce a modification between pseudo-natural transformations $\mathcal{B}\boldsymbol{\alpha}\colon \mathcal{B}\mathbf{A} \Rrightarrow \mathcal{B}\mathbf{B}$.

2.6 Functor pseudo double categories

Given two weak double categories $\mathbb{K}$ and $\mathbb{L}$, the horizontal transformations and modifications between them form a category $[\mathbb{K},\mathbb{L}]_h$:

- **Identity** at $\mathbf{A}\colon A_s \Rightarrow\!\!\!\!\!\!\shortmid\;\; A_t$ is given by $(\mathrm{id}_{\mathbf{A}})_s = \mathrm{id}_{A_s}$, $(\mathrm{id}_{\mathbf{A}})_t = \mathrm{id}_{A_t}$ and $(\mathrm{id}_{\mathbf{A}})_c = \mathrm{id}_{A_c}$;
- **Composition $\boldsymbol{\delta\gamma}$** is given by $(\delta\gamma)_s = \delta_s\gamma_s$, $(\delta\gamma)_t = \delta_t\gamma_t$, and $(\delta\gamma)_c = \delta_c\gamma_c$.

It's easy to check that this data satisfies axioms (MA1) and (MA2) as required. Further, there are two evident projections

$$\begin{tikzcd}
{[\mathbb{K},\mathbb{L}]_h} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & {[\mathbb{K},\mathbb{L}]_v},
\end{tikzcd}$$

which provide data for a double category $[\mathbb{K},\mathbb{L}]$, as follows:

- The horizontal composite

$$(\mathbf{C}\colon C_s \Rightarrow\!\!\!\!\!\!\shortmid\;\; C_t) \otimes (\mathbf{A}\colon A_s \Rightarrow\!\!\!\!\!\!\shortmid\;\; C_s)$$

has components functor $C_c(-) \otimes A_c(-)$, with pseudonaturality maps

$$(C \otimes A)_{\mathbf{X}} \colon C_t\mathbf{X} \otimes (\mathbf{C}X_s \otimes \mathbf{A}X_s) \to (\mathbf{C}X_t \otimes \mathbf{A}X_t) \otimes A_s\mathbf{X}$$

given by the pasting

$$\begin{tikzcd}
A_sX_s \arrow[r, "A_s\mathbf{X}", "\shortmid" marking] \arrow[d, "\mathbf{A}X_s"', "\shortmid" marking] & A_sX_t \arrow[d, "\mathbf{A}X_t", "\shortmid" marking] \\
C_sX_s \arrow[r, "C_s\mathbf{X}", "\shortmid" marking] \arrow[d, "\mathbf{C}X_s"', "\shortmid" marking] \arrow[ur, phantom, "\Uparrow A_{\mathbf{X}}"] & C_sX_t \arrow[d, "\mathbf{C}X_t", "\shortmid" marking] \\
C_tX_s \arrow[r, "C_t\mathbf{X}"', "\shortmid" marking] \arrow[ur, phantom, "\Uparrow C_{\mathbf{X}}"] & C_tX_t.
\end{tikzcd}$$

Given modifications

$$\begin{tikzcd}
A_s \arrow[r, Rightarrow, "\mathbf{A}", "\shortmid" marking] \arrow[d, Rightarrow, "\gamma_s"'] & C_s \arrow[d, Rightarrow, "\delta_s"] \\
B_s \arrow[r, Rightarrow, "\mathbf{B}"', "\shortmid" marking] \arrow[ur, phantom, "\Downarrow\boldsymbol{\gamma}"] & D_s
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
C_s \arrow[r, Rightarrow, "\mathbf{C}", "\shortmid" marking] \arrow[d, Rightarrow, "\delta_s"'] & C_t \arrow[d, Rightarrow, "\delta_t"] \\
D_s \arrow[r, Rightarrow, "\mathbf{D}"', "\shortmid" marking] \arrow[ur, phantom, "\Downarrow\boldsymbol{\delta}"] & D_t,
\end{tikzcd}$$

the composite modification $\boldsymbol{\delta} \otimes \boldsymbol{\gamma}$ has $(\delta \otimes \gamma)_s = \gamma_s$, $(\delta \otimes \gamma)_t = \delta_t$ and component at X given by

$$\boldsymbol{\delta}_X \otimes \boldsymbol{\gamma}_X \colon \mathbf{C}X \otimes \mathbf{A}X \to \mathbf{D}X \otimes \mathbf{B}X.$$

- The horizontal unit $\mathbf{I}_F \colon F \nRightarrow F$ at F has components functor $\mathbf{I}_{F(-)}$, and pseudonaturality maps $(I_F)_{\mathbf{X}}$ given by

$$(I_F)_{\mathbf{X}} = F\mathbf{X} \otimes \mathbf{I}_{FX_s} \xrightarrow{\mathfrak{r}_{F\mathbf{X}}^{-1}} F\mathbf{X} \xrightarrow{\mathfrak{l}_{F\mathbf{X}}} \mathbf{I}_{FX_t} \otimes F\mathbf{X}.$$

Given a vertical transformation $\alpha \colon F \Rightarrow G$, the modification $\mathbf{I}_\alpha$ has $(\mathbf{I}_\alpha)_s = \alpha = (\mathbf{I}_\alpha)_t$, and component at X given by

$$\mathbf{I}_{\alpha_X} \colon \mathbf{I}_{FX} \to \mathbf{I}_{GX}.$$

It remains to give the unit and associativity constraints for $[\mathbb{K}, \mathbb{L}]$. The special unit modifications $\mathfrak{l}_{\mathbf{A}}$ and $\mathfrak{r}_{\mathbf{A}}$ have components

$$\begin{aligned}(\mathfrak{l}_{\mathbf{A}})_X &= \mathfrak{l}_{\mathbf{A}X} \colon \mathbf{A}X \to \mathbf{I}_{A_t X} \otimes \mathbf{A}X \\ \text{and } (\mathfrak{r}_{\mathbf{A}})_X &= \mathfrak{r}_{\mathbf{A}X} \colon \mathbf{A}X \to \mathbf{A}X \otimes \mathbf{I}_{A_s X},\end{aligned}$$

and similarly, the special associativity modification $\mathfrak{a}_{\mathbf{A},\mathbf{B},\mathbf{C}}$ has components

$$(\mathfrak{a}_{\mathbf{A},\mathbf{B},\mathbf{C}})_X = \mathfrak{a}_{\mathbf{A}X,\mathbf{B}X,\mathbf{C}X} \colon \mathbf{A}X \otimes (\mathbf{B}X \otimes \mathbf{C}X) \to (\mathbf{A}X \otimes \mathbf{B}X) \otimes \mathbf{C}X.$$

This completes the definition of the pseudo double category $[\mathbb{K}, \mathbb{L}]$. We note that there is a sub-pseudo double category $[\mathbb{K}, \mathbb{L}]_\psi$, given by restricting to homomorphisms as objects, and taking all vertical transformations, horizontal transformations and modifications between them.

2.7 Whiskering of homomorphisms

In the theory of bicategories, given a strong transformation $\beta \colon H_1 \Rightarrow H_2 \colon \mathcal{M} \to \mathcal{N}$ and a morphism of bicategories $G \colon \mathcal{L} \to \mathcal{M}$, we can form a strong transformation $\beta G \colon H_1 G \Rightarrow H_2 G \colon \mathcal{L} \to \mathcal{N}$.

Similarly, given a strong transformation $\alpha \colon F_1 \Rightarrow F_2 \colon \mathcal{K} \to \mathcal{L}$ and a homomorphism of bicategories $G \colon \mathcal{L} \to \mathcal{M}$, we can form a strong transformation $G\alpha \colon GF_1 \Rightarrow GF_2 \colon \mathcal{K} \to \mathcal{M}$. However, we cannot in general weaken this to allow G to be a mere morphism.

This much is well known (see [Bén67, Gra74]); the goal of the following two sections is to given an extension of this 'whiskering' operation to pseudo double categories, and to show that it is very well-behaved with respect to the 2-categorical structure of **DblCat**.

Now, given a double morphism $G \colon \mathbb{L} \to \mathbb{M}$, we know by virtue of the 2-category structure of **DblCat** that we can whisker by G on either side; that is, given vertical transformations

$$\alpha \colon F_1 \Rightarrow F_2 \colon \mathbb{K} \to \mathbb{L} \quad \text{and} \quad \beta \colon H_1 \Rightarrow H_2 \colon \mathbb{M} \to \mathbb{N}$$

we can form vertical transformations

$$G\alpha\colon GF_1 \Rightarrow GF_2\colon \mathbb{K} \to \mathbb{M} \quad \text{and} \quad \beta G\colon H_1G \Rightarrow H_2G\colon \mathbb{L} \to \mathbb{N}.$$

What we shall do in this section is produce a similar whiskering operation on horizontal transformations, and show that it is compatible with the vertical whiskering:

Proposition 7. *Let $G\colon \mathbb{L} \to \mathbb{M}$ be a double morphism. Then 'precomposition with G' extends to a strict double homomorphism*

$$(\text{–})G\colon [\mathbb{M}, \mathbb{N}] \to [\mathbb{L}, \mathbb{N}].$$

Proof. We give $(\text{–})G$ as follows:

- $\big((\text{–})G\big)_0\colon [\mathbb{M}, \mathbb{N}]_v \to [\mathbb{L}, \mathbb{N}]_v$ is given by the whiskering operation in the 2-category **DblCat**. Thus we take the double morphism $H\colon \mathbb{M} \to \mathbb{N}$ to the double morphism $HG\colon \mathbb{L} \to \mathbb{N}$ and the vertical transformation $\alpha\colon H \Rightarrow H'$ to the vertical transformation $\alpha G\colon HG \Rightarrow H'G$.

- $\big((\text{–})G\big)_1\colon [\mathbb{M}, \mathbb{N}]_h \to [\mathbb{L}, \mathbb{N}]_h$ is given as follows. Given a horizontal transformation $\mathbf{A}\colon A_s \Rrightarrow A_t$, the horizontal transformation $\mathbf{A}G\colon A_sG \Rrightarrow A_tG$ has components functor A_cG_0 (and therefore component at X

  $$\mathbf{A}GX\colon A_sGX \to A_tGX \;)$$

  and pseudonaturality transformation $A_{G_1(\text{–})}$. Explicitly, the components of this pseudonaturality transformation are given by

  $$A_tG\mathbf{X} \otimes \mathbf{A}GX_s \xrightarrow{A_{G\mathbf{X}}} \mathbf{A}GX_t \otimes A_sG\mathbf{X}$$

  in N_1. The axiom (HTA1) follows from (HTA1) for $\mathbf{A}$ and (DMA1) for G, whilst (HTA2) and (HTA3) follow from (HTA2) and (HTA3) for $\mathbf{A}$ evaluated at $G(-)$. Next, given a modification

$$\begin{tikzcd} A_s \arrow[r, Rightarrow, "\mathbf{A}", "|" marking] \arrow[d, Rightarrow, "\gamma_s"'] \arrow[rd, phantom, "\Downarrow\boldsymbol{\gamma}"] & A_t \arrow[d, Rightarrow, "\gamma_t"] \\ B_s \arrow[r, Rightarrow, "\mathbf{B}"', "|" marking] & B_t, \end{tikzcd}$$

the modification $\boldsymbol{\gamma}G$ has $(\boldsymbol{\gamma}G)_s = \gamma_s G$, $(\boldsymbol{\gamma}G)_t = \gamma_t G$, and $(\boldsymbol{\gamma}G)_c = \gamma_c G_0$, and therefore component at X given by:

$$(\boldsymbol{\gamma}G)_X = \boldsymbol{\gamma}_{GX} \colon \mathbf{A}GX \to \mathbf{B}GX.$$

Now (MA1) follows from (MA1) for $\boldsymbol{\gamma}$ and (DMA1) for G, whilst (MA2) follows from (MA2) for $\boldsymbol{\gamma}$ evaluated at $G(-)$.

Visibly, $\big((-)G\big)_1$ and $\big((-)G\big)_0$ satisfy (DMA1), and we observe that $(\mathbf{A} \otimes \mathbf{B})G = \mathbf{A}G \otimes \mathbf{B}G$ and $\mathbf{I}_H G = \mathbf{I}_{HG}$, and thus (DMD3) and (DMD4) are trivial, and (DMA2) and (DMA3) are trivially satisfied. $\square$

We now move on to whiskerings on the left. As for bicategories, we cannot in general whisker *morphisms* with horizontal transformations on the left; we must instead restrict to homomorphisms.

Proposition 8. *Let $G\colon \mathbb{L} \to \mathbb{M}$ be a double homomorphism. Then 'postcomposition with G' induces a double homomorphism*

$$G(-)\colon [\mathbb{K}, \mathbb{L}] \to [\mathbb{K}, \mathbb{M}].$$

Proof. We give $G(-)$ as follows:

- $\big(G(-)\big)_0 \colon [\mathbb{K}, \mathbb{L}]_v \to [\mathbb{K}, \mathbb{M}]_v$ is given by the whiskering operation in the 2-category **DblCat**. Thus we take the double morphism $F\colon \mathbb{K} \to \mathbb{L}$ to the double morphism $GF\colon \mathbb{K} \to \mathbb{M}$ and the vertical transformation $\alpha\colon F \Rightarrow F'$ to the vertical transformation $G\alpha\colon GF \Rightarrow GF'$.

- $\big(G(-)\big)_1 \colon [\mathbb{K}, \mathbb{L}]_h \to [\mathbb{K}, \mathbb{M}]_h$ is given as follows. Given a horizontal transformation $\mathbf{A}\colon A_s \nRightarrow A_t$, the horizontal transformation $G\mathbf{A}\colon GA_s \nRightarrow GA_t$ has components functor G_1A_c (and therefore component at X given by

$$G\mathbf{A}X \colon GA_sX \twoheadrightarrow GA_tX \ )$$

and pseudonaturality transformation

$$\begin{tikzcd}
K_1 \arrow[r, "{[(A_t)_1, A_cs]}"] \arrow[d, swap, "{[A_ct, (A_s)_1]}"] \arrow[dr, phantom, "\Downarrow A"] & L_1 {}_s\times_t L_1 \arrow[d, "\otimes"] \arrow[dr, "G_{1s}\times_t G_1"] & \\
L_1 {}_s\times_t L_1 \arrow[r, swap, "\otimes"] \arrow[dr, swap, "G_{1s}\times_t G_1"] & L_1 \arrow[r, phantom, "\Downarrow \mathfrak{m}"] \arrow[d, phantom, "\Downarrow \mathfrak{m}^{-1}"] \arrow[dr, "G_1"] & M_1 {}_s\times_t M_1 \arrow[d, "\otimes"] \\
& M_1 {}_s\times_t M_1 \arrow[r, swap, "\otimes"] & M_1
\end{tikzcd}$$

where we observe that the composite along the top edge is $[G_1(A_t)_1, G_1A_cs]$, and that along the left edge $[G_1A_ct, G_1(A_s)_1]$ as required. Explicitly, the components of this pseudonaturality transformation are given by

$$GA_t\mathbf{X} \otimes G\mathbf{A}X_s \xrightarrow{\mathfrak{m}_{A_t\mathbf{X}, \mathbf{A}X_s}} G(A_t\mathbf{X} \otimes \mathbf{A}X_s) \xrightarrow{GA_{\mathbf{X}}} G(\mathbf{A}X_t \otimes A_s\mathbf{X}) \xrightarrow{\mathfrak{m}^{-1}_{\mathbf{A}X_t, A_s\mathbf{X}}} G\mathbf{A}X_t \otimes GA_s\mathbf{X}$$

in M_1. The axioms (HTA1)–(HTA3) follow from (HTA1)–(HTA3) for $\mathbf{A}$ and (DMA1)–(DMA3) for G. Finally, given a modification

$$\begin{tikzcd}
A_s \arrow[r, Rightarrow, "\mathbf{A}", "|" marking] \arrow[d, Rightarrow, swap, "\gamma_s"] \arrow[dr, phantom, "\Downarrow \boldsymbol{\gamma}"] & A_t \arrow[d, Rightarrow, "\gamma_t"] \\
B_s \arrow[r, Rightarrow, swap, "\mathbf{B}", "|" marking] & B_t,
\end{tikzcd}$$

the modification $G\boldsymbol{\gamma}$ has $(G\boldsymbol{\gamma})_s = G\gamma_s$, $(G\boldsymbol{\gamma})_t = G\gamma_t$ and $(G\boldsymbol{\gamma})_c = G_1\gamma_c$, and therefore components

$$(G\boldsymbol{\gamma})_X = G\boldsymbol{\gamma}_X \colon G\mathbf{A}X \to G\mathbf{B}X.$$

The axiom (MA1) follows from (MA1) for $\boldsymbol{\gamma}$ and (DMA1) for G, whilst (MA2) follows from (MA2) for $\boldsymbol{\gamma}$, naturality of $\mathfrak{m}_G$ and functoriality of G_1.

Again, it's clear that these definitions satisfy (DMA1); it remains to give the data

(DMD3) and (DMD4). So, the special invertible modification $\mathfrak{e}_A \colon \mathbf{I}_{GA} \Rrightarrow G\mathbf{I}_A$ has components

$$(\mathfrak{e}_A)_X = \mathfrak{e}_{AX} \colon \mathbf{I}_{GAX} \to G\mathbf{I}_{AX}.$$

whilst the special invertible modification $\mathfrak{m}_{\mathbf{A},\mathbf{B}} \colon G\mathbf{A} \otimes G\mathbf{B} \Rrightarrow G(\mathbf{A} \otimes \mathbf{B})$ has components

$$(\mathfrak{m}_{\mathbf{A},\mathbf{B}})_X = \mathfrak{m}_{\mathbf{A}X,\mathbf{B}X} \colon G\mathbf{A}X \otimes G\mathbf{B}X \to G(\mathbf{A}X \otimes \mathbf{B}X).$$

That these are modifications, satisfying (MA1) and (MA2), follows easily from (MA1) and (MA2) for $\mathbf{A}$ and $\mathbf{B}$ and (DMA1)–(DMA3) for G. It remains to check that this data $\mathfrak{m}$ and $\mathfrak{e}$ satisfies (DMA2) and (DMA3), but this follows immediately from (DMA2) and (DMA3) for G. □

We note before continuing that $G(–)$ and $(–)G$ restrict to respective homomorphisms

$$(–)G \colon [\mathbb{M}, \mathbb{N}]_\psi \to [\mathbb{L}, \mathbb{N}]_\psi \quad \text{and} \quad G(–) \colon [\mathbb{K}, \mathbb{L}]_\psi \to [\mathbb{K}, \mathbb{M}]_\psi.$$

2.8 Whiskering of vertical transformations

The above section gives us an 'action' of homomorphisms on functor pseudo double categories (we shall see below the precise sense in which this *is* an action). We can extend this action from homomorphisms to the maps between them. As before, we begin with whiskerings on the right:

Proposition 9. *Let G and $G' \colon \mathbb{L} \to \mathbb{M}$ be double morphisms, and let $\alpha \colon G \Rightarrow G'$ be a vertical transformation. Then precomposition with α induces a vertical transformation*

$$(–)\alpha \colon (–)G \Rightarrow (–)G' \colon [\mathbb{M}, \mathbb{N}] \to [\mathbb{L}, \mathbb{N}].$$

Proof. We give $(–)\alpha$ as follows:

- $\big((–)\alpha\big)_0$ has component at $H \in [\mathbb{M}, \mathbb{N}]_v$ given by the map $H\alpha \colon HG \to HG'$ in $[\mathbb{L}, \mathbb{N}]_v$. The naturality of these components in H is the equality

  $$HG \xRightarrow{H\alpha} HG' \xRightarrow{\beta G'} H'G' \;=\; HG \xRightarrow{\beta G} H'G \xRightarrow{H'\alpha} H'G'$$

  in **DblCat**;

- $\big((\text{–})\alpha\big)_1$ is given as follows. Its component at $\mathbf{A} \in [\mathbb{M}, \mathbb{N}]_h$ is the modification $\mathbf{A}\alpha \colon \mathbf{A}G \Rrightarrow \mathbf{A}G'$ with
$$(\mathbf{A}\alpha)_s = A_s\alpha, \quad (\mathbf{A}\alpha)_t = A_t\alpha \quad \text{and} \quad (\mathbf{A}\alpha)_c = A_c\alpha_0.$$
(MA1) follows from (VTA1) for α and (HTA1) for $\mathbf{A}$, whilst (MA2) follows from the naturality of $A_{(\text{–})}$. The naturality of the components of $\big((\text{–})\alpha\big)_1$ in $\mathbf{A}$ follows from the equality
$$A_cG_0 \xRightarrow{A_c\alpha_0} A_cG'_0 \xRightarrow{\beta_cG'_0} A'_cG'_0 \;=\; A_cG_0 \xRightarrow{\beta_cG_0} A'_cG_0 \xRightarrow{A'_c\alpha_0} A'_cG'_0$$
in **Cat**.

It's again visibly the case that this data satisfies (VTA1); thus it only remains to check (VTA2). We need diagrams of the following form to commute in $[\mathbb{M}, \mathbb{N}]_h$:

$$\begin{tikzcd}
\mathbf{A}G \otimes \mathbf{B}G \arrow[r, equal] \arrow[d, "\mathbf{A}\alpha\otimes\mathbf{B}\alpha"'] & (\mathbf{A} \otimes \mathbf{B})G \arrow[d, "(\mathbf{A}\otimes\mathbf{B})\alpha"] \\
\mathbf{A}G' \otimes \mathbf{B}G' \arrow[r, equal] & (\mathbf{A} \otimes \mathbf{B})G'.
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbf{I}_{HG} \arrow[r, equal] \arrow[d, "\mathbf{I}_{H\alpha}"'] & \mathbf{I}_H G \arrow[d, "\mathbf{I}_H\alpha"] \\
\mathbf{I}_{HG'} \arrow[r, equal] & \mathbf{I}_H G'.
\end{tikzcd}$$

But this is immediate since both $(\mathbf{A} \otimes \mathbf{B})\alpha$ and $\mathbf{A}\alpha \otimes \mathbf{B}\alpha$ have component at X given by
$$\mathbf{A}\alpha_X \otimes \mathbf{B}\alpha_X \colon \mathbf{A}GX \otimes \mathbf{B}GX \to \mathbf{A}G'X \otimes \mathbf{B}G'X;$$
and similarly, both $\mathbf{I}_{H\alpha}$ and $\mathbf{I}_H\alpha$ have component at X given by
$$\mathbf{I}_{H\alpha_X} \colon \mathbf{I}_{HGX} \to \mathbf{I}_{HG'X}.$$
□

Proposition 10. *Let G and $G' \colon \mathbb{L} \to \mathbb{M}$ be double homomorphisms, and let $\alpha \colon G \Rightarrow G'$ be a vertical transformation. Then postcomposition with α induces a vertical transformation*
$$\alpha(\text{–}) \colon G(\text{–}) \Rightarrow G'(\text{–}) \colon [\mathbb{K}, \mathbb{L}] \to [\mathbb{K}, \mathbb{M}].$$

Proof. We give the vertical transformation $\alpha(\text{–})$ as follows:

- $\big(\alpha(\text{–})\big)_0$ has component at $F \in [\mathbb{K}, \mathbb{L}]_v$ given by the map $\alpha F \colon GF \to G'F$ in

$[\mathbb{K},\mathbb{M}]_v$. The naturality of these components in F is the equality

$$GF \xRightarrow{\alpha F} G'F \xRightarrow{G'\beta} G'F' = GF \xRightarrow{G\beta} GF' \xRightarrow{\alpha F'} G'F'$$

in **DblCat**.

- $\big(\alpha(-)\big)_1$ is given as follows. Its component at $\mathbf{A} \in [\mathbb{K},\mathbb{L}]_h$ is the modification $\alpha\mathbf{A}\colon G\mathbf{A} \Rrightarrow G'\mathbf{A}$ with

$$(\alpha\mathbf{A})_s = \alpha A_s, \quad (\alpha\mathbf{A})_t = \alpha A_t \quad \text{and} \quad (\alpha\mathbf{A})_c = \alpha_1 A_c.$$

(MA1) follows from (HTA1) for $\mathbf{A}$ and (VTA1) for α, whilst (MA2) follows from the naturality of $A_{(-)}$ and $\mathfrak{m}_G$ and the functoriality of G. The naturality of the components of $\big(\alpha(-)\big)_1$ in $\mathbf{A}$ follows from the equality

$$G_1A_c \xRightarrow{\alpha_1 A_c} G'_1A_c \xRightarrow{G'_1\beta_c} G'_1A'_c = G_1A_c \xRightarrow{G_1\beta_c} G_1A'_c \xRightarrow{\alpha_1 A'_c} G'_1A'_c$$

in **Cat**.

As above this data straightforwardly satisfies (VTA1), whilst for (VTA2), the following diagrams must commute:

$$\begin{tikzcd} G\mathbf{A} \otimes G\mathbf{B} \arrow[r, "\mathfrak{m}_{\mathbf{A},\mathbf{B}}"] \arrow[d, "\alpha\mathbf{A}\otimes\alpha\mathbf{B}"'] & G(\mathbf{A}\otimes\mathbf{B}) \arrow[d, "\alpha(\mathbf{A}\otimes\mathbf{B})"] \\ G'\mathbf{A} \otimes G'\mathbf{B} \arrow[r, "\mathfrak{m}_{\mathbf{A},\mathbf{B}}"'] & G'(\mathbf{A}\otimes\mathbf{B}). \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbf{I}_{GH} \arrow[r, "\mathfrak{e}_H"] \arrow[d, "\mathbf{I}_{\alpha H}"'] & G\mathbf{I}_H \arrow[d, "\alpha\mathbf{I}_H"] \\ \mathbf{I}_{G'H} \arrow[r, "\mathfrak{e}_H"'] & G'\mathbf{I}_H. \end{tikzcd}$$

But they do, since taking components at X, we reduce to instances of (VTA2) for α. This completes the definition of $\alpha(-)$. □

Again, $\alpha(-)$ and $(-)\alpha$ restrict to respective vertical transformations

$$(-)\alpha\colon (-)G \Rightarrow (-)G'\colon [\mathbb{M},\mathbb{N}]_\psi \to [\mathbb{L},\mathbb{N}]_\psi$$

and

$$\alpha(-)\colon G(-) \Rightarrow G'(-)\colon [\mathbb{K},\mathbb{L}]_\psi \to [\mathbb{K},\mathbb{M}]_\psi.$$

We make one final remark; suppose we are given a vertical transformation $\alpha\colon G \Rightarrow$

G' in $[\mathbb{L}, \mathbb{M}]_\psi$ and a modification

$$\begin{tikzcd}
A_s \arrow[r, Rightarrow, "\mathbf{A}", "\shortmid"{marking}] \arrow[d, Rightarrow, "\gamma_s"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\gamma}"] & A_t \arrow[d, Rightarrow, "\gamma_t"] \\
B_s \arrow[r, Rightarrow, "\mathbf{B}"', "\shortmid"{marking}] & B_t
\end{tikzcd}$$

in $[\mathbb{M}, \mathbb{N}]$, then the two modifications

$$\begin{tikzcd}
A_sG \arrow[r, Rightarrow, "\mathbf{A}G", "\shortmid"{marking}] \arrow[d, Rightarrow, "\gamma_sG"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\gamma}G"] & A_tG \arrow[d, Rightarrow, "\gamma_tG"] \\
B_sG \arrow[r, Rightarrow, "\mathbf{B}G"', "\shortmid"{marking}] \arrow[d, Rightarrow, "B_s\alpha"'] \arrow[dr, phantom, "\Downarrow\mathbf{B}\alpha"] & B_tG \arrow[d, Rightarrow, "B_t\alpha"] \\
B_sG' \arrow[r, Rightarrow, "\mathbf{B}G'"', "\shortmid"{marking}] & B_tG'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A_sG \arrow[r, Rightarrow, "\mathbf{A}", "\shortmid"{marking}] \arrow[d, Rightarrow, "A_s\alpha"'] \arrow[dr, phantom, "\Downarrow\mathbf{A}\alpha"] & A_tG \arrow[d, Rightarrow, "A_t\alpha"] \\
A_sG' \arrow[r, Rightarrow, "\mathbf{B}G"', "\shortmid"{marking}] \arrow[d, Rightarrow, "\gamma_sG'"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\gamma}G'"] & A_tG' \arrow[d, Rightarrow, "\gamma_tG'"] \\
B_sG' \arrow[r, Rightarrow, "\mathbf{B}G'"', "\shortmid"{marking}] & B_tG'
\end{tikzcd}$$

in $[\mathbb{M}, \mathbb{N}]$ are the same by naturality of $\big((\text{–})\alpha\big)_1$; and hence we write this common value as $\boldsymbol{\gamma}\alpha$. Similarly, if we have $\boldsymbol{\gamma}\colon \mathbf{A} \Rrightarrow \mathbf{B}$ now in $[\mathbb{K}, \mathbb{L}]$ we write $\alpha\boldsymbol{\gamma}$ for the modification $\alpha\mathbf{B} \circ G\boldsymbol{\gamma} = G'\boldsymbol{\gamma} \circ \alpha\mathbf{A}$ in $[\mathbb{K}, \mathbb{M}]$.

2.9 The hom 2-functor on $\mathbf{DblCat}_\psi$

Now, it's not hard to see that the operations of the previous section are functorial with respect to vertical transformations. To be more precise, given double categories $\mathbb{K}$, $\mathbb{L}$, $\mathbb{M}$ and $\mathbb{N}$, the above operations induce functors

$$\begin{aligned}
[\mathbb{K}, \text{–}]\colon [\mathbb{L}, \mathbb{M}]_{v\psi} &\to \big[\, [\mathbb{K}, \mathbb{L}], [\mathbb{K}, \mathbb{M}] \,\big]_{v\psi} \\
\text{and } [\text{–}, \mathbb{N}]\colon [\mathbb{L}, \mathbb{M}]_{v\psi} &\to \big[\, [\mathbb{M}, \mathbb{N}], [\mathbb{L}, \mathbb{N}] \,\big]_{v\psi},
\end{aligned}$$

along with their 'pseudo' restrictions

$$\begin{aligned}
[\mathbb{K}, \text{–}]_\psi\colon [\mathbb{L}, \mathbb{M}]_{v\psi} &\to \big[\, [\mathbb{K}, \mathbb{L}]_\psi, [\mathbb{K}, \mathbb{M}]_\psi \,\big]_{v\psi} \\
\text{and } [\text{–}, \mathbb{N}]_\psi\colon [\mathbb{L}, \mathbb{M}]_{v\psi} &\to \big[\, [\mathbb{M}, \mathbb{N}]_\psi, [\mathbb{L}, \mathbb{N}]_\psi \,\big]_{v\psi}.
\end{aligned}$$

Moreover, it's straightforward to check that the following equalities hold:

$$\begin{aligned} \big((\text{–})G_1\big)G_2 &= (\text{–})(G_1G_2), & \big((\text{–})\alpha_1\big)\alpha_2 &= (\text{–})(\alpha_1\alpha_2), \\ G_1\big(G_2(\text{–})\big) &= (G_1G_2)(\text{–}), & \alpha_1\big(\alpha_2(\text{–})\big) &= (\alpha_1\alpha_2)(\text{–}), \\ \big(G_1(\text{–})\big)G_2 &= G_1\big((\text{–})G_2\big), \qquad \text{and} & \big(\alpha_1(\text{–})\big)\alpha_2 &= \alpha_1\big((\text{–})\alpha_2\big). \end{aligned}$$

which can be more succinctly stated as follows:

Proposition 11. *The functors* $[\mathbb{K}, \text{–}]$ *and* $[\text{–}, \mathbb{N}]$ *defined above provide data for 2-functors*

$$[\mathbb{K}, \text{–}]\colon \mathbf{DblCat}_\psi \to \mathbf{DblCat}_\psi \text{ and } [\text{–}, \mathbb{N}]\colon \mathbf{DblCat}_\psi^{\text{op}} \to \mathbf{DblCat}_\psi$$

which are compatible in the sense that they provide data for a 2-functor

$$[\,\text{–}, ?\,]\colon \mathbf{DblCat}_\psi^{\text{op}} \times \mathbf{DblCat}_\psi \to \mathbf{DblCat}_\psi.$$

Similarly, the functors $[\mathbb{K}, \text{–}]_\psi$ *and* $[\text{–}, \mathbb{N}]_\psi$ *defined above provide data for 2-functors*

$$[\mathbb{K}, \text{–}]_\psi\colon \mathbf{DblCat}_\psi \to \mathbf{DblCat}_\psi \text{ and } [\text{–}, \mathbb{N}]_\psi\colon \mathbf{DblCat}_\psi^{\text{op}} \to \mathbf{DblCat}_\psi$$

which are compatible in the sense that they provide data for a 2-functor

$$[\,\text{–}, ?\,]_\psi\colon \mathbf{DblCat}_\psi^{\text{op}} \times \mathbf{DblCat}_\psi \to \mathbf{DblCat}_\psi.$$

Now, what *are* these 2-functors? Does either of the bivariant 2-functors provide an 'internal hom' for $\mathbf{DblCat}_\psi$? Let us make this question precise: observe that $\mathbf{DblCat}_\psi$ has all finite products, and thus can be viewed as a monoidal bicategory, with the tensor product given by cartesian product. Then by an 'internal hom' for $\mathbf{DblCat}_\psi$, we mean a homomorphism of bicategories

$$\langle\text{–}, ?\,\rangle\colon \mathbf{DblCat}_\psi^{\text{op}} \times \mathbf{DblCat}_\psi \to \mathbf{DblCat}_\psi$$

such that for all pseudo double categories $\mathbb{K}$, we have a biadjunction

$$(\text{–}) \times \mathbb{K} \dashv \langle\mathbb{K}, \text{–}\rangle.$$

In other words, $\langle -, ?\, \rangle$, if it exists, exhibits $\mathbf{DblCat}_\psi$ as a *biclosed* monoidal bicategory in the sense of [DS97]. Though we do not intend to pursue this avenue in any detail in this thesis, it is worth making a few remarks.

Firstly, there is *no* good biadjunction for the 'lax hom' 2-functor $[-, ?\,]$, for the same reason as there is no good whiskering on the left by morphisms: at some point, we have to produce pseudo-naturality data for a horizontal transformation, and, due to the laxity of the morphisms involved, no choice of such data exists.

Secondly, the 'pseudo hom' 2-functor $[-, ?\,]_\psi$ *does* provide an internal hom in the above described sense. We don't intend to work through the rather messy details here, but we do note that although both $(-) \times \mathbb{K}$ and $[\,\mathbb{K}, -]$ are 2-functors, the adjunction between them is still only a *bi*adjunction rather than an honest 2-adjunction. We shall note further ramifications of this more conceptual view of the 'pseudo hom' 2-functor as we progress through the thesis.

Chapter 3

Clubs I

In this chapter, we gather together a collection of background material, recalling some concepts and elementary results of the theory of *clubs.* This theory has its genesis in work of Kelly's in the 1970's [Kel72a, Kel72b, Kel74b], work which he later revisited, leading to the more abstract formulation of [Kel92]. Another treatment of this material can be found in [Web05], whilst the related subject of 'cartesian monads' is treated in some detail by [Lei04a], for example.

We begin by recalling the concept of a *cartesian natural transformation* and an important proposition saying that such transformations are 'determined by their component at 1'. We then give the definition of an (abstract) *club*, together with a straightforward concrete characterisation of such gadgets. We finish by giving an important example of a club (indeed, the motivating example), the club for *symmetric strict monoidal categories.* For further details and a different perspective on the material of this chapter, we refer the reader to [Kel92] or [Web05].

3.1 Cartesian natural transformations

Definition 12. A natural transformation $\alpha\colon F \Rightarrow G\colon \mathbf{C} \to \mathbf{D}$ is called a **cartesian natural transformation** if all its naturality squares are pullbacks.

Proposition 13. *Suppose that* $\mathbf{C}$ *has a terminal object* 1. *Then a natural transformation* $\alpha\colon F \Rightarrow G\colon \mathbf{C} \to \mathbf{D}$ *is cartesian if and only if every naturality square* *of the form*

$$\begin{tikzcd}
FX \arrow[r, "F!"] \arrow[d, "\alpha_X"'] & F1 \arrow[d, "\alpha_1"] \\
GX \arrow[r, "G!"'] & G1
\end{tikzcd}$$

is a pullback.

Proof. The ‘only if’ direction is trivial. For the ‘if’ direction, suppose we are given a map $f\colon X \to Y$ in $\mathbf{C}$. We observe that in the diagram

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[r, "F!"] \arrow[d, "\alpha_Y"] & F1 \arrow[d, "\alpha_1"] \\
GX \arrow[r, "Gf"'] & GY \arrow[r, "G!"'] & G1
\end{tikzcd}$$

the outer edge and the right-hand square are pullbacks, and thus that the left-hand square is a pullback as required. ☐

3.2 The category of collections

Given an functor $S\colon \mathbf{C} \to \mathbf{D}$, we can form the slice category $[\mathbf{C}, \mathbf{D}]/S$, which we notate as follows:

- **Objects** are pairs (A, α), where $A\colon \mathbf{C} \to \mathbf{D}$ is a functor and $\alpha\colon A \Rightarrow S$ is a natural transformation;
- **Maps** $\gamma\colon (A, \alpha) \to (B, \beta)$ are natural transformations $\gamma\colon A \Rightarrow B$ satsifying $\beta\gamma = \alpha$.

We may consider the full subcategory of this given by the objects (A, α) where α is a *cartesian* natural transformations into S. We write $\mathit{Coll}(S)$ for this subcategory and call it the **category of collections** over S. Now, we have a functor $F\colon \mathit{Coll}(S) \to \mathbf{D}/S1$ which evaluates at 1:

$$\begin{aligned} F\colon \mathit{Coll}(S) &\to \mathbf{D}/S1 \\ (A, \alpha) &\mapsto (A1, \alpha_1) \\ \gamma &\mapsto \gamma_1, \end{aligned}$$

and the following proposition tells us that (for $\mathbf{D}$ sufficiently complete) we lose no real information in applying F:

Proposition 14. *Suppose* $\mathbf{D}$ *has enough pullbacks; then evaluation at* 1 *induces an equivalence of categories* $Coll(S) \simeq \mathbf{D}/S1$.

Proof. We construct a functor $G\colon \mathbf{D}/S1 \to Coll(S)$ which is pseudoinverse to F. Given an object (a, θ) of $\mathbf{D}/S1$, we give $G(a, \theta) = (A, \alpha)$ as follows. AX and α_X are given by the indicated object and arrow in the following (chosen) pullback diagram:

$$\begin{tikzcd}
AX \arrow[r] \arrow[d, "\alpha_X"'] & a \arrow[d, "\theta"] \\
SX \arrow[r, "S!"'] & S1,
\end{tikzcd}$$

whilst the value of A on a map $f\colon X \to Y$ of $\mathbf{C}$ is given by the unique map induced by the universal property of pullback in the following diagram

$$\begin{tikzcd}
AX \arrow[rr] \arrow[dr, dotted, "Af"] \arrow[dd, "\alpha_{\mathbf{X}}"'] & & A \arrow[dr, "\text{id}"] \arrow[dd, "\theta" near start] & \\
& AY \arrow[rr] \arrow[dd, "\alpha_Y" near start] & & A \arrow[dd, "\theta"] \\
SX \arrow[rr, "S!" near end] \arrow[dr, "Sf"'] & & S1 \arrow[dr, "\text{id}"] & \\
& SY \arrow[rr, "S!"'] & & S1
\end{tikzcd}$$

whose front and rear faces are pullbacks. Functoriality of A follows from the universal property of pullback and the functoriality of S, whilst the left face of the above diagram provides us with the naturality square for α at f. It remains to check that α is cartesian: but the front face is a pullback, as is the outer edge of the front and left-hand face together (since it is equal to the rear face), and thus the left-hand face is also pullback as required.

On maps, given $\psi\colon (a, \theta) \to (b, \phi)$ in $\mathbf{D}/S1$, we must give a map $G\psi = \gamma\colon (A, \alpha) \to (B, \beta)$ of $Coll(S)$. We give the component of γ at X as the unique map induced by the universal property of pullback in the diagram

$$\begin{tikzcd}
AX \arrow[rr] \arrow[dr, dotted, "\gamma_X"] \arrow[dd, "\alpha_X"'] & & a \arrow[dr, "\psi"] \arrow[dd, "\theta"] & \\
& BX \arrow[rr] \arrow[dd, "\beta_X"'] & & b \arrow[dd, "\phi"] \\
SX \arrow[rr, "S!"] \arrow[dr, "\mathrm{id}"'] & & S1 \arrow[dr, "\mathrm{id}"] & \\
& SX \arrow[rr, "S!"'] & & S1
\end{tikzcd}$$

whose front and rear faces are pullbacks. Visibly, we have $\beta\gamma = \alpha$; it remains only to check naturality of γ. So let $f\colon X \to Y$ be a map in $\mathbf{D}$ and consider the following diagram:

$$\begin{tikzcd}
AX \arrow[rr] \arrow[dr, dotted] \arrow[dd, "\alpha_X"'] & & a \arrow[dr, "\psi"] \arrow[dd, "\theta"] & \\
& BY \arrow[rr] \arrow[dd, "\beta_Y"'] & & b \arrow[dd, "\phi"] \\
SX \arrow[rr, "S!"] \arrow[dr, "f"'] & & S1 \arrow[dr, "\mathrm{id}"] & \\
& SY \arrow[rr, "S!"'] & & S1
\end{tikzcd}$$

whose front and rear faces are pullbacks. Then both $\gamma_Y \circ Af$ and $Bf \circ \gamma_X$ make the diagram commute when placed along the dotted arrow and hence by the universal property of pullback, they must coincide.

It remains to show that F and G are pseudo-inverse to each other. If we choose pullbacks such that the pullback of an identity is an identity, then we have $FG = \mathrm{id}_{\mathbf{D}/S1}$. Conversely, suppose we are given (A, α) in $Coll(S)$; let us write $(\hat{A}, \hat{\alpha})$ for $GF(A, \alpha)$. Then we have an invertible transformation

$$\begin{tikzcd}
A \arrow[rr, Rightarrow, "\eta_{(A,\alpha)}"] \arrow[dr, Rightarrow, "\alpha"'] & & \hat{A} \arrow[dl, Rightarrow, "\hat{\alpha}"] \\
& S &
\end{tikzcd}$$

whose components are the unique (invertible) maps induced in the diagram

$$\begin{tikzcd}
AX \arrow[rr, "A!"] \arrow[dr, dotted, "(\eta_{(A,\alpha)})_X"] \arrow[dd, "\alpha_X"'] & & A1 \arrow[dr, "\mathrm{id}"] \arrow[dd, "\alpha_1" near start] & \\
& \hat{A}X \arrow[rr, "\hat{A}!" near start] \arrow[dd, "\hat{\alpha}_X" near start] & & A1 \arrow[dd, "\alpha_1"] \\
SX \arrow[rr, "S!" near end] \arrow[dr, "\mathrm{id}"'] & & S1 \arrow[dr, "\mathrm{id}"] & \\
& SX \arrow[rr, "S!"'] & & S1
\end{tikzcd}$$

whose front and rear faces are pullbacks. Clearly $\hat{\alpha} \circ \eta_{(A,\alpha)} = \alpha$, and by the universal property of pullback, we see that $\eta_{(A,\alpha)}$ is natural in X. For the naturality of η in (A, α), suppose we are given a map $\gamma \colon (A, \alpha) \to (B, \beta)$ in $\mathit{Coll}(S)$, and let us write $\hat{\gamma}$ for $GF(\gamma)$. Then considering the diagram

$$\begin{tikzcd}
AX \arrow[rr, "A!"] \arrow[dr, dotted] \arrow[dd, "\alpha_X"'] & & A1 \arrow[dr, "\gamma_1"] \arrow[dd, "\alpha_1" near start] & \\
& \hat{B}X \arrow[rr, "\hat{B}!" near start] \arrow[dd, "\hat{\beta}_X" near start] & & B1 \arrow[dd, "\beta_1"] \\
SX \arrow[rr, "S!" near end] \arrow[dr, "\mathrm{id}"'] & & S1 \arrow[dr, "\mathrm{id}"] & \\
& S\mathbf{X} \arrow[rr, "S!"'] & & S1
\end{tikzcd}$$

whose front and rear faces are pullbacks, we see that both $(\eta_{(B,\beta)})_X \circ \gamma_X$ and $\hat{\gamma}_X \circ (\eta_{(A,\alpha)})_X$ make it commute when inserted for the dotted arrow, and hence must coincide. Thus we have $\eta_{(B,\beta)} \circ \gamma = \hat{\gamma} \circ \eta_{(A,\alpha)}$ as required. $\square$

3.3 Comma categories and monoidal comma categories

Given categories **C**, **D** and **E**, together with functors $F \colon \mathbf{C} \to \mathbf{E}$ and $G \colon \mathbf{D} \to \mathbf{E}$, we can form the comma category $(F \downarrow G)$, which we notate as follows:

- **Objects** are triples (U, X, f) where $U \in \mathbf{C}$, $X \in \mathbf{D}$ and $f \colon FU \to GX$;

- **Maps** $(U, X, f) \to (V, Y, g)$ are pairs (j, k) where $j\colon U \to V$ and $k\colon X \to Y$ such that

$$\begin{tikzcd} FU \arrow[r, "f"] \arrow[d, "Fj"'] & GX \arrow[d, "Gk"] \\ FV \arrow[r, "g"'] & GY \end{tikzcd}$$

  commutes.

Now, there is a natural 'monoidal enrichment' of the notion of comma categories, as follows:

Proposition 15. *Given monoidal categories* $\mathbf{C}$, $\mathbf{D}$ *and* $\mathbf{E}$, *together with an opmonoidal functor* $F\colon \mathbf{C} \to \mathbf{E}$ *and a monoidal functor* $G\colon \mathbf{D} \to \mathbf{E}$, *the comma category* $(F \downarrow G)$ *acquires a canonical monoidal structure.*

Proof. Suppose that F and G have (op)monoidal structure (F, e_F, m_F) and (G, e_G, m_G) respectively; then we equip $(F \downarrow G)$ with monoidal structure as follows. The unit is given by the object

$$F\mathbf{I} \xrightarrow{e_F} \mathbf{I} \xrightarrow{e_G} G\mathbf{I},$$

whilst the tensor product is given as follows:

- **On objects**, $(U, X, f) \otimes (V, Y, g)$ is $(U \otimes V, X \otimes Y, f \mathbin{\underline{\otimes}} g)$. where $f \mathbin{\underline{\otimes}} g$ is the composite

$$F(U \otimes V) \xrightarrow{m_F} FU \otimes FV \xrightarrow{f \otimes g} GX \otimes GY \xrightarrow{m_G} G(X \otimes Y).$$

- **On maps**, $(j, k) \otimes (m, n)$ is simply $(j \otimes m, k \otimes n)$; that the required square commutes follows from the functoriality of $\otimes$ and the naturality of m_F and m_G.

Unitality and associativity constraints are inherited in the evident way from $\mathbf{C}$ and $\mathbf{D}$:

$$\begin{aligned} \lambda_{(U,X,f)} &= (\lambda_U, \lambda_X) \\ \rho_{(U,X,f)} &= (\rho_U, \rho_X) \\ \text{and } \alpha_{(U,X,f),(V,Y,g),(W,Z,h)} &= (\alpha_{U,V,W}, \alpha_{X,Y,Z}). \end{aligned}$$

□

The canonicity of the monoidal structure given amounts to the fact that it has a 'comma object'-like universal property in a certain double category. We shall not go into the details here, but instead refer the reader to [GP04].

We shall be interested in a special case of the above, namely 'slicing over a monoid': we take F to be $\mathrm{id}_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{C}$ and we take G to be a monoidal functor $G \colon 1 \to \mathbf{C}$. Now, giving such a G amounts to giving a *monoid* in the monoidal category $\mathbf{C}$, and so the above result reduces to the following:

Corollary 16. *Let* $\mathbf{C}$ *be a monoidal category, and let* X *be a monoid in it. Then the slice category* $\mathbf{C}/X$ *acquires a canonical structure of monoidal category.*

3.4 Clubs

Suppose now that we are given a category $\mathbf{C}$ with all finite limits, together with a monad (S, η, μ) on $\mathbf{C}$. We can view S as a monoid in the strict monoidal category $[\mathbf{C}, \mathbf{C}]$, and so, applying the previous Corollary, we can equip the slice category $[\mathbf{C}, \mathbf{C}]/S$ with a strict monoidal structure, namely:

$$\mathbf{I} = (\mathrm{id} \xRightarrow{\eta} S) \quad \text{and} \quad (A, \alpha) \otimes (B, \beta) = (AB \xRightarrow{\alpha\beta} SS \xRightarrow{\mu} S).$$

Now, we may naturally ask whether the subcategory $\mathit{Coll}(S)$ of $[\mathbf{C}, \mathbf{C}]/S$ is closed under this monoidal structure. Explicitly:

Definition 17. We say that a subcategory $\mathbf{D}$ of $\mathbf{C}$ is a **monoidal subcategory** if $\mathbf{D}$ can be made into a monoidal category such that the inclusion $\mathbf{D} \hookrightarrow \mathbf{C}$ is a strict monoidal functor.

Definition 18. We say that (S, η, μ) is a **club** on $\mathbf{C}$ if $\mathit{Coll}(S)$ is a monoidal subcategory of $[\mathbf{C}, \mathbf{C}]/S$.

Proposition 19. (S, η, μ) *is a club on* $\mathbf{C}$ *if and only if:*

1. η *is a cartesian natural transformation;*
2. μ *is a cartesian natural transformation;*
3. S *preserves cartesian natural transformations into* S*: that is, given* $A \xRightarrow{\alpha} S$ *a cartesian natural transformation,* $SA \xRightarrow{S\alpha} SS$ *is also cartesian.*

Proof. We begin with th ‘if’ direction. Since *Coll*(S) is a full subcategory of $[\mathbf{C},\mathbf{C}]/S$, it suffices to check that the set of objects of *Coll*(S) is closed under the nullary and binary tensor products. Now,

$$\mathbf{I} = (\mathrm{id} \xRightarrow{\eta} S)$$

is cartesian by (1), and hence lies in *Coll*(S) as required, whilst given (A,α) and (B,β) cartesian, we have

$$(A,\alpha)\otimes(B,\beta) = (AB \xRightarrow{\alpha B} SB \xRightarrow{S\beta} SS \xRightarrow{\mu} S);$$

the first of the arrows in this composite is cartesian since α is, the second by (3) and the third by (2); thus the composite is itself cartesian. Thus (S,η,μ) is a club.

For the ‘only if’ direction, suppose that (S,η,μ) is a club. Then we have that

$$\mathbf{I} = (\mathrm{id} \xRightarrow{\eta} S)$$
$$\text{and } (S,\mathrm{id})\otimes(S,\mathrm{id}) = (SS \xRightarrow{\mathrm{id}} SS \xRightarrow{\mu} S)$$

lie in *Coll*(S), so that η and μ are cartesian natural transformations as required. Further, if $\alpha\colon A \Rightarrow S$ is a cartesian natural transformation, then

$$(S,\mathrm{id})\otimes(A,\alpha) = (SS \xRightarrow{S\alpha} SS \xRightarrow{\mu} S)$$

is also cartesian; and since μ is cartesian, we conclude that $S\alpha$ is also cartesian as required. ☐

We observe in passing that condition (3) of the above Proposition follows *a fortiori* if S happens to preserve pullbacks. Now, suppose we have a club S on a category $\mathbf{C}$; by Proposition 14, we have an equivalence of categories

$$\mathit{Coll}(S) \simeq \mathbf{C}/S1;$$

and thus the monoidal structure of the left-hand side transfers under this equivalence to give a monoidal structure on $\mathbf{C}/S1$. Let us examine more closely what this monoidal structure is. The unit is given by

$$1 \xrightarrow{\eta_1} S1$$

whilst given objects (a, θ) and (b, ϕ), their tensor product is given by the left-hand composite in the following diagram:

$$\begin{tikzcd}
a \otimes b \arrow[r] \arrow[d] & a \arrow[d, "\theta"] \\
Sb \arrow[r, "S!"'] \arrow[d, "S\phi"'] & S1 \\
SS1 \arrow[d, "\mu_1"'] & \\
S1 &
\end{tikzcd}$$

where the upper square is a (chosen) pullback. Finally, given maps $g\colon (a, \theta) \to (a', \theta')$ and $h\colon (b, \phi) \to (b, \phi')$ in $\mathbf{C}/S1$, the map $g \otimes h$ is given by the dotted arrow in

$$\begin{tikzcd}
a \otimes b \arrow[rrr] \arrow[dd] \arrow[drr, dotted] & & & a \arrow[dr, "g"] \arrow[dd, "\theta"] & \\
& & a' \otimes b' \arrow[rr] \arrow[dd] & & a' \arrow[dd, "\theta'"] \\
Sb \arrow[rrr, "S!"] \arrow[d, "S\phi"'] \arrow[drr, "Sh"'] & & & S1 \arrow[dr, "\mathrm{id}"] & \\
SS1 \arrow[d, "\mu_1"'] \arrow[drr, "\mathrm{id}"'] & & Sb' \arrow[rr, "S!"'] \arrow[d, "S\phi'"] & & S1 \\
S1 \arrow[drr, "\mathrm{id}"'] & & SS1 \arrow[d, "\mu_1"] & & \\
& & S1 & &
\end{tikzcd}$$

induced by the universal property of pullback; as evidenced by the left-hand face, this map is indeed a map of $\mathbf{C}/S1$.

3.5 The club for symmetric strict monoidal categories

We shall now give an example of a club on **Cat**, namely that for *symmetric strict monoidal categories*. This is the structure for which the concept of club was first brought into being (see [Kel72a, Kel72b]), and has been considered by many authors since – see [Lei04a], or [BD98] for an application very much in the spirit of this thesis.

Definition 20. We write $S1$ for the category of 'finite cardinals and bijections', with:

- **Objects** the natural numbers $0, 1, 2, \ldots$;
- **Maps** $\sigma \colon n \to m$ bijections of $\{1, \ldots, n\}$ with $\{1, \ldots, m\}$,

and with composition and identities given in the evident way.

Definition 21. The *free symmetric strict monoidal category* 2-functor $S \colon \mathbf{Cat} \to \mathbf{Cat}$ is given as follows:

- **On objects**: Given a small category $\mathbf{C}$, we give $S\mathbf{C}$ as follows:
  - **Objects** of $S\mathbf{C}$ are pairs $(n, \langle c_i \rangle)$, where $n \in S1$ and $c_1, \ldots, c_n \in \mathrm{ob}\,\mathbf{C}$;
  - **Arrows** of $S\mathbf{C}$ are

$$(\sigma, \langle g_i \rangle) \colon (n, \langle c_i \rangle) \to (m, \langle d_i \rangle),$$

  where $\sigma \in S1(n, m)$ and $g_i \colon c_i \to d_{\sigma(i)}$ (note that necessarily $n = m$).

  Composition and identities in $S\mathbf{C}$ are given in the evident way; namely,

$$\begin{aligned} \mathrm{id}_{(n, \langle c_i \rangle)} &= (\mathrm{id}_n, \langle \mathrm{id}_{c_i} \rangle) \\ \text{and } (\tau, \langle g_i \rangle) \circ (\sigma, \langle f_i \rangle) &= (\tau \circ \sigma, \big\langle g_{\sigma(i)} \circ f_i \big\rangle). \end{aligned}$$

- **On maps**: Given a functor $F \colon \mathbf{C} \to \mathbf{D}$, we give $SF \colon S\mathbf{C} \to S\mathbf{D}$ by

$$\begin{aligned} SF(n, \langle c_i \rangle) &= (n, \langle Fc_i \rangle) \\ SF(\sigma, \langle g_i \rangle) &= (\sigma, \langle Fg_i \rangle). \end{aligned}$$

- **On 2-cells**: Given a natural transformation $\alpha\colon F \Rightarrow G\colon \mathbf{C} \to \mathbf{D}$, we give $S\alpha\colon SF \Rightarrow SG\colon S\mathbf{C} \to S\mathbf{D}$ by
$$(S\alpha)_{(n,\langle c_i \rangle)} = (\mathrm{id}_n, \langle \alpha_{c_i} \rangle).$$

Now, although the above description suffices to describe the iterated functor $S^2\colon \mathbf{Cat} \to \mathbf{Cat}$, it will be much more pleasant to work with the following alternative presentation. We first describe $S^2 1$ as follows:

- **Objects** are order-preserving maps $\phi\colon n_\phi \to m_\phi$, where n_ϕ, $m_\phi \in \mathbb{N}$. We write such an object simply as ϕ, with the convention that ϕ has domain and codomain n_ϕ and m_ϕ respectively.

- **Maps** $f\colon \phi \to \psi$ are pairs of bijections $f_n\colon n_\phi \to n_\psi$ and $f_m\colon m_\phi \to m_\psi$ such that the following diagram commutes:
$$\begin{tikzcd} n_\phi \arrow[r, "f_n"] \arrow[d, "\phi"'] & n_\psi \arrow[d, "\psi"] \\ m_\phi \arrow[r, "f_m"'] & m_\psi . \end{tikzcd}$$

It may not be immediately obvious that this *is* a presentation of $S^2 1$. The picture is as follows: an object ϕ of $S^2 1$ is to be thought of as a collection of n_ϕ points partitioned into m_ϕ parts in accordance with ϕ. Given such an object, one can permute internally any of its m_ϕ parts, or can in fact permute the set of m_ϕ parts itself; and a typical map describes such a permutation. For example, the objects

$$\begin{array}{cc} \phi\colon 5 \to 4 & \psi\colon 5 \to 4 \\ 1,2,3,4,5 \mapsto 1,1,3,4,4 & 1,2,3,4,5 \mapsto 2,2,3,4,4 \end{array}$$

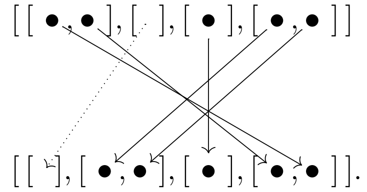
should be visualised as

$$[\,[\bullet,\bullet]\,,[\;]\,,[\bullet]\,,[\bullet,\bullet]\,] \qquad \text{and} \qquad [\,[\;]\,,[\bullet,\bullet]\,,[\bullet]\,,[\bullet,\bullet]\,]$$

respectively, whilst a typical map $\phi \to \psi$ is given by

$$\begin{aligned} f_n \colon 5 &\to 5 \\ 1,2,3,4,5 &\mapsto 5,4,3,1,2 \end{aligned} \qquad \begin{aligned} f_m \colon 4 &\to 4 \\ 1,2,3,4 &\mapsto 4,1,3,2 \end{aligned}$$

and should be visualised as



So now, given a category $\mathbf{C}$, we can present $S^2\mathbf{C}$ as follows:

- **Objects** of $S^2\mathbf{C}$ are pairs $(\phi, \langle c_i \rangle)$, where $\phi = n_\phi \to m_\phi \in S^2 1$ and $c_1, \ldots, c_{n_\phi} \in \operatorname{ob} \mathbf{C}$;

- **Arrows** of $S^2\mathbf{C}$ are

$$(f, \langle g_i \rangle) \colon (\phi, \langle c_i \rangle) \to (\psi, \langle d_i \rangle),$$

  where $f = (f_n, f_m) \in S^2 1(\phi, \psi)$ and $g_i \colon c_i \to d_{f_n(i)}$; composition and identities are given analogously to before.

We can extend the above in the obvious way to 1- and 2-cells of **Cat** to give a presentation of the 2-functor S^2. Using this alternate presentation of S^2, we may describe the rest of the 2-monad structure of S:

Definition 22. The 2-natural transformation $\eta \colon \mathrm{id}_{\mathbf{Cat}} \Rightarrow S$ has component at $\mathbf{C}$ given by

$$\begin{aligned} \eta_{\mathbf{C}} \colon \mathbf{C} &\to S\mathbf{C} \\ x &\mapsto (1, \langle x \rangle) \\ f &\mapsto (\mathrm{id}_1, \langle f \rangle), \end{aligned}$$

whilst the 2-natural transformation $\mu\colon S^2 \Rightarrow S$ has component at $\mathbf{C}$ given by

$$\begin{aligned} \eta_{\mathbf{C}}\colon SS\mathbf{C} &\to S\mathbf{C} \\ (\phi, \langle c_i \rangle) &\mapsto (n_\phi, \langle c_i \rangle) \\ (f, \langle g_i \rangle) &\mapsto (f_n, \langle g_i \rangle). \end{aligned}$$

Proposition 23. *(S, η, μ) is a club on* **Cat**.

Proof. It's a straightforward calculation to check that all the naturality diagrams for η and μ are pullbacks, and that S preserves *all* pullbacks. Hence, by Proposition 19, S is a club. ☐

Before moving on, let us note that we can give a presentation of S^3 in a similar style to above, which will come in useful later. We give $S^3 1$ as follows:

- **Objects** are diagrams $\phi = n_\phi \xrightarrow{\phi_1} m_\phi \xrightarrow{\phi_2} r_\phi$ in the category of finite ordinals and order preserving maps;
- **Maps** $f\colon \phi \to \psi$ are triples (f_n, f_m, f_r) of bijections making

$$\begin{tikzcd} n_\phi \arrow[r, "f_n"] \arrow[d, "\phi_1"'] & n_\psi \arrow[d, "\psi_1"] \\ m_\phi \arrow[r, "f_m"] \arrow[d, "\phi_2"'] & m_\psi \arrow[d, "\psi_2"] \\ r_\phi \arrow[r, "f_r"'] & r_\psi . \end{tikzcd}$$

  commute.

And then present $S^3\mathbf{C}$ as follows:

- **Objects** of $S^3\mathbf{C}$ are pairs $(\phi, \langle c_i \rangle)$, where $\phi = n_\phi \to m_\phi \to r_\phi \in S^3 1$ and $c_1, \ldots, c_{n_\phi} \in \operatorname{ob}\mathbf{C}$;
- **Arrows** of $S^3\mathbf{C}$ are

$$(f, \langle g_i \rangle)\colon (\phi, \langle c_i \rangle) \to (\psi, \langle d_i \rangle),$$

  where $f = (f_n, f_m, f_r) \in S^3 1(\phi, \psi)$ and $g_i\colon c_i \to d_{f_n(i)}$.

As before, we can now straightforwardly extend this definition to 1- and 2-cells of **Cat**.

Chapter 4

Pseudo double categories II

We wish to extend the theory of the previous chapter to pseudo double categories, but before we can do so, we shall need to establish double category analogues of the following notions:

- ‘slice category’;
- ‘cartesian natural transformation’;
- ‘category of collections’;
- ‘equivalence of categories’;
- ‘monoidal category’;
- ‘monoidal functor’;
- ‘monoidal structure of the endohom category’; and
- ‘monoidal slice category’.

In this chapter we shall tackle the first four of these. The details of ‘slice double categories’ and the more general ‘comma double category’ are already known, and can be found in [GP04], whilst the generalisation of ‘cartesian natural transformation’ is completely natural. Some care is needed for the concept of ‘category of collections’, whilst the characterisation of ‘equivalent pseudo double categories’ is almost self-evident, but does not appear to have been given explicitly before.

Henceforth, we shall assume without further mention that $\mathbb{K}$ and $\mathbb{L}$ are pseudo double categories such that:

- $\mathbb{K}$ has a **double terminal object**; that is, an object $1 \in K_0$ such that 1 is terminal in K_0 and $\mathbf{I}_1$ is terminal in K_1;
- L_1 and L_0 have all pullbacks and are equipped with a choice of such; and furthermore, that s and t preserve these choices of pullbacks *strictly*.

This strict preservation condition might appear rather strong at first, but as we shall see later, in all cases of interest to us, it is perfectly natural.

4.1 Comma double categories

We begin by extending the notion of comma category from plain categories to double categories. Like the notion of monoidal comma category, the notion of 'comma double category' enjoys a comma object-like universal property, which again is fully explored in [GP04]. We shall merely recap the details of the construction.

Given pseudo double categories $\mathbb{K}$, $\mathbb{L}$ and $\mathbb{M}$, together with a double opmorphism $F\colon \mathbb{K} \to \mathbb{M}$ and a double morphism $G\colon \mathbb{L} \to \mathbb{M}$, we may form the **comma double category** $(F \downarrow G)$ as follows:

- $(F \downarrow G)_1 = (F_1 \downarrow G_1)$;
- $(F \downarrow G)_0 = (F_0 \downarrow G_0)$;
- s and t are given by

$$s(\mathbf{U}, \mathbf{X}, \mathbf{f}) = (U_s, X_s, f_s), \qquad s(\mathbf{j}, \mathbf{k}) = (j_s, k_s),$$
$$t(\mathbf{U}, \mathbf{X}, \mathbf{f}) = (U_t, X_t, f_t), \qquad \text{and } t(\mathbf{j}, \mathbf{k}) = (j_t, k_t).$$

- $\mathbf{I}$ is given as follows:
  - On objects, $\mathbf{I}_{(U,X,f)} = (\mathbf{I}_U, \mathbf{I}_X, \underline{\mathbf{I}}_f)$, where $\underline{\mathbf{I}}_f$ is the composite

$$F\mathbf{I}_U \xrightarrow{\mathfrak{e}_X} \mathbf{I}_{FU} \xrightarrow{\mathbf{I}_f} \mathbf{I}_{GX} \xrightarrow{\mathfrak{e}_Y} G\mathbf{I}_X;$$

  - On maps, $\mathbf{I}_{(j,k)} = (\mathbf{I}_j, \mathbf{I}_k)$; the required square commutes by functoriality of $\mathbf{I}$ and naturality of $\mathfrak{e}$.

- $\otimes$ is given as follows:
  - On objects, $(\mathbf{U}, \mathbf{X}, \mathbf{f}) \otimes (\mathbf{V}, \mathbf{Y}, \mathbf{g})$ is given by $(\mathbf{U} \otimes \mathbf{V}, \mathbf{X} \otimes \mathbf{Y}, \mathbf{f} \underline{\otimes} \mathbf{g})$, where $\mathbf{f} \underline{\otimes} \mathbf{g}$ is the map
$$F(\mathbf{U} \otimes \mathbf{V}) \xrightarrow{\mathfrak{m}_{\mathbf{U},\mathbf{V}}} F\mathbf{U} \otimes F\mathbf{V} \xrightarrow{\mathbf{f} \otimes \mathbf{g}} G\mathbf{X} \otimes G\mathbf{Y} \xrightarrow{\mathfrak{m}_{\mathbf{X},\mathbf{Y}}} G(\mathbf{X} \otimes \mathbf{Y});$$
  - On maps, $(\mathbf{j}, \mathbf{k}) \otimes (\mathbf{m}, \mathbf{n})$ is given by $(\mathbf{j} \otimes \mathbf{m}, \mathbf{k} \otimes \mathbf{n})$; the required square commutes by the functoriality of $\otimes$ and the naturality of $\mathfrak{m}$.
- The natural transformations $\mathfrak{l}$, $\mathfrak{r}$ and $\mathfrak{a}$ providing (DD6)–(DD7) are specified by
$$\begin{aligned} \mathfrak{l}_{(\mathbf{U},\mathbf{X},\mathbf{f})} &= (\mathfrak{l}_{\mathbf{U}}, \mathfrak{l}_{\mathbf{X}}) \\ \mathfrak{r}_{(\mathbf{U},\mathbf{X},\mathbf{f})} &= (\mathfrak{r}_{\mathbf{U}}, \mathfrak{r}_{\mathbf{X}}) \\ \text{and } \mathfrak{a}_{(\mathbf{U},\mathbf{X},\mathbf{f}),(\mathbf{V},\mathbf{Y},\mathbf{g}),(\mathbf{W},\mathbf{Z},\mathbf{h})} &= (\mathfrak{a}_{\mathbf{U},\mathbf{V},\mathbf{W}}, \mathfrak{a}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}) \end{aligned}$$
  That the required squares commute for these to be maps follows straightforwardly using (DMA2) and (DMA3) for F and G and the functoriality of $\mathbf{I}$ and $\otimes$ for $\mathbb{M}$.

It's immediate that this data satisfies (DA1) and (DA2), whilst (DA3) and (DA4) follows from (DA3) and (DA4) for $\mathbb{K}$ and $\mathbb{L}$ together with the functoriality of F and G.

Again, we shall be interested in a special case of the above, this time where F is the identity homomorphism $\mathrm{id}_{\mathbb{K}} \colon \mathbb{K} \to \mathbb{K}$ and G is a double morphism $1 \to \mathbb{K}$ (for 1 is the terminal double category). Now, such a functor G amounts to giving a *monad* in the double category $\mathbb{K}$. Explicitly:

Definition 24. A **monad** in the pseudo double category $\mathbb{K}$ consists of:

- An object X in K_0;
- An object $\mathbf{X} \colon X \nrightarrow X$ in K_1;
- Special maps
$$\mathfrak{m} \colon \mathbf{X} \otimes \mathbf{X} \to \mathbf{X} \quad \text{and} \quad \mathfrak{e} \colon \mathbf{I}_X \to \mathbf{X}$$

subject to the commutativity of the usual unitality and associativity diagrams:

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "\mathfrak{l}_{\mathbf{X}}"] \arrow[d, "\mathrm{id}_{\mathbf{X}}"'] & \mathbf{I}_X \otimes \mathbf{X} \arrow[d, "\mathfrak{e} \otimes \mathrm{id}_{\mathbf{X}}"] \\
\mathbf{X} & \mathbf{X} \otimes \mathbf{X} \arrow[l, "\mathfrak{m}"]
\end{tikzcd} \quad , \quad
\begin{tikzcd}
\mathbf{X} \arrow[r, "\mathfrak{r}_{\mathbf{X}}"] \arrow[d, "\mathrm{id}_{\mathbf{X}}"'] & \mathbf{X} \otimes \mathbf{I}_X \arrow[d, "\mathrm{id}_{\mathbf{X}} \otimes \mathfrak{e}"] \\
\mathbf{X} & \mathbf{X} \otimes \mathbf{X} \arrow[l, "\mathfrak{m}"]
\end{tikzcd}$$

and
$$\begin{tikzcd}
\mathbf{X} \otimes (\mathbf{X} \otimes \mathbf{X}) \arrow[rr, "\mathfrak{a}_{\mathbf{X},\mathbf{X},\mathbf{X}}"] \arrow[d, "\mathrm{id}_{\mathbf{X}} \otimes \mathfrak{m}"'] & & (\mathbf{X} \otimes \mathbf{X}) \otimes \mathbf{X}. \arrow[d, "\mathfrak{m} \otimes \mathrm{id}_{\mathbf{X}}"] \\
\mathbf{X} \otimes \mathbf{X} \arrow[dr, "\mathfrak{m}"'] & & \mathbf{X} \otimes \mathbf{X} \arrow[dl, "\mathfrak{m}"] \\
& \mathbf{X} &
\end{tikzcd}$$

(Note that this is the same as giving a monad in the bicategory $\mathcal{B}\mathbb{K}$.) It follows from the previous section that we can form the comma double category $(\mathrm{id} \downarrow \mathbf{X})$, which we shall notate as the *slice double category* $\mathbb{K}/\mathbf{X}$. Let us now describe the monad we shall need for the theory of double clubs. We begin with the following straightforward result:

Proposition 25. *Given a pseudo double category $\mathbb{K}$ and an object $X \in K_0$, the functor $\ulcorner X \urcorner : 1 \to K_0$ extends to a double homomorphism $\ulcorner \mathbf{I}_X \urcorner : 1 \to \mathbb{K}$.*

Proof. To give $\ulcorner \mathbf{I}_X \urcorner$ is to give an 'iso-monad' in $\mathbb{K}$ whose multiplication and unit are invertible; for this we take $\mathbf{I}_X \colon X \twoheadrightarrow X$, with multiplication and unit given by

$$\mathfrak{m} = \mathfrak{l}^{-1}_{\mathbf{I}_X} = \mathfrak{r}^{-1}_{\mathbf{I}_X} \colon \mathbf{I}_X \otimes \mathbf{I}_X \to \mathbf{I}_X \quad \text{and} \quad \mathfrak{e} = \mathrm{id}_{\mathbf{I}_X} \colon \mathbf{I}_X \to \mathbf{I}_X. \qquad \square$$

In particular, given a double homomorphism $S \colon \mathbb{K} \to \mathbb{L}$, we can consider the object $\mathrm{id}_{\mathbb{K}} \in [\mathbb{K}, \mathbb{K}]_\psi$, and therefore form the double homomorphism

$$1 \xrightarrow{\ulcorner \mathbf{I}_{\mathrm{id}_{\mathbb{K}}} \urcorner} [\mathbb{K}, \mathbb{K}]_\psi \xrightarrow{S(-)} [\mathbb{K}, \mathbb{L}]_\psi.$$

This gives us a monad $S\mathbf{I}$ in $[\mathbb{K}, \mathbb{L}]_\psi$, and so we can form the slice double category $[\mathbb{K}, \mathbb{L}]_\psi / S\mathbf{I}$. Similarly, we can form the double monad $S\mathbf{I}_1$ corresponding to the homomorphism

$$1 \xrightarrow{\ulcorner \mathbf{I}_1 \urcorner} \mathbb{K} \xrightarrow{S} \mathbb{L},$$

and therefore the slice double category $\mathbb{L}/S\mathbf{I}_1$. Now, since we shall be using the double category $[\mathbb{K},\mathbb{L}]_\psi/S\mathbf{I}$ extensively in the next chapter, it's probably worth giving an elementary description of it here. It has:

- **Objects** (A,α) given by a double homomorphism $A\colon \mathbb{K}\to\mathbb{L}$ together with a vertical transformation $\alpha\colon A\Rightarrow S$;

- **Vertical maps** $\gamma\colon (A,\alpha)\to(B,\beta)$ given by vertical transformations $\gamma\colon A\Rightarrow B$ such that the diagram

$$\begin{tikzcd}
A \arrow[rr, Rightarrow, "\gamma"] \arrow[dr, Rightarrow, "\alpha"'] & & B \arrow[dl, Rightarrow, "\beta"] \\
& S &
\end{tikzcd}$$

  commutes;

- **Horizontal maps** $(\mathbf{A},\boldsymbol{\alpha})\colon (A_s,\alpha_s)\looparrowright(A_t,\alpha_t)$ given by pairs $(\mathbf{A},\boldsymbol{\alpha})$ where $\mathbf{A}$ is a horizontal transformation and $\boldsymbol{\alpha}$ a modification as follows:

$$\begin{tikzcd}
A_s \arrow[r, Rightarrow, "\mathbf{A}", "\shortmid" marking] \arrow[d, Rightarrow, "\alpha_s"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\alpha}"] & A_t \arrow[d, Rightarrow, "\alpha_t"] \\
S \arrow[r, Rightarrow, "S\mathbf{I}"', "\shortmid" marking] & S;
\end{tikzcd}$$

- **Cells**

$$\begin{tikzcd}
(A_s,\alpha_s) \arrow[r, "(\mathbf{A},\boldsymbol{\alpha})", "\shortmid" marking] \arrow[d, "\gamma_s"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\gamma}"] & (A_t,\alpha_t) \arrow[d, "\gamma_t"] \\
(B_s,\beta_s) \arrow[r, "(\mathbf{B},\boldsymbol{\beta})"', "\shortmid" marking] & (B_t,\beta_t)
\end{tikzcd}$$

  are transformations $\boldsymbol{\gamma}\colon \mathbf{A}\Rrightarrow\mathbf{B}$ such that the diagram

$$\begin{tikzcd}
\mathbf{A} \arrow[rr, Rrightarrow, "\boldsymbol{\gamma}"] \arrow[dr, Rrightarrow, "\boldsymbol{\alpha}"'] & & \mathbf{B} \arrow[dl, Rrightarrow, "\boldsymbol{\beta}"] \\
& S\mathbf{I} &
\end{tikzcd}$$

  commutes;

- **Horizontal identities** given on objects (A, α) by

$$\mathbf{I}_{(A,\alpha)} = \mathbf{I}_A \xRightarrow{\mathbf{I}_\alpha} \mathbf{I}_S \xRightarrow{\mathfrak{e}} S\mathbf{I}$$

  (where $\mathfrak{e}$ is the unit of the monad $S\mathbf{I}$, with components $\mathfrak{e}_X \colon \mathbf{I}_{SX} \to S\mathbf{I}_X$), and on maps $\gamma \colon (A, \alpha) \to (B, \beta)$ by

$$\begin{tikzcd}
\mathbf{I}_A \arrow[rr, Rightarrow, "\mathbf{I}_\gamma"] \arrow[dr, Rightarrow, "\mathfrak{e}\circ\mathbf{I}_\alpha"'] & & \mathbf{B} \arrow[dl, Rightarrow, "\mathfrak{e}\circ\mathbf{I}_\beta"] \\
& S\mathbf{I}; &
\end{tikzcd}$$

- **Horizontal composition** given on objects by

$$(\mathbf{A}, \boldsymbol{\alpha}) \otimes (\mathbf{A}', \boldsymbol{\alpha}') = \big(\mathbf{A} \otimes \mathbf{A}' \xRightarrow{\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}'} S\mathbf{I} \otimes S\mathbf{I} \xRightarrow{\mathfrak{m}} S\mathbf{I}\big)$$

  (where $\mathfrak{m}$ is the multiplication of the monad $S\mathbf{I}$, with components

$$\mathfrak{m}_X = S\mathbf{I}_X \otimes S\mathbf{I}_X \xrightarrow{\mathfrak{m}_{\mathbf{I}_X, \mathbf{I}_X}} S(\mathbf{I}_X \otimes \mathbf{I}_X) \xrightarrow{S\mathfrak{l}^{-1}_{\mathbf{I}_X}} S\mathbf{I}_X \ ),$$

  and on maps by

$$\begin{tikzcd}
\mathbf{A} \otimes \mathbf{A}' \arrow[rr, Rightarrow, "\boldsymbol{\gamma}\otimes\boldsymbol{\gamma}'"] \arrow[dr, Rightarrow, "\mathfrak{m}\circ(\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}')"'] & & \mathbf{B} \otimes \mathbf{B}' \arrow[dl, Rightarrow, "\mathfrak{m}\circ(\boldsymbol{\beta}\otimes\boldsymbol{\beta}')"] \\
& S\mathbf{I}. &
\end{tikzcd}$$

4.2 The double category of collections

We should now like to restrict from the full slice category $[\mathbb{K}, \mathbb{L}]_\psi / S\mathbf{I}$ to something aping the category of collections. However, before we can do this, we need to know what we should be restricting to: in other words, we need an analogue of cartesian natural transformation:

Definition 26.

- A vertical transformation $\alpha \colon F \Rightarrow G \colon \mathbb{K} \to \mathbb{L}$ is called a **cartesian vertical transformation** if the natural transformations $\alpha_1 \colon F_1 \Rightarrow G_1$ and $\alpha_0 \colon F_0 \Rightarrow G_0$ are cartesian;

- A modification $\boldsymbol{\gamma}\colon \mathbf{A} \Rrightarrow \mathbf{B}$ is called a **cartesian modification** if γ_s and γ_t are cartesian vertical transformations and the natural transformation $\gamma_c\colon A_c \Rightarrow B_c$ is cartesian.

So, the double category of collections $\mathbb{C}oll(S)$ should have:

- $\mathbb{C}oll(S)_0$ being the full subcategory of $\big([\mathbb{K},\mathbb{L}]_\psi/S\mathbf{I}\big)_0$ whose objects are the *cartesian* vertical transformations into S;
- $\mathbb{C}oll(S)_1$ being the full subcategory of $\big([\mathbb{K},\mathbb{L}]_\psi/S\mathbf{I}\big)_1$ whose objects are the *cartesian* modifications into $S\mathbf{I}$,

with the remaining data inherited from the double category $[\mathbb{K},\mathbb{L}]_\psi/S\mathbf{I}$. In order for this to make sense, we need $\mathbb{C}oll(S)$ to be closed under the horizontal units and composition of $[\mathbb{K},\mathbb{L}]_\psi/S\mathbf{I}$, and this is not automatic. In fact, it requires S to have the following property:

Definition 27. Let $S\colon \mathbb{K} \to \mathbb{L}$ be a double homomorphism; we say that S has **property (hps)** (horizontal pullback stability) if it satisfies:

- **Property (hps1)**: given horizontally composable pullbacks

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathbf{p}_1"] \arrow[d, "\mathbf{p}_2"'] & \mathbf{B} \arrow[d, "\mathbf{f}"] \\ S\mathbf{C} \arrow[r, "S!"'] & S\mathbf{I}_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbf{A}' \arrow[r, "\mathbf{p}_1'"] \arrow[d, "\mathbf{p}_2'"'] & \mathbf{B}' \arrow[d, "\mathbf{f}'"] \\ S\mathbf{C}' \arrow[r, "S!"'] & S\mathbf{I}_1, \end{tikzcd}$$

  in L_1, the diagram

$$\begin{tikzcd} \mathbf{A}' \otimes \mathbf{A} \arrow[r, "\mathbf{p}_1'\otimes\mathbf{p}_1"] \arrow[d, "\mathbf{p}_2'\otimes\mathbf{p}_2"'] & \mathbf{B}' \otimes \mathbf{B} \arrow[d, "\mathbf{f}'\otimes\mathbf{f}"] \\ S\mathbf{C}' \otimes S\mathbf{C} \arrow[r, "S!\otimes S!"'] & S\mathbf{I}_1 \otimes S\mathbf{I}_1 \end{tikzcd}$$

  is a pullback in L_1; and

- **Property (hps2)**: given a pullback

$$\begin{tikzcd} A \arrow[r, "p_1"] \arrow[d, "p_2"'] & B \arrow[d, "f"] \\ SC \arrow[r, "S!"'] & S1 \end{tikzcd}$$

  in L_0, the diagram

$$\begin{tikzcd} \mathbf{I}_A \arrow[r, "\mathbf{I}_{p_1}"] \arrow[d, "\mathbf{I}_{p_2}"'] & \mathbf{I}_B \arrow[d, "\mathbf{I}_f"] \\ \mathbf{I}_{SC} \arrow[r, "\mathbf{I}_{S!}"'] & \mathbf{I}_{S1} \end{tikzcd}$$

  is a pullback in L_1.

Proposition 28. *Given a homomorphism* $S\colon \mathbb{K} \to \mathbb{L}$ *with property (hps), the categories* $\mathbb{C}oll(S)_0$ *and* $\mathbb{C}oll(S)_1$ *provide data for a pseudo double category whose remaining data is inherited from* $[\mathbb{K}, \mathbb{L}]_\psi/S\mathbf{I}$.

Proof. We must check that the horizontal units of $[\mathbb{K}, \mathbb{L}]_\psi/S\mathbf{I}$ are cartesian modifications, and that the horizontal composition of two cartesian modifications is another cartesian modification. For the first of these, given $(A, \alpha) \in \mathbb{C}oll(S)_0$, we have $\mathbf{I}_{(A,\alpha)}$ given by the modification

$$\mathbf{I}_{(A,\alpha)} = \mathbf{I}_A \xRightarrow{\mathbf{I}_\alpha} \mathbf{I}_S \xRightarrow{\mathfrak{e}} S\mathbf{I};$$

so consider the diagram

$$\begin{tikzcd} \mathbf{I}_{AX} \arrow[r, "\mathbf{I}_{A!}"] \arrow[d, "\mathbf{I}_{\alpha_X}"'] & \mathbf{I}_{A1} \arrow[d, "\mathbf{I}_{\alpha_1}"] \\ \mathbf{I}_{SX} \arrow[r, "\mathbf{I}_{S!}"] \arrow[d, "\mathfrak{e}_X"'] & \mathbf{I}_{S1} \arrow[d, "\mathfrak{e}_1"] \\ S\mathbf{I}_X \arrow[r, "S\mathbf{I}_!"'] & S\mathbf{I}_1. \end{tikzcd}$$

It follows from property (hps2) and the cartesianness of α that the top square is a pullback; and the lower square commutes, and so is a pullback since both vertical arrows are isomorphisms. Thus the outer edge is again a pullback, and so $\mathbf{I}_\alpha$ is cartesian as required.

For the second, suppose we are given horizontally composable objects $(\mathbf{A}, \boldsymbol{\alpha})$ and $(\mathbf{B}, \boldsymbol{\beta})$ of $\mathbb{C}oll(S)_1$; we must show that the modification

$$\mathbf{A} \otimes \mathbf{B} \xRightarrow{\boldsymbol{\alpha} \otimes \boldsymbol{\beta}} S\mathbf{I} \otimes S\mathbf{I} \xRightarrow{\mathfrak{m}} S\mathbf{I}$$

is also cartesian. So consider the diagram:

$$\begin{tikzcd}
\mathbf{A}X \otimes \mathbf{B}X \arrow[r, "\mathbf{A}! \otimes \mathbf{B}!"] \arrow[d, "\boldsymbol{\alpha}_X \otimes \boldsymbol{\beta}_X"'] & \mathbf{A}1 \otimes \mathbf{B}1 \arrow[d, "\boldsymbol{\alpha}_1 \otimes \boldsymbol{\beta}_1"] \\
S\mathbf{I}_X \otimes S\mathbf{I}_X \arrow[r, "S\mathbf{I}_! \otimes S\mathbf{I}_!"'] \arrow[d, "\mathfrak{m}_X"'] & S\mathbf{I}_1 \otimes S\mathbf{I}_1 \arrow[d, "\mathfrak{m}_1"] \\
S\mathbf{I}_X \arrow[r, "S\mathbf{I}_!"'] & S\mathbf{I}_1 .
\end{tikzcd}$$

The upper square is a pullback by property (hps2) and the cartesianness of α and β; the lower square commutes and has isomorphisms down the sides, and hence is a pullback. So the outer edge is also a pullback as required. □

4.3 Adjunctions and equivalences

Now, we aim to imitate the equivalence of categories $Coll(S) \simeq \mathbf{D}/S1$ at the pseudo double category level, and to do this, we need a suitable notion of 'equivalence of double categories'. There is an obvious candidate for this, namely equivalence in the 2-category $\mathbf{DblCat}_\psi$; so in this section, we give an elementary characterisation of such equivalences.

In fact, for very little extra effort, we can garner significant extra generality by giving a characterisation of adjunctions in **DblCat**. A well-known result in the theory of monoidal categories [Kel74a] says that to give an adjunction in **MonCat**, the 2-category of monoidal categories, lax monoidal functors and monoidal transformations, is to give an adjunction between the underlying ordinary categories in

Cat for which the left adjoint is strong monoidal.

We shall produce a direct generalisation of this to pseudo double categories, for which we need an analogue of 'underlying ordinary category'; more precisely, we need an appropriate analogue of the 2-category **Cat**:

Definition 29. We write **DblGph** for the 2-category $[\bullet \rightrightarrows \bullet, \mathbf{Cat}]$.

Explicitly, **DblGph** has:

- **Objects** being 'double graphs' $\mathbb{K}$, that is, diagrams of the form $K_1 \overset{s}{\underset{t}{\rightrightarrows}} K_0$ in **Cat**, subject to no further conditions;
- **Maps** $F\colon \mathbb{K} \to \mathbb{L}$ being 'maps of double graphs', that is, pairs of functors $F_0\colon K_0 \to L_0$ and $F_1\colon K_1 \to L_1$ compatible with source and target:
$$sF_1 = F_0 s \quad \text{and} \quad tF_1 = F_0 t;$$
- **2-cells** $\alpha\colon F \Rightarrow G$ being 'transformations of double graphs', that is, pairs of natural transformations $\alpha_0\colon F_0 \Rightarrow G_0$ and $\alpha_1\colon F_1 \Rightarrow G_1$, compatible with source and target:
$$s\alpha_1 = \alpha_0 s \quad \text{and} \quad t\alpha_1 = \alpha_0 t.$$

There is an evident 2-functor $U\colon \mathbf{DblCat} \to \mathbf{DblGph}$ which 'forgets horizontal structure'.

Proposition 30. *Giving an adjunction $F \dashv G\colon \mathbb{L} \to \mathbb{K}$ in* **DblCat** *is equivalent to giving an adjunction $F \dashv G\colon U\mathbb{L} \to U\mathbb{K}$ in* **DblGph** *together with the structure of a double homomorphism on F.*

Let us spell out explicitly what the right hand side of the above amounts to:

- A double homomorphism $F\colon \mathbb{K} \to \mathbb{L}$;
- A map of double graphs $G\colon \mathbb{L} \to \mathbb{K}$;
- Adjunctions $F_0 \dashv G_0$ and $F_1 \dashv G_1$ with unit and counit (η_0, ϵ_0) and (η_1, ϵ_1) respectively,

such that

$$\begin{tikzcd}
L_1 \arrow[r, bend left, "F_1G_1"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow\epsilon_1"] & L_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & L_0 & = & L_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & L_0 \arrow[r, bend left, "F_0G_0"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow\epsilon_0"] & L_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
K_1 \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "G_1F_1"'] \arrow[r, phantom, "\Downarrow\eta_1"] & K_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & K_0 & = & K_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & K_0 \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "G_0F_0"'] \arrow[r, phantom, "\Downarrow\eta_0"] & K_0.
\end{tikzcd}$$

Proof. On an abstract level, this proof runs as follows: the 2-functor $U\colon \mathbf{DblCat} \to \mathbf{DblGph}$ has a left 2-adjoint F, which gives the 'free double category' on a given double graph. Now, the 2-category of strict algebras and strict algebra maps for the induced monad UF on **DblGph** is precisely the 2-category of *strict* double categories, whilst the 2-category of *pseudo*-algebras and lax algebra maps is *almost* the 2-category **DblCat**; more precisely, it is the 2-category of 'unbiased' (in the sense of [Lei04a]) pseudo double categories, which come equipped with n-ary horizontal composition functors for all n. As in the bicategorical case, it is not too hard to show that this notion is essentially equivalent to the 'biased' notion of pseudo double category that we have adopted.

Now, the 2-category **DblGph** is complete and cocomplete as a 2-category, and hence by Section 6.4 of [BKP89], there is a 2-monad T' on **DblGph** whose *strict* algebras are precisely the *pseudo* algebras for the composite monad $T = UF$. Thus, we have a 2-monad T' on **DblGph** whose category of strict algebras and lax algebra maps can be identified with **DblCat**.

But now we are in a position to apply Kelly's 'doctrinal adjunction'; by Theorem 1.5 of [Kel74a], to give an adjunction in **DblCat** is precisely to give an adjunction between the underlying objects of **DblGph** for which the left adjoint is a pseudo map of T'-algebras; and to give such a map is essentially the same thing as giving a homomorphism of pseudo double categories.

Now, there are many details missing from the above, and rather than attempt to fill them in, it will be easier to give a direct proof following [Kel74a]. So, suppose first we are given an adjunction $UF \dashv UG$ in **DblGph** for which the left adjoint is a double homomorphism; then it suffices to equip G with data (DMD3) and (DMD4), satisfying (DMA2) and (DMA3), and to show that $\eta = (\eta_0, \eta_1)$ and

$\epsilon = (\epsilon_0, \epsilon_1)$ satisfy (VTA2) with respect to it. So, suppose that F has comparison transformations

$$\begin{tikzcd}
K_1 {}_s\!\times_t K_1 \arrow[r, "F_{1\,s}\times_t F_1"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Downarrow\mathfrak{m}"] & L_1 {}_s\!\times_t L_1 \arrow[d, "\otimes"] & \text{and} & K_0 \arrow[r, "F_0"] \arrow[d, "\mathbf{I}"'] \arrow[dr, phantom, "\Downarrow\mathfrak{e}"] & L_0 \arrow[d, "\mathbf{I}"] \\
K_1 \arrow[r, "F_1"'] & L_1 & & K_1 \arrow[r, "F_1"'] & L_1
\end{tikzcd}$$

Then we give the comparison transformations for G as the mates

$$\begin{tikzcd}
L_1 {}_s\!\times_t L_1 \arrow[r, "G_{1\,s}\times_t G_1"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Downarrow\overline{\mathfrak{m}^{-1}}"] & K_1 {}_s\!\times_t K_1 \arrow[d, "\otimes"] & \text{and} & L_0 \arrow[r, "G_0"] \arrow[d, "\mathbf{I}"'] \arrow[dr, phantom, "\Downarrow\overline{\mathfrak{e}^{-1}}"] & K_0 \arrow[d, "\mathbf{I}"] \\
L_1 \arrow[r, "G_1"'] & K_1 & & L_1 \arrow[r, "G_1"'] & K_1
\end{tikzcd}$$

of $\mathfrak{m}^{-1}$ and $\mathfrak{e}^{-1}$ under the adjunctions $F_0 \dashv G_0$, $F_1 \dashv G_1$ and $F_1 {}_s\!\times_t F_1 \dashv G_1 {}_s\!\times_t G_1$. Explicitly, the components of these transformations at $(\mathbf{X}, \mathbf{Y})$ and X respectively are given as follows:-

$$\begin{tikzcd}
G\mathbf{X} \otimes G\mathbf{Y} \arrow[d, "\eta_{G\mathbf{X}\otimes G\mathbf{Y}}"] & & \mathbf{I}_{GX} \arrow[d, "\eta_{\mathbf{I}_{GX}}"] \\
GF(G\mathbf{X} \otimes G\mathbf{Y}) \arrow[d, "G\mathfrak{m}^{-1}_{G\mathbf{X},G\mathbf{Y}}"] & & GF(\mathbf{I}_{GX}) \arrow[d, "G\mathfrak{e}_X^{-1}"] \\
G(FG\mathbf{X} \otimes FG\mathbf{Y}) \arrow[d, "G(\epsilon_{\mathbf{X}}\otimes\epsilon_{\mathbf{Y}})"] & \text{and} & GFG\mathbf{I}_X \arrow[d, "G\epsilon_{\mathbf{I}_X}"] \\
G(\mathbf{X} \otimes \mathbf{Y}) & & G\mathbf{I}_X.
\end{tikzcd}$$

That this data satisfies (DMA2) and (DMA3) follows automatically from (DMA2) and (DMA3) for F and the functoriality of mates, and it's now a straightforward exercise in the calculus of mates, following [Kel74a], to show that $\eta = (\eta_0, \eta_1)$ and $\epsilon = (\epsilon_0, \epsilon_1)$ satisfy (VTA2) with respect to this data. Thus G becomes a double morphism and η and ϵ become vertical transformations, and so we can conclude that we have an adjunction in **DblCat** as required.

Conversely, any adjunction (F, G, η, ϵ) in **DblCat** gives rise to the data specified above; we need only check that F is a homomorphism, i.e., that its special comparison maps are invertible. Suppose that the comparison maps for G are $\mathfrak{m}'$ and $\mathfrak{e}'$; then it's easy to check that their mates $\overline{\mathfrak{m}'}$ and $\overline{\mathfrak{e}'}$ furnish us with inverses for $\mathfrak{m}'$ and $\mathfrak{e}'$ (explicitly, these inverses are given by

$$\begin{tikzcd}
F(\mathbf{X} \otimes \mathbf{Y}) \arrow[d, "F(\eta_{\mathbf{X}} \otimes \eta_{\mathbf{Y}})"] & & F\mathbf{I}_X \arrow[d, "F\mathbf{I}_{\eta_X}"] \\
F(GF\mathbf{X} \otimes GF\mathbf{Y}) \arrow[d, "F\mathfrak{m}'_{F\mathbf{X},F\mathbf{Y}}"] & \text{and} & F\mathbf{I}_{GFX} \arrow[d, "F\mathfrak{e}'_X"] \\
FG(F\mathbf{X} \otimes F\mathbf{Y}) \arrow[d, "\epsilon_{F\mathbf{X} \otimes F\mathbf{Y}}"] & & FG\mathbf{I}_{FX} \arrow[d, "\epsilon_{\mathbf{I}_{FX}}"] \\
F\mathbf{X} \otimes F\mathbf{Y} & & \mathbf{I}_{FX}\text{.)}
\end{tikzcd}$$

The only thing remaining to check is that these two processes are mutually inverse. Suppose we are given an adjunction (F, G, η, ϵ) in **DblCat**; then we must show that we can reconstruct this adjunction from the underlying adjunction in **DblGph** together with the data for F.

This amounts to checking that the special comparison maps we produce for G are the ones we started with; but this is immediate, since we take them to be $\overline{\mathfrak{m}^{-1}}$ and $\overline{\mathfrak{e}^{-1}}$, which are $\overline{\overline{\mathfrak{m}'}} = \mathfrak{m}'$ and $\overline{\overline{\mathfrak{e}'}} = \mathfrak{e}'$ as required. ☐

Corollary 31. *Suppose we are given double categories $\mathbb{K}$ and $\mathbb{L}$, and:*

- *A double homomorphism $F \colon \mathbb{K} \to \mathbb{L}$;*
- *A map of double graphs $G \colon \mathbb{L} \to \mathbb{K}$*

together with natural isomorphisms $\eta_i \colon \mathrm{id}_{K_i} \cong G_iF_i$ and $\epsilon_i \colon F_iG_i \cong \mathrm{id}_{K_i}$ ($i = 0, 1$), such that

$$\begin{tikzcd}
L_1 \arrow[r, bend left, "F_1G_1"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow\epsilon_1"] & L_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & L_0 & = & L_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & L_0 \arrow[r, bend left, "F_0G_0"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow\epsilon_0"] & L_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
K_1 \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "G_1F_1"'] \arrow[r, phantom, "\Downarrow\eta_1"] & K_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & K_0 & = & K_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & K_0 \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "G_0F_0"'] \arrow[r, phantom, "\Downarrow\eta_0"] & K_0.
\end{tikzcd}$$

Then $\mathbb{K}$ and $\mathbb{L}$ are equivalent in $\mathbf{DblCat}_\psi$.

Proof. To give this data is to give an equivalence in **DblGph**, so by replacing ϵ_1 and ϵ_0, we can make this into an *adjoint* equivalence in **DblGph**. Now, applying the previous result, we get an (adjoint) equivalence in **DblCat**; but now we note that the comparison special maps for G will be invertible, since they are constructed from a composite of invertible maps, and hence that our equivalence is an equivalence in $\mathbf{DblCat}_\psi$ as well. ☐

Chapter 5

Pseudo double categories III

We now move on to give double category analogues of the remaining notions listed in the previous chapter We begin by defining 'monoidal double category' and 'monoidal double morphism'. From an abstract viewpoint, we can view these as being derived from the theory of *bicategories enriched in a monoidal bicategory* as developed by [Car95] and [Lac95]: indeed, a monoidal double category can be seen as a one-object bicategory enriched in the (cartesian) monoidal bicategory $\mathbf{DblCat}_\psi$, and a monoidal double morphism as a suitable map between such.

Next, we show that the 'endohom' double category $[\mathbb{K}, \mathbb{K}]_\psi$ is a canonical example of a monoidal double category. Again, there is a more abstract view available: recalling the remarks following Proposition 11, the monoidal bicategory $\mathbf{DblCat}_\psi$ is in fact a *biclosed* monoidal bicategory, and thus becomes a monoidal bicategory 'enriched over itself' (see [Lac95]). From this viewpoint, the double category $[\mathbb{K}, \mathbb{K}]_\psi$ automatically acquires a monoidal structure, since it is the hom-object from $\mathbb{K}$ to $\mathbb{K}$ in the $\mathbf{DblCat}_\psi$-enriched bicategory $\mathbf{DblCat}_\psi$. However, we shall not pursue this more abstract level here, partly for reasons of brevity, and partly because the extra abstraction would be more of a hindrance than a help later on, when we will need to utilise these constructions in a hands-on manner.

Lastly in this chapter, we deal with the notion of 'monoidal slice double category' and its more general relative, the 'monoidal comma double category'. In particular, we shall see that, given a 'monad on a pseudo double category $\mathbb{K}$', by which we mean a monad (S, η, μ) on $\mathbb{K}$ in the 2-category $\mathbf{DblCat}_\psi$, the slice double category $[\mathbb{K}, \mathbb{K}]_\psi / S\mathbf{I}$ acquires a natural structure of monoidal double category.

5.1 Monoidal double categories

Recall that the 2-category $\mathbf{DblCat}_\psi$ has finite products, given in the obvious way, and hence is a (cartesian) monoidal bicategory. Thus we can define:

Definition 32. A **monoidal double category** is a pseudomonoid in $\mathbf{DblCat}_\psi$.

Proposition 33. *Giving a monoidal double category* $\mathbb{K}$ *is equivalent to giving a double category* $\mathbb{K}$ *such that*

- K_0 *is a (not necessarily strict) monoidal category, with data* $(\bullet_0, \ulcorner e \urcorner, \alpha_0, \lambda_0, \rho_0)$*;*
- K_1 *is a (not necessarily strict) monoidal category, with data* $(\bullet_1, \ulcorner \mathbf{e} \urcorner, \alpha_1, \lambda_1, \rho_1)$*;*
- *The functors* s *and* $t\colon K_1 \to K_0$ *are strict monoidal;*
- *The functors* $\mathbf{I}\colon K_0 \to K_1$ *and* $\otimes\colon K_1 \,{}_s\!\times_t K_1 \to K_1$ *are strong monoidal (where we observe that* $K_1 \,{}_s\!\times_t K_1$ *acquires a monoidal structure via pullback along the strict monoidal functors* s *and* t*);*
- *The associativity and unitality natural transformations* $\mathfrak{a}$*,* $\mathfrak{l}$ *and* $\mathfrak{r}$ *for* $\mathbb{K}$ *are monoidal natural transformations.*

Proof. Giving a pseudomonoid in $\mathbf{DblCat}_\psi$ is equivalent to giving a double category $\mathbb{K}$ equipped with homomorphisms

$$\bullet\colon \mathbb{K}\times\mathbb{K} \to \mathbb{K} \quad \text{and} \quad e\colon 1 \to \mathbb{K}$$

and vertical transformations

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "e\times\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] \arrow[dr, phantom, shift left=3ex, "\overset{\lambda}{\Rightarrow}"] & \mathbb{K}\times\mathbb{K} \arrow[d, "\bullet"] & \mathbb{K} \arrow[l, "\mathrm{id}\times e"'] \arrow[dl, "\mathrm{id}"] \arrow[dl, phantom, shift right=3ex, "\overset{\rho}{\Leftarrow}"'] \\
& \mathbb{K} &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{K}\times\mathbb{K}\times\mathbb{K} \arrow[r, "\mathrm{id}\times\otimes"] \arrow[d, "\otimes\times\mathrm{id}"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathbb{K}\times\mathbb{K} \arrow[d, "\bullet"] \\
\mathbb{K}\times\mathbb{K} \arrow[r, "\bullet"'] & \mathbb{K}
\end{tikzcd}$$

satisfying pseudomonoid coherence laws. Let us work through the data and axioms that this involves:

- The data (DMD1) and (DMD2) for $\bullet$ and e and the data (VTD1) and (VTD2) for α, ρ and λ together with the pseudomonoid coherence laws are equivalent to giving data

$$(K_0, \bullet_0, \ulcorner e \urcorner, \alpha_0, \lambda_0, \rho_0) \quad \text{and} \quad (K_1, \bullet_1, \ulcorner \mathbf{e} \urcorner, \alpha_1, \lambda_1, \rho_1)$$

for two monoidal categories;

- The axiom (DMA1) for $\bullet$ and e and the axiom (VTA1) for α, ρ and λ amount to the following equalities:

$$\begin{aligned}
s(\mathbf{X} \bullet_1 \mathbf{Y}) &= X_s \bullet_0 Y_s & t(\mathbf{X} \bullet_1 \mathbf{Y}) &= X_t \bullet_0 Y_t \\
s(\mathbf{F} \bullet_1 \mathbf{G}) &= F_s \bullet_0 G_s & t(\mathbf{F} \bullet_1 \mathbf{G}) &= F_t \bullet_0 G_t \\
s(\mathbf{e}) &= e & t(\mathbf{e}) &= e \\
s\big((\alpha_1)_{\mathbf{X},\mathbf{Y},\mathbf{Z}}\big) &= (\alpha_0)_{X_s,Y_s,Z_s} & t\big((\alpha_1)_{\mathbf{X},\mathbf{Y},\mathbf{Z}}\big) &= (\alpha_0)_{X_t,Y_t,Z_t} \\
s\big((\lambda_1)_{\mathbf{X}}\big) &= (\lambda_0)_{X_s} & t\big((\lambda_1)_{\mathbf{X}}\big) &= (\lambda_0)_{X_t} \\
s\big((\rho_1)_{\mathbf{X}}\big) &= (\rho_0)_{X_s} & t\big((\rho_1)_{\mathbf{X}}\big) &= (\rho_0)_{X_t}
\end{aligned}$$

  which say that s and t are strict monoidal functors;

- The data (DMD3) and (DMD4) for $\bullet$ and e amount to giving invertible special maps

$$k_{\mathbf{W},\mathbf{X},\mathbf{Y},\mathbf{Z}} \colon (\mathbf{W} \bullet \mathbf{X}) \otimes (\mathbf{Y} \bullet \mathbf{Z}) \to (\mathbf{W} \otimes \mathbf{Y}) \bullet (\mathbf{X} \otimes \mathbf{Z})$$

  in K_1, natural in $(\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \in (K_1 \times K_1) \;{}_{s \times s}{\times}_{t \times t}\; (K_1 \times K_1)$, invertible special maps

$$u_{X,Y} \colon \mathbf{I}_{W \bullet X} \to \mathbf{I}_W \bullet \mathbf{I}_X$$

  in K_1, natural in $(X, Y) \in K_0 \times K_0$, and invertible special maps

$$k_{\mathbf{e}} \colon \mathbf{e} \otimes \mathbf{e} \to \mathbf{e} \quad \text{and} \quad u_e \colon \mathbf{I}_e \to \mathbf{e},$$

  whilst the axioms (VTA2) and (VTA3) for α, ρ and λ amount to the commutativity of the following diagrams:

$$\begin{tikzcd}
\big(\mathbf{X}\bullet(\mathbf{Y}\bullet\mathbf{Z})\big)\otimes\big(\mathbf{X}'\bullet(\mathbf{Y}'\bullet\mathbf{Z}')\big) \arrow[r, "{\alpha_{\mathbf{X},\mathbf{Y},\mathbf{Z}}\otimes\alpha_{\mathbf{X}',\mathbf{Y}',\mathbf{Z}'}}"] \arrow[d, "{k_{\mathbf{X},(\mathbf{Y}\bullet\mathbf{Z}),\mathbf{X}',(\mathbf{Y}'\bullet\mathbf{Z}')}}"'] & \big((\mathbf{X}\bullet\mathbf{Y})\bullet\mathbf{Z}\big)\otimes\big((\mathbf{X}'\bullet\mathbf{Y}')\bullet\mathbf{Z}'\big) \arrow[d, "{k_{(\mathbf{X}\bullet\mathbf{Y}),\mathbf{Z},(\mathbf{X}'\bullet\mathbf{Y}'),\mathbf{Z}'}}"] \\
\big(\mathbf{X}\otimes\mathbf{X}'\big)\bullet\big((\mathbf{Y}\bullet\mathbf{Z})\otimes(\mathbf{Y}'\bullet\mathbf{Z}')\big) \arrow[d, "{(\mathbf{X}\otimes\mathbf{X}')\bullet k_{\mathbf{Y},\mathbf{Z},\mathbf{Y}',\mathbf{Z}'}}"'] & \big((\mathbf{X}\bullet\mathbf{Y})\otimes(\mathbf{X}'\bullet\mathbf{Y}')\big)\bullet\big(\mathbf{Z}\otimes\mathbf{Z}'\big) \arrow[d, "{k_{\mathbf{X},\mathbf{Y},\mathbf{X}',\mathbf{Y}'}\bullet(\mathbf{Z}\otimes\mathbf{Z}')}"] \\
\big(\mathbf{X}\otimes\mathbf{X}'\big)\bullet\big((\mathbf{Y}\otimes\mathbf{Y}')\bullet(\mathbf{Z}\otimes\mathbf{Z}')\big) \arrow[r, "{\alpha_{(\mathbf{X}\otimes\mathbf{X}'),(\mathbf{Y}\otimes\mathbf{Y}'),(\mathbf{Z}\otimes\mathbf{Z}')}}"'] & \big((\mathbf{X}\otimes\mathbf{X}')\bullet(\mathbf{Y}\otimes\mathbf{Y}')\big)\bullet\big(\mathbf{Z}\otimes\mathbf{Z}'\big)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{I}_{X\bullet(Y\bullet Z)} \arrow[r, "\mathbf{I}_{\alpha_{X,Y,Z}}"] \arrow[d, "{u_{X,Y\bullet Z}}"'] & \mathbf{I}_{(X\bullet Y)\bullet Z} \arrow[d, "{u_{X\bullet Y,Z}}"] \\
\mathbf{I}_X\bullet\mathbf{I}_{Y\bullet Z} \arrow[d, "{\mathbf{I}_X\bullet u_{Y,Z}}"'] & \mathbf{I}_{X\bullet Y}\bullet\mathbf{I}_Z \arrow[d, "{u_{X,Y}\bullet\mathbf{I}_Z}"] \\
\mathbf{I}_X\bullet(\mathbf{I}_Y\bullet\mathbf{I}_Z) \arrow[r, "{\alpha_{\mathbf{I}_X,\mathbf{I}_Y,\mathbf{I}_Z}}"'] & (\mathbf{I}_X\bullet\mathbf{I}_Y)\bullet\mathbf{I}_Z
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{X}\otimes\mathbf{Y} \arrow[r, "\lambda_{\mathbf{X}}\otimes\lambda_{\mathbf{Y}}"] \arrow[d, "\lambda_{\mathbf{X}\otimes\mathbf{Y}}"'] & (\mathbf{e}\bullet\mathbf{X})\otimes(\mathbf{e}\bullet\mathbf{Y}) \arrow[d, "{k_{\mathbf{e},\mathbf{X},\mathbf{e},\mathbf{Y}}}"] \\
\mathbf{e}\bullet(\mathbf{X}\otimes\mathbf{Y}) & (\mathbf{e}\otimes\mathbf{e})\bullet(\mathbf{X}\otimes\mathbf{Y}) \arrow[l, "u_{\mathbf{e}}\bullet(\mathbf{X}\otimes Y)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{X}\otimes\mathbf{Y} \arrow[r, "\rho_{\mathbf{X}}\otimes\rho_{\mathbf{Y}}"] \arrow[d, "\rho_{\mathbf{X}\otimes\mathbf{Y}}"'] & (\mathbf{X}\bullet\mathbf{e})\otimes(\mathbf{Y}\bullet\mathbf{e}) \arrow[d, "{k_{\mathbf{X},\mathbf{e},\mathbf{Y},\mathbf{e}}}"] \\
(\mathbf{X}\otimes\mathbf{Y})\bullet\mathbf{e} & (\mathbf{X}\otimes\mathbf{Y})\bullet(\mathbf{e}\otimes\mathbf{e}) \arrow[l, "(\mathbf{X}\otimes Y)\bullet u_{\mathbf{e}}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{I}_X \arrow[r, "\mathbf{I}_{\lambda_X}"] \arrow[d, "\lambda_{\mathbf{I}_X}"'] & \mathbf{I}_{e\bullet X} \arrow[d, "{u_{e,X}}"] \\
\mathbf{e}\bullet\mathbf{I}_X & \mathbf{I}_e\bullet\mathbf{I}_X \arrow[l, "u_e\bullet\mathbf{I}_X"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{I}_X \arrow[r, "\mathbf{I}_{\rho_X}"] \arrow[d, "\rho_{\mathbf{I}_X}"'] & \mathbf{I}_{X\bullet e} \arrow[d, "{u_{X,e}}"] \\
\mathbf{I}_X\bullet\mathbf{e} & \mathbf{I}_X\bullet\mathbf{I}_e. \arrow[l, "\mathbf{I}_X\bullet u_e"]
\end{tikzcd}$$

Taken together, this data is equivalent to saying that $\mathbf{I}$ and $\otimes$ are strong monoidal functors with respect to the monoidal structures on K_0 and K_1. (To be precise, the information presented above says that $\mathbf{I}$ and $\otimes$ are strong *op*monoidal functors; but this is an equivalent notion, since we may pass from k and u to k^{-1} and u^{-1}.)

- The axioms (DMA2) and (DMA3) for $\bullet$ and e amount to the commutativity of the following diagrams:

$$\begin{tikzcd}
\mathbf{X} \bullet \mathbf{Y} \arrow[r, "\mathfrak{l}_{\mathbf{X} \bullet \mathbf{Y}}"] \arrow[d, "\mathfrak{l}_{\mathbf{X}} \bullet \mathfrak{l}_{\mathbf{Y}}"'] & \mathbf{I}_{X_t \bullet Y_t} \otimes (\mathbf{X} \bullet \mathbf{Y}) \arrow[d, "u_{X_t, Y_t} \otimes (\mathbf{X} \bullet \mathbf{Y})"] \\
(\mathbf{I}_{X_t} \otimes \mathbf{X}) \bullet (\mathbf{I}_{Y_t} \otimes \mathbf{Y}) & (\mathbf{I}_{X_t} \bullet \mathbf{I}_{Y_t}) \otimes (\mathbf{X} \bullet \mathbf{Y}) \arrow[l, "k_{\mathbf{I}_{X_t}, \mathbf{I}_{Y_t}, \mathbf{X}, \mathbf{Y}}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{X} \bullet \mathbf{Y} \arrow[r, "\mathfrak{r}_{\mathbf{X} \bullet \mathbf{Y}}"] \arrow[d, "\mathfrak{r}_{\mathbf{X}} \bullet \mathfrak{r}_{\mathbf{Y}}"'] & (\mathbf{X} \bullet \mathbf{Y}) \otimes \mathbf{I}_{X_s \bullet Y_s} \arrow[d, "(\mathbf{X} \bullet \mathbf{Y}) \otimes u_{X_s, Y_s}"] \\
(\mathbf{X} \otimes \mathbf{I}_{X_s}) \bullet (\mathbf{Y} \otimes \mathbf{I}_{Y_s}) & (\mathbf{X} \bullet \mathbf{Y}) \otimes (\mathbf{I}_{X_s} \bullet \mathbf{I}_{Y_s}) \arrow[l, "k_{\mathbf{X}, \mathbf{Y}, \mathbf{I}_{X_s}, \mathbf{I}_{Y_s}}"]
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathbf{X} \bullet \mathbf{X}') \otimes \big((\mathbf{Y} \bullet \mathbf{Y}') \otimes (\mathbf{Z} \bullet \mathbf{Z}')\big) \arrow[r, "\mathfrak{a}_{\mathbf{X} \bullet \mathbf{X}', \mathbf{Y} \bullet \mathbf{Y}', \mathbf{Z} \bullet \mathbf{Z}'}"] \arrow[d, "(\mathbf{X} \bullet \mathbf{X}') \otimes k_{\mathbf{Y}, \mathbf{Y}', \mathbf{Z}, \mathbf{Z}'}"'] & \big((\mathbf{X} \bullet \mathbf{X}') \otimes (\mathbf{Y} \bullet \mathbf{Y}')\big) \otimes (\mathbf{Z} \bullet \mathbf{Z}') \arrow[d, "k_{\mathbf{X}, \mathbf{X}', \mathbf{Y}, \mathbf{Y}'} \otimes (\mathbf{Z} \bullet \mathbf{Z}')"] \\
(\mathbf{X} \bullet \mathbf{X}') \otimes \big((\mathbf{Y} \otimes \mathbf{Z}) \bullet (\mathbf{Y}' \otimes \mathbf{Z}')\big) \arrow[d, "k_{\mathbf{X}, \mathbf{X}', (\mathbf{Y} \otimes \mathbf{Z}), (\mathbf{Y}' \otimes \mathbf{Z}')}"'] & \big((\mathbf{X} \otimes \mathbf{Y}) \bullet (\mathbf{X}' \otimes \mathbf{Y}')\big) \otimes (\mathbf{Z} \bullet \mathbf{Z}') \arrow[d, "k_{(\mathbf{X} \otimes \mathbf{Y}), (\mathbf{X}' \otimes \mathbf{Y}'), \mathbf{Z}, \mathbf{Z}'}"] \\
\big(\mathbf{X} \otimes (\mathbf{Y} \otimes \mathbf{Z})\big) \bullet \big(\mathbf{X}' \otimes (\mathbf{Y}' \otimes \mathbf{Z}')\big) \arrow[r, "\mathfrak{a}_{\mathbf{X}, \mathbf{Y}, \mathbf{Z}} \bullet \mathfrak{a}_{\mathbf{X}', \mathbf{Y}', \mathbf{Z}'}"'] & \big((\mathbf{X} \otimes \mathbf{Y}) \otimes \mathbf{Z}\big) \bullet \big((\mathbf{X}' \otimes \mathbf{Y}') \otimes \mathbf{Z}'\big)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{e} \arrow[r, "\mathfrak{l}_{\mathbf{e}}"] \arrow[d, "\mathrm{id}_{\mathbf{e}}"'] & \mathbf{I}_e \otimes \mathbf{e} \arrow[d, "u_e \otimes \mathbf{e}"] & & \mathbf{e} \arrow[r, "\mathfrak{r}_{\mathbf{e}}"] \arrow[d, "\mathrm{id}_{\mathbf{e}}"'] & \mathbf{e} \otimes \mathbf{I}_e \arrow[d, "\mathbf{e} \otimes u_e"] \\
\mathbf{e} & \mathbf{e} \otimes \mathbf{e} \arrow[l, "k_{\mathbf{e}}"] & & \mathbf{e} & \mathbf{e} \otimes \mathbf{e} \arrow[l, "k_{\mathbf{e}}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{e} \otimes (\mathbf{e} \otimes \mathbf{e}) \arrow[rr, "\mathfrak{a}_{\mathbf{e}, \mathbf{e}, \mathbf{e}}"] \arrow[d, "\mathbf{e} \otimes k_{\mathbf{e}}"'] & & (\mathbf{e} \otimes \mathbf{e}) \otimes \mathbf{e} \arrow[d, "k_{\mathbf{e}} \otimes \mathbf{e}"'] \\
\mathbf{e} \otimes \mathbf{e} \arrow[dr, "k_{\mathbf{e}}"'] & & \mathbf{e} \otimes \mathbf{e} \arrow[dl, "k_{\mathbf{e}}"] \\
& \mathbf{e} &
\end{tikzcd}$$

which taken together say that $\mathfrak{a}$, $\mathfrak{l}$ and $\mathfrak{r}$ are strong monoidal transformations with respect to the data given above. (Again, more precisely this data says that they are strong opmonoidal transformations; but as before these are equivalent notions, since we may pass from k and u to k^{-1} and u^{-1}.) ☐

5.2 Monoidal double morphisms

We now consider the most appropriate notion of map between monoidal double categories. Such a map need not be a *homo*morphism of pseudo double categories, and therefore it will not do to ask for a map of pseudomonoids in $\mathbf{DblCat}_\psi$.

However, we observe that the 2-category **DblCat** also has finite products, and that the inclusion $\mathbf{DblCat}_\psi \to \mathbf{DblCat}$ preserves them. Hence we can view a monoidal double category *a fortiori* as a pseudomonoid in **DblCat**, and thus define:

Definition 34. A **monoidal double morphism** between monoidal double categories $\mathbb{K}$ and $\mathbb{L}$ is a (lax) map of pseudomonoids $\mathbb{K} \to \mathbb{L}$ in **DblCat**.

Proposition 35. *Giving a monoidal double morphism $F\colon \mathbb{K} \to \mathbb{L}$ is equivalent to giving a double morphism $F\colon \mathbb{K} \to \mathbb{L}$ such that*

- *F_0 and F_1 are lax monoidal functors;*
- *The equalities $sF_1 = F_0 s$ and $tF_1 = F_0 t$ hold as equalities of lax monoidal functors;*
- *The natural transformations*

  $$\mathfrak{m}\colon F_1(\text{–}) \otimes F_1(?) \to F_1(\text{–} \otimes ?) \quad \textit{and} \quad \mathfrak{e}\colon \mathbf{I}_{F_0(\text{–})} \to F_1(\mathbf{I}_{(\text{–})})$$

  are lax monoidal natural transformations (where we observe that all the functors in question are indeed lax monoidal functors; for instance, $F_1(\text{–}) \otimes F_1(?)$ is the composite

  $$K_1 \, {}_s{\times}_t \, K_1 \xrightarrow{F_1 {}_s{\times}_t F_1} L_1 \, {}_s{\times}_t \, L_1 \xrightarrow{\otimes} L_1$$

  which is the composite of a lax monoidal and a strong monoidal functor as required).

Proof. Giving a lax map of pseudomonoids $\mathbb{K} \to \mathbb{L}$ in **DblCat** is equivalent to giving a morphism of double categories $F\colon \mathbb{K} \to \mathbb{L}$ equipped with vertical transformations:

$$\begin{tikzcd}
1 \arrow[r, equal] \arrow[d, "e"'] \arrow[dr, phantom, "\Downarrow m_e"] & 1 \arrow[d, "e"] \\
\mathbb{K} \arrow[r, "F"'] & \mathbb{L}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{K} \times \mathbb{K} \arrow[r, "F \times F"] \arrow[d, "\bullet"'] \arrow[dr, phantom, "\Downarrow m"] & \mathbb{L} \times \mathbb{L} \arrow[d, "\bullet"] \\
\mathbb{K} \arrow[r, "F"'] & \mathbb{L},
\end{tikzcd}$$

satisfying pseudomonoid map coherence axioms. Let us work through this data and see what it amounts to:

- The data (DMD1) and (DMD2) for F, the data (VTD1) and (VTD2) for m_e and m, and the pseudomonoid map coherence laws are together equivalent to giving coherent data

  $$(F_0, m_e, m_0) \quad \text{and} \quad (F_1, m_{\mathbf{e}}, m_1)$$

  for lax monoidal functors $F_0 \colon K_0 \to L_0$ and $F_1 \colon K_1 \to L_1$ respectively;

- The axiom (VTA1) for m_e and m corresponds to the equalities

  $$\begin{aligned} s\big((m_1)_{\mathbf{X},\mathbf{Y}}\big) &= (m_0)_{X_s,Y_s} & \qquad t\big((m_1)_{\mathbf{X},\mathbf{Y}}\big) &= (m_0)_{X_t,Y_t} \\ s(m_{\mathbf{e}}) &= m_e & t(m_{\mathbf{e}}) &= m_e \end{aligned}$$

  which say that the equalities $sF_1 = F_0 s$ and $tF_1 = F_0 t$ hold as equalities of lax monoidal functors.

- The axioms (VTA2) and (VTA3) for m_e and m correspond to the commutativity of the following diagrams:

$$\begin{tikzcd}
(F\mathbf{W} \bullet F\mathbf{X}) \otimes (F\mathbf{Y} \bullet F\mathbf{Z}) \arrow[r, "k_{F\mathbf{W},F\mathbf{X},F\mathbf{Y},F\mathbf{Z}}"] \arrow[d, "m_{\mathbf{W},\mathbf{X}} \otimes m_{\mathbf{Y},\mathbf{Z}}"'] & (F\mathbf{W} \otimes F\mathbf{Y}) \bullet (F\mathbf{X} \otimes F\mathbf{Z}) \arrow[d, "\mathfrak{m}_{\mathbf{W},\mathbf{Y}} \bullet \mathfrak{m}_{\mathbf{X},\mathbf{Z}}"] \\
F(\mathbf{W} \bullet \mathbf{X}) \otimes F(\mathbf{Y} \bullet \mathbf{Z}) \arrow[d, "\mathfrak{m}_{\mathbf{W}\bullet\mathbf{X},\mathbf{Y}\bullet\mathbf{Z}}"'] & F(\mathbf{W} \otimes \mathbf{Y}) \bullet F(\mathbf{X} \otimes \mathbf{Z}) \arrow[d, "m_{\mathbf{W}\otimes\mathbf{Y},\mathbf{X}\otimes\mathbf{Z}}"] \\
F\big((\mathbf{W} \bullet \mathbf{X}) \otimes (\mathbf{Y} \bullet \mathbf{Z})\big) \arrow[r, "Fk_{\mathbf{W},\mathbf{X},\mathbf{Y},\mathbf{Z}}"'] & F\big((\mathbf{W} \otimes \mathbf{Y}) \bullet (\mathbf{X} \otimes \mathbf{Z})\big)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{I}_{FX\bullet FY} \arrow[r, "u_{FX,FY}"] \arrow[d, "\mathbf{I}_{m_{X,Y}}"'] & \mathbf{I}_{FX} \bullet \mathbf{I}_{FY} \arrow[d, "\mathfrak{e}_X \bullet \mathfrak{e}_Y"] \\
\mathbf{I}_{F(X\bullet Y)} \arrow[d, "\mathfrak{e}_{X\bullet Y}"'] & F\mathbf{I}_X \bullet F\mathbf{I}_Y \arrow[d, "m_{\mathbf{I}_X,\mathbf{I}_Y}"] \\
F\mathbf{I}_{X\bullet Y} \arrow[r, "Fu_{X,Y}"'] & F(\mathbf{I}_X \bullet \mathbf{I}_Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{e} \otimes \mathbf{e} \arrow[r, "k_{\mathbf{e}}"] \arrow[d, "m_{\mathbf{e}} \otimes m_{\mathbf{e}}"'] & \mathbf{e} \arrow[dd, "m_{\mathbf{e}}"] \\
F\mathbf{e} \otimes F\mathbf{e} \arrow[d, "\mathfrak{m}_{\mathbf{e},\mathbf{e}}"'] & \\
F(\mathbf{e} \otimes \mathbf{e}) \arrow[r, "Fk_{\mathbf{e}}"'] & F\mathbf{e}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{I}_e \arrow[r, "u_e"] \arrow[d, "\mathbf{I}_{m_e}"'] & \mathbf{e} \arrow[dd, "m_{\mathbf{e}}"] \\
\mathbf{I}_{Fe} \arrow[d, "\mathfrak{e}_e"'] & \\
F\mathbf{I}_e \arrow[r, "Fu_e"'] & F\mathbf{e}
\end{tikzcd}$$

which say precisely that $\mathfrak{m}$ and $\mathfrak{e}$ are lax monoidal transformations (again, once we have first passed from k and u to k^{-1} and u^{-1}). $\square$

We can define in the evident way notions of *monoidal double homomorphism*, *opmonoidal double morphism*, *opmonoidal double opmorphism*, and so on. Let us also note the correct notion of vertical transformation between monoidal double morphisms:

Definition 36. A **monoidal vertical transformation** between monoidal double morphisms F, $G\colon \mathbb{K} \to \mathbb{L}$ is a pseudomonoid transformation $F \to G$ in **DblCat**.

Proposition 37. *Giving a monoidal vertical transformation $\alpha\colon F \Rightarrow G$ is equivalent to giving a vertical transformation $\alpha\colon F \Rightarrow G$ such that α_0 and α_1 are monoidal transformations.*

Proof. The equalities of pastings required for α to be a pseudomonoid transformation are easily seen to be equivalent to the equalities of pastings required for α_0 and α_1 to be monoidal transformations. $\square$

Straightforwardly, monoidal double categories, monoidal double morphisms and monoidal vertical transformations form a 2-category **MonDblCat**; and we have obvious variant 2-categories $\mathbf{MonDblCat}_\psi$, **OpMonDblCat**, etc.

5.3 The monoidal double category $[\mathbb{K}, \mathbb{K}]_\psi$

Given a small category $\mathbf{C}$, the endofunctor category $[\mathbf{C}, \mathbf{C}]$ acquires a structure of monoidal category. We shall see in this section that a similar result holds for pseudo double categories, namely, that the endohom double category $[\mathbb{K}, \mathbb{K}]_\psi$ is naturally a monoidal double category.

As we noted above, if we were to prove that the 2-functor $[-, ?\,]_\psi \colon \mathbf{DblCat}_\psi^{\mathrm{op}} \times \mathbf{DblCat}_\psi \to \mathbf{DblCat}_\psi$ was indeed an 'internal hom' for the monoidal bicategory $\mathbf{DblCat}_\psi$, then this result would follow from general principles (see [DS97]). However, since we have not proved this, and since it will be useful to see an explicit description of the monoidal structure on $[\mathbb{K}, \mathbb{K}]_\psi$, we proceed by a 'bare hands' approach. Analogous with the bicategorical case, there are two canonical choices for the composite of two horizontal transformations

$$\mathbf{A} \colon A_s \mathrel{\Rightarrow\!\!\!\!\!\shortmid}\; A_t \colon \mathbb{K} \to \mathbb{K} \quad \text{and} \quad \mathbf{B} \colon B_s \mathrel{\Rightarrow\!\!\!\!\!\shortmid}\; B_t \colon \mathbb{K} \to \mathbb{K},$$

namely

$$A_t\mathbf{B} \otimes \mathbf{A}B_s \quad \text{and} \quad \mathbf{A}B_s \otimes A_t\mathbf{B}.$$

And, as for bicategories, it makes no material difference which of these we choose:

Proposition 38. *There are canonical invertible special modifications*

$$i_{\mathbf{A},\mathbf{B}} \colon A_t\mathbf{B} \otimes \mathbf{A}B_s \Rrightarrow \mathbf{A}B_s \otimes A_t\mathbf{B},$$

natural in $\mathbf{A}$ *and* $\mathbf{B}$.

Proof. We take $i_{\mathbf{A},\mathbf{B}}$ to have central natural transformation $A_{B_c(-)}$; so the component of $i_{\mathbf{A},\mathbf{B}}$ at X is given by

$$A_{\mathbf{B}X} \colon A_t\mathbf{B}X \otimes \mathbf{A}B_sX \to \mathbf{A}B_sX \otimes A_t\mathbf{B}X.$$

Visibly this satisfies (MA1), whilst (MA2) is a long diagram chase using the axioms (HTA2) and (HTA3) for $\mathbf{A}$ and $\mathbf{B}$. For the naturality of these maps in $\mathbf{A}$ and $\mathbf{B}$, suppose we are given modifications $\boldsymbol{\alpha} \colon \mathbf{A} \Rrightarrow \mathbf{C}$ and $\boldsymbol{\beta} \colon \mathbf{B} \Rrightarrow \mathbf{D}$. Then we require the following diagrams to commute for all $X \in K_0$:

$$\begin{tikzcd}
A_t\mathbf{B}X \otimes \mathbf{A}B_sX \arrow[r, "A_t\boldsymbol{\beta}_X \otimes \mathbf{A}(\beta_s)_X"] \arrow[d, "A_{\mathbf{B}X}"'] & A_t\mathbf{D}X \otimes \mathbf{A}D_sX \arrow[r, "(\alpha_t)_{\mathbf{D}X} \otimes \boldsymbol{\alpha}_{D_sX}"] \arrow[d, "A_{\mathbf{D}X}"] & C_t\mathbf{D}X \otimes \mathbf{C}D_sX \arrow[d, "C_{\mathbf{D}X}"] \\
\mathbf{A}B_sX \otimes A_t\mathbf{B}X \arrow[r, "\mathbf{A}(\beta_s)_X \otimes A_t\boldsymbol{\beta}_X"'] & \mathbf{A}D_sX \otimes A_t\mathbf{D}X \arrow[r, "\boldsymbol{\alpha}_{D_sX} \otimes (\alpha_t)_{\mathbf{D}X}"'] & \mathbf{C}D_sX \otimes C_t\mathbf{D}X.
\end{tikzcd}$$

But the left-hand square is a naturality square for $A_{(-)}$ whilst the right-hand square is axiom (MA2) for $\boldsymbol{\alpha}$; and hence we are done. $\square$

These transformations are canonical in the sense that they satisfy pasting equalities formally similar to those for the 'middle-4 interchanger' in a **Gray**-category. Though we shall not spell out these pasting equalities in their full generality, we shall be using them implicitly in what follows to assert the commutativity of certain diagrams.

Proposition 39. *The double category $[\mathbb{K}, \mathbb{K}]_\psi$ is a monoidal double category.*

Proof.

- **Monoidal structure on** $[\mathbb{K}, \mathbb{K}]_{v\psi}$: Observe that this is the hom-category $\mathbf{DblCat}_\psi(\mathbb{K}, \mathbb{K})$ in the 2-category $\mathbf{DblCat}_\psi$, and hence is equipped with a strict monoidal structure.
- **Monoidal structure on** $[\mathbb{K}, \mathbb{K}]_{h\psi}$: We take for the tensor unit $\mathbf{e}$, the object

  $$\mathbf{e} = \mathbf{I}_{\mathrm{id}} \colon \mathrm{id} \Rrightarrow\!\!\!\!\!\mid\;\; \mathrm{id}.$$

  The tensor product is given as follows:

  - **On objects**: given $\mathbf{A} \colon A_s \Rrightarrow\!\!\!\!\!\mid\;\; A_t$ and $\mathbf{B} \colon B_s \Rrightarrow\!\!\!\!\!\mid\;\; B_t$, we take

    $$\mathbf{A} \bullet \mathbf{B} = \mathbf{A}B_t \otimes A_s\mathbf{B} \colon A_sB_s \Rrightarrow\!\!\!\!\!\mid\;\; A_tB_t.$$

    Explicitly, this has components

    $$(\mathbf{A} \bullet \mathbf{B})(X) = A_sB_sX \xrightarrow{A_s\mathbf{B}X} A_sB_tX \xrightarrow{\mathbf{A}B_tX} A_tB_tX.$$

  - **On maps**: Given $\boldsymbol{\alpha} \colon \mathbf{A} \Rrightarrow \mathbf{C}$ and $\boldsymbol{\beta} \colon \mathbf{B} \Rrightarrow \mathbf{D}$, we take

    $$\boldsymbol{\alpha} \bullet \boldsymbol{\beta} = \boldsymbol{\alpha}\beta_t \otimes \alpha_s\boldsymbol{\beta} \colon \mathbf{A}B_t \otimes A_s\mathbf{B} \Rrightarrow \mathbf{C}D_t \otimes C_s\mathbf{D}.$$

  The functoriality of $\bullet$ is immediate from the functoriality of $\otimes$ and of the whiskering operations. We must now exhibit the unitality and associativity coherence constraints in $[\mathbb{K},\mathbb{K}]_{v\psi}$. For unitality, we have that

$$\mathbf{e} \bullet \mathbf{A} = \mathbf{I}_{\mathrm{id}} A_t \otimes \mathbf{A}$$
$$\mathbf{A} \bullet \mathbf{e} = \mathbf{A} \otimes A_s \mathbf{I}_{\mathrm{id}}$$

and hence we give $\rho_{\mathbf{A}}$ and $\lambda_{\mathbf{A}}$ by the special invertible modifications

$$\begin{tikzcd}
\mathbf{A} \arrow[d, Rrightarrow, "\mathfrak{l}_{\mathbf{A}}"] & & \mathbf{A} \arrow[d, Rrightarrow, "\mathfrak{r}_{\mathbf{A}}"] \\
\mathbf{I}_{A_t} \otimes \mathbf{A} \arrow[d, Rrightarrow, "\mathrm{id}"] & \text{and} & \mathbf{A} \otimes \mathbf{I}_{A_s} \arrow[d, Rrightarrow, "\mathbf{A} \otimes \mathfrak{e}_{\mathbf{A}}"] \\
\mathbf{I}_{\mathrm{id}} A_t \otimes \mathbf{A} & & \mathbf{A} \otimes A_s \mathbf{I}_{\mathrm{id}}
\end{tikzcd}$$

respectively. The naturality of these in $\mathbf{A}$ follows from the naturality of $\mathfrak{l}$, $\mathfrak{r}$ and $\mathfrak{e}$. For the associativity modifications, suppose we are given $\mathbf{A}\colon A_s \nRightarrow A_t$, $\mathbf{B}\colon B_s \nRightarrow B_t$ and $\mathbf{C}\colon C_s \nRightarrow C_t$. Now we have

$$\mathbf{A} \bullet (\mathbf{B} \bullet \mathbf{C}) = \mathbf{A}(B_t C_t) \otimes A_s(\mathbf{B} C_t \otimes B_s \mathbf{C})$$
$$(\mathbf{A} \bullet \mathbf{B}) \bullet \mathbf{C} = (\mathbf{A} B_t \otimes A_s \mathbf{B}) C_t \otimes (A_s B_s) \mathbf{C}$$

Hence we take $\alpha_{\mathbf{A},\mathbf{B},\mathbf{C}}$ to be the special modification

$$\begin{tikzcd}
\mathbf{A}(B_t C_t) \otimes A_s(\mathbf{B} C_t \otimes B_s \mathbf{C}) \arrow[d, Rrightarrow, "\mathbf{A}(B_t C_t) \otimes \mathfrak{m}^{-1}_{\mathbf{B} C_t, B_s \mathbf{C}}"] \\
\mathbf{A}(B_t C_t) \otimes \big(A_s(\mathbf{B} C_t) \otimes A_s(B_s \mathbf{C})\big) \arrow[d, Rrightarrow, "\mathrm{id}"] \\
(\mathbf{A} B_t) C_t \otimes \big((A_s \mathbf{B}) C_t \otimes (A_s B_s) \mathbf{C}\big) \arrow[d, Rrightarrow, "\mathfrak{a}_{(\mathbf{A} B_t) C_t, (A_s \mathbf{B}) C_t, (A_s B_s) \mathbf{C}}"] \\
\big((\mathbf{A} B_t) C_t \otimes (A_s \mathbf{B}) C_t\big) \otimes (A_s B_s) \mathbf{C} \arrow[d, Rrightarrow, "\mathrm{id}"] \\
(\mathbf{A} B_t \otimes A_s \mathbf{B}) C_t \otimes (A_s B_s) \mathbf{C}.
\end{tikzcd}$$

The naturality of these components in $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ follows from the naturality of $\mathfrak{m}$ and $\mathfrak{a}$; and a routine diagram chase using the coherence axioms for $\mathfrak{l}$, $\mathfrak{r}$, $\mathfrak{a}$, $\mathfrak{m}$ and $\mathfrak{e}$ shows that α, ρ and λ satisfy the associativity pentagon and the unit triangles.

- s **and** $t\colon [\mathbb{K},\mathbb{K}]_{h\psi} \to [\mathbb{K},\mathbb{K}]_{v\psi}$ **are strict monoidal**: this is immediate from above.

- $\mathbf{I}\colon [\mathbb{K},\mathbb{K}]_{v\psi} \to [\mathbb{K},\mathbb{K}]_{h\psi}$ **is strong monoidal**: We observe that $\mathbf{I}_e = \mathbf{e}$, so that $\mathbf{I}$ is strict monoidal with respect to the unit. For the binary tensor $\bullet$, we have

$$\mathbf{I}_F \bullet \mathbf{I}_G = \mathbf{I}_F G \otimes F\mathbf{I}_G$$

and hence we take $u_{F,G}\colon \mathbf{I}_{FG} \Rrightarrow \mathbf{I}_F \bullet \mathbf{I}_G$ to be the special invertible modification

$$\begin{tikzcd}
\mathbf{I}_{FG} \arrow[d, Rightarrow, "\mathfrak{l}_{\mathbf{I}_{FG}}"] \\
\mathbf{I}_{FG} \otimes \mathbf{I}_{FG} \arrow[d, Rightarrow, "\mathrm{id}\otimes\mathfrak{e}_G"] \\
\mathbf{I}_F G \otimes F\mathbf{I}_G
\end{tikzcd}$$

Again, naturality in F and G follows from naturality of $\mathfrak{e}$, and it's easy to check that the three diagrams making $\mathbf{I}$ strong monoidal do commute.

- $\otimes\colon [\mathbb{K},\mathbb{K}]_{h\psi}\ {}_s\!\times_t [\mathbb{K},\mathbb{K}]_{h\psi} \to [\mathbb{K},\mathbb{K}]_{h\psi}$ **is strong monoidal**: Since $\mathbf{I}_e = \mathbf{e}$, we can take

$$k_{\mathbf{e}}\colon \mathbf{e} \otimes \mathbf{e} \to \mathbf{e}$$

to be the canonical map $\mathfrak{r}^{-1}_{\mathbf{I_e}} = \mathfrak{l}^{-1}_{\mathbf{I_e}}$. Now, suppose we are given horizontal transformations

$$\begin{aligned}
&\mathbf{A}\colon A_1 \nRightarrow A_2, && \mathbf{A}'\colon A_2 \nRightarrow A_3,\\
&\mathbf{B}\colon B_1 \nRightarrow B_2, && \text{and } \mathbf{B}'\colon B_2 \nRightarrow B_3.
\end{aligned}$$

Then

$$(\mathbf{A}' \bullet \mathbf{B}') \otimes (\mathbf{A} \bullet \mathbf{B}) = (\mathbf{A}'B_3 \otimes A_2\mathbf{B}') \otimes (\mathbf{A}B_2 \otimes A_1\mathbf{B})$$

whilst

$$(\mathbf{A}' \otimes \mathbf{A}) \bullet (\mathbf{B}' \otimes \mathbf{B}) = (\mathbf{A}' \otimes \mathbf{A})B_3 \otimes A_1(\mathbf{B}' \otimes \mathbf{B}).$$

Therefore we take for $k_{\mathbf{A}',\mathbf{B}',\mathbf{A},\mathbf{B}} \colon (\mathbf{A}' \bullet \mathbf{B}') \otimes (\mathbf{A} \bullet \mathbf{B}) \Rrightarrow (\mathbf{A}' \otimes \mathbf{A}) \bullet (\mathbf{B}' \otimes \mathbf{B})$ the special invertible modification

$$\begin{tikzcd}
(\mathbf{A}'B_3 \otimes A_2\mathbf{B}') \otimes (\mathbf{A}B_2 \otimes A_1\mathbf{B}) \arrow[d, Rrightarrow, "\mathfrak{a}"] \\
\big(\mathbf{A}'B_3 \otimes (A_2\mathbf{B}' \otimes \mathbf{A}B_2)\big) \otimes A_1\mathbf{B} \arrow[d, Rrightarrow, "(\mathbf{A}'B_3 \otimes i_{\mathbf{A},\mathbf{B}'}) \otimes A_1\mathbf{B}"] \\
\big(\mathbf{A}'B_3 \otimes (\mathbf{A}B_3 \otimes A_1\mathbf{B}')\big) \otimes A_1\mathbf{B}s \arrow[d, Rrightarrow, "\mathfrak{a}"] \\
(\mathbf{A}'B_3 \otimes \mathbf{A}B_3) \otimes (A_1\mathbf{B}' \otimes A_1\mathbf{B}) \arrow[d, Rrightarrow, "\mathrm{id} \otimes \mathfrak{m}_{\mathbf{B}',\mathbf{B}}"] \\
(\mathbf{A}' \otimes \mathbf{A})B_3 \otimes A_1(\mathbf{B}' \otimes \mathbf{B})
\end{tikzcd}$$

where the maps labelled $\mathfrak{a}$ are appropriate composites of associativity maps. The naturality of the displayed map in all variables follows from the naturality of $\mathfrak{a}$, i and $\mathfrak{m}$. It's now a diagram chase to check that the required coherence laws hold to make $\otimes$ strong monoidal;

- **The natural transformations $\mathfrak{a}$, $\mathfrak{l}$ and $\mathfrak{r}$ are strong monoidal transformations**: This is another routine diagram chase. ☐

5.4 Monoidal comma double categories

The following result extends the notion of comma double category to a notion of 'monoidal comma double category'. Again, this construction has a universal property in a suitable double category, namely the double category which looks like **MonDblCat** in the vertical direction and like $\mathbf{OpMonDblCat}_o$ in the horizontal (following [GP04]); but we content ourselves with simply giving the construction here.

Proposition 40. *Let $\mathbb{K}$, $\mathbb{L}$ and $\mathbb{M}$ be monoidal double categories, let F be an* *opmonoidal double opmorphism, and let G be a monoidal double morphism. Then the comma double category $(F \downarrow G)$ becomes a monoidal double category.*

Proof. We know by Proposition 35 that we can view F_0 and F_1 as opmonoidal functors, and G_0 and G_1 as monoidal functors. Therefore, applying Proposition 15, we see that $(F_1 \downarrow G_1)$ and $(F_0 \downarrow G_0)$ can be equipped with the structure of monoidal categories. It is straightforward to check that s and t are strict monoidal with respect to this; for example, given $(\mathbf{U}, \mathbf{X}, \mathbf{f})$ and $(\mathbf{U}', \mathbf{X}', \mathbf{f}')$ in $(F_1 \downarrow G_1)$, we have $(\mathbf{U}, \mathbf{X}, \mathbf{f}) \bullet (\mathbf{U}', \mathbf{X}', \mathbf{f}')$ given by

$$F(\mathbf{U} \bullet \mathbf{U}') \xrightarrow{m_{\mathbf{U},\mathbf{U}'}} F\mathbf{U} \bullet F\mathbf{U}' \xrightarrow{\mathbf{f} \bullet \mathbf{f}'} G\mathbf{X} \bullet G\mathbf{X}' \xrightarrow{m_{\mathbf{X},\mathbf{X}'}} G(\mathbf{X} \bullet \mathbf{X}'),$$

whose image under s is the object

$$F(U_s \bullet U'_s) \xrightarrow{m_{U_s,U'_s}} FU_s \bullet FU'_s \xrightarrow{f_s \bullet f'_s} GX_s \bullet GX'_s \xrightarrow{m_{X_s,X'_s}} G(X_s \bullet X'_s)$$

which is $(U_s, X_s, f_s) \bullet (U'_s, X'_s, f'_s)$ as required. It remains to specify the invertible transformations k and u and the invertible maps $k_{\mathbf{e}}$ and u_e, which we do as follows:

$$\begin{aligned} k_{(\mathbf{U},\mathbf{X},\mathbf{f}),(\mathbf{U}',\mathbf{X}',\mathbf{f}'),(\mathbf{V},\mathbf{Y},\mathbf{g}),(\mathbf{V}',\mathbf{Y}',\mathbf{g}')} &= (k_{\mathbf{U},\mathbf{U}',\mathbf{V},\mathbf{V}'}, k_{\mathbf{X},\mathbf{X}',\mathbf{Y},\mathbf{Y}'}), & k_{\mathbf{e}} &= (k_{\mathbf{e}}, k_{\mathbf{e}}), \\ u_{(U,X,f),(V,Y,g)} &= (u_{U,V}, u_{X,Y}), & \text{and } u_e &= (u_e, u_e). \end{aligned}$$

That the required squares commute for these to be maps follows straightforwardly using the coherence diagrams for F and G. Their naturality follows from the naturality of k and u for F and G; and finally the coherence diagrams that they are required to satisfy follow using the coherence diagrams for F, G, $\mathbb{K}$ and $\mathbb{L}$. ☐

Once more, we specialise to the case where $F = \mathrm{id}_{\mathbb{K}} \colon \mathbb{K} \to \mathbb{K}$ and $G \colon 1 \to \mathbb{K}$, where 1 is the terminal double category, viewed as a strict monoidal double category in the evident way. Now, such a functor G amounts to giving a *monoidal monad* in the double category $\mathbb{K}$. Explicitly:

Definition 41. A **monoidal monad** in the monoidal double category $\mathbb{K}$ consists of:

- A monad $(\mathbf{X} \colon X \twoheadrightarrow X, \mathfrak{m}, \mathfrak{e})$ in $\mathbb{K}$;

- Maps

$$\boldsymbol{\mu}\colon \mathbf{X} \bullet \mathbf{X} \to \mathbf{X}, \quad \boldsymbol{\eta}\colon \mathbf{e} \to \mathbf{X}$$

$$\mu\colon X \bullet X \to X, \quad \text{and} \quad \eta\colon e \to X$$

such that:

- $s(\boldsymbol{\mu}) = t(\boldsymbol{\mu}) = \mu$ and $s(\boldsymbol{\eta}) = t(\boldsymbol{\eta}) = \eta$;
- $(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\eta})$ is a monoid in the monoidal category K_1;
- (X, μ, η) is a monoid in the monoidal category K_0;
- The following diagrams commute:

$$\begin{tikzcd}
(\mathbf{X} \bullet \mathbf{X}) \otimes (\mathbf{X} \bullet \mathbf{X}) \arrow[r, "k_{\mathbf{X},\mathbf{X},\mathbf{X},\mathbf{X}}"] \arrow[d, "\boldsymbol{\mu} \otimes \boldsymbol{\mu}"'] & (\mathbf{X} \otimes \mathbf{X}) \bullet (\mathbf{X} \otimes \mathbf{X}) \arrow[d, "\mathfrak{m} \bullet \mathfrak{m}"] \\
\mathbf{X} \otimes \mathbf{X} \arrow[d, "\mathfrak{m}"'] & \mathbf{X} \bullet \mathbf{X} \arrow[d, "\boldsymbol{\mu}"] \\
\mathbf{X} \arrow[r, "\mathrm{id}"'] & \mathbf{X}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{I}_{X \bullet X} \arrow[r, "u_{X,X}"] \arrow[d, "\mathbf{I}_{\mu}"'] & \mathbf{I}_X \bullet \mathbf{I}_X \arrow[d, "\mathfrak{e} \bullet \mathfrak{e}"] \\
\mathbf{I}_X \arrow[d, "\mathfrak{e}"'] & \mathbf{X} \bullet \mathbf{X} \arrow[d, "\boldsymbol{\mu}"] \\
\mathbf{X} \arrow[r, "\mathrm{id}"'] & \mathbf{X}
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{e} \otimes \mathbf{e} \arrow[r, "k_{\mathbf{e}}"] \arrow[d, "\boldsymbol{\eta} \otimes \boldsymbol{\eta}"'] & \mathbf{e} \arrow[dd, "\boldsymbol{\eta}"] \\
\mathbf{X} \otimes \mathbf{X} \arrow[d, "\mathfrak{m}"'] & \\
\mathbf{X} \arrow[r, "\mathrm{id}"'] & \mathbf{X}
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{I}_e \arrow[r, "u_e"] \arrow[d, "\mathbf{I}_{\eta}"'] & \mathbf{e} \arrow[dd, "\boldsymbol{\eta}"] \\
\mathbf{I}_X \arrow[d, "\mathfrak{e}"'] & \\
\mathbf{X} \arrow[r, "\mathrm{id}"'] & \mathbf{X}.
\end{tikzcd}$$

Thus, given a monoidal monad $\mathbf{X}$ in a double category $\mathbb{K}$, we can apply Proposition 40 to see that the slice double category $\mathbb{K}/\mathbf{X}$ becomes a monoidal double category.

5.5 Monads on a pseudo double category

Definition 42. Let $\mathbb{K}$ be a double category. A **double monad** on $\mathbb{K}$ is a monad on the object $\mathbb{K}$ in the 2-category $\mathbf{DblCat}_{\psi}$.

Proposition 43. *Let $\mathbb{K}$ be a double category, and let (S, μ, η) be a double monad* *on* $\mathbb{K}$*. Then the monad* $S\mathbf{I}$ *in the monoidal double category* $[\mathbb{K},\mathbb{K}]_\psi$ *becomes a monoidal monad.*

Proof. S is a monad in $\mathbf{DblCat}_\psi$, and thus a monoid in $\mathbf{DblCat}_\psi(\mathbb{K},\mathbb{K}) = [\mathbb{K},\mathbb{K}]_{v\psi}$. We equip the object $S\mathbf{I} \in [\mathbb{K},\mathbb{K}]_{h\psi}$ with monoid structure as follows. Recall that $S\mathbf{I}$ is in fact the monad $S\mathbf{I}_{\mathrm{id}_\mathbb{K}}$; then the unit $\boldsymbol{\eta}\colon \mathbf{I}_{\mathrm{id}_\mathbb{K}} \Rrightarrow S\mathbf{I}$ is given by the modification

$$\begin{tikzcd}
\mathrm{id}_\mathbb{K} \arrow[r, Rightarrow, "\mathbf{I}_{\mathrm{id}_\mathbb{K}}", "|" marking] \arrow[d, Rightarrow, "\eta"'] \arrow[dr, phantom, "\Downarrow \eta\mathbf{I}_{\mathrm{id}_\mathbb{K}}"] & \mathrm{id}_\mathbb{K} \arrow[d, Rightarrow, "\eta"] \\
S \arrow[r, Rightarrow, "S\mathbf{I}_{\mathrm{id}_\mathbb{K}}"', "|" marking] & S.
\end{tikzcd}$$

For the multiplication, observe first that we have

$$S\mathbf{I} \bullet S\mathbf{I} = (S\mathbf{I}_{\mathrm{id}_\mathbb{K}})S \otimes S(S\mathbf{I}_{\mathrm{id}_\mathbb{K}}) = S\mathbf{I}_S \otimes S(S\mathbf{I}_{\mathrm{id}_\mathbb{K}}).$$

Therefore we take for $\boldsymbol{\mu}\colon S\mathbf{I} \bullet S\mathbf{I} \Rrightarrow S\mathbf{I}$ the modification

$$\begin{tikzcd}
S\mathbf{I}_S \otimes S(S\mathbf{I}_{\mathrm{id}_\mathbb{K}}) \arrow[d, Rightarrow, "\mathfrak{m}_{\mathbf{I}_S, S\mathbf{I}_{\mathrm{id}_\mathbb{K}}}"] \\
S(\mathbf{I}_S \otimes S\mathbf{I}_{\mathrm{id}_\mathbb{K}}) \arrow[d, Rightarrow, "S\mathfrak{l}^{-1}_{S\mathbf{I}_{\mathrm{id}_\mathbb{K}}}"] \\
SS\mathbf{I}_{\mathrm{id}_\mathbb{K}} \arrow[d, Rightarrow, "\mu\mathbf{I}_{\mathrm{id}_\mathbb{K}}"] \\
S\mathbf{I}_{\mathrm{id}_\mathbb{K}}.
\end{tikzcd}$$

It's straightforward to work through the definitions and see that this does indeed make $S\mathbf{I}$ into a monoid in $[\mathbb{K},\mathbb{K}]_{h\psi}$. Further, s and t send this monoid to the monoid S in $[\mathbb{K},\mathbb{K}]_{v\psi}$ as required, whilst it's another diagram chase to check that the diagrams expressing the compatibility of the monoid and monad structure on S are satisfied. □

Thus assembling all of the above, we have:

Proposition 44. *Given a double monad* (S,η,μ) *on a double category* $\mathbb{K}$*, the slice* *double category* $[\mathbb{K},\mathbb{K}]_\psi/S\mathbf{I}$ *has a natural structure of monoidal double category.*

Chapter 6

Clubs II

We are now ready to extend the concept of club, as given in Chapter 3 to a concept of 'double club'. As we now have all the necessary theory at our disposal, this is a straightforward step. However, the definition of (plain) club loses its force without the important Proposition 14, which gives us an equivalence of categories $Coll(S) \simeq \mathbf{C}/S1$. Our first task, therefore, is to establish an analogue of this result. We then give our definition of double club; however, this definition is rather hard to work with in practice, and so we give a useful equivalent definition.

6.1 Evaluation at 1 in the double category of collections

Let $S\colon \mathbb{K} \to \mathbb{L}$ be a double homomorphism and consider the double category of collections $\mathbb{C}oll(S)$. We have a homomorphism $F\colon \mathbb{C}oll(S) \to \mathbb{L}/S\mathbf{I}_1$ which 'evaluates at 1':

$$\begin{aligned} F_0\colon \mathbb{C}oll(S)_0 &\to L_0/S1 \\ (A,\alpha) &\mapsto (A1,\alpha_1) \\ \gamma &\mapsto \gamma_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} F_1\colon \mathbb{C}oll(S)_1 &\to L_1/S\mathbf{I}_1 \\ (\mathbf{A},\boldsymbol{\alpha}) &\mapsto (\mathbf{A}1,\boldsymbol{\alpha}_1) \\ \boldsymbol{\gamma} &\mapsto \boldsymbol{\gamma}_1. \end{aligned}$$

And in fact, this is a *strict* homomorphism, since we have

$$\begin{aligned} F\mathbf{I}_{(A,\alpha)} &= F\big(\mathbf{I}_A \xRightarrow{\mathbf{I}_\alpha} \mathbf{I}_S \xRightarrow{\mathfrak{e}} S\mathbf{I}\big) \\ &= \big(\mathbf{I}_{A1} \xRightarrow{\mathbf{I}_{\alpha_1}} \mathbf{I}_{S1} \xRightarrow{\mathfrak{e}} S\mathbf{I}_1\big) \\ &= \mathbf{I}_{(A1,\alpha_1)} \\ &= \mathbf{I}_{F(A,\alpha)} \end{aligned}$$

and

$$\begin{aligned} F\big((\mathbf{A},\boldsymbol{\alpha}) \otimes (\mathbf{B},\boldsymbol{\beta})\big) &= F\big(\mathbf{A} \otimes \mathbf{B} \xRightarrow{\boldsymbol{\alpha}\otimes\boldsymbol{\beta}} S\mathbf{I} \otimes S\mathbf{I} \xRightarrow{\mathfrak{m}} S\mathbf{I}\big) \\ &= \big(\mathbf{A}1 \otimes \mathbf{B}1 \xRightarrow{\boldsymbol{\alpha}_1\otimes\boldsymbol{\beta}_1} S\mathbf{I}_1 \otimes S\mathbf{I}_1 \xRightarrow{\mathfrak{m}_1} S\mathbf{I}_1\big) \\ &= (\mathbf{A}1, \boldsymbol{\alpha}_1) \otimes (\mathbf{B}1, \boldsymbol{\beta}_1) \\ &= F(\mathbf{A},\boldsymbol{\alpha}) \otimes F(\mathbf{B},\boldsymbol{\beta}). \end{aligned}$$

Now, just as in the plain category case, we have the following proposition which tells us that we essentially lose no information in applying F:

Proposition 45. *Let S be a homomorphism $\mathbb{K} \to \mathbb{L}$ satisfying property (hps). Then evaluation at 1 induces an equivalence of double categories*

$$\mathbb{C}oll(S) \simeq \mathbb{L}/S\mathbf{I}_1.$$

Proof. We seek to apply Corollary 31, and thus we must gather all the data required for this. We have the strict homomorphism $F\colon \mathbb{C}oll(S) \to \mathbb{L}/S\mathbf{I}_1$ as above; in the opposite direction, we must exhibit a map of double graphs $G\colon \mathbb{L}/S\mathbf{I}_1 \to \mathbb{C}oll(S)$. Now, we can form categories of collections $Coll(S_0)$ and $Coll(S_1)$, and by Proposition 14 we have equivalences of categories

$$Coll(S_0) \simeq L_0/S1 \quad \text{and} \quad Coll(S_1) \simeq L_1/S\mathbf{I}_1$$

where the rightward direction of these equivalences is given by evaluation at 1 and $\mathbf{I}_1$ respectively. We are now ready to give G_0:

- **On objects**: given an object $(a,\theta) \in L_0/S1$, under the first equivalence we produce an object $(A_0, \alpha_0) \in \mathit{Coll}(S_0)$. We can also form the object $\mathbf{I}_{(a,\theta)} \in L_1/S\mathbf{I}_1$, and under the second equivalence this produces an object $(A_1, \alpha_1) \in \mathit{Coll}(S_1)$. Explicitly, A_0, α_0, A_1 and α_1 are given by the labelled objects and arrows in the following pullback diagrams:

$$\begin{tikzcd}
A_0X \arrow[r] \arrow[d, "(\alpha_0)_X"'] & a \arrow[d, "\theta"] & & A_1\mathbf{X} \arrow[r] \arrow[d, "(\alpha_1)_\mathbf{X}"'] & \mathbf{I}_a \arrow[d, "\mathfrak{e}_1\circ\mathbf{I}_\theta"] \\
SX \arrow[r, "S!"'] & S1 & \text{and} & S\mathbf{X} \arrow[r, "S!"'] & S\mathbf{I}_1
\end{tikzcd}$$

We aim to equip $A = (A_0, A_1)$ with the structure of a double homomorphism, and to show that $\alpha = (\alpha_0, \alpha_1)$ becomes a cartesian vertical transformation with respect to this structure. To do this, we must produce the data (DMD3) and (DMD4); that is, special natural isomorphisms

$$\mathfrak{m}_{\mathbf{X},\mathbf{Y}} \colon A\mathbf{X} \otimes A\mathbf{Y} \to A(\mathbf{X} \otimes \mathbf{Y}) \quad \text{and} \quad \mathfrak{e}_X \colon \mathbf{I}_{AX} \to A\mathbf{I}_X.$$

So consider the diagram:

$$\begin{tikzcd}
A\mathbf{X} \otimes A\mathbf{Y} \arrow[rr, "A!\otimes A!"] \arrow[dd, "\alpha_\mathbf{X}\otimes\alpha_\mathbf{Y}"'] & & \mathbf{I}_a \otimes \mathbf{I}_a \arrow[dr, "\mathfrak{r}_{\mathbf{I}_a}^{-1}"] \arrow[dd, "(\mathfrak{e}\circ\mathbf{I}_\theta)\otimes(\mathfrak{e}\circ\mathbf{I}_\theta)" near start] & \\
& A(\mathbf{X} \otimes \mathbf{Y}) \arrow[rr, "A!" near start] \arrow[dd, "\alpha_{\mathbf{X}\otimes\mathbf{Y}}"' near end] & & \mathbf{I}_a \arrow[dd, "\mathfrak{e}\circ\mathbf{I}_\theta"] \\
S\mathbf{X} \otimes S\mathbf{Y} \arrow[rr, "S!\otimes S!" near end] \arrow[dr, "\mathfrak{m}_{\mathbf{X},\mathbf{Y}}"'] & & S\mathbf{I}_1 \otimes S\mathbf{I}_1 \arrow[dr, "\mathfrak{m}_1"] & \\
& S(\mathbf{X} \otimes \mathbf{Y}) \arrow[rr, "S!"'] & & S\mathbf{I}_1
\end{tikzcd}$$

The front face is a pullback by definition; the back face by property (hps1). All the diagonal maps are isomorphisms, and the bottom and right faces commute by the coherence axioms for S and $\mathbb{L}$. Thus we induce a unique isomorphism $A\mathbf{X} \otimes A\mathbf{Y} \to A(\mathbf{X} \otimes \mathbf{Y})$ along the missing diagonal. Arguing identically with the diagram

$$\begin{tikzcd}
\mathbf{I}_{AX} \arrow[rr, "\mathbf{I}_{A!}"] \arrow[dd, "\mathbf{I}_{\alpha_X}"'] & & \mathbf{I}_a \arrow[dr, "\mathrm{id}"] \arrow[dd, "\mathbf{I}_\theta"] & \\
& A\mathbf{I}_X \arrow[rr, "A!"] \arrow[dd, "\alpha_{\mathbf{I}_X}"'] & & \mathbf{I}_a \arrow[dd, "\mathfrak{e}_1 \circ \mathbf{I}_\theta"] \\
\mathbf{I}_{SX} \arrow[rr, "\mathbf{I}_{S!}"] \arrow[dr, "\mathfrak{e}_X"'] & & \mathbf{I}_{S1} \arrow[dr, "\mathfrak{e}_1"] & \\
& S\mathbf{I}_X \arrow[rr, "S\mathbf{I}_!"'] & & S\mathbf{I}_1 .
\end{tikzcd}$$

we induce a unique isomorphism $\mathbf{I}_{AX} \to A\mathbf{I}_X$. It's straightforward, using the naturality of $\mathfrak{e}$ and $\mathfrak{m}$ together with the universal property of pullback, to see that these isomorphisms are natural in $(\mathbf{X}, \mathbf{Y})$ and X respectively.

It remains to check that A satisfies (DMA1)–(DMA3) and that α satisfies (VTA1)–(VTA3). For (DMA1) and (VTA1), we need that

$$sA_1 = A_0 s, \quad tA_1 = A_1 t, \quad s\alpha_1 = \alpha_0 s \quad \text{and} \quad t\alpha_1 = \alpha_0 t,$$

but this is straightforward. Indeed

$$s\left(\begin{tikzcd}
A_1\mathbf{X} \arrow[r, "A_1!"] \arrow[dd, "(\alpha_1)_{\mathbf{X}}"'] & \mathbf{I}_a \arrow[d, "\mathbf{I}_\theta"] \\
& \mathbf{I}_{S1} \arrow[d, "\mathfrak{e}_1"] \\
S\mathbf{X} \arrow[r, "S!"'] & S\mathbf{I}_1
\end{tikzcd}\right) = \begin{tikzcd}
A_0 X_s \arrow[r, "A_0!"] \arrow[d, "(\alpha_0)_{X_s}"'] & a \arrow[d, "\theta"] \\
SX_s \arrow[r, "S!"'] & S1
\end{tikzcd}$$

by (DMA1) for S and the fact that s strictly preserves the chosen pullbacks; and the same argument works for t. (VTA2) and (VTA3) are also easy, since we observe that the required diagrams are the left hand faces of the two commutative cubes above; further, α_0 and α_1 are cartesian natural transformations, and hence α is a cartesian vertical transformation as required. It remains to show that A satisfies (DMA2) and (DMA3), and this follows from the coherence axioms for S and $\mathbb{K}$, and the universal property of pullback.

- **On maps**: suppose we have a map $\psi\colon (a,\theta) \to (b,\phi)$ in $L_0/S1$, with $G_0(a,\theta) = (A,\alpha)$ and $G_0(b,\phi) = (B,\beta)$. Then we must produce a map $\gamma\colon (A,\alpha) \to (B,\beta)$; that is, a vertical transformation $\gamma\colon A \Rightarrow B$ making the diagram

  $$\begin{tikzcd} A \arrow[rr, Rightarrow, "\gamma"] \arrow[dr, Rightarrow, "\alpha"'] & & B \arrow[dl, Rightarrow, "\beta"] \\ & S & \end{tikzcd}$$

  commute. Now, using the equivalences $L_0/S1 \simeq \mathit{Coll}(S_0)$ and $L_1/S\mathbf{I}_1 \simeq \mathit{Coll}(S_1)$ as before, we produce natural transformations γ_0 and γ_1 making

  $$\begin{tikzcd} A_0 \arrow[rr, Rightarrow, "\gamma_0"] \arrow[dr, Rightarrow, "\alpha_0"'] & & B_0 \arrow[dl, Rightarrow, "\beta_0"] \\ & S_0 & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_1 \arrow[rr, Rightarrow, "\gamma_1"] \arrow[dr, Rightarrow, "\alpha_1"'] & & B_1 \arrow[dl, Rightarrow, "\beta_1"] \\ & S_1 & \end{tikzcd}$$

  commute. We aim to show that $\gamma = (\gamma_0, \gamma_1)$ becomes a vertical transformation. (VTA1) follows as before; so it remains only to check (VTA2) and (VTA3), and these follow from the naturality of $\mathfrak{t}^{-1}$ and the universal property of pullback.

We now move on to G_1. Suppose we have an object

$$\begin{tikzcd} a_s \arrow[r, "\mathbf{a}", "\shortmid"{marking}] \arrow[d, "\theta_s"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\theta}"] & a_t \arrow[d, "\theta_t"] \\ S1 \arrow[r, "S\mathbf{I}_1"', "\shortmid"{marking}] & S1 \end{tikzcd}$$

of $L_1/S\mathbf{I}_1$, with $G_0(a_s,\theta_s) = (A_s,\alpha_s)$ and $G_0(a_t,\theta_t) = (A_t,\alpha_t)$, say. Then we must produce an object $(\mathbf{A},\boldsymbol{\alpha}) \in \mathbb{C}\mathit{oll}(S)_1$ as follows:

$$\begin{tikzcd} A_s \arrow[r, Rightarrow, "\mathbf{A}", "\shortmid"{marking}] \arrow[d, Rightarrow, "\alpha_s"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\alpha}"] & A_t \arrow[d, Rightarrow, "\alpha_t"] \\ S \arrow[r, Rightarrow, "S\mathbf{I}"', "\shortmid"{marking}] & S. \end{tikzcd}$$

Under the equivalence $L_1/S\mathbf{I}_1 \simeq Coll(S_1)$, we take $(\mathbf{a}, \boldsymbol{\theta})$ to a functor $A\colon K_1 \to L_1$ and a cartesian natural transformation $\alpha\colon A \Rightarrow S_1$. Thus we specify the horizontal transformation $\mathbf{A}$ to have source A_s, target A_t and components functor

$$A_c = A\mathbf{I}\colon K_0 \to L_1.$$

Similarly, we take the modification $\boldsymbol{\alpha}$ to have source α_s, target α_t and central natural transformation

$$\alpha_c = \alpha\mathbf{I}\colon A\mathbf{I} \Rightarrow S\mathbf{I}\colon K_0 \to L_1.$$

Explicitly, $\mathbf{A}X$ and α_X will be the indicated arrows in the following pullback diagram:

$$\begin{tikzcd}
\mathbf{A}X \arrow[r, "\mathbf{A}!"] \arrow[d, "\alpha_X"'] & \mathbf{a} \arrow[d, "\boldsymbol{\theta}"] \\
S\mathbf{I}_X \arrow[r, "S!"'] & S\mathbf{I}_1
\end{tikzcd} .$$

We must now specify the pseudonaturality data (HTD2) for $\mathbf{A}$. So consider the diagram

$$\begin{tikzcd}
A_t\mathbf{X} \otimes \mathbf{A}X_s \arrow[rr, "A_t!\otimes\mathbf{A}!"] \arrow[dd, "(\alpha_t)_{\mathbf{X}}\otimes\boldsymbol{\alpha}_{X_s}"'] & & \mathbf{I}_{a_t} \otimes \mathbf{a} \arrow[dr, "\mathfrak{r}_{\mathbf{a}}\circ\mathfrak{l}_{\mathbf{a}}^{-1}"] \arrow[dd, "(\mathfrak{e}_1\circ\mathbf{I}_{\theta_t})\otimes\boldsymbol{\theta}"] & \\
& \mathbf{A}X_t \otimes A_s\mathbf{X} \arrow[rr, "\mathbf{A}!\otimes A_s!"] \arrow[dd, "\boldsymbol{\alpha}_{X_t}\otimes(\alpha_s)_{\mathbf{X}}"'] & & \mathbf{a} \otimes \mathbf{I}_{a_s} \arrow[dd, "\boldsymbol{\theta}\otimes(\mathfrak{e}_1\circ\mathbf{I}_{\theta_s})"] \\
S\mathbf{X} \otimes S\mathbf{I}_{X_s} \arrow[rr, "S!\otimes S!"] \arrow[dr, "(S\mathbf{I})_{\mathbf{X}}"'] & & S\mathbf{I}_1 \otimes S\mathbf{I}_1 \arrow[dr, "\mathrm{id}"] & \\
& S\mathbf{I}_{X_t} \otimes S\mathbf{X} \arrow[rr, "S!\otimes S!"'] & & S\mathbf{I}_1 \otimes S\mathbf{I}_1.
\end{tikzcd}$$

The front and back faces are pullbacks by property (hps1) and the diagonal maps are all isomorphisms. It's easy to check that the bottom and right faces commute, and thus we induce a unique isomorphism along the missing diagonal, which will be the component $A_{\mathbf{X}}$ of the pseudonaturality natural transformation. That these components are natural in $\mathbf{X}$ follows from the naturality of $(S\mathbf{I})_{(-)}$ and the universal property of pullback.

We must now check (HTA1)–(HTA3) and (MA1)–(MA2). For (HTA1) and (MA1), we observe that

$$s\left(\begin{tikzcd} \mathbf{A}X \arrow[r, "\mathbf{A}!"] \arrow[d, "\boldsymbol{\alpha}_X"'] & \mathbf{a} \arrow[d, "\boldsymbol{\theta}"] \\ S\mathbf{I}_X \arrow[r, "S!"'] & S\mathbf{I}_1 \end{tikzcd}\right) = \begin{tikzcd} A_sX \arrow[r, "A_s!"] \arrow[d, "(\alpha_s)_X"'] & a_s \arrow[d, "\theta_s"] \\ SX \arrow[r, "S!"'] & S1 \end{tikzcd}$$

and

$$t\left(\begin{tikzcd} \mathbf{A}X \arrow[r, "\mathbf{A}!"] \arrow[d, "\boldsymbol{\alpha}_X"'] & \mathbf{a} \arrow[d, "\boldsymbol{\theta}"] \\ S\mathbf{I}_X \arrow[r, "S!"'] & S\mathbf{I}_1 \end{tikzcd}\right) = \begin{tikzcd} A_tX \arrow[r, "A_t!"] \arrow[d, "(\alpha_t)_X"'] & a_t \arrow[d, "\theta_t"] \\ SX \arrow[r, "S!"'] & S1 \end{tikzcd}$$

since s and t strictly preserve chosen pullbacks; thus $sA_c = (A_s)_0$, $tA_c = (A_t)_0$, $s\alpha_c = (\alpha_s)_0$ and $t\alpha_c = (\alpha_t)_0$ as required. For (MA2), we observe that the required diagram at $\mathbf{X}$ is just the left-hand face of the above cube; further, $\alpha_c = \alpha\mathbf{I}$ is a cartesian natural transformation, and hence $\boldsymbol{\alpha}$ is a cartesian modification as required. Finally, (HTA2) and (HTA3) follow from the coherence axioms for S and $\mathbb{L}$ and the universal property of pullback.

We now give G_1 on maps. Given a map $\boldsymbol{\psi}\colon (\mathbf{a}, \boldsymbol{\theta}) \to (\mathbf{b}, \boldsymbol{\phi})$ in $K_1/S\mathbf{I}_1$, we must produce a map $\boldsymbol{\gamma}\colon (\mathbf{A}, \boldsymbol{\alpha}) \to (\mathbf{B}, \boldsymbol{\beta})$ of $\mathbb{C}oll(S)_1$, and thus a modification $\boldsymbol{\gamma}\colon \mathbf{A} \Rrightarrow \mathbf{B}$ fitting into the diagram

$$\begin{tikzcd} \mathbf{A} \arrow[rr, Rrightarrow, "\gamma"] \arrow[dr, Rrightarrow, "\boldsymbol{\alpha}"'] & & \mathbf{B} \arrow[dl, Rrightarrow, "\boldsymbol{\beta}"] \\ & S\mathbf{I}. & \end{tikzcd}$$

For its source and target, we take the vertical transformations

$$\gamma_s = G_0(\psi_s)\colon A_s \Rightarrow B_s \quad \text{and} \quad \gamma_t = G_0(\psi_t)\colon A_t \Rightarrow B_t.$$

For the central natural transformation, we apply once more the equivalence $L_1/S\mathbf{I}_1 \cong$

$Coll(S_1)$ to get a commuting diagram

$$\begin{tikzcd}A \arrow[rr, Rightarrow, "\gamma"] \arrow[dr, Rightarrow, "\alpha"'] & & B \arrow[dl, Rightarrow, "\beta"] \\ & S_1, & \end{tikzcd}$$

in the functor category $[L_1, L_1]$. We need a natural transformation $\gamma_c \colon A_c \Rightarrow B_c$, and from above we have $A_c = A\mathbf{I}$ and $B_c = B\mathbf{I}$; so we take $\gamma_c = \gamma\mathbf{I}$. It remains to show that this data satisfies (MA1) and (MA2), which we do by an argument similar to above. Finally, we note that we have

$$\alpha_c = \alpha\mathbf{I} = (\beta \circ \gamma)\mathbf{I} = \beta\mathbf{I} \circ \gamma\mathbf{I} = \beta_c \circ \gamma_c$$

as required. This completes the definition of G_1.

By construction, it is immediate that $G = (G_0, G_1)$ becomes a map of double graphs; so next we show that (F_0, G_0) and (F_1, G_1) provide data for equivalences of categories. First note that if we choose pullbacks in L_0 and L_1 such that the pullback of identity arrows are identity arrows then we have

$$F_0G_0 = \mathrm{id}_{L_0/S1} \quad \text{and} \quad F_1G_1 = \mathrm{id}_{L_1/S\mathbf{I}_1}.$$

In the other direction, we construct a natural isomorphism $\mathrm{id}_{\mathbb{C}oll(S)_0} \Rightarrow G_0F_0$ as follows. Given (A, α) in $\mathbb{C}oll(S)_0$, let us write $(\hat{A}, \hat{\alpha})$ for $G_0F_0(A, \alpha)$: then we seek an invertible vertical transformation $\eta_{(A,\alpha)}$ making the diagram

$$\begin{tikzcd}A \arrow[rr, Rightarrow, "\eta_{(A,\alpha)}"] \arrow[dr, Rightarrow, "\alpha"'] & & \hat{A} \arrow[dl, Rightarrow, "\hat{\alpha}"] \\ & S & \end{tikzcd}$$

commute. So consider the diagrams:

$$\begin{tikzcd}
AX \arrow[rr, "A!"] \arrow[dr, dotted] \arrow[dd, swap, "\alpha_X"] & & A1 \arrow[dr, "\mathrm{id}"] \arrow[dd, near start, "\alpha_1"] & \\
& \hat{A}X \arrow[rr, near start, "\hat{A}!"] \arrow[dd, near start, swap, "\hat{\alpha}_X"] & & A1 \arrow[dd, "\alpha_1"] \\
SX \arrow[rr, near end, "S!"] \arrow[dr, swap, "\mathrm{id}"] & & S1 \arrow[dr, "\mathrm{id}"] & \\
& SX \arrow[rr, swap, "S!"] & & S1
\end{tikzcd}$$

and

$$\begin{tikzcd}
A\mathbf{X} \arrow[rr, "A!"] \arrow[dr, dotted] \arrow[dd, swap, "\alpha_{\mathbf{X}}"] & & A\mathbf{I}_1 \arrow[dr, "\mathfrak{e}_1^{-1}"] \arrow[dd, near start, "\alpha_{\mathbf{I}_1}"] & \\
& \hat{A}\mathbf{X} \arrow[rr, near start, "\hat{A}!"] \arrow[dd, near start, swap, "\hat{\alpha}_{\mathbf{X}}"] & & \mathbf{I}_{A1} \arrow[dd, "\mathfrak{e}_1 \circ \mathbf{I}_{\alpha_1}"] \\
S\mathbf{X} \arrow[rr, near end, "S!"] \arrow[dr, swap, "\mathrm{id}"] & & S\mathbf{I}_1 \arrow[dr, "\mathrm{id}"] & \\
& S\mathbf{X} \arrow[rr, swap, "S!"] & & S\mathbf{I}_1.
\end{tikzcd}$$

In these diagrams, the rear face is a pullback by cartesianness of α, the front face is a pullback by definition, and the diagonal maps are all isomorphisms. Hence we induce unique isomorphisms along the dotted diagonals which we take as the components of $\eta_{(A,\alpha)}$. Clearly we have $\hat{\alpha} \circ \eta_{(A,\alpha)} = \alpha$, and as before these components are natural in X and $\mathbf{X}$ respectively, by the universal property of pullback. It remains to check (VTA1)–(VTA3), but these follow from the universal property of pullback and coherence axioms for A.

For the naturality of η in (A, α), suppose we are given a map $\gamma\colon (A, \alpha) \to (B, \beta)$ in $\mathbb{C}oll(S)_0$, and let us write $\hat{\gamma}$ for $G_0F_0(\gamma)$. Then considering the diagrams

$$\begin{tikzcd}
AX \arrow[rr, "A!"] \arrow[dr, dotted] \arrow[dd, swap, "\alpha_X"] & & A1 \arrow[dr, "\gamma_1"] \arrow[dd, near start, "\alpha_1"] & \\
& \hat{B}X \arrow[rr, near start, "\hat{B}!"] \arrow[dd, near start, swap, "\hat{\beta}_X"] & & B1 \arrow[dd, "\beta_1"] \\
SX \arrow[rr, near end, "S!"] \arrow[dr, swap, "\mathrm{id}"] & & S1 \arrow[dr, "\mathrm{id}"] & \\
& SX \arrow[rr, swap, "S!"] & & S1
\end{tikzcd}$$

and

$$\begin{tikzcd}
A\mathbf{X} \arrow[rr, "A!"] \arrow[dr, dotted] \arrow[dd, swap, "\alpha_{\mathbf{X}}"] & & A\mathbf{I}_1 \arrow[dr, "\mathfrak{e}_1^{-1} \circ \gamma_{\mathbf{I}_1}"] \arrow[dd, near start, "\alpha_{\mathbf{I}_1}"] & \\
& \hat{B}\mathbf{X} \arrow[rr, near start, "\hat{B}!"] \arrow[dd, near start, swap, "\hat{\beta}_{\mathbf{X}}"] & & \mathbf{I}_{B1} \arrow[dd, "\mathfrak{e}_1 \circ \mathbf{I}_{\beta_1}"] \\
S\mathbf{X} \arrow[rr, near end, "S!"] \arrow[dr, swap, "\mathrm{id}"] & & S\mathbf{I}_1 \arrow[dr, "\mathrm{id}"] & \\
& S\mathbf{X} \arrow[rr, swap, "S!"] & & S\mathbf{I}_1
\end{tikzcd}$$

whose front and rear faces are pullbacks, we see that:

- For the left-hand diagram, both $(\eta_{(B,\beta)})_X \circ \gamma_X$ and $\hat{\gamma}_X \circ (\eta_{(A,\alpha)})_X$ make it commute when inserted for the dotted arrow, and hence must coincide;
- For the right-hand diagram, both $(\eta_{(B,\beta)})_{\mathbf{X}} \circ \gamma_{\mathbf{X}}$ and $\hat{\gamma}_{\mathbf{X}} \circ (\eta_{(A,\alpha)})_{\mathbf{X}}$ make it

commute when inserted for the dotted arrow, and hence must coincide.

Thus we conclude that $\eta_{(B,\beta)} \circ \gamma = \hat{\gamma} \circ \eta_{(A,\alpha)}$, and so we have a natural isomorphism $\eta_0 \colon \mathrm{id}_{\mathbb{C}oll(S)_0} \Rightarrow G_0 F_0$ as required.

Next, we exhibit a natural isomorphism $\mathrm{id}_{\mathbb{C}oll(S)_1} \Rightarrow G_1 F_1$. So suppose we are given an object $(\mathbf{A}, \boldsymbol{\alpha}) \in \mathbb{C}oll(S)_1$; let us write $(\hat{\mathbf{A}}, \hat{\boldsymbol{\alpha}})$ for $G_0 F_0(\mathbf{A}, \boldsymbol{\alpha})$. Then we have an invertible modification

$$\begin{tikzcd}
\mathbf{A} \arrow[rr, Rightarrow, "\boldsymbol{\eta}_{(\mathbf{A},\boldsymbol{\alpha})}"] \arrow[dr, Rightarrow, "\boldsymbol{\alpha}"'] & & \hat{\mathbf{A}} \arrow[dl, Rightarrow, "\hat{\boldsymbol{\alpha}}"] \\
& S\mathbf{I} &
\end{tikzcd}$$

with source $\eta_{(A_s,\alpha_s)}$, target $\eta_{(A_t,\alpha_t)}$, and with component at X given by the dotted arrow

$$\begin{tikzcd}
\mathbf{A}X \arrow[rr, "\mathbf{A}!"] \arrow[dr, dotted] \arrow[dd, "\boldsymbol{\alpha}_X"'] & & \mathbf{A}1 \arrow[dr, "\mathrm{id}"] \arrow[dd, "\boldsymbol{\alpha}_1" near end] & \\
& \hat{\mathbf{A}}X \arrow[rr, "\hat{\mathbf{A}}!" near start] \arrow[dd, "\hat{\boldsymbol{\alpha}}_X" near start] & & \mathbf{A}1 \arrow[dd, "\boldsymbol{\alpha}_1"] \\
S\mathbf{I}_X \arrow[rr, "S!" near end] \arrow[dr, "\mathrm{id}"'] & & S\mathbf{I}_1 \arrow[dr, "\mathrm{id}"] & \\
& S\mathbf{I}_X \arrow[rr, "S!"'] & & S\mathbf{I}_1,
\end{tikzcd}$$

induced by the universal property of pullback. By definition and the fact that s and t strictly preserve pullbacks, we have (MA1) satisfied; and we argue as before using the universal property of pullback to see that these components are natural in X and satisfy (MA2), and similarly to see that the maps $\boldsymbol{\eta}_{(\mathbf{A},\boldsymbol{\alpha})}$ are natural in $(\mathbf{A}, \boldsymbol{\alpha})$. Thus we have a natural isomorphism $\eta_1 \colon \mathrm{id}_{\mathbb{C}oll(S)_1} \Rightarrow G_1 F_1$ as required.

The final requirement is that η_0 and η_1 are compatible with the source and target maps:

$$s\eta_1 = \eta_0 s \quad \text{and} \quad t\eta_1 = \eta_1 t$$

and this follows from the definitions and the fact that s and t strictly preserve pullbacks. Thus we have all the requirements for Corollary 31, and so have an equivalence of double categories $\mathbb{C}oll(S) \simeq \mathbb{K}/S\mathbf{I}_1$. □

6.2 Double clubs

Definition 46. Let $\mathbb{K}$ and $\mathbb{L}$ be double categories.

- We say that $\mathbb{K}$ is a **vertically full sub-double category** of $\mathbb{L}$ if there is a strict homomorphism $F\colon \mathbb{K} \to \mathbb{L}$ such that F_0 and F_1 exhibit K_0 and K_1 as full subcategories of L_0 and L_1.
- If $\mathbb{K}$ and $\mathbb{L}$ are monoidal double categories, we say that $\mathbb{K}$ is a **sub-monoidal double category** of $\mathbb{L}$ if there is a strict monoidal strict homomorphism $F\colon \mathbb{K} \to \mathbb{L}$ exhibiting K_0 and K_1 as subcategories of L_0 and L_1.

In particular, if $\mathbb{K}$ is a vertically full sub-double category of a monoidal double category $\mathbb{L}$, then $\mathbb{K}$ can be made into a sub-monoidal double category of $\mathbb{L}$ if and only the object sets of K_0 and K_1 are closed under the binary and nullary tensors on L_0 and L_1 respectively.

Definition 47. Let (S, η, μ) be a double monad on a double category $\mathbb{K}$. We say that S is a **double club** if:

- S has property (hps);
- $\mathbb{C}oll(S)$ is a sub-monoidal double category of $[\mathbb{K}, \mathbb{K}]_\psi/S\mathbf{I}$.

Note that this is simply the natural generalisation of Definition 18: the extra requirement that condition (hps) be satisfied is necessary to ensure that $\mathbb{C}oll(S)$ exists in the first place; in the plain category case, the existence of the 'category of collections' is automatic.

Now, by the results above, there is an equivalence of double categories $\mathbb{C}oll(S) \simeq \mathbb{K}/S\mathbf{I}_1$; therefore if S is a double club, then the monoidal structure on $\mathbb{C}oll(S)$ transfers under the equivalence to a monoidal structure on $\mathbb{K}/S\mathbf{I}_1$, such that the equivalence becomes a monoidal equivalence (i.e., an equivalence in **MonDblCat**.

Let us spell out what this monoidal structure is. On $K_0/S1$, we have tensor unit $(1 \xrightarrow{\eta_1} S1)$, and the tensor product

$$(a \xrightarrow{\theta} S1) \bullet (b \xrightarrow{\phi} S1)$$

is given by the composite down the left-hand side in the diagram

$$\begin{tikzcd}
a \bullet b \arrow[r] \arrow[d] & a \arrow[d, "\theta"] \\
Sb \arrow[r, "S!"'] \arrow[d, "S\phi"'] & S1 \\
SS1 \arrow[d, "\mu_1"'] & \\
S1 &
\end{tikzcd}$$

where the upper square is a (chosen) pullback. On $K_1/S\mathbf{I}_1$, we have tensor unit $(\mathbf{I}_1 \xrightarrow{\eta_{\mathbf{I}_1}} S\mathbf{I}_1)$ and the tensor product

$$\begin{tikzcd}
a_s \arrow[r, "\mathbf{a}", "+" marking] \arrow[d, "\theta_s"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\theta}"] & a_t \arrow[d, "\theta_t"] & & b_s \arrow[r, "\mathbf{b}", "+" marking] \arrow[d, "\phi_s"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\phi}"] & b_t \arrow[d, "\phi_t"] \\
S1 \arrow[r, "S\mathbf{I}_1"', "+" marking] & S1 & \bullet & S1 \arrow[r, "S\mathbf{I}_1"', "+" marking] & S1
\end{tikzcd}$$

is given by horizontally composing the left-hand arrows from the two diagrams

$$\begin{tikzcd}
\mathbf{a}b_t \arrow[r] \arrow[d] & \mathbf{a} \arrow[d, "\boldsymbol{\theta}"] & & a_s\mathbf{b} \arrow[r] \arrow[d] & \mathbf{I}_{a_s} \arrow[d, "\mathfrak{e}_1 \circ \mathbf{I}_{\theta_s}"] \\
S\mathbf{I}_{b_t} \arrow[r, "S!"'] \arrow[d, "S\mathbf{I}_{\phi_t}"'] & S\mathbf{I}_1 & \text{and} & S\mathbf{b} \arrow[r, "S!"'] \arrow[d, "S\boldsymbol{\phi}"'] & S\mathbf{I}_1 \\
S\mathbf{I}_{S1} \arrow[d, "\mu_{\mathbf{I}_1} \circ S\mathfrak{e}_1"'] & & & SS\mathbf{I}_1 \arrow[d, "\mu_{\mathbf{I}_1}"'] & \\
S\mathbf{I}_1 & & & S\mathbf{I}_1, &
\end{tikzcd}$$

where the top squares are again pullbacks. However, we can do better; indeed, we claim that we can take the binary tensor product to be given by the pullback

$$\begin{tikzcd}
\mathbf{a} \bullet' \mathbf{b} \arrow[r] \arrow[d] & \mathbf{a} \arrow[d, "\boldsymbol{\theta}"] \\
S\mathbf{b} \arrow[r, "S!"'] \arrow[d, "S\boldsymbol{\phi}"'] & S\mathbf{I}_1 \\
SS\mathbf{I}_1 \arrow[d, "\mu_{\mathbf{I}_1}"'] & \\
S\mathbf{I}_1. &
\end{tikzcd}$$

To see this, consider the following diagram:

$$\begin{tikzcd}
\mathbf{a} \bullet' \mathbf{b} \arrow[rr] \arrow[dd] \arrow[dr, dotted] & & \mathbf{a} \arrow[dd, "\boldsymbol{\theta}"] \arrow[dr, "\mathfrak{r}_{\mathbf{a}}"] & \\
& \mathbf{a}b_t \otimes a_s\mathbf{b} \arrow[rr] \arrow[dd] & & \mathbf{a} \otimes \mathbf{I}_{a_s} \arrow[dd, "\boldsymbol{\theta}\otimes(\mathfrak{e}_1 \circ \mathbf{I}_{\theta_s})"] \\
S\mathbf{b} \arrow[rr, "S!"] \arrow[dr, "\mathfrak{m}^{-1}_{\mathbf{I}_{b_t},\mathbf{b}} \circ S\mathfrak{l}_{\mathbf{b}}"'] & & S\mathbf{I}_1 \arrow[dr, "\mathfrak{m}^{-1}_{\mathbf{I}_1,\mathbf{I}_1} \circ S\mathfrak{l}_{\mathbf{I}_1}"] & \\
& S\mathbf{I}_{b_t} \otimes S\mathbf{b} \arrow[rr, "S! \otimes S!"'] & & S\mathbf{I}_1 \otimes S\mathbf{I}_1,
\end{tikzcd}$$

whose rear face is a chosen pullback and whose front face is obtained from horizontally composing the above two pullbacks. By property (hps1), the front face is again a pullback, and it's easy to check that the bottom and right faces commute. Since the diagonal maps are isomorphisms, we induce an isomorphism along the dotted arrow.

It's now a matter of diagram chasing to see that this isomorphism is compatible with the maps into $S\mathbf{I}_1$, and that it is compatible with the associativity and unitality constraints for the two monoidal structures just described. In other words, the identity functor on $K_1/S\mathbf{I}_1$ can be extended to a monoidal equivalence

$$(K_1/S\mathbf{I}_1, \bullet, \mathbf{e}) \simeq (K_1/S\mathbf{I}_1, \bullet', \mathbf{e}).$$

Thus we may legitimately take the tensor product to be given by $\bullet'$ on $K_1/S\mathbf{I}_1$ and still be left with a monoidal equivalence $\mathbb{K}/S\mathbf{I}_1 \simeq \mathbb{C}oll(S)$. We shall not spell out all the details in full here, since we shall not explicitly need to use the tensor product on $\mathbb{K}/S\mathbf{I}_1$ in what shall follow.

6.3 An alternative description

The above definition of a double club, though compact, is not very easy to work with; the following alternate description will make it much easier to prove that a double monad on a double category has the structure of a double club.

We begin by observing that if (S, η, μ) is a double monad on $\mathbb{K}$, then (S_0, η_0, μ_0) is a monad on K_0 and (S_1, η_1, μ_1) a monad on K_1. Therefore it makes sense to ask whether or not S_0 and S_1 are clubs in the sense of Chapter 3 on their respective categories, and once we have asked this, we may naturally ask whether this is sufficient to make S into a double club. In fact, as long as S has property (hps), the answer is yes:

Proposition 48. *If (S, η, μ) is a double monad on $\mathbb{K}$ such that:*

- *S has property (hps);*
- *S_0 and S_1 are clubs on the categories K_0 and K_1 respectively,*

then S is a double club.

Proof. We must check that $\mathbb{C}oll(S)$ is a sub-monoidal double category of $[\mathbb{K}, \mathbb{K}]_\psi/S\mathbf{I}$. Since $\mathbb{C}oll(S)$ is a vertically full sub-double category of $[\mathbb{K}, \mathbb{K}]_\psi/S\mathbf{I}$, it suffices to check that:

- $\mathbb{C}oll(S)_0$ is closed under the monoidal structure on $[\mathbb{K}, \mathbb{K}]_{v\psi}/S$;
- $\mathbb{C}oll(S)_1$ is closed under the monoidal structure on $[\mathbb{K}, \mathbb{K}]_{h\psi}/S\mathbf{I}$.

We begin with $\mathbb{C}oll(S)_0$. Now, we have evident forgetful functors

$$\pi_i \colon [\mathbb{K}, \mathbb{K}]_{v\psi}/S \to [K_i, K_i]/S_i \quad (\text{for } i = 0, 1)$$

which are strict monoidal. Since S_0 and S_1 are clubs, $Coll(S_i)$ is closed under the monoidal structure on $[K_i, K_i]/S_i$. But an object A of $[\mathbb{K}, \mathbb{K}]_{v\psi}$ lies in $\mathbb{C}oll(S)_0$

just when its projections $\pi_i(A)$ lie in $Coll(S_i)$; and hence we see that $\mathbb{C}oll(S)_0$ is closed under the monoidal structure on $[\mathbb{K}, \mathbb{K}]_{v\psi}$ as required.

Moving on to $\mathbb{C}oll(S)_1$, we first show that the unit object

$$\begin{tikzcd}
\mathrm{id}_{\mathbb{K}} \arrow[r, Rightarrow, "\mathbf{I}_{\mathrm{id}_{\mathbb{K}}}", "|" marking] \arrow[d, Rightarrow, "\eta"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\eta}"] & \mathrm{id}_{\mathbb{K}} \arrow[d, Rightarrow, "\eta"] \\
S \arrow[r, Rightarrow, "S\mathbf{I}"', "|" marking] & S
\end{tikzcd}$$

of $[\mathbb{K}, \mathbb{K}]_{h\psi}$ lies in $\mathbb{C}oll(S)_1$. By Proposition 19 and the fact that S_0 and S_1 are clubs, we have that η_0 and η_1 are cartesian natural transformations; hence $\eta \colon \mathrm{id}_{\mathbb{K}} \Rightarrow S$ is a cartesian vertical transformation. It remains to show that the central natural transformation of $\boldsymbol{\eta}$ is cartesian, i.e., that diagrams of the following form are pullbacks:

$$\begin{tikzcd}
\mathbf{I}_X \arrow[r, "\mathbf{I}_!"] \arrow[d, "\eta_{\mathbf{I}_X}"'] & \mathbf{I}_1 \arrow[d, "\eta_{\mathbf{I}_1}"] \\
S\mathbf{I}_X \arrow[r, "S\mathbf{I}_!"'] & S\mathbf{I}_1,
\end{tikzcd}$$

which is just the cartesianness of η_0. We now show that $\mathbb{C}oll(S)_1$ is closed under the binary tensor product on $[\mathbb{K}, \mathbb{K}]_{h\psi}$. So suppose we are given

$$\begin{tikzcd}
A_s \arrow[r, Rightarrow, "\mathbf{A}", "|" marking] \arrow[d, Rightarrow, "\alpha_s"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\alpha}"] & A_t \arrow[d, Rightarrow, "\alpha_t"] & & B_s \arrow[r, Rightarrow, "\mathbf{B}", "|" marking] \arrow[d, Rightarrow, "\beta"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\beta}"] & B_t \arrow[d, Rightarrow, "\beta_t"] \\
S \arrow[r, Rightarrow, "S\mathbf{I}_{\mathrm{id}}"', "|" marking] & S & \text{and} & S \arrow[r, Rightarrow, "S\mathbf{I}_{\mathrm{id}}"', "|" marking] & S
\end{tikzcd}$$

cartesian modifications; then their tensor product is the composite modification

$$\begin{tikzcd}
A_sB_s \arrow[r, Rightarrow, "\mathbf{A} \bullet \mathbf{B}", "|" marking] \arrow[d, Rightarrow, "\alpha_s\beta_s"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\alpha}\bullet\boldsymbol{\beta}"] & A_tB_t \arrow[d, Rightarrow, "\alpha_t\beta_t"] \\
SS \arrow[r, Rightarrow, "S\mathbf{I} \bullet S\mathbf{I}", "|" marking] \arrow[d, Rightarrow, "\mu"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\mu}"] & SS \arrow[d, Rightarrow, "\mu"] \\
S \arrow[r, Rightarrow, "S\mathbf{I}"', "|" marking] & S,
\end{tikzcd}$$

so it suffices to show that $\boldsymbol{\alpha} \bullet \boldsymbol{\beta}$ and $\boldsymbol{\mu}$ are cartesian modifications. We begin with $\boldsymbol{\alpha} \bullet \boldsymbol{\beta}$; the cartesianness of $\alpha_s\beta_s$ and $\alpha_t\beta_t$ follows from the fact that S_1 and S_0 are clubs on K_1 and K_0, and so it suffices to check that the central natural transformation of $\boldsymbol{\alpha} \bullet \boldsymbol{\beta}$ is cartesian. This central natural transformation has components

$$\mathbf{A}B_tX \otimes A_s\mathbf{B}X \xrightarrow{\boldsymbol{\alpha}_{B_tX}\otimes(\alpha_s)_{\mathbf{B}X}} S\mathbf{I}_{B_tX} \otimes S\mathbf{B}X \xrightarrow{S\mathbf{I}_{(\beta_t)_X}\otimes S\boldsymbol{\beta}_X} S\mathbf{I}_{SX} \otimes SS\mathbf{I}_X\,.$$

So, consider the following diagram:

$$\begin{tikzcd}
\mathbf{A}B_tX \arrow[r, "\mathbf{A}B_t!"] \arrow[d, "\boldsymbol{\alpha}_{B_tX}"'] & \mathbf{A}B_t1 \arrow[d, "\boldsymbol{\alpha}_{B_t1}"] \\
S\mathbf{I}_{B_tX} \arrow[r, "S\mathbf{I}_{B_t!}"'] \arrow[d, "S\mathfrak{e}_X"'] \arrow[dddd, bend right=60, "S\mathbf{I}_{(\beta_t)_X}"'] & S\mathbf{I}_{B_t1} \arrow[d, "S\mathfrak{e}_1"] \arrow[dddd, bend left=60, "S\mathbf{I}_{(\beta_t)_1}"] \\
SB_t\mathbf{I}_X \arrow[r, "SB_t\mathbf{I}_!"'] \arrow[d, "S(\beta_t)_{\mathbf{I}_X}"'] & SB_t\mathbf{I}_1 \arrow[d, "S(\beta_t)_{\mathbf{I}_1}"] \\
SS\mathbf{I}_X \arrow[r, "SS\mathbf{I}_!"'] \arrow[d, "S\mathfrak{e}_X^{-1}"'] & SS\mathbf{I}_1 \arrow[d, "S\mathfrak{e}_1^{-1}"] \\
S\mathbf{I}_{SX} \arrow[r, "S\mathbf{I}_{S!}"'] & S\mathbf{I}_{S1}.
\end{tikzcd}$$

The top square is a pullback by cartesianness of α, the second and fourth are pullbacks since their vertical sides are isomorphisms, and the third square is a pullback by cartesianness of β_t and because S_1 preserves cartesian natural transformations into S_1. Therefore the outside edge of this diagram is a pullback. Similarly, considering the diagram

$$\begin{tikzcd}
A_s\mathbf{B}X \arrow[r, "A_s\mathbf{B}!"] \arrow[d, "(\alpha_s)_{\mathbf{B}X}"'] & A_s\mathbf{B}1 \arrow[d, "(\alpha_s)_{\mathbf{B}1}"] \\
S\mathbf{B}X \arrow[r, "S\mathbf{B}!"'] \arrow[d, "S\boldsymbol{\beta}_X"'] & S\mathbf{B}1 \arrow[d, "S\boldsymbol{\beta}_1"] \\
SS\mathbf{I}_X \arrow[r, "SS\mathbf{I}_!"'] & SS\mathbf{I}_1,
\end{tikzcd}$$

the top square is a pullback by cartesianness of α_s, whilst the bottom square is a pullback by cartesianness of β and the fact that S_1 preserves cartesian transformations into S_1. Thus, forming the tensor product of these two diagrams and applying condition (hps1), we see therefore that the naturality squares for $\boldsymbol{\alpha} \bullet \boldsymbol{\beta}$ are pullbacks as required.

Finally, we check that $\boldsymbol{\mu}$ is a cartesian modification. By Proposition 19 and the fact that S_0 and S_1 are clubs, we have that μ_0 and μ_1 are cartesian natural transformations; hence $\mu \colon SS \Rightarrow S$ is a cartesian vertical transformation. So we need only check that the central natural transformation of $\boldsymbol{\mu}$ is cartesian, for which we must check that the outer edge of the following diagram is a pullback:

$$\begin{tikzcd}
S\mathbf{I}_{SX} \otimes SS\mathbf{I}_X \arrow[r, "S\mathbf{I}_{S!} \otimes SS!"] \arrow[d, "\mathfrak{m}_{\mathbf{I}_{SX}, S\mathbf{I}_X}"'] & S\mathbf{I}_{S1} \otimes SS\mathbf{I}_1 \arrow[d, "\mathfrak{m}_{\mathbf{I}_{S1}, S\mathbf{I}_1}"] \\
S(\mathbf{I}_{SX} \otimes S\mathbf{I}_X) \arrow[r, "S(\mathbf{I}_{S!} \otimes S!)"] \arrow[d, "S\mathfrak{l}^{-1}_{S\mathbf{I}_X}"'] & S(\mathbf{I}_{S1} \otimes S\mathbf{I}_1) \arrow[d, "S\mathfrak{l}^{-1}_{S\mathbf{I}_1}"] \\
SS\mathbf{I}_X \arrow[r, "SS!"] \arrow[d, "\mu_{\mathbf{I}_X}"'] & SS\mathbf{I}_1 \arrow[d, "\mu_{\mathbf{I}_1}"] \\
S\mathbf{I}_X \arrow[r, "S!"'] & S\mathbf{I}_1.
\end{tikzcd}$$

Now, the bottom square is a pullback by cartesianness of μ, whilst all other squares are pullbacks since they have isomorphisms along their vertical edges; hence the outer edge is a pullback as required. $\square$

Part II

The double club for symmetric monoidal categories

Chapter 7

The pseudo double category $\mathbb{C}at$

Having developed a theory of double clubs, we should now like to give an example of such. Above, we gave an example of a plain club on **Cat**, that for symmetric strict monoidal categories. What we shall do over the next two chapters is illustrate how we may extend this club to a double club. The first step in this process is to define the pseudo double category $\mathbb{C}at$ which this is to be a double club *on*.

$\mathbb{C}at$, the double category of 'categories, functors, profunctors and modifications', is one of the better-known pseudo double categories, explored in [GP99] and (in the guise of a 'fc-multicategory') [Lei04a]. It can be viewed as a generalisation of the bicategory **Mod** of 'categories, profunctors and modifications'. This bicategory, of course, dates back to the earliest days of category theory; for a concise modern reference, see [Bor94]. We begin, therefore, by describing this bicategory.

7.1 The bicategory Mod

We fix a presentation of the bicategory **Mod**, as follows:

- **Objects** are small categories $\mathbf{C}, \mathbf{D}, \ldots$;
- **Maps** $F\colon \mathbf{C} \twoheadrightarrow \mathbf{D}$ are functors $F\colon \mathbf{D}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$. We write a typical value of this functor as $F(d;c)$, and given maps $h\colon c \to c'$ in $\mathbf{C}$ and $f\colon d' \to d$ in $\mathbf{D}$, write the action of F on maps as

$$\begin{aligned} h \bullet (\text{–})\colon F(d;c) &\to F(d;c') \\ \text{and } (\text{–}) \bullet f\colon F(d;c) &\to F(d';c). \end{aligned}$$

Sometimes it will be useful to use 'arrow' notation; we write a typical element g of $F(d;c)$ as $g\colon d \to\!\!\!\!\!\mapsto c$, and given such an element, write the elements $h \bullet g$ and $g \bullet f$ as

$$\begin{tikzcd} d \arrow[r, "g", "\shortmid" marking] & c \arrow[r, "h"] & c' \end{tikzcd}$$

$$\text{and } \begin{tikzcd} d' \arrow[r, "f"] & d \arrow[r, "g", "\shortmid" marking] & c \end{tikzcd}$$

respectively. Analogously with categorical composition, we'll tend to drop the '$\bullet$' symbol where convenient, and denote these actions simply by juxtaposition;

- **2-cells** $\alpha\colon F \Rightarrow G$ are natural transformations $F \Rightarrow G\colon \mathbf{D}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$; we write the components of this transformation as

  $$\alpha_{d,c}\colon F(d;c) \to G(d;c).$$

  In practice, we drop the suffixes and use α indifferently for all these components; so given $g \in F(d;c)$, we write its image under $\alpha_{d,c}$ as $\alpha(g)$.

Recall that we have homomorphisms

$$(-)_*\colon \mathbf{Cat} \to \mathbf{Mod} \quad \text{and} \quad (-)^*\colon \mathbf{Cat} \to \mathbf{Mod}^{\mathrm{coop}}$$

which are the identity on objects, take a functor $F\colon \mathbf{C} \to \mathbf{D}$ to the respective profunctors

$$\begin{aligned} F_*\colon \mathbf{C} &\to\!\!\!\!\!\mapsto \mathbf{D} & \qquad & F^*\colon \mathbf{D} \to\!\!\!\!\!\mapsto \mathbf{C} \\ F_*(d;c) &= \mathbf{D}(d;Fc) & \text{and} \qquad & F^*(c;d) = \mathbf{D}(Fc;d) \\ F_*(g;f) &= \mathbf{D}(g;Ff) & & F^*(f;g) = \mathbf{D}(Ff;g), \end{aligned}$$

and take a natural transformation $\alpha\colon F \Rightarrow G$ to transformations $\alpha_*\colon F_* \Rightarrow G_*$ and $\alpha^*\colon G^* \Rightarrow F^*$ with respective components

$$(\alpha_*)_{d,c}\colon F_*(d;c) \Rightarrow G_*(d;c) = \alpha_c \circ (-)\colon \mathbf{D}(d;Fc) \to \mathbf{D}(d;Gc)$$

and

$$(\alpha^*)_{d,c} \colon G^*(d;c) \Rightarrow F^*(d;c) = (-) \circ \alpha_c \colon \mathbf{D}(Gc;d) \to \mathbf{D}(Fc;d).$$

7.2 The pseudo double category $\mathbb{C}at$

We are now ready for:

Definition 49. The pseudo double category $\mathbb{C}at$ is given as follows:

- $\mathbb{C}at_0 = \mathbf{Cat}$, the category of small categories;

- $\mathbb{C}at_1$ is the category with

  - **Objects** $\mathbf{X} = (X_s, X_t, X)$ made up of a pair of small categories X_s and X_t together with a profunctor $X \colon X_s \nrightarrow X_t$;

  - **Maps** $\mathbf{f} = (f_s, f_t, f) \colon \mathbf{X} \to \mathbf{Y}$ made up of a pair of functors $f_s \colon X_s \to Y_s$ and $f_t \colon X_t \to Y_t$ together with a 2-cell

$$\begin{tikzcd}
X_s \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "(f_s)_*"', "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow f"] & X_t \arrow[d, "(f_t)_*", "\shortmid"{marking}] \\
Y_s \arrow[r, "Y"', "\shortmid"{marking}] & Y_t
\end{tikzcd}$$

    of **Mod**. Equivalently, we can give a 2-cell

$$\begin{tikzcd}
X_t^{\mathrm{op}} \times X_s \arrow[rr, "f_t^{\mathrm{op}} \times f_s"] \arrow[dr, "X"'] & \overset{f}{\Rightarrow} & Y_t^{\mathrm{op}} \times Y_s \arrow[dl, "Y"] \\
& \mathbf{Set} &
\end{tikzcd}$$

    in **Cat**, and therefore natural families of maps

$$f_{x_t,x_s} \colon X(x_t;x_s) \to Y(f_t x_t; f_s x_s);$$

  - Identity maps $\mathrm{id}_{\mathbf{X}} \colon \mathbf{X} \to \mathbf{X}$ given by $(\mathrm{id}_{X_s}, \mathrm{id}_{X_t}, \mathrm{id}_X)$ where id_X is the

2-cell given by the pasting

$$\begin{tikzcd}
X_s \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "\mathbf{I}_{X_s}"', "\shortmid"{marking}] \arrow[d, bend right=70, "(\mathrm{id}_{X_s})_*"', "\shortmid"{marking}] & X_t \arrow[d, "\mathbf{I}_{X_t}", "\shortmid"{marking}] \arrow[d, bend left=70, "(id_{X_t})_*", "\shortmid"{marking}] \\
X_s \arrow[r, "X"', "\shortmid"{marking}] & X_t
\end{tikzcd}$$

- Composition $\mathbf{g} \circ \mathbf{f}$ given by $(g_s f_s, g_t f_t, g \circ f)$, where $g \circ f$ is the pasting

$$\begin{tikzcd}
X_s \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "(f_s)_*"', "\shortmid"{marking}] \arrow[dd, bend right=70, "(g_s f_s)_*"'] & X_t \arrow[d, "(f_t)_*", "\shortmid"{marking}] \arrow[dd, bend left=70, "(g_t f_t)_*", "\shortmid"{marking}] \arrow[dl, phantom, "\Downarrow f"] \\
Y_s \arrow[r, "Y"', "\shortmid"{marking}] \arrow[d, "(g_s)_*"', "\shortmid"{marking}] & Y_t \arrow[d, "(g_t)_*", "\shortmid"{marking}] \arrow[dl, phantom, "\Downarrow g"] \\
Z_s \arrow[r, "\mathbf{Z}"', "\shortmid"{marking}] & Z_t.
\end{tikzcd}$$

- We have evident functors $s, t \colon \mathbb{C}at_1 \to \mathbb{C}at_0$;

- The horizontal composition functor

$$\otimes \colon \mathbb{C}at_1 \ {}_s\!\times_t \ \mathbb{C}at_1 \to \mathbb{C}at_1$$

  is given as follows:

  - **On objects**: given $\mathbf{X} = X \colon A \mathrel{+\!\!\!\to} B$ and $\mathbf{Y} = Y \colon B \mathrel{+\!\!\!\to} C$, we take their composite in **Mod**:

$$\mathbf{Y} \otimes \mathbf{X} = A \xrightarrow{X} B \xrightarrow{Y} C;$$

  - **On maps**: given $\mathbf{f} \colon \mathbf{X} \to \mathbf{X}'$ and $\mathbf{g} \colon \mathbf{Y} \to \mathbf{Y}'$ as follows:

$$\begin{tikzcd}
A \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "(f_A)_*"', "\shortmid"{marking}] & B \arrow[d, "(f_B)_*", "\shortmid"{marking}] \arrow[dl, phantom, "\Downarrow f"] & \text{and} & B \arrow[r, "Y", "\shortmid"{marking}] \arrow[d, "(f_B)_*"', "\shortmid"{marking}] & C \arrow[d, "(f_C)_*", "\shortmid"{marking}] \arrow[dl, phantom, "\Downarrow g"] \\
A' \arrow[r, "X'"', "\shortmid"{marking}] & B' & & B' \arrow[r, "Y'"', "\shortmid"{marking}] & C'
\end{tikzcd}$$

we give $\mathbf{g} \otimes \mathbf{f} \colon \mathbf{Y} \otimes \mathbf{X} \to \mathbf{Y}' \otimes \mathbf{X}'$ by the pasting

$$\begin{tikzcd}
A \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "(f_A)_*"', "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow f"] & B \arrow[r, "Y", "\shortmid"{marking}] \arrow[d, "(f_B)_*", "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow g"] & C \arrow[d, "(f_C)_*", "\shortmid"{marking}] \\
A' \arrow[r, "X'"', "\shortmid"{marking}] & B' \arrow[r, "Y'"'] & C'
\end{tikzcd}$$

in **Mod**;

- The units functor $\mathbf{I} \colon \mathbb{C}at_0 \to \mathbb{C}at_1$ is given by:
  - **On objects**: given $X \in \mathbb{C}at_0$, we take for $\mathbf{I}_X$ the identity profunctor $\mathbf{I}_X \colon X \;\mathrlap{\to}{\,+}\; X$;
  - **On maps**: given $f \colon X \to Y \in \mathbb{C}at_0$, we take $\mathbf{I}_f \colon \mathbf{I}_X \to \mathbf{I}_Y$ to be given by

$$\begin{tikzcd}
X \arrow[r, "\mathbf{I}_X", "\shortmid"{marking}] \arrow[d, "f_*"', "\shortmid"{marking}] \arrow[dr, "f_*", "\shortmid"{marking}] & X \arrow[d, "f_*", "\shortmid"{marking}] \\
Y \arrow[r, "\mathbf{I}_Y"', "\shortmid"{marking}] & Y
\end{tikzcd}$$

This data clearly satisfies (DA1) and (DA2). The data (DD6) and (DD7) making composition pseudo-associative and pseudo-unital is given by the associativity and unitality 2-cells from **Mod**. That these components are natural in maps of $\mathbb{C}at_1$ is a straightforward application of the pasting theorem for bicategories, and that they satisfy (DA3) and (DA4) follows immediately from the coherence of the bicategory **Mod**.

Since we want to apply the theory of double clubs in $\mathbb{C}at$, we should check that $\mathbb{C}at_0$ and $\mathbb{C}at_1$ are sufficiently complete for our purposes. Evidently $\mathbb{C}at_0$ is finitely complete, whilst for $\mathbb{C}at_1$, we observe that it is isomorphic to the category $\mathbf{Cat}/\mathbf{2}$, where $\mathbf{2}$ is the arrow category $0 \to 1$, and hence finitely complete.

Indeed, we can give an explicit description of finite limits in $\mathbb{C}at_1$: the terminal object is $\mathbf{I}_1$, where 1 is the terminal category, whilst the pullback

$$\begin{tikzcd} \mathbf{D} \arrow[r, "\mathbf{j}"] \arrow[d, "\mathbf{k}"'] & \mathbf{C} \arrow[d, "\mathbf{f}"] \\ \mathbf{B} \arrow[r, "\mathbf{g}"'] & \mathbf{A} \end{tikzcd}$$

is given as follows: D_s, D_t, j_s, j_t, k_s and k_t arise from the pullbacks:

$$\begin{tikzcd} D_s \arrow[r, "j_s"] \arrow[d, "k_s"'] & C_s \arrow[d, "f_s"] \\ B_s \arrow[r, "g_s"'] & A_s \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D_t \arrow[r, "j_t"] \arrow[d, "k_t"'] & C_t \arrow[d, "f_t"] \\ B_t \arrow[r, "g_t"'] & A_t \end{tikzcd}$$

in **Cat**; whilst given $d_t = (c_t, b_t) \in D_t$ and $d_s = (c_s, b_s) \in D_s$, we have $D(d_t; d_s)$, j_{d_t,d_s} and k_{d_t,d_s} given by the pullback

$$\begin{tikzcd} D(d_t; d_s) \arrow[r, "j_{d_t,d_s}"] \arrow[d, "k_{d_t,d_s}"'] & C(c_t; c_s) \arrow[d, "f_{c_t,c_s}"] \\ B(b_t; b_s) \arrow[r, "g_{b_t,b_s}"'] & A(a_t; a_s) \end{tikzcd}$$

(where $a_s = f_s c_s = g_s b_s$ and $a_t = f_t c_t = g_t b_t$). Evidently, given a choice of pullbacks in $\mathbb{C}at_0$, we can choose pullbacks in $\mathbb{C}at_1$ such that s and t strictly preserve them. Thus $\mathbb{C}at$ satisfies the completeness properties we required in Chapter 3.

Now, $\mathbb{C}at$ also has the following property, of which we shall make use later:

Proposition 50. *The functor* $[s, t] \colon \mathbb{C}at_1 \to \mathbb{C}at_0 \times \mathbb{C}at_0$ *is a fibration.*

Proof. Suppose we are given $\mathbf{Y} = Y \colon Y_s \to\!\!\!\!\!\!+\, Y_t$ in $\mathbb{C}at_1$ and functors $f_s \colon X_s \to Y_s$ and $f_t \colon X_t \to Y_t$ in $\mathbb{C}at_0$; then we must construct a map $\langle f_s, f_t\rangle \colon \langle f_s, f_t\rangle_*(\mathbf{Y}) \to \mathbf{Y}$ in $\mathbb{C}at_1$ as follows:-

$$\begin{tikzcd} X_s \arrow[r, "\langle f_s,f_t\rangle_*(Y)", "\shortmid"{marking}] \arrow[d, "(f_s)_*"', "\shortmid"{marking}] & X_t \arrow[d, "(f_t)_*", "\shortmid"{marking}] \arrow[dl, phantom, "\Downarrow\langle f_s,f_t\rangle"] \\ Y_s \arrow[r, "Y"', "\shortmid"{marking}] & Y_t. \end{tikzcd}$$

So we take for $\langle f_s, f_t\rangle_*(Y)$ the profunctor given by

$$X_t^{\mathrm{op}} \times X_s \xrightarrow{f_t^{\mathrm{op}}\times f_s} Y_t^{\mathrm{op}} \times Y_s \xrightarrow{Y} \mathbf{Set},$$

and for the 2-cell $\langle f_s, f_t\rangle$, we take the identity natural transformation:

$$\begin{tikzcd}
X_t^{\mathrm{op}} \times X_s \arrow[rr, "f_t^{\mathrm{op}}\times f_s"] \arrow[dr, "Y\circ(f_t^{\mathrm{op}}\times f_s)"'] & \overset{\mathrm{id}}{\Rightarrow} & Y_t^{\mathrm{op}} \times Y_s \arrow[dl, "Y"] \\
& \mathbf{Set}. &
\end{tikzcd}$$

To see that $\langle f_s, f_t\rangle$ is a cartesian arrow, suppose we have an arrow $\mathbf{g}\colon \mathbf{W} \to \mathbf{Y}$:-

$$\begin{tikzcd}
W_s \arrow[r, "W", "\shortmid"{marking}] \arrow[d, "(g_s)_*"', "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow g"] & W_t \arrow[d, "(g_t)_*"] \\
Y_s \arrow[r, "Y"', "\shortmid"{marking}] & Y_t
\end{tikzcd}$$

of $\mathbb{C}at_1$ together with factorisations $g_s = f_s h_s$ and $g_t = f_t h_t$; then we must exhibit a factorisation $\mathbf{g} = \langle f_s, f_t\rangle \circ \mathbf{h}$ in $\mathbb{C}at_1$; so we give $\mathbf{h}$ by

$$\begin{tikzcd}
W_s \arrow[r, "W", "\shortmid"{marking}] \arrow[d, "(h_s)_*"', "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow h"] & W_t \arrow[d, "(h_t)_*", "\shortmid"{marking}] \\
X_s \arrow[r, "\langle f_s, f_t\rangle_*(Y)"', "\shortmid"{marking}] & X_t
\end{tikzcd}$$

where h is simply the 2-cell g:-

$$\begin{tikzcd}
W_t^{\mathrm{op}} \times W_s \arrow[rr, "h_t^{\mathrm{op}}\times h_s"] \arrow[dr, "W"'] & \overset{g}{\Rightarrow} & X_t^{\mathrm{op}} \times X_s \arrow[dl, "Y\circ(f_t^{\mathrm{op}}\times f_s)"] \\
& \mathbf{Set}. &
\end{tikzcd}$$

Easily we have $\mathbf{g} = \langle f_s, f_t\rangle \circ \mathbf{h}$ as required; and furthermore, any such factorisation is necessarily unique. $\square$

Chapter 8

The double club S

We wish to show that the symmetric strict monoidal category 2-monad S, as described Chapter 3, extends from a club on **Cat** to a double club on $\mathbb{C}at$, and for this we shall use the fact that we can lift it from a 2-monad on **Cat** to a pseudomonad (see Appendix B) on **Mod**.

This is a very special example of the theory of *pseudo-distributive laws*, as developed by [Mar99], [ECP] and [Tan04], a theory that we shall not venture into at present; instead we shall simply describe what is necessary in order for a pseudomonad on **Mod** to be a 'lifting' of a 2-monad on **Cat**. Although this latter information is implicit in the work of [Tan04], the details have not been worked out before.

Then, using the fact that such a lifting is possible for the symmetric strict monoidal category 2-monad S, we can show that S can be extended from a 2-monad on **Cat** to a double monad on $\mathbb{C}at$. The final step is to show that this double monad is in fact a double *club*, for which we use the characterisation of double clubs given in Proposition 48.

8.1 Lifting monads from Cat to Mod

Suppose we are given *any* 2-monad S on **Cat** and a pseudomonad $\hat{S}$ on **Mod** (see Appendix B for the notation used for pseudomonads). We should like to know what it means for $\hat{S}$ to be a 'lifting' of S. The work of [Tan04] tells us what this involves at an abstract level: namely, the existence of a 'pseudo-distributive law' between S and the free small cocompletion '2-monad' on **Cat** (for size reasons, this fails to be a genuine 2-monad, but we shall not worry about this here). However, we are rather more interested in what this amounts to at a concrete level:

Definition 51. Given a 2-monad (S, η, μ) on **Cat** and a pseudomonad $(\hat{S}, \hat{\eta}, \hat{\mu}, \lambda, \rho, \tau)$ on **Mod**, we say that $\hat{S}$ is a **lifting** of S if:

- $\hat{S}\mathbf{C} = S\mathbf{C}$ on objects;

- $\hat{S}$ and S are compatible on 1- and 2-cells in the following sense: observe that, as in Chapter 1, we may view **Cat** and **Mod** as pseudo double categories with only identity vertical arrows, and that further, we may view S, $\hat{S}$ and $(-)_*$ as homomorphisms between these double categories. Then we demand that there is an invertible vertical transformation

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, "(-)_*"] \arrow[d, "S"'] \arrow[dr, phantom, "\Downarrow\theta"] & \mathbf{Mod} \arrow[d, "\hat{S}"] \\
\mathbf{Cat} \arrow[r, "(-)_*"'] & \mathbf{Mod}.
\end{tikzcd}$$

  Explicitly, we have for all **C** and **D**, natural isomorphisms

$$\theta \colon \hat{S}\big((-)_*\big) \Rightarrow \big(S(-)\big)_* \colon \mathbf{Cat}(\mathbf{C}, \mathbf{D}) \to \mathbf{Mod}(S\mathbf{C}, S\mathbf{D})$$

  such that the diagrams

$$\begin{tikzcd}
\hat{S}(G_*) \otimes \hat{S}(F_*) \arrow[r, "\theta_G \otimes \theta_F"] \arrow[d, "\mathfrak{m}_{G_*,F_*}"'] & (SG)_* \otimes (SF)_* \arrow[d, "\mathfrak{m}_{SG,SF}"] \\
\hat{S}(G_* \otimes F_*) \arrow[d, "\hat{S}\mathfrak{m}_{G,F}"'] & (SG \circ SF)_* \arrow[d, "\mathrm{id}"] \\
\hat{S}\big((G \circ F)_*\big) \arrow[r, "\theta_{G \circ F}"'] & \big(S(G \circ F)\big)_*
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{I}_{\hat{S}\mathbf{C}} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathfrak{e}_{\mathbf{C}}"'] & \mathbf{I}_{S\mathbf{C}} \arrow[d, "\mathfrak{e}_{S\mathbf{C}}"] \\
\hat{S}(\mathbf{I}_{\mathbf{C}}) \arrow[d, "\hat{S}\mathfrak{e}_{\mathbf{C}}"'] & (\mathrm{id}_{S\mathbf{C}})_* \arrow[d, "\mathrm{id}"] \\
\hat{S}\big((\mathrm{id}_{\mathbf{C}})_*\big) \arrow[r, "\theta_{\mathrm{id}_{\mathbf{C}}}"'] & \big(S(\mathrm{id}_{\mathbf{C}})\big)_*
\end{tikzcd}$$

commute.

- The pseudo-natural transformations $\hat{\eta}$ and $\hat{\mu}$ have components given by

$$\hat{\eta}_{\mathbf{C}} = (\eta_{\mathbf{C}})_* \colon \mathbf{C} \mathrel{\ooalign{\hfil$\mapstochar$\hfil\cr$\to$\cr}} \hat{S}\mathbf{C}$$
$$\hat{\mu}_{\mathbf{C}} = (\mu_{\mathbf{C}})_* \colon \hat{S}\hat{S}\mathbf{C} \mathrel{\ooalign{\hfil$\mapstochar$\hfil\cr$\to$\cr}} \hat{S}\mathbf{C};$$

- The transformation θ is compatible with the pseudo-naturality 2-cells for $\hat{\eta}$ and $\hat{\mu}$ in the sense that, given a functor $F \colon \mathbf{C} \to \mathbf{D}$, the following diagrams commute:

$$\begin{tikzcd}
\hat{S}(F_*) \otimes (\eta_{\mathbf{C}})_* \arrow[r, "\theta_F \otimes \mathrm{id}"] \arrow[d, "\hat{\eta}_{F_*}"'] & (SF)_* \otimes (\eta_{\mathbf{C}})_* \arrow[r, "\mathfrak{m}_{SF,\eta_{\mathbf{C}}}"] & (SF \circ \eta_{\mathbf{C}})_* \arrow[d, "\mathrm{id}"] \\
(\eta_{\mathbf{D}})_* \otimes F_* \arrow[rr, "\mathfrak{m}_{\eta_{\mathbf{D}},F}"'] & & (\eta_{\mathbf{D}} \circ F)_*
\end{tikzcd}$$

and

$$\begin{tikzcd}
\hat{S}(F_*) \otimes (\mu_{\mathbf{C}})_* \arrow[r, "\theta_F \otimes \mathrm{id}"] \arrow[d, "\hat{\mu}_{F_*}"'] & (SF)_* \otimes (\mu_{\mathbf{C}})_* \arrow[r, "\mathfrak{m}_{SF,\mu_{\mathbf{C}}}"] & (SF \circ \mu_{\mathbf{C}})_* \arrow[d, "\mathrm{id}"] \\
(\mu_{\mathbf{D}})_* \otimes \hat{S}\hat{S}(F_*) \arrow[r, "\mathrm{id} \otimes (\theta_{SF} \circ \hat{S}\theta_F)"'] & (\mu_{\mathbf{D}})_* \otimes (SSF)_* \arrow[r, "\mathfrak{m}_{\eta_{\mathbf{D}},SSF}"'] & (\mu_{\mathbf{D}} \circ SSF)_*.
\end{tikzcd}$$

- The invertible modifications λ, ρ and τ have component 2-cells given by:

$$\begin{tikzcd}
\lambda_{\mathbf{C}} = (\mu_{\mathbf{C}})_* \otimes \hat{S}\big((\eta_{\mathbf{C}})_*\big), \arrow[d, Rightarrow, "\mathrm{id}\otimes\theta_{\eta_{\mathbf{C}}}"] & \rho_{\mathbf{C}} = (\mu_{\mathbf{C}})_* \otimes (\eta_{S\mathbf{C}})_*, \arrow[d, Rightarrow, "\mathfrak{m}_{\mu_{\mathbf{C}},\eta_{S\mathbf{C}}}"] & \text{and} & \tau_{\mathbf{C}} = (\mu_{\mathbf{C}})_* \otimes (\mu_{S\mathbf{C}})_* \,. \arrow[d, Rightarrow, "\mathfrak{m}_{\mu_{\mathbf{C}},\mu_{S\mathbf{C}}}"] \\
(\mu_{\mathbf{C}})_* \otimes (S\eta_{\mathbf{C}})_* \arrow[d, Rightarrow, "\mathfrak{m}_{\mu_{\mathbf{C}},S\eta_{\mathbf{C}}}"] & (\mu_{\mathbf{C}} \circ \eta_{S\mathbf{C}})_* \arrow[d, Rightarrow, "\mathrm{id}"] & & (\mu_{\mathbf{C}} \circ \mu_{S\mathbf{C}})_* \arrow[d, Rightarrow, "\mathrm{id}"] \\
(\mu_{\mathbf{C}} \circ S\eta_{\mathbf{C}})_* \arrow[d, Rightarrow, "\mathrm{id}"] & (\mathrm{id}_{S\mathbf{C}})_* \arrow[d, Rightarrow, "\mathfrak{e}_{S\mathbf{C}}"] & & (\mu_{\mathbf{C}} \circ S\mu_{\mathbf{C}})_* \arrow[d, Rightarrow, "\mathfrak{m}^{-1}_{\mu_{\mathbf{C}},S\mu_{\mathbf{C}}}"] \\
(\mathrm{id}_{S\mathbf{C}})_* \arrow[d, Rightarrow, "\mathfrak{e}_{S\mathbf{C}}"] & \mathbf{I}_{S\mathbf{C}} & & (\mu_{\mathbf{C}})_* \otimes (S\mu_{\mathbf{C}})_* \arrow[d, Rightarrow, "\mathrm{id}\otimes\theta^{-1}_{\mu_{\mathbf{C}}}"] \\
\mathbf{I}_{S\mathbf{C}} & & & (\mu_{\mathbf{C}})_* \otimes \hat{S}(\mu_{\mathbf{C}})_*
\end{tikzcd}$$

In the sequel, we shall often need to produce pasting diagrams involving the coherence 2-cells for $(-)_*$ and $\hat{S}$, or some of the 2-cells θ_F as exhibited above; for the sake of a clearer presentation we shall leave such 2-cells unlabelled where it is clear how they should be filled in.

8.2 The pseudomonad $\hat{S}$ on Mod

Now, in the case of interest to us, it happens that the 2-monad (S, η, μ) of Chapter 3 can be lifted to a pseudomonad $(\hat{S}, \hat{\eta}, \hat{\mu}, \lambda, \rho, \tau)$ on **Mod**, in the sense of Definition 51. A more thorough look at this particular lifting may be found in [Tan04]; therefore we shall not check the details here, but merely describe those parts of the pseudomonad $\hat{S}$ which will be of use to us later.

In particular, we shall omit describing the 'coherence' data for $\hat{S}$, namely the comparison isomorphisms $\hat{S}\mathrm{id}_{\mathbf{C}} \cong \mathrm{id}_{S\mathbf{C}}$ and $\hat{S}(G \otimes F) \cong \hat{S}G \otimes \hat{S}F$ for $\hat{S}$, the pseudo-naturality isomorphisms for $\hat{\eta}$ and $\hat{\mu}$, and the invertible modifications λ, ρ and τ. However, we shall describe the remaining data:

Definition 52. The homomorphism $\hat{S}\colon \mathbf{Mod} \to \mathbf{Mod}$ is given as follows:

- **On objects**: Given a small category $\mathbf{C}$, we take $\hat{S}\mathbf{C} = S\mathbf{C}$;
- **On maps**: Given a map $F\colon \mathbf{C} \twoheadrightarrow \mathbf{D}$, the map $\hat{S}F\colon S\mathbf{C} \twoheadrightarrow S\mathbf{D}$ is the following profunctor: an element of $\hat{S}F\big((n, \langle d_i \rangle); (m, \langle c_i \rangle)\big)$ is given by

$$(\sigma, \langle g_i \rangle)\colon (n, \langle d_i \rangle) \twoheadrightarrow (m, \langle c_i \rangle),$$

where $\sigma \in S1(n,m)$ and $g_i \in F(d_i; c_{\sigma(i)})$, whilst the action of maps $(\tau, \langle h_i \rangle) \colon (m, \langle c_i \rangle) \to (m', \langle c'_i \rangle)$ and $(\upsilon, \langle f_i \rangle) \colon (n', \langle d'_i \rangle) \to (n, \langle d_i \rangle)$ is given by

$$(\sigma, \langle g_i \rangle) \bullet (\upsilon, \langle f_i \rangle) = (\sigma \circ \upsilon, \left\langle g_{\upsilon(i)} \bullet f_i \right\rangle)$$
$$(\tau, \langle h_i \rangle) \bullet (\sigma, \langle g_i \rangle) = (\tau \circ \sigma, \left\langle h_{\sigma(i)} \bullet g_i \right\rangle);$$

- **On 2-cells**: Given a transformation $\alpha \colon F \Rightarrow G \colon \mathbf{C} \to\!\!\!\!\!\!+\; \mathbf{D}$, we give $S\alpha \colon SF \Rightarrow SG \colon S\mathbf{C} \to\!\!\!\!\!\!+\; S\mathbf{D}$ by
$$(S\alpha)(\sigma, \langle g_i \rangle) = (\sigma, \langle \alpha(g_i) \rangle).$$

Further, the pseudo-natural transformations

$$\hat{\eta} \colon \mathrm{id} \Rightarrow \hat{S} \colon \mathbf{Mod} \to \mathbf{Mod}$$
$$\text{and } \hat{\mu} \colon \hat{S}^2 \Rightarrow \hat{S} \colon \mathbf{Mod} \to \mathbf{Mod}$$

have respective components

$$\hat{\eta}_X = (\eta_X)_* \quad \text{and} \quad \hat{\mu}_X = (\mu_X)_*.$$

8.3 Lifting S to $\mathbb{C}at$

We are now ready to show that the symmetric monoidal category 2-monad (S, η, μ) on **Cat** extends to a double monad on $\mathbb{C}at$. For this section only, let us change our notation slightly, and write (S_0, η_0, μ_0) for the free symmetric strict monoidal category monad on $\mathbf{Cat} = \mathbb{C}at_0$.

Proposition 53. *We can extend S_0 to a homomorphism of double categories $S \colon \mathbb{C}at \to \mathbb{C}at$.*

Proof. We use S_0 as the data (DMD1) for S; for (DMD2), we need to give a functor $S_1 \colon \mathbb{C}at_1 \to \mathbb{C}at_1$, which we do as follows:

- **On objects:** $S_1(X_s \xrightarrow{X} X_t) = S_0 X_s \xrightarrow{\hat{S}X} S_0 X_t$;

- **On maps:** given $\mathbf{f} \colon \mathbf{X} \to \mathbf{Y}$, we let $S_1\mathbf{f}$ be given by $(S_0 f_s, S_0 f_t, Sf)$, where

where Sf is the 2-cell

$$\begin{tikzcd}
S_0X_s \arrow[r, "\hat{S}X", "\shortmid"{marking}] \arrow[d, "\hat{S}((f_s)_*)", "\shortmid"{marking}] \arrow[d, bend right=60, "(S_0f_s)_*"', "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow \hat{S}f"] & S_0X_t \arrow[d, "\hat{S}((f_t)_*)"', "\shortmid"{marking}] \arrow[d, bend left=60, "(S_0f_t)_*", "\shortmid"{marking}] \\
S_0Y_s \arrow[r, "\hat{S}Y"', "\shortmid"{marking}] & S_0Y_t
\end{tikzcd}$$

(note that we are abusively treating $\hat{S}$ as a 2-functor here, omitting the canonical 2-cells such as $\hat{S}((f_t)_* \otimes X) \cong \hat{S}(f_t)_* \otimes \hat{S}X$; that we may do so is a consequence of the coherence theorem for bicategories).

It is straightforward to check that S_1 thus defined is a functor. Now the diagram (DMA1):

$$\begin{tikzcd}
& \mathbb{C}at_1 \arrow[dl, "s"'] \arrow[d, "S_1"] \arrow[dr, "t"] & \\
\mathbb{C}at_0 \arrow[d, "S_0"'] & \mathbb{C}at_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \mathbb{C}at_0 \arrow[d, "S_0"] \\
\mathbb{C}at_0 & & \mathbb{C}at_0
\end{tikzcd}$$

commutes as required, and thus we write S interchangeably for S_0 and S_1. We now need to give the data (DMD3) and (DMD4), that is, special natural isomorphisms

$$\mathfrak{e}_X \colon \mathbf{I}_{SX} \to S\mathbf{I}_X$$
$$\text{and } \mathfrak{m}_{\mathbf{Y},\mathbf{X}} \colon S\mathbf{Y} \otimes S\mathbf{X} \to S(\mathbf{Y} \otimes \mathbf{X}).$$

But this is straightforward; we simply take the respective isomorphisms

$$\begin{tikzcd}
SX \arrow[r, "\mathbf{I}_{SX}", "\shortmid"{marking}] \arrow[d, "(\mathrm{id}_{SX})_*"', "\shortmid"{marking}] & SX \arrow[d, "(\mathrm{id}_{SX})_*", "\shortmid"{marking}] \\
SX \arrow[r, "\mathbf{I}_{SX}", "\shortmid"{marking}] \arrow[r, bend right=50, "\hat{S}\mathbf{I}_X"', "\shortmid"{marking}] & SX
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
SA \arrow[r, "\hat{S}X", "\shortmid"{marking}] \arrow[d, "(\mathrm{id}_{SA})_*"', "\shortmid"{marking}] & SB \arrow[r, "\hat{S}Y", "\shortmid"{marking}] \arrow[d, "(\mathrm{id}_{SB})_*", "\shortmid"{marking}] & SC \arrow[d, "(\mathrm{id}_{SC})_*", "\shortmid"{marking}] \\
SA \arrow[r, "\hat{S}X", "\shortmid"{marking}] \arrow[rr, bend right=30, "\hat{S}(Y \otimes X)"', "\shortmid"{marking}] & SB \arrow[r, "\hat{S}Y", "\shortmid"{marking}] & SC.
\end{tikzcd}$$

That these are natural and satisfy (DMA2) and (DMA3) follows from the naturality and coherence of the mediating 2-cells of $\hat{S} \colon \mathbf{Mod} \to \mathbf{Mod}$ from which they are derived. $\square$

Proposition 54. *We can extend the natural transformations* $\eta_0\colon \mathrm{id}_{\mathbf{Cat}} \Rightarrow S_0$ *and* $\mu_0\colon S_0S_0 \Rightarrow S_0$ *to vertical transformations* $\eta\colon \mathrm{id}_{\mathbb{C}at} \Rightarrow S$ *and* $\mu\colon SS \Rightarrow S$*.*

Proof. We must produce the data (VTD2), that is, natural transformations $\eta_1\colon \mathrm{id}_{\mathbb{C}at_1} \Rightarrow S_1$ and $\mu_1\colon S_1S_1 \Rightarrow S_1$. Now, we know that η_0 lifts to a pseudo-natural transformation

$$\hat{\eta}\colon \mathrm{id}_{\mathbf{Mod}} \Rightarrow \hat{S}\colon \mathbf{Mod} \to \mathbf{Mod}$$

with components $\hat{\eta}_X = (\eta_X)_*$; thus we take $(\eta_1)_{\mathbf{X}} = (\eta_{X_s}, \eta_{X_t}, \eta_X)$, where η_X is the pseudo-naturality 2-cell

$$\begin{tikzcd}
X_s \arrow[r, "X", "|" marking] \arrow[d, "(\eta_{X_s})_*"', "|" marking] \arrow[dr, phantom, "\Downarrow \hat{\eta}_X"] & X_t \arrow[d, "(\eta_{X_t})_*", "|" marking] \\
SX_s \arrow[r, "\hat{S}X"', "|" marking] & SX_t.
\end{tikzcd}$$

To check that η_1 is natural in $\mathbf{X}$, we need to show that the two pastings

$$\begin{tikzcd}
X_s \arrow[r, "X", "|" marking] \arrow[d, "(f_s)_*" description] \arrow[dr, phantom, "\Downarrow f"] \arrow[dd, bend right=60, "(\eta_{Y_s}\circ f_s)_*"', "|" marking] & X_t \arrow[d, "(f_t)_*" description] \arrow[dd, bend left=60, "(\eta_{Y_t}\circ f_t)_*", "|" marking] \\
Y_s \arrow[r, "Y"', "|" marking] \arrow[d, "(\eta_{Y_s})_*" description] \arrow[dr, phantom, "\Downarrow \hat{\eta}_Y"] & Y_t \arrow[d, "(\eta_{Y_t})_*" description] \\
SY_s \arrow[r, "\hat{S}Y"', "|" marking] & SY_t
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X_s \arrow[r, "X", "|" marking] \arrow[d, "(\eta_{X_s})_*" description] \arrow[dr, phantom, "\Downarrow \hat{\eta}_X"] \arrow[dd, bend right=60, "(Sf_s\circ \eta_{X_s})_*"', "|" marking] & X_t \arrow[d, "(\eta_{X_t})_*" description] \arrow[dd, bend left=60, "(Sf_t\circ \eta_{X_t})_*", "|" marking] \\
SX_s \arrow[r, "\hat{S}X"', "|" marking] \arrow[d, "(Sf_s)_*" description] \arrow[dr, phantom, "\Downarrow Sf"] & SX_t \arrow[d, "(Sf_t)_*" description] \\
SY_s \arrow[r, "\hat{S}Y"', "|" marking] & SY_t
\end{tikzcd}$$

agree; but observe that

$$\begin{tikzcd}
 & & X_t \arrow[rrd, "(f_t)_*", "\shortmid"{marking}] & & \\
X_s \arrow[rru, "X", "\shortmid"{marking}] \arrow[rrd, "(f_s)_*", "\shortmid"{marking}] \arrow[d, "(\eta_{X_s})_*"', "\shortmid"{marking}] & & \Downarrow f & & Y_t \arrow[d, "(\eta_{Y_t})_*", "\shortmid"{marking}] \\
SX_s \arrow[rrd, "\hat{S}((f_s)_*)"', "\shortmid"{marking}] & \Downarrow \hat{\eta}_{(f_s)_*} & Y_s \arrow[rru, "Y", "\shortmid"{marking}] \arrow[d, "(\eta_{Y_s})_*", "\shortmid"{marking}] & \Downarrow \hat{\eta}_Y & SY_t \\
 & & SY_s \arrow[rru, "\hat{S}Y"', "\shortmid"{marking}] & &
\end{tikzcd}
=
\begin{tikzcd}
 & & X_t \arrow[d, "(\eta_{X_t})_*", "\shortmid"{marking}] \arrow[rrd, "(f_t)_*", "\shortmid"{marking}] & & \\
X_s \arrow[rru, "X", "\shortmid"{marking}] \arrow[d, "(\eta_{X_s})_*"', "\shortmid"{marking}] & \Downarrow \hat{\eta}_X & SX_t \arrow[rrd, "\hat{S}((f_t)_*)", "\shortmid"{marking}] & \Downarrow \hat{\eta}_{(f_t)_*} & Y_t \arrow[d, "(\eta_{Y_t})_*", "\shortmid"{marking}] \\
SX_s \arrow[rru, "\hat{S}X", "\shortmid"{marking}] \arrow[rrd, "\hat{S}((f_s)_*)"', "\shortmid"{marking}] & & \Downarrow \hat{S}f & & SY_t \\
 & & SY_s \arrow[rru, "\hat{S}Y"', "\shortmid"{marking}] & &
\end{tikzcd}$$

by the pseudo-naturality of $\hat{\eta}$, whence the result follows easily. We argue entirely analogously to define μ_1. It remains to check (VTA1) and (VTA2). For (VTA1), it's evident that we have $\eta_0 s = s\eta_1$, $\eta_0 t = t\eta_1$, $\mu_0 s = s\mu_1$ and $\mu_0 t = t\mu_1$ as required. For (VTA2) for η, we must show that the diagram

$$\begin{tikzcd}
\mathbf{Y} \otimes \mathbf{X} \arrow[r, "\mathrm{id}"] \arrow[d, "\eta_{\mathbf{Y}} \otimes \eta_{\mathbf{X}}"'] & \mathbf{Y} \otimes \mathbf{X} \arrow[d, "\eta_{\mathbf{Y} \otimes \mathbf{X}}"] \\
S\mathbf{Y} \otimes S\mathbf{X} \arrow[r, "\mathfrak{m}_{\mathbf{Y},\mathbf{X}}"'] & S(\mathbf{Y} \otimes \mathbf{X})
\end{tikzcd}$$

commutes; but this is to say that the following 2-cells are equal:

$$\begin{tikzcd}
A \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "(\eta_A)_*"', "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow \hat{\eta}_X"] & B \arrow[r, "Y", "\shortmid"{marking}] \arrow[d, "(\eta_B)_*"', "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow \hat{\eta}_Y"] & C \arrow[d, "(\eta_C)_*", "\shortmid"{marking}] \\
SA \arrow[r, "\hat{S}X"', "\shortmid"{marking}] \arrow[rr, bend right, "\hat{S}(Y \otimes X)"', "\shortmid"{marking}] & SB \arrow[r, "\hat{S}Y"', "\shortmid"{marking}] & SC
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "X", "\shortmid"{marking}] \arrow[d, "(\eta_A)_*"', "\shortmid"{marking}] & B \arrow[r, "Y", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \hat{\eta}_{Y \otimes X}"] & C \arrow[d, "(\eta_C)_*", "\shortmid"{marking}] \\
SA \arrow[rr, "\hat{S}(Y \otimes X)"', "\shortmid"{marking}] & & SC,
\end{tikzcd}$$

which indeed they are by the axioms making $\hat{\eta}$ into a pseudo-natural transformation. Similarly, for (VTA2) for μ, we need to check that the 2-cells

$$\begin{tikzcd}
SSA \arrow[r, "\hat{S}\hat{S}X", "\shortmid"{marking}] \arrow[d, "(\mu_A)_*"', "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow\hat{\mu}_X"] & SSB \arrow[r, "\hat{S}\hat{S}X", "\shortmid"{marking}] \arrow[d, "(\mu_B)_*"] \arrow[dr, phantom, "\Downarrow\hat{\mu}_Y"] & C \arrow[d, "(\mu_C)_*", "\shortmid"{marking}] \\
SA \arrow[r, "\hat{S}X"', "\shortmid"{marking}] \arrow[rr, bend right, "\hat{S}(Y\otimes X)"', "\shortmid"{marking}] & SB \arrow[r, "\hat{S}Y"'] & SC
\end{tikzcd}$$

and

$$\begin{tikzcd}
SSA \arrow[r, "\hat{S}\hat{S}X", "\shortmid"{marking}] \arrow[rr, bend right, "\hat{S}\hat{S}(Y\otimes X)"', "\shortmid"{marking}] \arrow[d, "(\mu_A)_*"', "\shortmid"{marking}] & SSB \arrow[r, "\hat{S}\hat{S}Y", "\shortmid"{marking}] & SSC \arrow[d, "(\mu_C)_*", "\shortmid"{marking}] \\
SA \arrow[rr, bend right, "\hat{S}(Y\otimes X)"', "\shortmid"{marking}] & \Downarrow\hat{\mu}_{Y\otimes X} & SC
\end{tikzcd}$$

agree, which they do by the axioms for $\hat{\mu}$. ☐

Proposition 55. *(S, η, μ) is a double monad on $\mathbb{C}at$.*

Proof. We need to check that the monad laws

$$\mu \circ \mu S = \mu \circ S\mu$$
$$\mu \circ S\eta = \mathrm{id}_S$$
$$\mu \circ \eta S = \mathrm{id}_S$$

hold in $\mathbf{DblCat}_\psi(\mathbb{C}at, \mathbb{C}at)$. Since we already know that (S_0, η_0, μ_0) is a monad on $\mathbb{C}at_0$, it suffices to check that the monad laws hold for (S_1, η_1, μ_1) on $\mathbb{C}at_1$. So let us demonstrate that $\mu_1 \circ S_1\eta_1 = \mathrm{id}_{\mathbb{C}at_1}$; we need that the two pastings

$$\begin{tikzcd}
SX_s \arrow[r, "\hat{S}X", "\shortmid"{marking}] \arrow[d, "(S\eta_{X_s})_*"] \arrow[dr, phantom, "\Downarrow\hat{S}\hat{\eta}_X"] \arrow[dd, bend right=60, "(\mu_{X_s}S\eta_{X_s})_*"', "\shortmid"{marking}] & SX_t \arrow[d, "(S\eta_{X_t})_*"] \arrow[dd, bend left=60, "(\mu_{X_t}S\eta_{X_t})_*", "\shortmid"{marking}] \\
SSX_s \arrow[r, "\hat{S}\hat{S}X"', "\shortmid"{marking}] \arrow[d, "(\mu_{X_s})_*"] \arrow[dr, phantom, "\Downarrow\hat{\mu}_X"] & SSX_t \arrow[d, "(\mu_{X_t})_*"] \\
SX_s \arrow[r, "\hat{S}X"', "\shortmid"{marking}] & SX_t
\end{tikzcd}$$

and

$$\begin{tikzcd}
SX_s \arrow[r, "\hat{S}X", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_{SX_s}"] \arrow[d, bend right=60, "(\mathrm{id}_{SX_s})_*"', "\shortmid"{marking}] & SX_t \arrow[d, "\mathrm{id}_{SX_t}"'] \arrow[d, bend left=60, "(id_{SX_t})_*", "\shortmid"{marking}] \\
SX_s \arrow[r, "\hat{S}X"', "\shortmid"{marking}] & SX_t
\end{tikzcd}$$

are the same. For this, we consider the invertible modification

$$\lambda\colon \hat{\mu} \circ \hat{S}\hat{\eta} \Rrightarrow \mathrm{id}_{\hat{S}}\colon \hat{S} \Rightarrow \hat{S}\colon \mathbf{Mod} \to \mathbf{Mod}$$

which is part of the pseudomonad structure of $\hat{S}$. This satisfies

$$\begin{tikzcd}
SX_s \arrow[r, "\hat{S}X", "\shortmid" marking] \arrow[d, "\hat{S}(\eta_{X_s})_*"'] \arrow[dr, phantom, "\Downarrow \hat{S}\hat{\eta}_X"] & SX_t \arrow[d, "\hat{S}(\eta_{X_t})_*"'] \arrow[dd, bend left=60, "\mathrm{id}_{SX_t}", "\shortmid" marking] \\
SSX_s \arrow[r, "\hat{S}\hat{S}X"', "\shortmid" marking] \arrow[d, "(\mu_{X_s})_*"'] \arrow[dr, phantom, "\Downarrow \hat{\mu}_X"] & SSX_t \arrow[d, "(\mu_{X_t})_*"'] \arrow[r, phantom, "\overset{\lambda_{X_t}}{\Rightarrow}"] & {} \\
SX_s \arrow[r, "\hat{S}X"', "\shortmid" marking] & SX_t
\end{tikzcd}
=
\begin{tikzcd}
SX_s \arrow[r, "\hat{S}X", "\shortmid" marking] \arrow[d, "\hat{S}(\eta_{X_s})_*"'] \arrow[dd, bend left=60, "\mathrm{id}_{SX_s}", "\shortmid" marking] & SX_t \arrow[dd, bend left=60, "\mathrm{id}_{SX_t}", "\shortmid" marking] \\
SSX_s \arrow[d, "(\mu_{X_s})_*"'] \arrow[r, phantom, "\overset{\lambda_{X_s}}{\Rightarrow}"] & {} \\
SX_s \arrow[r, "\hat{S}X"', "\shortmid" marking] & SX_t.
\end{tikzcd}$$

But now from the previous section, we know that the 2-cell λ_X is given by the pasting

$$\begin{tikzcd}
SX \arrow[d, "\hat{S}(\eta_X)_*"'] \arrow[d, bend left, "(S\eta_X)_*"] \arrow[dd, bend left=60, "(\mu_X S\eta_X)_*"] \arrow[dd, bend left=90, "\mathrm{id}_{SX}"] \\
SSX \arrow[d, "(\mu_X)_*"'] \\
SX
\end{tikzcd}$$

and therefore the result follows easily. We proceed similarly for the other unit law and the associativity law. □

8.4 S is a double club

Now we know that (S, η, μ) is a double monad on $\mathbb{C}at$, we are ready to check that it is also a double *club*. Using Proposition 48, it suffices to check two things for this: firstly, that S has property (hps), and secondly, that S_0 and S_1 are clubs on their respective categories.

8.4.1 Property (hps)

To show that S satisfies property (hps), we shall use the following two propositions:

Proposition 56. *Suppose that*

$$\begin{tikzcd} D \arrow[r, "j"] \arrow[d, "k"'] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

is a pullback in $\mathbb{C}at_0$*; then so is*

$$\begin{tikzcd} \mathbf{I}_D \arrow[r, "\mathbf{I}_j"] \arrow[d, "\mathbf{I}_k"'] & \mathbf{I}_C \arrow[d, "\mathbf{I}_g"] \\ \mathbf{I}_B \arrow[r, "\mathbf{I}_f"'] & \mathbf{I}_A \end{tikzcd}$$

in $\mathbb{C}at_1$*.*

Proof. Viewing $\mathbb{C}at_1$ as $\mathbf{Cat}/\mathbf{2}$, we see that the functor $\mathbf{I}_{(\ )} \colon \mathbb{C}at_0 \to \mathbb{C}at_1$ sends D to $(D \times \mathbf{2}) \xrightarrow{\pi_2} \mathbf{2}$; now it's easy to see that this functor preserves small limits and so *a fortiori* the result. □

Proposition 57. *Let* A *be a small groupoidal category and suppose we are given pullback diagrams*

$$(23) := \begin{tikzcd} \mathbf{D}_{23} \arrow[r, "\mathbf{j}_{23}"] \arrow[d, "\mathbf{k}_{23}"'] & \mathbf{C}_{23} \arrow[d, "\mathbf{g}_{23}"] \\ \mathbf{B}_{23} \arrow[r, "\mathbf{f}_{23}"'] & \mathbf{I}_A \end{tikzcd} \quad \textit{and} \quad (12) := \begin{tikzcd} \mathbf{D}_{12} \arrow[r, "\mathbf{j}_{12}"] \arrow[d, "\mathbf{k}_{12}"'] & \mathbf{C}_{12} \arrow[d, "\mathbf{g}_{12}"] \\ \mathbf{B}_{12} \arrow[r, "\mathbf{f}_{12}"'] & \mathbf{I}_A \end{tikzcd}$$

in $\mathbb{C}at_1$ *with*

$$s(12) = \begin{tikzcd} D_1 \arrow[r, "j_1"] \arrow[d, "k_1"'] & C_1 \arrow[d, "g_1"] \\ B_1 \arrow[r, "f_1"'] & A, \end{tikzcd} \qquad t(12) = \begin{tikzcd} D_2 \arrow[r, "j_2"] \arrow[d, "k_2"'] & C_2 \arrow[d, "g_2"] \\ B_2 \arrow[r, "f_2"'] & A, \end{tikzcd}$$

$$s(23) = \begin{tikzcd} D_2 \arrow[r, "j_2"] \arrow[d, "k_2"'] & C_2 \arrow[d, "g_2"] \\ B_2 \arrow[r, "f_2"'] & A, \end{tikzcd} \quad \textit{and} \quad t(23) = \begin{tikzcd} D_3 \arrow[r, "j_3"] \arrow[d, "k_3"'] & C_3 \arrow[d, "g_3"] \\ B_3 \arrow[r, "f_3"'] & A. \end{tikzcd}$$

Suppose further that the arrow $f_2\colon B_2 \to A$ is a fibration; then the diagram

$$(13) := \begin{tikzcd} \mathbf{D}_{23} \otimes \mathbf{D}_{12} \arrow[r, "\mathbf{j}_{23}\otimes\mathbf{j}_{12}"] \arrow[d, "\mathbf{k}_{23}\otimes\mathbf{k}_{12}"'] & \mathbf{C}_{23} \otimes \mathbf{C}_{12} \arrow[d, "\mathbf{g}_{23}\otimes\mathbf{g}_{12}"] \\ \mathbf{B}_{23} \otimes \mathbf{B}_{12} \arrow[r, "\mathbf{f}_{23}\otimes\mathbf{f}_{12}"'] & \mathbf{I}_A \otimes \mathbf{I}_A \end{tikzcd}$$

is also a pullback.

Proof. First some notation; we shall use b_i, c_i and d_i to denote typical elements of B_i, C_i and D_i (for $i = 1, \ldots, 3$), and similarly use a_i to denote elements of A, with the convention that

$$k_i(d_i) = b_i, \quad j_i(d_i) = c_i, \quad \text{and} \quad f_i(b_i) = a_i = g_i(c_i).$$

So now, let $\mathbf{E} = (E_1, E_2, E)$ be the pullback

$$\begin{tikzcd} \mathbf{E} \arrow[r, "\mathbf{j}'"] \arrow[d, "\mathbf{k}'"'] & \mathbf{C}_{23} \otimes \mathbf{C}_{12} \arrow[d, "\mathbf{g}_{23}\otimes\mathbf{g}_{12}"] \\ \mathbf{B}_{23} \otimes \mathbf{B}_{12} \arrow[r, "\mathbf{f}_{23}\otimes\mathbf{f}_{12}"'] & \mathbf{I}_A \otimes \mathbf{I}_A. \end{tikzcd}$$

The universal property of pullback induces a canonical arrow

$$\mathbf{u} = (u_1, u_2, u)\colon \mathbf{D}_{23} \otimes \mathbf{D}_{12} \to \mathbf{E}$$

in $\mathbb{C}at_1$. It suffices to show that this map is an isomorphism. Observe first that $s(13) = s(12)$ and $t(13) = t(23)$, and thus that these projections are pullback diagrams in **Cat**. Thus we may take it that $E_1 = D_1$ and $E_2 = D_3$, and that $u_1 = \mathrm{id}_{D_1}$ and $u_2 = \mathrm{id}_{D_3}$. Thus we need only concern ourselves with the 2-cell u; we shall exhibit an inverse v for this 2-cell. First, let us describe explicitly what u does. A typical element of $\mathbf{D}_{23} \otimes \mathbf{D}_{12}(d_3; d_1)$ looks like

$$\big((\alpha, \gamma) \otimes (\beta, \delta)\big) = \big((b_3, c_3) \xrightarrow{(\alpha,\gamma)} (b_2, c_2)\big) \otimes \big((b_2, c_2) \xrightarrow{(\beta,\delta)} (b_1, c_1)\big)$$

where $\alpha\colon b_3 \mathrel{\mapstochar\relbar\joinrel\rightarrow} b_2$, $\beta\colon b_2 \mathrel{\mapstochar\relbar\joinrel\rightarrow} b_1$, $\gamma\colon c_3 \mathrel{\mapstochar\relbar\joinrel\rightarrow} c_2$, and $\delta\colon c_2 \mathrel{\mapstochar\relbar\joinrel\rightarrow} c_1$ satisfy

$$a_3 \xrightarrow{f_{23}(\alpha)} a_2 = a_3 \xrightarrow{g_{23}(\gamma)} a_2 \quad \text{and} \quad a_2 \xrightarrow{f_{12}(\beta)} a_1 = a_2 \xrightarrow{g_{23}(\delta)} a_1,$$

whilst a typical element of $\mathbf{E}(d_3; d_1)$ looks like

$$\big((\alpha \otimes \beta), (\gamma \otimes \delta)\big) = \big((b_3 \overset{\alpha}{\mapstochar\longrightarrow} b) \otimes (b \overset{\beta}{\mapstochar\longrightarrow} b_1), (c_3 \overset{\gamma}{\mapstochar\longrightarrow} c) \otimes (c \overset{\delta}{\mapstochar\longrightarrow} c_1)\big)$$

where

$$a_3 \xrightarrow{f_{23}(\alpha)} f_2(b) \xrightarrow{f_{12}(\beta)} a_1 = a_3 \xrightarrow{g_{23}(\gamma)} g_2(c) \xrightarrow{g_{12}(\delta)} a_1$$

in A. Then the 2-cell u has components given by

$$\begin{aligned} u_{d_3,d_1} \colon \mathbf{D}_{23} \otimes \mathbf{D}_{12}(d_3; d_1) &\to \mathbf{E}(d_3; d_1) \\ \big((\alpha, \gamma) \otimes (\beta, \delta)\big) &\mapsto (\alpha \otimes \beta, \gamma \otimes \delta). \end{aligned}$$

Now let us construct the promised inverse v for this 2-cell. Suppose we are given an element $(\alpha \otimes \beta, \gamma \otimes \delta) \in \mathbf{E}(d_3; d_1)$; we must send this to an element of $\mathbf{D}_{23} \otimes \mathbf{D}_{12}(d_3; d_1)$. So consider the map

$$\psi := f_2(b) \xrightarrow{f_{23}(\alpha)^{-1}} a_3 \xrightarrow{g_{23}(\gamma)} g_2(c)$$

in A. The functor $f_2\colon B_2 \to A$ is a fibration and A is a groupoid; thus f_2 is also a *co*fibration, and so we can lift the displayed map to a cocartesian arrow $\hat{\psi}\colon b \to \psi^* b$ in B_2; and since ψ is invertible, so is $\hat{\psi}$. So now we set

$$v\big((\alpha \otimes \beta, \gamma \otimes \delta)\big) := \big((b_3, c_3) \overset{(\hat{\psi}\alpha, \gamma)}{\mapstochar\longrightarrow} (\psi^* b, c)\big) \otimes \big((\psi^* b, c) \overset{(\beta\hat{\psi}^{-1}, \delta)}{\mapstochar\longrightarrow} (b_1, c_1)\big).$$

For this to be well-defined we need to check firstly that it does indeed map into $\mathbf{D}_{23} \otimes \mathbf{D}_{12}(d_3; d_1)$; and secondly that it is independent of the choice of representative for $(\alpha \otimes \beta, \gamma \otimes \delta)$. For the first of these, we simply observe that

$$\begin{aligned}
f_{23}(b_3 \overset{\alpha}{\relbar\joinrel\mapstochar\rightarrow} b \xrightarrow{\hat{\psi}} \psi^* b) &= \big(a_3 \xrightarrow{f_{23}(\alpha)} f_2(b) \xrightarrow{f_{23}(\alpha)^{-1}} a_3 \xrightarrow{g_{23}(\gamma)} g_2(c)\big) \\
&= \big(a_3 \xrightarrow{g_{23}(\gamma)} g_2(c)\big) \\
&= g_{23}(c_3 \overset{\gamma}{\relbar\joinrel\mapstochar\rightarrow} c) \\
\text{and } f_{12}(\psi^* b \xrightarrow{\hat{\psi}^{-1}} b \overset{\beta}{\relbar\joinrel\mapstochar\rightarrow} b_1) &= (g_2(c) \xrightarrow{g_{23}(\gamma)^{-1}} a_3 \xrightarrow{f_{23}(\alpha)} f_2(b) \xrightarrow{f_{12}(\beta)} a_1) \\
&= (g_2(c) \xrightarrow{g_{23}(\gamma)^{-1}} a_3 \xrightarrow{g_{23}(\gamma)} g_2(c) \xrightarrow{g_{12}(\delta)} a_1) \\
&= (g_2(c) \xrightarrow{g_{12}(\delta)} a_1) \\
&= g_{12}(c \overset{\delta}{\relbar\joinrel\mapstochar\rightarrow} c_1)
\end{aligned}$$

and so we map into $\mathbf{D}_{23} \otimes \mathbf{D}_{12}(d_3; d_1)$ as required. For the second, it suffices to check that two equalities hold:

$$\begin{aligned}
v\big((\alpha \otimes \beta), (c_3 \overset{\gamma}{\relbar\joinrel\mapstochar\rightarrow} c) \otimes (c \xrightarrow{\epsilon} c' \overset{\delta}{\relbar\joinrel\mapstochar\rightarrow} c_1)\big) &= v\big((\alpha, \beta), (c_3 \overset{\gamma}{\relbar\joinrel\mapstochar\rightarrow} c \xrightarrow{\epsilon} c') \otimes (c' \overset{\delta}{\relbar\joinrel\mapstochar\rightarrow} c_1)\big) \\
v\big((b_3 \overset{\alpha}{\relbar\joinrel\mapstochar\rightarrow} b) \otimes (b \xrightarrow{\epsilon} b' \overset{\beta}{\relbar\joinrel\mapstochar\rightarrow} b_1), (\gamma \otimes \delta)\big) &= v\big((b_3 \overset{\alpha}{\relbar\joinrel\mapstochar\rightarrow} b \xrightarrow{\epsilon} b') \otimes (b' \overset{\beta}{\relbar\joinrel\mapstochar\rightarrow} b_1), (\gamma \otimes \delta)\big).
\end{aligned}$$

We begin with the first of these. Let us write

$$\begin{aligned}
\psi &:= f_2(b) \xrightarrow{f_{23}(\alpha)^{-1}} a_3 \xrightarrow{g_{23}(\gamma)} g_2(c) \\
\text{and } \phi &:= f_2(b) \xrightarrow{f_{23}(\alpha)^{-1}} a_3 \xrightarrow{g_{23}(\gamma)} g_2(c) \xrightarrow{g_2(\epsilon)} g_2(c');
\end{aligned}$$

then we have

$$\begin{aligned}
v(\alpha \otimes \beta, \gamma \otimes \delta\epsilon) &= (\hat{\psi}\alpha, \gamma) \otimes (\beta\hat{\psi}^{-1}, \delta\epsilon), \\
\text{and } v(\alpha \otimes \beta, \epsilon\gamma \otimes \delta) &= (\hat{\phi}\alpha, \epsilon\gamma) \otimes (\beta\hat{\phi}^{-1}, \delta).
\end{aligned}$$

Now, let us write η for $g_2(\epsilon)$; then we have $\hat{\eta} \colon \psi^* b \to \eta^* \psi^* b$ which satisfies

$$f_2(\hat{\eta}) = g_2(\epsilon)$$

so that $(\hat{\eta}, \epsilon)$ is a map in D_2; further, we have that

$$\big(b \xrightarrow{\hat{\psi}} \psi^* b \xrightarrow{\hat{\eta}} \eta^* \psi^* b\big) = \big(b \xrightarrow{\hat{\phi}} \phi^* b\big);$$

and thus we have

$$\begin{aligned}
(\hat{\phi}\alpha, \epsilon\gamma) \otimes (\beta\hat{\phi}^{-1}, \delta) &= (\hat{\eta}\hat{\psi}\alpha, \epsilon\gamma) \otimes (\beta\hat{\psi}^{-1}\hat{\eta}^{-1}, \delta) \\
&= \big((\hat{\eta}, \epsilon) \bullet (\hat{\psi}\alpha, \gamma)\big) \otimes (\beta\hat{\psi}^{-1}\hat{\eta}^{-1}, \delta) \\
&= (\hat{\psi}\alpha, \gamma) \otimes \big((\beta\hat{\psi}^{-1}\hat{\eta}^{-1}, \delta) \bullet (\hat{\eta}, \epsilon)\big) \\
&= (\hat{\psi}\alpha, \gamma) \otimes (\beta\hat{\psi}^{-1}, \delta\epsilon)
\end{aligned}$$

as required. Similarly, we must compare

$$\begin{aligned}
&\big((b_3 \overset{\alpha}{\relbar\mapstochar\rightarrow} b) \otimes (b \xrightarrow{\epsilon} b' \overset{\beta}{\relbar\mapstochar\rightarrow} b_1), (c_3 \overset{\gamma}{\relbar\mapstochar\rightarrow} c) \otimes (c \overset{\delta}{\relbar\mapstochar\rightarrow} c_1)\big) \\
\text{and } &\big((b_3 \overset{\alpha}{\relbar\mapstochar\rightarrow} b \xrightarrow{\epsilon} b') \otimes (b' \overset{\beta}{\relbar\mapstochar\rightarrow} b_1), (c_3 \overset{\gamma}{\relbar\mapstochar\rightarrow} c) \otimes (c \overset{\delta}{\relbar\mapstochar\rightarrow} c_1)\big).
\end{aligned}$$

Let us write

$$\begin{aligned}
\psi &:= f_2(b) \xrightarrow{f_{23}(\alpha)^{-1}} a_3 \xrightarrow{g_{23}(\gamma)} g_2(c) \\
\phi &:= f_2(b') \xrightarrow{f_2(\epsilon)^{-1}} f_2(b) \xrightarrow{f_{23}(\alpha)^{-1}} a_3 \xrightarrow{g_{23}(\gamma)} g_2(c);
\end{aligned}$$

then

$$\begin{aligned}
v\big((\alpha \otimes \beta\epsilon, \gamma \otimes \delta)\big) &= (\hat{\psi}\alpha, \gamma) \otimes (\beta\epsilon\hat{\psi}^{-1}, \delta), \\
\text{and } v\big((\epsilon\alpha \otimes \beta, \gamma \otimes \delta)\big) &= (\hat{\phi}\epsilon\alpha, \gamma) \otimes (\beta\hat{\phi}^{-1}, \delta)\big).
\end{aligned}$$

Now, we certainly have $f_2(\hat{\psi}) = \psi$; but also we have $f_2(\hat{\phi}\epsilon) = \phi f_2(\epsilon) = \psi$. Hence by cocartesianness of $\hat{\psi}$, there is a map θ making the following diagram commute:

$$\begin{tikzcd}
b \arrow[r, "\hat{\psi}"] \arrow[d, "\epsilon"'] & \psi^* b \arrow[d, dotted, "\theta"] \\
b' \arrow[r, "\hat{\phi}"'] & \phi^* b'
\end{tikzcd}$$

and with $f_2(\theta) = \mathrm{id}_{g_2(c)}$. But now we calculate:

$$\begin{aligned}(\hat{\phi}\epsilon\alpha,\gamma)\otimes(\beta\hat{\phi}^{-1},\delta) &= (\theta\hat{\psi}\alpha,\gamma)\otimes(\beta\hat{\phi}^{-1},\delta)\\ &= (\hat{\psi}\alpha,\gamma)\otimes(\beta\hat{\phi}^{-1}\theta,\delta)\\ &= (\hat{\psi}\alpha,\gamma)\otimes(\beta\epsilon\hat{\psi}^{-1},\delta)\end{aligned}$$

as required. Thus the 2-cell v is well-defined; it remains to show that it is inverse to the 2-cell u. We have $u\big((\alpha,\gamma)\otimes(\beta,\delta)\big) = (\alpha\otimes\beta,\gamma\otimes\delta)$, and thus

$$v\big(u\big((\alpha,\gamma)\otimes(\beta,\delta)\big)\big) = \big((b_3,c_3)\overset{(\hat{\psi}\alpha,\gamma)}{\relbar\joinrel\mapstochar\joinrel\rightarrow}(\psi^*b_2,c_2)\big)\otimes\big((\psi^*b_2,c_2)\overset{(\beta\hat{\psi}^{-1},\delta)}{\relbar\joinrel\mapstochar\joinrel\rightarrow}(b_1,c_1)\big),$$

where $\psi := g_{23}(\gamma)\circ f_{23}(\alpha)^{-1}$. But by definition of $\mathbf{D}_{23}\otimes\mathbf{D}_{12}$, we have $f_{23}(\alpha) = g_{23}(\gamma)\colon a_3 \to a_2$, and thus

$$vu\big((\alpha,\gamma)\otimes(\beta,\delta)\big) = \big((\alpha,\gamma)\otimes(\beta,\delta)\big)$$

as required. Conversely, given $(\alpha\otimes\beta,\gamma\otimes\delta)$ in $\mathbf{E}(d_3;d_1)$, we have that

$$\begin{aligned}uv(\alpha\otimes\beta,\gamma\otimes\delta) &= \big((\hat{\psi}^{-1}\alpha)\otimes(\beta\hat{\psi}),\gamma\otimes\delta\big)\\ &= \big(\alpha\otimes\beta,\gamma\otimes\delta\big)\end{aligned}$$

as required. □

Corollary 58. *The homomorphism S satisfies property (hps).*

Proof. Condition (hps2) follows trivially from Proposition 56. For (hps1), suppose we are given horizontally composable pullbacks

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathbf{p}_1"] \arrow[d, "\mathbf{p}_2"'] & \mathbf{B} \arrow[d, "\mathbf{f}"] \\ S\mathbf{C} \arrow[r, "S!"'] & S\mathbf{I}_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbf{A}' \arrow[r, "\mathbf{p}_1'"] \arrow[d, "\mathbf{p}_2'"'] & \mathbf{B}' \arrow[d, "\mathbf{f}'"] \\ S\mathbf{C}' \arrow[r, "S!"'] & S\mathbf{I}_1, \end{tikzcd}$$

in $\mathbb{C}at_1$. Then consider the diagram

$$\begin{tikzcd}
\mathbf{A}' \otimes \mathbf{A} \arrow[r, "\mathbf{p}_1' \otimes \mathbf{p}_1"] \arrow[d, "\mathbf{p}_2' \otimes \mathbf{p}_2"'] & \mathbf{B}' \otimes \mathbf{B} \arrow[d, "\mathbf{f}' \otimes \mathbf{f}"] \\
S\mathbf{C}' \otimes S\mathbf{C} \arrow[r, "S! \otimes S!"'] & S\mathbf{I}_1 \otimes S\mathbf{I}_1
\end{tikzcd}$$

We observe that $S1$ is a groupoid in **Cat**, and that the arrow $S!\colon SC_t \to S1$ in **Cat** is a fibration. Since we have an isomorphism $S\mathbf{I}_1 \cong \mathbf{I}_{S1}$, we can now apply Proposition 57, thus making this square a pullback as required. □

8.4.2 S_0 and S_1 are clubs

We already know from Proposition 23 that (S_0, η_0, μ_0) is a club, but we need to check that the same is true of (S_1, η_1, μ_1):

Proposition 59. *The monad (S_1, η_1, μ_1) is a club on $\mathbb{C}at_1$.*

Proof. First we prove that the naturality squares for η_1 are pullbacks, for which it suffices to show that the squares

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "!"] \arrow[d, "\eta_{\mathbf{X}}"'] & \mathbf{I}_1 \arrow[d, "\eta_{\mathbf{I}_1}"] \\
S\mathbf{X} \arrow[r, "S!"'] & S\mathbf{I}_1
\end{tikzcd}$$

are pullbacks. Evidently, we have that the squares

$$\begin{tikzcd}
X_s \arrow[r, "!"] \arrow[d, "\eta_{X_s}"'] & 1 \arrow[d, "\eta_1"] & & X_t \arrow[r, "!"] \arrow[d, "\eta_{X_t}"'] & 1 \arrow[d, "\eta_1"] \\
SX_s \arrow[r, "S!"'] & S1 & \text{and} & SX_t \arrow[r, "S!"'] & S1
\end{tikzcd}$$

are pullbacks since (S_0, η_0, μ_0) is a club; thus it suffices to check that

$$\begin{tikzcd}
X(x_t; x_s) \arrow[r, "!"] \arrow[d, "\hat{\eta}"'] & 1(*, *) \arrow[d, "\hat{\eta}"] \\
\hat{S}X\big((1, \langle x_t \rangle); (1, \langle x_s \rangle)\big) \arrow[r, "\hat{S}!"'] & S1(1, 1)
\end{tikzcd}$$

is a pullback in **Set**; but this is evident since both the vertical arrows are isomorphisms. Proceeding similarly, for the naturality squares of μ_1 to be pullbacks, it suffices to check that the squares

$$\begin{tikzcd}
\hat{S}\hat{S}X\big((\phi, \langle x_{ti}\rangle); (\psi, \langle x_{si}\rangle)\big) \arrow[r, "\hat{S}\hat{S}!"] \arrow[d, "\hat{\mu}"'] & SS1(\phi, \psi) \arrow[d, "\hat{\mu}"] \\
\hat{S}X\big((n_\phi, \langle x_{ti}\rangle); (n_\psi, \langle x_{si}\rangle)\big) \arrow[r, "\hat{S}!"'] & S1(n_\phi, n_\psi)
\end{tikzcd}$$

are pullbacks; but this is straightforward merely by working through the definitions of these sets from above. Finally, we must check that S_1 preserves cartesian natural transformations into S_1; we shall in fact show that S_1 preserves *all* pullbacks from which the result follows *a fortiori*. So suppose that

$$\begin{tikzcd}
\mathbf{D} \arrow[r, "\mathbf{j}"] \arrow[d, "\mathbf{k}"'] & \mathbf{C} \arrow[d, "\mathbf{f}"] \\
\mathbf{B} \arrow[r, "\mathbf{g}"'] & \mathbf{A}
\end{tikzcd}$$

is a pullback. Since we have that $D_i = C_i \times_{A_i} B_i$, and S_0 preserves pullbacks, we have $S_0 D_i = S_0 C_i \times_{S_0 A_i} S_0 B_i$ (for $i \in \{s, t\}$). So it suffices to check that the squares

$$\begin{tikzcd}
\hat{S}D\big((n, \langle d_{ti}\rangle); (m, \langle d_{si}\rangle)\big) \arrow[r, "\hat{S}j"] \arrow[d, "\hat{S}k"'] & \hat{S}C\big((n, \langle c_{ti}\rangle); (m, \langle c_{si}\rangle)\big) \arrow[d, "\hat{S}f"] \\
\hat{S}B\big((n, \langle b_{ti}\rangle); (m, \langle b_{si}\rangle)\big) \arrow[r, "\hat{S}g"'] & \hat{S}A\big((n, \langle a_{ti}\rangle); (m, \langle a_{si}\rangle)\big)
\end{tikzcd}$$

are pullbacks; and again, this is easy working through from the definitions. □

And thus we can conclude with the main result of this part of the thesis:

Corollary 60. *The double monad (S, η, μ) is a double club on $\mathbb{C}at$.*

Proof. By Proposition 58, S has property (hps); and by Propositions 23 and 59, S_0 and S_1 are clubs on their respective categories. Therefore, by Proposition 48, (S, η, μ) is a double club on $\mathbb{C}at$. □

Part III

Polycategories

Chapter 9

Multicategories and polycategories

We now wish to put the double club (S, η, μ) constructed above to useful work. We shall use it to provide an abstract description of the theory of *polycategories*.

In this chapter, we recap this theory. We begin by looking at their more straightforward cousins, *multicategories*, as introduced by [Lam69], and developed by, amongst others, [Bur71], [Her00], [Lei04a] and [BD98]. We also describe our preferred abstract presentation of the theory of multicategories, the presentation adopted by [BD98] and [CT03].

We then move on to describe the theory of *polycategories*, as introduced by [Sza75] and pursued in [CS97]. We give a novel abstract presentation of the theory of polycategories, generalising that given for multicategories; although something similar has been attempted by [Kos03], the formalism used here seems to be somewhat neater.

Finally, we lay out what will be necessary to realise this putative new presentation: the establishment of a *pseudo-distributive law* between a pseudomonad and a pseudocomonad on the bicategory **Mod**, the construction of which is the work of the remainder of this thesis.

9.1 Multicategories

We begin by re-examining the theory of multicategories: the material here summarises [BD98], [Hyl02] and [Lei04b], amongst others. Note that throughout, we shall only be interested in the theory of *symmetric* multicategories, and, later, of *symmetric* polycategories; that is, we allow ourselves to reorder freely the 'inputs' and ‘outputs’ of our maps. The non-symmetric case is considered in more detail by [Kos03].

First a little notation. We write X^* for the free monoid on a set X, and $\Gamma, \Delta, \Sigma, \Lambda$ for typical elements thereof. We will use comma to denote the concatenation operation on X^*, as in “Γ, Δ”; and we will tend to conflate elements of X with their image in X^*. Given $\Gamma = x_1, \ldots, x_n \in X^*$, we define $|\Gamma| = n$, and given $\sigma \in S_n$, write $\sigma\Gamma$ for the element $x_{\sigma(1)}, \ldots, x_{\sigma(n)} \in X^*$.

Definition 61. A **symmetric multicategory** $\mathbb{M}$ consists of:

- A set $\mathrm{ob}\,\mathbb{M}$ of **objects**;
- For every $\Gamma \in (\mathrm{ob}\,\mathbb{M})^*$ and $y \in \mathrm{ob}\,\mathbb{M}$, a set $\mathbb{M}(\Gamma; y)$ of **multimaps** from Γ to y (we write a typical element of such as $f\colon \Gamma \to y$); further, for every $\sigma \in S_{|\Gamma|}$, an **exchange isomorphism**

$$\mathbb{M}(\Gamma; y) \to \mathbb{M}(\sigma\Gamma; y).$$

This data satisfies axioms expressing the fact that exchange isomorphisms compose as expected. Furthermore, we have:

- For every $x \in \mathrm{ob}\,\mathbb{M}$, an **identity map** $\mathrm{id}_x \in \mathbb{M}(x; x)$;
- For every $\Gamma, \Delta_1, \Delta_2 \in (\mathrm{ob}\,\mathbb{M})^*$ and $y, z \in \mathrm{ob}\,\mathbb{M}$, a **composition map**

$$\mathbb{M}(\Gamma; y) \times \mathbb{M}(\Delta_1, y, \Delta_2; z) \to \mathbb{M}(\Delta_1, \Gamma, \Delta_2; z),$$

all subject to axioms expressing that composition is associative, unital, and compatible with exchange isomorphisms.

(See [Lam69] for the full details of this definition.) Now, this data expresses composition as a *binary* operation performed between two multimaps; however, there is another view, where we ‘multicompose’ a family of multimaps $g_i\colon \Gamma_i \to y_i$ with a multimap $f\colon y_1, \ldots, y_n \to z$.

The transit from one view to the other is straightforward: we recover the multicomposition from the binary composition by performing, in any order, the binary compositions of the g_i's with f – and the axioms for binary composition ensure that this gives a uniquely defined composite. Conversely, we can recover binary composition from multicomposition by setting all but one of the g_i's to be the identity.

We can express the operation of multicomposition as follows: fix the object set $X = \mathrm{ob}\,\mathbb{M}$, and consider it as a discrete category. As before, we write S for the free symmetric strict monoidal category monad on **Cat**; so consider now the functor category $[(SX)^{\mathrm{op}} \times X, \mathbf{Set}]$. To give an object F of this is to give sets of multimaps as above, together with coherent exchange isomorphisms. Further, this category has a 'substitution' monoidal structure given by

$$(G \otimes F)(\Gamma; z) = \sum_{\substack{k \in \mathbb{N} \\ y_1, \ldots, y_k \in X}} \int^{\Delta_1, \ldots, \Delta_k \in SX} G(y_1, \ldots, y_k; z) \times \prod_{i=1}^{k} F(\Delta_i; y_i) \times SX(\Gamma, \bigotimes_{i=1}^{k} \Delta_i),$$

and

$$\mathbf{I}(\Gamma; x) = \begin{cases} \{*\} & \text{if } \Gamma = x \\ \emptyset & \text{otherwise;} \end{cases}$$

and to give a multicategory is precisely to give a monoid with respect to this monoidal structure. Indeed, suppose we have a monoid $F \in [(SX)^{\mathrm{op}} \times X, \mathbf{Set}]$. Then the unit map $j \colon \mathbf{I} \to F$ picks out for each $x \in X$ an element of $F(x; x)$, which will correspond to the identity multimap $\mathrm{id}_x \colon x \to x$. What about the multiplication map $m \colon F \otimes F \to F$? Unpacking the above definition, we see that $(F \otimes F)(\Gamma; z)$ can be described as follows. Let $\Delta_1, \ldots, \Delta_k \in (\mathrm{ob}\,\mathbb{M})^*$ be such that

- $|\Gamma| = n = \sum |\Delta_i|$;

- there exists $\sigma \in S_n$ such that $\sigma\Gamma = \Delta_1, \ldots, \Delta_k$,

and let $f_i \colon \Delta_i \to y_i$ (for $i = 1, \ldots, k$), and $g \colon y_1, \ldots, y_k \to z$ be multimaps in F.

Then this gives us a typical element of $(F \otimes F)(\Gamma; z)$, which we visualise as

$$\begin{tikzcd}
\Gamma \arrow[d, "\sigma"] \\
\Delta_1, \ldots, \Delta_k \arrow[d, "f_1, \ldots, f_k"] \\
y_1, \ldots, y_k \arrow[d, "g"] \\
z.
\end{tikzcd}$$

Now, the map $m \colon F \otimes F \to F$ sends this element to an element of $F(\Gamma; z)$; in other words, it specifies the result of this 'multicomposition'. The associativity and unitality laws for a monoid ensure that this composition process is associative and unital as required.

At this point we observe that we can express this more abstractly. Indeed, we have seen that S lifts to a pseudomonad $\hat{S}$ on **Mod**, and thus we can form the 'Kleisli bicategory' $Kl(\hat{S})$ of the pseudomonad $\hat{S}$. This gadget makes its only other published appearance in [ECP]; we leave the phrase 'Kleisli bicategory' in quotes for now, since no-one has yet attempted to work through the details of the coherence it involves, and we do not intend to do so here. However, we can describe it very simply:

Definition 62. Let $\mathcal{B}$ be a bicategory, let $(S, \eta, \mu, \lambda, \rho, \tau)$ be a pseudomonad on $\mathcal{B}$. Then the Kleisli bicategory $Kl(S)$ of the pseudomonad S has:

- **Objects** those of $\mathcal{B}$;

- **Hom-categories** given by $Kl(S)(X, Y) = \mathcal{B}(X, SY)$;

- **Identity map** at X given by the component $\eta_X \colon X \to SX$;

- **Composition**

$$Kl(S)(Y, Z) \times Kl(S)(X, Y) \to Kl(S)(X, Z)$$

given by

$$\begin{tikzcd}
\mathcal{B}(Y,SZ)\times\mathcal{B}(X,SY) \arrow[d, "\cong"] \\
1\times\mathcal{B}(Y,SZ)\times\mathcal{B}(X,SY) \arrow[d, "\ulcorner\mu_Z\urcorner\times S\times\mathrm{id}"] \\
\mathcal{B}(SSZ,SZ)\times\mathcal{B}(SY,SSZ)\times\mathcal{B}(X,SY) \arrow[d, "\otimes"] \\
\mathcal{B}(X,SZ)
\end{tikzcd}$$

where we use $\otimes$ to stand for some choice of order of composition for this threefold composite. Explicitly, on maps, this composition is given by

$$(Y\xrightarrow{G}SZ)\otimes(X\xrightarrow{F}SY)=X\xrightarrow{F}SY\xrightarrow{SG}SSZ\xrightarrow{\mu_Z}SZ$$

for some choice of bracketing for this composite.

The remaining data to make this a bicategory – namely, the associativity and unitality constraints – can be constructed in an obvious way using the associativity and unitality constraints for $\mathcal{B}$ and the coherence modifications for the pseudomonad S. We shall not check the details required to show that this data does indeed satisfy the required coherence axioms for a bicategory.

Applying this to the pseudomonad $\hat{S}$ on **Mod**, we see that the monoidal structure on $[(SX)^{\mathrm{op}}\times X,\mathbf{Set}]$ described above is just horizontal composition in $Kl(\hat{S})(X,X)$. Hence we arrive at an alternative, but equivalent, definition of multicategory:

Definition 63. A symmetric multicategory is a monad on a discrete object X in the bicategory $Kl(\hat{S})$.

This description is well known, though not often stated in precisely this form: it is the approach of [BD98] and [CT03].

9.2 Polycategories

We recall now the notion of symmetric *polycategory*:

Definition 64. A **symmetric polycategory** $\mathbb{P}$ consists of

- A set $\mathrm{ob}\,\mathbb{P}$ of **objects**;
- For each pair (Γ, Δ) of elements of $(\mathrm{ob}\,\mathbb{P})^*$, a set $\mathbb{P}(\Gamma; \Delta)$ of **polymaps** from Γ to Δ;
- For each Γ, $\Delta \in (\mathrm{ob}\,\mathbb{P})^*$, each $\sigma \in S_{|\Gamma|}$ and $\tau \in S_{|\Delta|}$, **exchange isomorphisms**
$$\mathbb{P}(\Gamma; \Delta) \to \mathbb{P}(\sigma\Gamma; \tau\Delta);$$
- For each $x \in \mathrm{ob}\,\mathbb{P}$, an **identity map** $\mathrm{id}_x \in \mathbb{P}(x; x)$;
- For $\Gamma, \Delta_1, \Delta_2, \Lambda_1, \Lambda_2, \Sigma \in (\mathrm{ob}\,\mathbb{P})^*$, and $x \in \mathrm{ob}\,\mathbb{P}$, **composition maps**
$$\mathbb{P}(\Gamma; \Delta_1, x, \Delta_2) \times \mathbb{P}(\Lambda_1, x, \Lambda_2; \Sigma) \to \mathbb{P}(\Lambda_1, \Gamma, \Lambda_2; \Delta_1, \Sigma, \Delta_2),$$

subject to laws expressing the associativity and unitality of composition, expressing that the exchange isomorphisms compose as expected, and that they are compatible with composition.

For the full details of this, we refer the reader to [Sza75] or [CS97]. We recover the notion of a multicategory if we assert that $\mathbb{P}(\Gamma; \Delta)$ is empty unless Δ is a singleton.

Now, as before, we may shift from giving a 'binary composition' of two polymaps to giving a 'polycomposition' operation on two *families* of composable polymaps. First, we need to say what we mean by *composable*.

Definition 65. Let $\mathbf{f} := \{f_m \colon \Lambda_m \to \Sigma_m\}_{1 \leqslant m \leqslant j}$ and $\mathbf{g} := \{g_n \colon \Gamma_n \to \Delta_n\}_{1 \leqslant n \leqslant k}$ be families of polymaps, such that
$$\sum |\Sigma_m| = \sum |\Gamma_n| = l.$$
We say that a permutation $\sigma \in S_l$ is a **matching** if $\sigma(\Sigma_1, \ldots, \Sigma_j) = \Gamma_1, \ldots, \Gamma_k$.

Informally, this matching shows 'which output has been plugged into which input', and so we can define a composite map $\mathbf{g} \circ_\sigma \mathbf{f}$. However, we would like our notion of polycomposition to coincide with notion of binary composition; hence, we should be able to perform polycomposition by repeated binary compositions.

However, not all matchings have this property. Let us define what the 'suitable' matchings are:

Definition 66. Given a matching σ for $\mathbf{f}$ and $\mathbf{g}$, form the bipartite multigraph graph G as follows. Its two vertex sets are labelled by $f_1, \ldots, f_m$ and $g_1, \ldots, g_n$, and we add one edge between f_i and g_j for every element of Σ_i which is paired with an element of Γ_j under the matching σ. We shall say that the matching σ is **suitable** just when G is acyclic, connected and has no multiple edges.

Proposition 67. *A matching σ is suitable if and only if the associated composite map $\mathbf{g} \circ_\sigma \mathbf{f}$ can be formed by repeated binary compositions.*

In fact, to prove this we shall need to prove something slightly stronger. A little more notation: given a list $\Sigma = x_1, \ldots, x_k \in X^*$, by a *sublist* of Σ we shall mean a list $\Gamma = x_{i_1}, \ldots, x_{i_j}$ where $1 \leqslant x_1 < x_2 < \cdots < x_{i_j} \leqslant k$. In our eyes, sublists of Σ are in bijection with subsets of $\{1, \ldots, |\Sigma|\}$; for example, the list x, x has *two* distinct sublists of size 1.

Definition 68. Let $\mathbf{f} := \{f_m \colon \Lambda_m \to \Sigma_m\}_{1 \leqslant m \leqslant j}$ and $\mathbf{g} := \{g_n \colon \Gamma_n \to \Delta_n\}_{1 \leqslant n \leqslant k}$ be families of polymaps. Let Σ be a sublist of $\Sigma_1, \ldots, \Sigma_m$ and let Γ be a sublist of $\Gamma_1, \ldots, \Gamma_n$, such that $|\Sigma| = |\Gamma| = l$. We say that a permutation $\sigma \in S_l$ is a **partial matching** if $\sigma(\Sigma) = \Gamma$.

Now, as before, we can define the notion of the associated graph G for a partial matching, and thus the notion of a *suitable* partial matching. Also, we can define the notion of the associated composite map $\mathbf{g} \circ_\sigma \mathbf{f}$ for a partial matching. Now the previous proposition follows *a fortiori* from the following:

Proposition 69. *A partial matching σ is suitable if and only if the associated composite map $\mathbf{g} \circ_\sigma \mathbf{f}$ can be formed by repeated binary compositions.*

Proof. The 'if' direction is a straightforward induction:

- The empty succession of binary compositions certainly gives rise to a suitable matching;
- Suppose we have already performed a series of binary compositions whose associated partial matching is suitable; then performing a further binary com-

position will add one new edge and one new vertex to the associated graph, retaining its tree structure.

For the 'only if' direction, we observe that if the partial matching σ is suitable, then the associated graph G is a tree, and so in particular will have a vertex of degree 1. Choose any such vertex: it corresponds to one of our polymaps f_i or g_i, without loss of generality to f_i, say. We begin by forming the binary composition of f_i with the polymap g_j which is connected to f_i in G. Suppose

$$f_i \colon \Lambda_i \to \Sigma_i, x, \Sigma'_i \quad \text{and} \quad g_j \colon \Gamma_j, x, \Gamma'_j \to \Delta_j,$$

where the two x's are matched under σ. Then the resultant composite map will be

$$g_i \circ f_j \colon \Gamma_j, \Lambda_i, \Gamma'_j \to \Sigma_i, \Delta_j, \Sigma'_i.$$

Note that f_i has no other outputs taking part in the partial matching σ. Thus we can now form a partial matching σ' of $\mathbf{f} \setminus \{f_i\}$ with $\mathbf{g} \setminus \{g_j\} \cup \{g_j \circ f_i\}$, which simply matches elements in the same way as σ except for the no-longer present matching of x. Now it's easy to see that the associated graph of σ' will be the same as that of σ, but with the vertex corresponding to f_i and the single adjacent edge removed. We continue by induction on the size of the tree G.

Note that we may at each stage have several possible choices of vertices of degree 1 which we may take as the next binary composition to perform. However, the associativity laws for a polycategory ensure that the resultant composite will be independent of the choice we make at each stage. $\square$

Hence our global notion of composition of polymaps is given by composing a family $\mathbf{f}$ with a family $\mathbf{g}$ along a suitable matching σ. How can we express this more abstractly? We would like to imitate the previous section; given a set X of objects, we may view it as a discrete category and consider the functor category $[(SX)^{\mathrm{op}} \times SX, \mathbf{Set}]$. To give an element of this is to give sets of polymaps together with coherent exchange isomorphisms. What we should now like to do is to set up a monoidal structure on this category such that a monoid in it is precisely a polycategory. The unit is straightforward:

$$I(\Gamma;\Delta) = \begin{cases} \{*\} & \text{if } \Gamma = x = \Delta \\ \emptyset & \text{otherwise;} \end{cases}$$

and we can describe what a typical element of $(F \otimes F)(\Gamma;\Delta)$ should look like. Let

$$\Psi_1, \ldots, \Psi_k \ , \quad \Lambda_1, \ldots, \Lambda_k \ , \quad \Sigma_1, \ldots, \Sigma_l \ , \quad \text{and} \quad \Phi_1, \ldots, \Phi_l$$

be elements of $(\operatorname{ob}\mathbb{M})^*$, such that

- $|\Gamma| = n = \sum |\Psi_i|$;
- $\sum |\Lambda_i| = m = \sum |\Sigma_j|$;
- $\sum |\Phi_j| = p = |\Gamma|$;
- there exists $\sigma \in S_n$ such that $\sigma\Gamma = \Psi_1, \ldots, \Psi_k$;
- there exists $\tau \in S_m$ such that τ is a suitable matching of $\{\Lambda_i\}$ with $\{\Sigma_i\}$;
- there exists $\upsilon \in S_p$ such that $\upsilon(\Phi_1, \ldots, \Phi_k) = \Delta$;

and let $f_i \colon \Psi_i \to \Lambda_i$ (for $i = 1, \ldots, k$), and $g_j \colon \Sigma_j \to \Phi_j$ (for $j = 1, \ldots, l$) be polymaps in F. Then this gives us a typical element of $(F \otimes F)(\Gamma;\Delta)$, which we visualise as

$$\begin{tikzcd}
\Gamma \arrow[d, "\sigma"] \\
\Psi_1, \ldots, \Psi_k \arrow[d, "f_1,\ldots,f_k"] \\
\Lambda_1, \ldots, \Lambda_k \arrow[d, "\tau"] \\
\Sigma_1, \ldots, \Sigma_l \arrow[d, "g_1,\ldots,g_l"] \\
\Phi_1, \ldots, \Phi_l \arrow[d, "\upsilon"] \\
\Delta.
\end{tikzcd}$$

Then as for the multicategory case, the multiplication map $m\colon F \otimes F \to F$ should specify a composite map for this 'formal polycomposite', and the associativity and unitality conditions for a monoid should ensure that this polycomposition is associative and unital.

So our problem is reduced to finding a suitable way of expressing this monoidal structure; and in fact we skip straight over this stage and view polycategories as monads in a suitable bicategory. To see what this bicategory is, we shall need the following fact, to be proven later:

Proposition 70. *The 2-monad* (S, η, μ) *on* **Cat** *lifts to a pseudocomonad* $(\hat{T}, \hat{\epsilon}, \hat{\Delta})$ *as well as a pseudomonad* $(\hat{S}, \hat{\eta}, \hat{\mu})$ *on* **Mod***, such that we have*

$$\hat{T} = \hat{S}, \quad \hat{\epsilon}_{\mathbf{C}} = (\eta_{\mathbf{C}})^* \quad \textit{and} \quad \hat{\Delta}_{\mathbf{C}} = (\mu_{\mathbf{C}})^*.$$

The key idea is to produce a *pseudo-distributive law* $(\delta, \overline{\eta}, \overline{\epsilon}, \overline{\mu}, \overline{\Delta})$ of the pseudocomonad $\hat{T}$ over the pseudomonad $\hat{S}$; that is, there should be a pseudo-natural transformation

$$\delta\colon \hat{T}\hat{S} \Rightarrow \hat{S}\hat{T}$$

along with invertible modifications $\overline{\eta}$, $\overline{\epsilon}$, $\overline{\mu}$ and $\overline{\Delta}$, replacing the equalities for a standard distributive law, all subject to ten coherence laws – for full details, see Section 10.2. Given such a pseudo-distributive law, polycategories will emerge as monads in the 'two-sided Kleisli bicategory' of this pseudo-distributive law. Since this construction may not be familiar, we describe it first one dimension down:

Definition 71. Let $\mathbf{C}$ be a category, let (S, η, μ) be a monad and (T, ϵ, Δ) a comonad on $\mathbf{C}$, and let $\delta\colon TS \Rightarrow ST$ be a distributive law of the comonad over the monad; so we have the four equalities:

$$\begin{aligned} \epsilon S &= S\epsilon \circ \delta \\ \eta T &= \delta \circ T\eta \\ S\Delta \circ \delta &= \delta T \circ T\delta \circ \Delta S \\ \text{and } \delta \circ T\mu &= \mu T \circ S\delta \circ \delta S \end{aligned}$$

Then the two-sided Kleisli category $Kl(\delta)$ of the distributive law δ has:

- **Objects** those of $\mathbf{C}$;
- **Maps** $A \to B$ in $Kl(\delta)$ given by maps $TA \to SB$ in $\mathbf{C}$,

with

- **Identity maps** $\mathrm{id}_A \colon A \to A$ in $Kl(\delta)$ given by the map

  $$TA \xrightarrow{\epsilon_A} A \xrightarrow{\eta_A} SA$$

  in $\mathbf{C}$;
- **Composition** for maps $f \colon A \to B$ and $g \colon B \to C$ in $Kl(\delta)$ given by the map

  $$TA \xrightarrow{\Delta_A} TTA \xrightarrow{Tf} TSB \xrightarrow{\delta_B} STB \xrightarrow{Sg} SSC \xrightarrow{\mu_C} SC$$

  in $\mathbf{C}$.

Now, we can emulate such a construction one dimension up:

Definition 72. Let $\mathcal{B}$ be a bicategory, let $(S, \eta, \mu, \lambda, \rho, \tau)$ be a pseudomonad and $(T, \epsilon, \Delta, \lambda', \rho', \tau')$ a pseudocomonad on $\mathcal{B}$, and let $(\delta, \overline{\eta}, \overline{\epsilon}, \overline{\mu}, \overline{\Delta})$ be a pseudo-distributive law of the pseudocomonad over the pseudomonad. Then the two-sided Kleisli bicategory $Kl(\delta)$ of the pseudo-distributive law δ has:

- **Objects** those of $\mathcal{B}$;
- **Hom-categories** given by $Kl(\delta)(X, Y) = \mathcal{B}(TX, SY)$;
- **Identity map** at X given by the composite

  $$TX \xrightarrow{\epsilon_X} X \xrightarrow{\eta_X} SX;$$

- **Composition**

  $$Kl(\delta)(Y, Z) \times Kl(\delta)(X, Y) \to Kl(\delta)(X, Z)$$

given by

$$\begin{tikzcd}
\mathcal{B}(TY, SZ) \times \mathcal{B}(TX, SY) \arrow[d, "\cong"] \\
1 \times \mathcal{B}(TY, SZ) \times 1 \times \mathcal{B}(TX, SY) \times 1 \arrow[d, "\ulcorner \mu_Z \urcorner \times S \times \ulcorner \delta_Y \urcorner \times T \times \ulcorner \epsilon_X \urcorner"] \\
\mathcal{B}(SSZ, SZ) \times \mathcal{B}(STY, SSZ) \times \mathcal{B}(TSY, STY) \times \mathcal{B}(TTX, TSY) \times \mathcal{B}(TX, TTX) \arrow[d, "\otimes"] \\
\mathcal{B}(TX, SZ)
\end{tikzcd}$$

where we use $\otimes$ to stand for some choice of order of composition for the displayed fivefold composite. Explicitly, on maps, this composition is given by taking for $(TY \xrightarrow{G} SZ) \otimes (TX \xrightarrow{F} SY)$ (some choice of bracketing for) the composite

$$TX \xrightarrow{\Delta_X} TTX \xrightarrow{TF} TSY \xrightarrow{\delta_Y} STY \xrightarrow{SG} SSZ \xrightarrow{\mu_Z} SZ.$$

Again, we shall not provide the remaining pseudoassociativity and pseudounitality data to make this into a bicategory: they are now constructed from the pseudomonad structure of S, the pseudocomonad structure of T and the pseudo-distributive structure of δ. Again, it's a long and gory diagram chase using the coherence for S, T and δ to check that this data is coherent as required.

Returning to the case under consideration, we claim there is a pseudo-distributive law $\delta\colon \hat{T}\hat{S} \Rightarrow \hat{S}\hat{T}$, which should function as follows. Recall that we are taking $\hat{T} = \hat{S}$, and thus the component $\delta_{\mathbf{C}}\colon \hat{T}\hat{S}\mathbf{C} \twoheadrightarrow \hat{S}\hat{T}\mathbf{C}$ at $\mathbf{C}$ is given by a functor $(SS\mathbf{C})^{\mathrm{op}} \times SS\mathbf{C} \to \mathbf{Set}$. So, given a *discrete* category X, we wish to take $\delta_X(\{\Sigma_m\}_{1 \leqslant m \leqslant j}; \{\Gamma_n\}_{1 \leqslant n \leqslant k})$ to be the set of admissible matchings of $\{\Sigma_m\}$ with $\{\Gamma_n\}$. If we unwrap the definition of two-sided Kleisli bicategory above, we now see that the desired monoidal structure on $[(SX)^{\mathrm{op}} \times SX, \mathbf{Set}]$ is given precisely by horizontal composition in $Kl(\delta)(X, X)$.

Thus we should *like* to define a polycategory to be a monad on a discrete object X in the bicategory $Kl(\delta)$; but to do this, we must first establish the existence of the pseudo-distributive law δ. It is the task of the remainder of this thesis to do this.

Chapter 10

Deriving the pseudo-distributive law δ

Before we can construct the pseudo-distributive law δ, we must first prove Proposition 70 of the previous chapter and show that (S, η, μ) lifts to a pseudocomonad $(\hat{T}, \hat{\epsilon}, \hat{\Delta})$ on **Mod**. For this, we shall identify the *dual 2-monad* of a 2-monad on **Cat** and the *dual pseudocomonad* of a pseudomonad on **Mod**, and then see how these relate to the 'liftings' of Definition 51.

We then begin construction of the pseudo-distributive law δ between $\hat{T}$ and $\hat{S}$, starting by spelling out explicitly the data and axioms that are required for this. This is essentially drawn from [Tan04], with some minor modifications to deal with the fact that we are looking at a pseudo-distributive law of a pseudo*co*monad over a pseudomonad rather than of one pseudomonad over another.

Now, this definition involves giving a prodigious amount of data and coherence, and we therefore devote the remainder of the chapter to a discussion of how we may use the theory of double clubs to reduce to something much simpler.

10.1 Dual 2-monads and dual pseudocomonads

There is a 2-monad on **Cat** which freely adds finite products to a category, and another which freely adds finite coproducts. These two monads should provide an example of a pair of 'dual' 2-monads on **Cat**, and the following definition gives substance to this intuition:

Definition 73. Let (S, η, μ) be a 2-monad on **Cat**. Then the *dual 2-monad*

(T, ϵ, Δ) of S is given as follows. There is a 3-functor

$$(-)^{\mathrm{co}} \colon \mathbf{2\text{-}CAT} \to \mathbf{2\text{-}CAT}^{*},$$

(where $\mathbf{2\text{-}CAT}^{*}$ is $\mathbf{2\text{-}CAT}$ with the 3-cells reversed), and so, given the 2-monad (S, η, μ) on $\mathbf{Cat}$, we have a 2-monad $(S^{\mathrm{co}}, \eta^{\mathrm{co}}, \mu^{\mathrm{co}})$ on $\mathbf{Cat}^{\mathrm{co}}$. Furthermore, we have a 2-functor

$$O = (-)^{\mathrm{op}} \colon \mathbf{Cat} \to \mathbf{Cat}^{\mathrm{co}}.$$

So we take (T, ϵ, Δ) to be given by

$$\begin{aligned} T &= O^{-1} M^{\mathrm{co}} O \colon \mathbf{Cat} \to \mathbf{Cat}, \\ \epsilon &= O^{-1} \eta^{\mathrm{co}} O \colon O^{-1} \mathrm{id}_{\mathbf{Cat}^{\mathrm{co}}} O \Rightarrow O^{-1} M^{\mathrm{co}} O \\ \text{and } \Delta &= O^{-1} \mu^{\mathrm{co}} O \colon O^{-1} M^{\mathrm{co}} M^{\mathrm{co}} O \Rightarrow O^{-1} M^{\mathrm{co}} O. \end{aligned}$$

We must check that (T, ϵ, Δ) so defined really *is* a 2-monad. Observe that

$$O^{-1} \mathrm{id}_{\mathbf{Cat}^{\mathrm{co}}} O = \mathrm{id}_{\mathbf{Cat}}$$

and

$$O^{-1} S^{\mathrm{co}} S^{\mathrm{co}} O = (O^{-1} S^{\mathrm{co}} O)(O^{-1} S^{\mathrm{co}} O) = TT$$

and therefore that we have $\epsilon \colon \mathrm{id}_{\mathbf{Cat}} \Rightarrow T$ and $\Delta \colon TT \Rightarrow T$; and it's similarly straightforward to see that the monad laws will hold for (T, ϵ, Δ) – essentially we are just 'conjugating by O'.

We now wish to do something similar with pseudomonads on **Mod**. However, in this case, the dual structure will not be a pseudomonad but rather a *pseudocomonad* (see Appendix B for the notation used for pseudocomonads).

Definition 74. Let $(\hat{S}, \hat{\eta}, \hat{\mu}, \lambda, \rho, \tau)$ be a pseudomonad on **Mod**. Then we give the *dual pseudocomonad* $(\hat{T}, \hat{\epsilon}, \hat{\Delta}, \lambda', \rho', \tau')$ on **Mod** as follows. There is a strict trihomomorphism

$$(-)^{\mathrm{op}} \colon \mathbf{BICAT} \to \mathbf{BICAT}^{\mathrm{co}},$$

and so, given a pseudomonad $(\hat{S}, \hat{\eta}, \hat{\mu}, \lambda, \rho, \tau)$ on **Mod**, we have a pseudocomonad $(\hat{S}^{\mathrm{op}}, \hat{\eta}^{\mathrm{op}}, \hat{\mu}^{\mathrm{op}}, (\lambda^{\mathrm{op}})^{-1}, (\rho^{\mathrm{op}})^{-1}, (\tau^{\mathrm{op}})^{-1})$ on $\mathbf{Mod}^{\mathrm{op}}$. Furthermore, we have a strict homomorphism

$$O = (-)^{\mathrm{op}} \colon \mathbf{Mod} \to \mathbf{Mod}^{\mathrm{op}}.$$

So now proceeding as above, we set

$$\hat{T} = O^{-1}\hat{S}^{\mathrm{op}}O \colon \mathbf{Mod} \to \mathbf{Mod},$$

define pseudo-natural transformations

$$\begin{aligned} \hat{\epsilon} &= O^{-1}\hat{\eta}^{\mathrm{op}}O \colon \hat{T} \Rightarrow \mathrm{id}_{\mathbf{Mod}} \\ \text{and } \hat{\Delta} &= O^{-1}\mu^{\mathrm{co}}O \colon \hat{T} \Rightarrow \hat{T}\hat{T}, \end{aligned}$$

and invertible modifications

$$\begin{aligned} \lambda' &= O^{-1}(\lambda^{\mathrm{op}})^{-1}O \colon \hat{T}\hat{\epsilon} \circ \hat{\Delta} \Rrightarrow \mathrm{id}_{\hat{T}} \\ \rho' &= O^{-1}(\rho^{\mathrm{op}})^{-1}O \colon \hat{\epsilon}\hat{T} \circ \hat{\Delta} \Rrightarrow \mathrm{id}_{\hat{T}} \\ \text{and } \tau' &= O^{-1}(\tau^{\mathrm{op}})^{-1}O \colon \hat{\Delta}\hat{T} \circ \hat{\Delta} \Rrightarrow \hat{T}\hat{\Delta} \circ \hat{\mu}^{\circ}. \end{aligned}$$

In an analogous manner to above, these will satisfy the coherence laws for a pseudocomonad on **Mod**.

Now, suppose that the pseudomonad $(\hat{S}, \hat{\eta}, \hat{\mu}, \lambda, \rho, \tau)$ on **Mod** lifts the 2-monad (S, η, μ) on **Cat**. Then the dual pseudocomonad $(\hat{T}, \hat{\epsilon}, \hat{\Delta}, \lambda', \rho', \tau')$ lifts the dual 2-monad (T, ϵ, Δ) in the following sense. Consider the diagram:

$$\begin{tikzcd}
\mathbf{Cat}^{\mathrm{coop}} \arrow[r, "(-)^*"] \arrow[d, "(-)^{\mathrm{op}}"'] & \mathbf{Mod} \arrow[d, "(-)^{\mathrm{op}}"] \\
\mathbf{Cat}^{\mathrm{op}} \arrow[r, "(-)_*"] \arrow[d, "S"'] \arrow[dr, phantom, "\Downarrow\theta^{\mathrm{op}}"] & \mathbf{Mod}^{\mathrm{op}} \arrow[d, "\hat{S}"] \\
\mathbf{Cat}^{\mathrm{op}} \arrow[r, "(-)_*"'] \arrow[d, "(-)^{\mathrm{op}}"'] & \mathbf{Mod}^{\mathrm{op}} \arrow[d, "(-)^{\mathrm{op}}"] \\
\mathbf{Cat}^{\mathrm{coop}} \arrow[r, "(-)^*"'] & \mathbf{Mod}.
\end{tikzcd}$$

The upper and lower squares commute on the nose, and thus we obtain a vertical transformation

$$\begin{tikzcd}
\mathbf{Cat}^{\mathrm{coop}} \arrow[r, "(-)^*"] \arrow[d, "T^{\mathrm{coop}}"'] \arrow[dr, phantom, "\Downarrow\theta'"] & \mathbf{Mod} \arrow[d, "\hat{T}"] \\
\mathbf{Cat}^{\mathrm{coop}} \arrow[r, "(-)^*"'] & \mathbf{Mod}.
\end{tikzcd}$$

Furthermore, the components of $\hat{\epsilon}$ and $\hat{\Delta}$ are given by

$$\hat{\epsilon}_X = ((\eta_{X^{\mathrm{op}}})_*)^{\mathrm{op}} = ((\eta_{X^{\mathrm{op}}})^{\mathrm{op}})^* = (\epsilon_X)^*$$
$$\text{and } \hat{\Delta}_X = ((\mu_{X^{\mathrm{op}}})_*)^{\mathrm{op}} = ((\mu_{X^{\mathrm{op}}})^{\mathrm{op}})^* = (\Delta_X)^*,$$

and it's easy to compute that the transformation θ' is compatible with the pseudonaturality 2-cells for $\hat{\epsilon}$ and $\hat{\Delta}$ in the sense that, given a functor $F\colon \mathbf{C} \to \mathbf{D}$, the following diagrams commute:

$$\begin{tikzcd}
F^* \otimes (\epsilon_{\mathbf{D}})^* \arrow[rr, "\mathfrak{m}_{F,\epsilon_{\mathbf{D}}}"] \arrow[d, "\hat{\epsilon}_{F^*}"'] & & (\epsilon_{\mathbf{D}} \circ F)^* \arrow[d, "\mathrm{id}"] \\
(\epsilon_{\mathbf{C}})^* \otimes \hat{T}(F^*) \arrow[r, "\mathrm{id}\otimes\theta'_F"'] & (\epsilon_{\mathbf{C}})^* \otimes (NF)^* \arrow[r, "\mathfrak{m}_{\epsilon_{\mathbf{C}},NF}"'] & (NF \circ \epsilon_{\mathbf{C}})^*
\end{tikzcd}$$

and

$$\begin{tikzcd}
\hat{T}\hat{T}(F^*) \otimes (\Delta_{\mathbf{D}})^* \arrow[r, "(\theta'_{NF}\circ\hat{T}\theta'_F)\otimes\mathrm{id}"] \arrow[d, "\hat{\Delta}_{F^*}"'] & (NNF)^* \otimes (\Delta_{\mathbf{D}})^* \arrow[r, "\mathfrak{m}_{NNF,\Delta_{\mathbf{D}}}"] & (\Delta_{\mathbf{D}} \circ NNF)^* \arrow[d, "\mathrm{id}"] \\
(\mu_{\mathbf{C}})^* \otimes \hat{T}(F^*) \arrow[r, "\mathrm{id}\otimes\theta'_F"'] & (\mu_{\mathbf{C}})^* \otimes (NF)^* \arrow[r, "\mathfrak{m}_{\Delta_{\mathbf{C}},NF}"'] & (NF \circ \Delta_{\mathbf{C}})^*.
\end{tikzcd}$$

Finally, the components of the invertible modifications τ', λ' and ρ' are obtained as for τ, λ and ρ, but this time using θ' and the coherence data for the homomorphism $(-)^*$.

Now, let us examine in detail the dual 2-monad (T, ϵ, Δ) of the free symmetric strict monoidal category 2-monad (S, η, μ) on **Cat**. The 2-functor T has its action on objects given by $T\mathbf{C} = \big(S(\mathbf{C}^{\mathrm{op}})\big)^{\mathrm{op}}$, and this category has:

- **Objects** pairs $(n, \langle c_i \rangle)$, where $n \in S1$ and $x_1, \ldots, x_n \in \mathrm{ob}\,\mathbf{C}$;

- **Arrows**
$$(\sigma, \langle g_i \rangle) \colon (n, \langle c_i \rangle) \to (m, \langle d_i \rangle),$$
where $\sigma \in S1(n,m)$ and $g_i \colon c_{\sigma(i)} \to d_i$ in $\mathbf{C}$,

with composition and identities given in the evident way. There is an obvious isomorphism of categories $\gamma_{\mathbf{C}} \colon S\mathbf{C} \cong T\mathbf{C}$, which is the identity on objects and sends the map $(\sigma, \langle g_i \rangle)$ to $(\sigma^{-1}, \langle g_{\sigma^{-1}(i)} \rangle)$.

In fact, if we go on to describe the action of T on 1- and 2-cells, and then it's easy to see that the isomorphisms $\gamma_{\mathbf{C}}$ become the components of a 2-natural isomorphism $\gamma \colon S \Rightarrow T$. Furthermore, we have

$$\epsilon = \gamma \circ \eta \quad \text{and} \quad \Delta = \gamma \circ \mu \circ \gamma^{-1}\gamma^{-1},$$

and so we see that S and T are isomorphic as 2-monads. Thus, we may, without loss of generality take it that in fact $(S, \eta, \mu) = (T, \epsilon, \Delta)$.

Arguing similarly, we may apply Proposition 74 to the pseudomonad $(\hat{S}, \hat{\eta}, \hat{\mu}, \lambda, \rho, \tau)$ on **Mod**, to form the dual pseudocomonad $(\hat{T}, \hat{\epsilon}, \hat{\Delta}, \lambda', \rho', \tau')$. By spelling out the action of $\hat{T}$ on 1- and 2-cells of **Mod**, we find that as above, we may take it without loss of generality that $\hat{T} = \hat{S}$. Clearly, we may not take $\hat{\epsilon} = \hat{\eta}$ or $\hat{\Delta} = \hat{\mu}$, but nonetheless we will have

$$\hat{\epsilon}_X = (\eta_X)^* \quad \text{and} \quad \hat{\Delta}_X = (\mu_X)^*.$$

10.2 Pseudo-distributive laws

We are now ready to begin constructing our desired pseudo-distributive law; we begin by spelling out the requirements for such a pseudo-distributive law:

Definition 75. Let $(S, \eta, \mu, \lambda, \rho, \tau)$ be a pseudomonad and $(T, \epsilon, \Delta, \lambda', \rho', \tau')$ a pseudocomonad on a bicategory $\mathcal{B}$. Then a **pseudo-distributive law** δ of T over S is given by the following data:

(PDD1) A pseudo-natural transformation $\delta \colon TS \Rightarrow ST$;

(PDD2) Invertible modifications

$$\begin{tikzcd}
T \arrow[d, "T\eta"'] \arrow[dr, "\eta T"] \arrow[d, phantom, shift left=20, "\overset{\overline{\eta}}{\Longrightarrow}"] & \\
TS \arrow[r, "\delta"'] & ST
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
TS \arrow[r, "\delta"] \arrow[d, "\epsilon S"'] \arrow[r, phantom, shift right=15, "\overset{\overline{\epsilon}}{\Longrightarrow}"] & ST; \arrow[dl, "S\epsilon"] \\
S &
\end{tikzcd}$$

(PDD3) Invertible modifications

$$\begin{tikzcd}
TSS \arrow[r, "\delta S"] \arrow[d, "T\mu"'] & STS \arrow[r, "S\delta"] \arrow[d, phantom, "\Downarrow\overline{\mu}"] & SST \arrow[d, "\mu T"] \\
TS \arrow[rr, "\delta"'] & & ST
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
TS \arrow[rr, "\delta"] \arrow[d, "\Delta S"'] & & ST \arrow[d, "S\Delta"] \\
TTS \arrow[r, "T\delta"'] & TST \arrow[r, "\delta T"'] \arrow[u, phantom, "\Downarrow\overline{\Delta}"] & STT,
\end{tikzcd}$$

subject to the following axioms

$$\begin{tikzcd}
TS \arrow[r, "\epsilon S"] \arrow[dr, "\delta"] & S \arrow[dl, phantom, shift left=15, "\Downarrow\overline{\epsilon}"] \\
T \arrow[u, "T\eta"] \arrow[r, "\eta T"'] \arrow[ur, phantom, shift right=15, "\Downarrow\overline{\eta}"] & ST \arrow[u, "S\epsilon"']
\end{tikzcd}
=
\begin{tikzcd}
TS \arrow[rr, "\epsilon S"] & \Downarrow\cong & S \\
& \mathrm{id}_{\mathcal{B}} \arrow[ur, "\eta"'] & \\
T \arrow[uu, "T\eta"] \arrow[ur, "\epsilon"'] \arrow[rr, "\eta T"'] & \Downarrow\cong & ST \arrow[uu, "S\epsilon"']
\end{tikzcd}
\qquad \text{(PDA1)}$$

$$\begin{tikzcd}
TSS \arrow[r, "\delta S"] \arrow[dr, "T\mu"] & STS \arrow[r, "S\delta"] \arrow[dr, phantom, "\Downarrow\overline{\mu}"] & SST \arrow[d, "\mu T"] \\
TS \arrow[u, "T\eta S"] \arrow[r, "\mathrm{id}_{TS}"'] \arrow[ur, phantom, shift right=10, "\Downarrow T\rho"] & TS \arrow[r, "\delta"'] & ST
\end{tikzcd}
=
\begin{tikzcd}
TSS \arrow[r, "\delta S"] \arrow[dr, phantom, shift left=10, "\Downarrow\overline{\eta}S"] & STS \arrow[r, "S\delta"] \arrow[d, phantom, "\Downarrow\cong"] & SST \arrow[d, "\mu T"] \\
TS \arrow[u, "T\eta S"] \arrow[ur, "\eta TS"'] \arrow[r, "\delta"'] & ST \arrow[ur, "\eta ST"] \arrow[r, "\mathrm{id}_{ST}"'] \arrow[ur, phantom, shift right=15, "\Downarrow\rho T"] & ST
\end{tikzcd}$$

(PDA2)

$$\begin{tikzcd}
TSS \arrow[r, "\delta S"] \arrow[dr, "T\mu"] & STS \arrow[r, "S\delta"] \arrow[dr, phantom, "\Downarrow\overline{\mu}"] & SST \arrow[d, "\mu T"] \\
TS \arrow[u, "TS\eta"] \arrow[r, "\mathrm{id}_{TS}"'] \arrow[ur, phantom, shift right=10, "\Downarrow T\lambda"] & TS \arrow[r, "\delta"'] & ST
\end{tikzcd}
=
\begin{tikzcd}
TSS \arrow[r, "\delta S"] \arrow[dr, phantom, "\Downarrow\cong"] & STS \arrow[r, "S\delta"] \arrow[dr, phantom, shift left=10, "\Downarrow S\overline{\eta}"] & SST \arrow[d, "\mu T"] \\
TS \arrow[u, "TS\eta"] \arrow[r, "\delta"'] & ST \arrow[u, "ST\eta"'] \arrow[ur, "S\eta T"] \arrow[r, "\mathrm{id}_{ST}"'] \arrow[ur, phantom, shift right=15, "\Downarrow\lambda T"] & ST
\end{tikzcd}$$

(PDA3)

$$\begin{tikzcd}
TS \arrow[r, "\delta"] \arrow[d, "\Delta S"'] \arrow[dr, phantom, "\Downarrow\overline{\Delta}"] & ST \arrow[r, "\mathrm{id}_{ST}"] \arrow[dr, "S\Delta"'] & ST \arrow[dl, phantom, "\Downarrow S\rho'"] \\
TTS \arrow[r, "T\delta"'] & TST \arrow[r, "\delta T"'] & STT \arrow[u, "S\epsilon T"']
\end{tikzcd}
=
\begin{tikzcd}
TS \arrow[r, "\mathrm{id}_{TS}"] \arrow[d, "\Delta S"'] \arrow[dr, phantom, "\Downarrow\rho' S"] & TS \arrow[r, "\delta"] \arrow[d, phantom, "\Downarrow\cong"] & ST \arrow[d, phantom, "\Downarrow\overline{\epsilon}T"] \\
TTS \arrow[ur, "\epsilon TS"'] \arrow[r, "T\delta"'] & TST \arrow[ur, "\epsilon ST"] \arrow[r, "\delta T"'] & STT \arrow[u, "S\epsilon T"']
\end{tikzcd}$$

(PDA4)

$$\begin{tikzcd}
TS \arrow[r, "\delta"] \arrow[d, "\Delta S"'] \arrow[dr, phantom, "\Downarrow\overline{\Delta}"] & ST \arrow[r, "\mathrm{id}_{ST}"] \arrow[dr, "S\Delta"'] & ST \arrow[dl, phantom, "\Downarrow S\lambda'"] \\
TTS \arrow[r, "T\delta"'] & TST \arrow[r, "\delta T"'] & STT \arrow[u, "ST\epsilon"']
\end{tikzcd}
=
\begin{tikzcd}
TS \arrow[r, "\mathrm{id}_{TS}"] \arrow[d, "\Delta S"'] \arrow[dr, phantom, "\Downarrow\lambda' S"] & TS \arrow[r, "\delta"] \arrow[dr, phantom, "\Downarrow\cong"] & ST \\
TTS \arrow[ur, "T\epsilon S"] \arrow[ur, phantom, "\Downarrow T\overline{\epsilon}"'] \arrow[r, "T\delta"'] & TST \arrow[u, "TS\epsilon"'] \arrow[r, "\delta T"'] & STT \arrow[u, "ST\epsilon"']
\end{tikzcd}$$

(PDA5)

$$\begin{tikzcd}
& TSSS \arrow[r, "\delta SS"] \arrow[dl, "TS\mu"'] \arrow[dr, "T\mu S"] & STSS \arrow[r, "S\delta S"] \arrow[dr, phantom, "\Downarrow\overline{\mu}S"] & SSTS \arrow[r, "SS\delta"] \arrow[dr, "\mu TS"] & SSST \arrow[dr, "\mu ST"] \arrow[d, phantom, "\Downarrow\cong"] & \\
TSS \arrow[dr, "T\mu"'] \arrow[rr, phantom, "\overset{T\tau}{\Leftarrow}"] & & TSS \arrow[rr, "\delta S"'] \arrow[dl, "T\mu"] \arrow[drr, phantom, "\Downarrow\overline{\mu}"] & & STS \arrow[r, "S\delta"'] & SST \arrow[dl, "\mu T"] \\
& TS \arrow[rrr, "\delta"'] & & & ST &
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
& TSSS \arrow[r, "\delta SS"] \arrow[dl, "TS\mu"'] \arrow[d, phantom, "\Downarrow\cong"] & STSS \arrow[r, "S\delta S"] \arrow[dl, "ST\mu"] \arrow[dr, phantom, "\Downarrow S\overline{\mu}"] & SSTS \arrow[r, "SS\delta"] & SSST \arrow[dl, "S\mu T"'] \arrow[dr, "\mu ST"] & \\
TSS \arrow[r, "\delta S"'] \arrow[dr, "T\mu"'] & STS \arrow[rr, "S\delta"'] \arrow[dr, phantom, "\Downarrow\overline{\mu}"] & & SST \arrow[dr, "\mu T"'] \arrow[rr, phantom, "\overset{\tau T}{\Leftarrow}"] & & SST \arrow[dl, "\mu T"] \\
& TS \arrow[rrr, "\delta"'] & & & ST &
\end{tikzcd}$$

(PDA6)

$$\begin{tikzcd}
& TS \arrow[rrr, "\delta"'] \arrow[dl, "\Delta S"'] \arrow[dr, "\Delta S"] \arrow[from=1-2, to=2-5, phantom, "\Downarrow\overline{\Delta}"] & & & ST \arrow[dr, "S\Delta"] & \\
TTS \arrow[dr, "\Delta TS"'] \arrow[rr, phantom, "\overset{\tau' S}{\Longleftarrow}"] & & TTS \arrow[rr, "T\delta"'] \arrow[dl, "T\Delta S"] \arrow[dr, phantom, "\Downarrow T\overline{\Delta}"] & & TST \arrow[r, "\delta T"'] \arrow[dl, "TS\Delta"] \arrow[d, phantom, "\Downarrow\cong"] & STT \arrow[dl, "ST\Delta"] \\
& TTTS \arrow[r, "TT\delta"] & TTST \arrow[r, "T\delta T"] & TSTT \arrow[r, "\delta TT"] & STTT &
\end{tikzcd}$$

$$\Big\|$$

$$\begin{tikzcd}
& TS \arrow[rrr, "\delta"'] \arrow[dl, "\Delta S"'] \arrow[from=1-2, to=2-4, phantom, "\Downarrow\overline{\Delta}"] & & & ST \arrow[dl, "S\Delta"'] \arrow[dr, "S\Delta"] & \\
TTS \arrow[r, "T\delta"'] \arrow[dr, "\Delta TS"'] \arrow[from=2-1, to=3-3, phantom, "\Downarrow\cong"] & TST \arrow[rr, "\delta T"'] \arrow[dr, "\Delta ST"] \arrow[from=2-2, to=3-5, phantom, "\Downarrow\overline{\Delta}T"] & & STT \arrow[dr, "S\Delta T"'] \arrow[rr, phantom, "\overset{S\tau'}{\Longleftarrow}"] & & STT \arrow[dl, "ST\Delta"] \\
& TTTS \arrow[r, "TT\delta"] & TTST \arrow[r, "T\delta T"] & TSTT \arrow[r, "\delta TT"] & STTT &
\end{tikzcd}$$

(PDA7)

$$\begin{tikzcd}
TSS \arrow[rrr, "\epsilon SS"] \arrow[dr, "\delta S"'] \arrow[ddd, "T\mu"'] & \arrow[d, phantom, "\Downarrow\overline{\epsilon}S"] & \arrow[dd, phantom, "\Downarrow S\overline{\epsilon}"] & SS \arrow[ddd, "\mu"] \\
& STS \arrow[urr, "S\epsilon S"] \arrow[dr, "S\delta"'] \arrow[dd, phantom, "\Downarrow\overline{\mu}"] & & \\
& & SST \arrow[uur, "SS\epsilon"'] \arrow[d, "\mu T"'] \arrow[r, phantom, "\Downarrow\cong"] & {} \\
TS \arrow[rr, "\delta"'] & & ST \arrow[r, "S\epsilon"'] & S
\end{tikzcd}
=
\begin{tikzcd}
TSS \arrow[r, "\epsilon SS"] \arrow[d, "T\mu"'] & SS \arrow[dd, "\mu"] \\
TS \arrow[d, "\delta"'] \arrow[dr, "\epsilon S"] \arrow[ur, phantom, "\Downarrow\cong"] \arrow[dr, phantom, "\Downarrow\overline{\epsilon}"'] & \\
ST \arrow[r, "S\epsilon"'] & S
\end{tikzcd}$$

(PDA8)

$$
\begin{tikzcd}
T \arrow[rr, "T\eta"] \arrow[dddd, "\Delta"'] & & TS \arrow[rrr, "\delta"] \arrow[d, "\Delta S"] & & & ST \arrow[dddd, "S\Delta"] \\
& \Downarrow\cong & TTS \arrow[drr, "T\delta"] & \Downarrow\overline{\Delta} & & \\
& & \Downarrow T\overline{\eta} & & TST \arrow[ddr, "\delta T"] & \\
& & & & \Downarrow\overline{\eta}T & \\
TT \arrow[uuurr, "TT\eta"] \arrow[uurrrr, "T\eta T"'] \arrow[rrrrr, "\eta TT"'] & & & & & STT
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
T \arrow[rr, "T\eta"] \arrow[ddrr, "\eta T"'] \arrow[dddd, "\Delta"'] & & TS \arrow[dd, "\delta"] \\
& \Downarrow\overline{\eta} & \\
& & ST \arrow[dd, "S\Delta"] \\
& \Downarrow\cong & \\
TT \arrow[rr, "\eta TT"'] & & STT
\end{tikzcd}
\tag{PDA9}
$$

$$\begin{tikzcd}
 & TSS \arrow[rrr, "\delta S"] \arrow[ddl, "T\mu"'] \arrow[ddr, "\Delta SS"] & & & STS \arrow[rr, "S\delta"] \arrow[d, "S\Delta S"] & & SST \arrow[ddr, "SS\Delta"] & \\
 & & \Downarrow\overline{\Delta}S & & STTS \arrow[dr, "ST\delta"] & & \Downarrow S\overline{\Delta} & \\
TS \arrow[ddr, "\Delta S"'] & \overset{\cong}{\Leftarrow} & TTSS \arrow[r, "T\delta S"'] \arrow[ddl, "TT\mu"] & TSTS \arrow[ur, "\delta TS"] \arrow[dr, "TS\delta"'] & \Downarrow\cong & STST \arrow[rr, "S\delta T"] & & SSTT \arrow[ddl, "\mu TT"] \\
 & & \Downarrow T\overline{\mu} & & TSST \arrow[ur, "\delta ST"'] \arrow[d, "T\mu T"] & & \Downarrow\overline{\mu}T & \\
 & TTS \arrow[rrr, "T\delta"'] & & & TST \arrow[rr, "\delta T"'] & & STT &
\end{tikzcd}$$

$$\Vert$$

$$\begin{tikzcd}
 & TSS \arrow[rr, "\delta S"] \arrow[ddl, "T\mu"'] & & STS \arrow[rr, "S\delta"] & & SST \arrow[ddl, "\mu T"'] \arrow[ddr, "SS\Delta"] & \\
 & & \Downarrow\overline{\mu} & & & & \\
TS \arrow[rrrr, "\delta"'] \arrow[ddr, "\Delta S"'] & & & & ST \arrow[ddr, "S\Delta"'] & \overset{\cong}{\Leftarrow} & SSTT \arrow[ddl, "\mu TT"] \\
 & & \Downarrow\overline{\Delta} & & & & \\
 & TTS \arrow[rr, "T\delta"'] & & TST \arrow[rr, "\delta T"'] & & STT &
\end{tikzcd}$$

(PDA10)

10.3 Lifting to $\mathbb{C}oll(S)$

We would like now to apply the theory of double clubs to reduce the above definition to something more tractable. Explicitly, by exploiting the equivalence of double categories

$$\mathbb{C}oll(S) \simeq \mathbb{C}at/S\mathbf{I}_1,$$

we shall need only to specify data and coherence for our pseudo-distributive law ‘at 1’. However, as it stands, our pseudo-distributive law is not specified in terms of data and axioms in $\mathbb{C}oll(S)$, but rather in terms of data and axioms in the bicategory $[\mathbf{Mod}, \mathbf{Mod}]_\psi$. Observe, however, that the definitions of pseudomonad, pseudocomonad and pseudo-distributive law make sense in any bicategory equipped with a suitable 'whiskering' operations.

Now, we know that S is a double club on $\mathbb{C}at$, and thus that $\mathbb{C}oll(S)$ is a monoidal double category. Furthermore, it follows from Appendix A that $\mathbb{C}oll(S)$ also comes equipped with a suitable notion of 'whiskering', and thus so also does the bicategory $\mathcal{B}\big(\mathbb{C}oll(S)\big)$. Thus we may talk about pseudomonads, pseudocomonads and pseudo-distributive laws in $\mathcal{B}\big(\mathbb{C}oll(S)\big)$.

So we seek to establish a pseudo-distributive law in $\mathcal{B}\big(\mathbb{C}oll(S)\big)$ which lifts the desired pseudo-distributive law δ between $\hat{S}$ and $\hat{T}$ in $[\mathbf{Mod}, \mathbf{Mod}]_\psi$. To see what we mean by 'lifts' in this context, we need to know what we are intending to lift along; that is, we need to produce a homomorphism of bicategories $V\colon \mathcal{B}\big(\mathbb{C}oll(S)\big) \to [\mathbf{Mod}, \mathbf{Mod}]_\psi$. To do this, we recall the following:

- Every pseudo double category $\mathbb{K}$ contains a bicategory $\mathcal{B}\mathbb{K}$, consisting of the objects, horizontal maps and special cells of $\mathbb{K}$;
- Any double homomorphism $F\colon \mathbb{K} \to \mathbb{L}$ induces a homomorphism of bicategories $\mathcal{B}F\colon \mathcal{B}\mathbb{K} \to \mathcal{B}\mathbb{L}$;
- Any horizontal transformation $\mathbf{A}\colon A_s \Rrightarrow A_t$ induces a pseudo-natural transformation $\mathcal{B}\mathbf{A}\colon \mathcal{B}A_s \Rightarrow \mathcal{B}A_t$;
- Any special modification $\boldsymbol{\phi}\colon \mathbf{A} \Rrightarrow \mathbf{B}$ induces a modification $\mathcal{B}\boldsymbol{\phi}\colon \mathcal{B}\mathbf{A} \Rrightarrow \mathcal{B}\mathbf{B}$.

Furthermore, this operation $\mathcal{B}$ respects all forms of composition strictly, and therefore we have:

Proposition 76. *Given pseudo double categories $\mathbb{K}$ and $\mathbb{L}$, 'ignoring vertical arrows' induces a strict homomorphism of bicategories*

$$\mathcal{B}(\text{–})\colon \mathcal{B}\big([\mathbb{K}, \mathbb{L}]_\psi\big) \to [\mathcal{B}\mathbb{K}, \mathcal{B}\mathbb{L}]_\psi.$$

Now, in the case of interest to us, we have $\mathcal{B}\mathbb{C}at = \mathbf{Mod}$, and therefore a strict homomorphism $\mathcal{B}[\mathbb{C}at, \mathbb{C}at]_\psi \to [\mathbf{Mod}, \mathbf{Mod}]_\psi$. Moreover, we have a strict homomorphism of pseudo double categories

$$U\colon \mathbb{C}oll(S) \to [\mathbb{C}at, \mathbb{C}at]_\psi$$

which forgets the projection onto $S\mathbf{I}$; therefore we induce a strict homomorphism of bicategories

$$\mathcal{B}U\colon \mathcal{B}\big(\mathbb{C}oll(S)\big) \to \mathcal{B}\big([\mathbb{C}at, \mathbb{C}at]_\psi\big),$$

and composing this with the previous strict homomorphism, we obtain a strict homomorphism

$$V := \mathcal{B}\big(\mathbb{C}oll(S)\big) \xrightarrow{\mathcal{B}U} \mathcal{B}\big([\mathbb{C}at, \mathbb{C}at]_\psi\big) \xrightarrow{\mathcal{B}(-)} [\mathbf{Mod}, \mathbf{Mod}]_\psi.$$

Definition 77. We shall say that a datum in $\mathcal{B}\big(\mathbb{C}oll(S)\big)$ *lifts* a datum in $[\mathbf{Mod}, \mathbf{Mod}]_\psi$ if applying V to the former yields the latter.

So our line of attack will be to first lift $\hat{S}$ and $\hat{T}$ to $\mathbb{C}oll(S)$; once we have done this, we can give coherent data for a pseudo-distributive law in $\mathbb{C}oll(S)$ between these liftings, and then, applying the homomorphism V, obtain a pseudo-distributive law between $\hat{S}$ and $\hat{T}$ as desired. We note that in order for this to work, we use the fact that the strict homomorphism V also respects the 'whiskering' operations on $\mathcal{B}\big(\mathbb{C}oll(S)\big)$ and $[\mathbf{Mod}, \mathbf{Mod}]_\psi$.

10.3.1 Lifting $\hat{S}$ and $\hat{T}$

We now need to lift all the data for $\hat{S}$ and $\hat{T}$. The first stage is straightforward: for (PMD1) and (PCD1), we take the objects (S, id_S) and (T, id_S) of $\mathbb{C}oll(S)$, and have $V(S, \mathrm{id}_S) = \hat{S}$ and $V(T, \mathrm{id}_S) = \hat{T}$ as required.

For the remaining data, we shall perform the lifting in two stages; first we lift to $\mathcal{B}\big([\mathbb{C}at, \mathbb{C}at]_\psi\big)$, and thence to $\mathcal{B}\big(\mathbb{C}oll(S)\big)$. We start with (PMD2) and (PCD2):

Proposition 78. *We can lift the pseudo-natural transformations*

$$\hat{\eta}\colon \mathrm{id}_{\mathbf{Mod}} \Rightarrow \hat{S}, \quad \hat{\mu}\colon \hat{S}\hat{S} \Rightarrow \hat{S}, \quad \hat{\epsilon}\colon \hat{S} \Rightarrow \mathrm{id}_{\mathbf{Mod}} \quad \textit{and} \quad \hat{\Delta}\colon \hat{S} \Rightarrow \hat{S}\hat{S}$$

to respective horizontal transformations

$$\boldsymbol{\eta}\colon \mathrm{id}_{\mathbb{C}at} \Rrightarrow S, \quad \boldsymbol{\mu}\colon SS \Rrightarrow S, \quad \boldsymbol{\epsilon}\colon S \Rrightarrow \mathrm{id}_{\mathbb{C}at} \quad \textit{and} \quad \boldsymbol{\Delta}\colon S \Rrightarrow SS.$$

Proof. We shall illustrate the case for $\hat{\eta}$ and $\hat{\epsilon}$ only; $\hat{\mu}$ and $\hat{\Delta}$ follow in identical fashion. So, the component functors of $\boldsymbol{\eta}$ and $\boldsymbol{\epsilon}$ are given on objects by

$$\boldsymbol{\eta}X = \hat{\eta}_X = (\eta_X)_*\colon X \to\!\!\!\!\!\shortmid\; SX \quad \text{and} \quad \boldsymbol{\epsilon}X = \hat{\epsilon}_X = (\eta_X)^*\colon SX \to\!\!\!\!\!\shortmid\; X$$

and on a map $f\colon X \to Y$ by $\boldsymbol{\eta}f$ and $\boldsymbol{\epsilon}f$ given as follows:

$$\begin{tikzcd}
X \arrow[r, "\hat{\eta}_X", "|" marking] \arrow[d, "f_*"', "-" marking] \arrow[dr, phantom, "\Downarrow \hat{\eta}^{-1}_{(f_*)}"] & SX \arrow[d, "\hat{S}(f_*)", "-" marking] \arrow[d, bend left=60, "(Sf)_*", "-" marking] \\
Y \arrow[r, "\hat{\eta}_Y"', "|" marking] & SY
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
SX \arrow[r, "\hat{\epsilon}_X", "|" marking] \arrow[d, "\hat{S}(f_*)"', "-" marking] \arrow[d, bend right=60, "(Sf)_*"', "-" marking] \arrow[dr, phantom, "\Downarrow \hat{\epsilon}^{-1}_{(f_*)}"] & X \arrow[d, "f_*", "-" marking] \\
SY \arrow[r, "\hat{\epsilon}_Y"', "|" marking] & Y.
\end{tikzcd}$$

Easily this data satisfies (HTA1). It remains to give the pseudonaturality data (HTD2), which we do as follows:

$$\eta_{\mathbf{X}} = \begin{tikzcd}
X_s \arrow[r, "\mathbf{X}", "|" marking] \arrow[d, "\hat{\eta}_{X_s}"', "-" marking] \arrow[dr, phantom, "\Uparrow \hat{\eta}_{\mathbf{X}}^{-1}"] & X_t \arrow[d, "\hat{\eta}_{X_t}", "-" marking] \\
SX_s \arrow[r, "\hat{S}\mathbf{X}"', "|" marking] & SX_t.
\end{tikzcd}
\quad \text{and} \quad
\epsilon_{\mathbf{X}} = \begin{tikzcd}
SX_s \arrow[r, "\hat{S}\mathbf{X}", "|" marking] \arrow[d, "\hat{\epsilon}_{X_s}"', "-" marking] \arrow[dr, phantom, "\Uparrow \hat{\epsilon}_{\mathbf{X}}^{-1}"] & SX_t \arrow[d, "\hat{\epsilon}_{X_t}", "-" marking] \\
X_s \arrow[r, "\mathbf{X}"', "|" marking] & X_t.
\end{tikzcd}$$

We must check the naturality of these components in maps of $\mathbb{C}at_1$. So, given a map $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ of $\mathbb{C}at_1$, we need the equality

$$\begin{tikzcd}
X_s \arrow[r, "\boldsymbol{\eta}X_s", "|" marking] \arrow[d, "f_s"'] \arrow[dr, phantom, "\Downarrow \boldsymbol{\eta}f_s"] & SX_s \arrow[r, "S\mathbf{X}", "|" marking] \arrow[d, "Sf_s"] \arrow[dr, phantom, "\Downarrow S\mathbf{f}"] & SX_t \arrow[d, "Sf_t"] \\
Y_s \arrow[r, "\boldsymbol{\eta}Y_s", "|" marking] \arrow[d, equal] & SY_s \arrow[r, "S\mathbf{Y}", "|" marking] \arrow[d, phantom, "\Downarrow \eta_{\mathbf{Y}}"] & SY_t \arrow[d, equal] \\
Y_s \arrow[r, "\mathbf{Y}"', "|" marking] & Y_t \arrow[r, "\hat{\eta}_{Y_t}"', "|" marking] & SY_t
\end{tikzcd}
=
\begin{tikzcd}
X_s \arrow[r, "\boldsymbol{\eta}X_s", "|" marking] \arrow[d, equal] & SX_s \arrow[r, "S\mathbf{X}", "|" marking] \arrow[d, phantom, "\Downarrow \eta_{\mathbf{X}}"] & SX_t \arrow[d, equal] \\
X_s \arrow[r, "\mathbf{X}", "|" marking] \arrow[d, "f_s"'] \arrow[dr, phantom, "\Downarrow \mathbf{f}"] & X_t \arrow[r, "\boldsymbol{\eta}X_t", "|" marking] \arrow[d, "f_t"] \arrow[dr, phantom, "\Downarrow \boldsymbol{\eta}f_t"] & SX_t \arrow[d, "Sf_t"] \\
Y_s \arrow[r, "\mathbf{Y}"', "|" marking] & Y_t \arrow[r, "\boldsymbol{\eta}Y_t"', "|" marking] & SY_t
\end{tikzcd}$$

to hold. But this follows from the following equality of pastings

$$\begin{tikzcd}
X_s \arrow[rrr, "\hat{\eta}_{X_s}", "\shortmid"{marking}] \arrow[d, "(f_s)_*"', "\shortmid"{marking}] \arrow[rrd, phantom, "\Downarrow \hat{\eta}^{-1}_{(f_s)_*}"] & & & SX_s \arrow[dl, "\hat{S}(f_s)_*"', "\shortmid"{marking}] \arrow[dd, bend right=40, "\hat{S}(\mathbf{Y}\otimes(f_s)_*)"', "\shortmid"{marking}] \arrow[dd, bend left=40, "\hat{S}((f_t)_*\otimes\mathbf{X})", "\shortmid"{marking}] \arrow[dd, phantom, "\overset{\hat{S}f}{\Leftarrow}"] \arrow[dr, "\hat{S}\mathbf{X}", "\shortmid"{marking}] & \\
Y_s \arrow[rr, "\hat{\eta}_{Y_s}", "\shortmid"{marking}] \arrow[d, "\mathbf{Y}"', "\shortmid"{marking}] \arrow[rrrd, phantom, "\Downarrow \hat{\eta}^{-1}_{\mathbf{Y}}"] & & SY_s \arrow[dr, "\hat{S}\mathbf{Y}"', "\shortmid"{marking}] & & SX_t \arrow[dl, "(Sf_t)_*", "\shortmid"{marking}] \\
Y_t \arrow[rrr, "\hat{\eta}_{Y_t}"', "\shortmid"{marking}] & & & SY_t &
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
X_s \arrow[rrr, "\hat{\eta}_{X_s}", "\shortmid"{marking}] \arrow[dd, "\mathbf{Y}\otimes(f_s)_*"', "\shortmid"{marking}] \arrow[rrrdd, phantom, "\Downarrow \hat{\eta}^{-1}_{\mathbf{Y}\otimes(f_s)_*}"] & & & SX_s \arrow[dd, bend right=40, "\hat{S}(\mathbf{Y}\otimes(f_s)_*)"', "\shortmid"{marking}] \arrow[dd, bend left=40, "\hat{S}((f_t)_*\otimes\mathbf{X})", "\shortmid"{marking}] \arrow[dd, phantom, "\overset{\hat{S}f}{\Leftarrow}"] \arrow[dr, "\hat{S}\mathbf{X}", "\shortmid"{marking}] & \\
 & & & & SX_t \arrow[dl, "(Sf_t)_*", "\shortmid"{marking}] \\
Y_t \arrow[rrr, "\hat{\eta}_{Y_t}"', "\shortmid"{marking}] & & & SY_t &
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
X_s \arrow[rrr, "\hat{\eta}_{X_s}", "\shortmid"{marking}] \arrow[dd, bend right=40, "\mathbf{Y}\otimes(f_s)_*"', "\shortmid"{marking}] \arrow[dd, bend left=40, "(f_t)_*\otimes\mathbf{X}", "\shortmid"{marking}] \arrow[dd, phantom, "\overset{f}{\Leftarrow}"] \arrow[rrrdd, phantom, "\Downarrow \hat{\eta}^{-1}_{(f_t)_*\otimes\mathbf{X}}"] & & & SX_s \arrow[dd, "\hat{S}((f_t)_*\otimes\mathbf{X})", "\shortmid"{marking}] \arrow[dr, "\hat{S}\mathbf{X}", "\shortmid"{marking}] & \\
 & & & & SX_t \arrow[dl, "(Sf_t)_*", "\shortmid"{marking}] \\
Y_t \arrow[rrr, "\hat{\eta}_{Y_t}"', "\shortmid"{marking}] & & & SY_t &
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
X_s \arrow[rrr, "\hat{\eta}_{X_s}", "\shortmid"{marking}] \arrow[d, "\mathbf{X}", "\shortmid"{marking}] \arrow[dd, bend right=60, "\mathbf{Y}\otimes(f_s)_*"', "\shortmid"{marking}] \arrow[rrrd, phantom, "\Downarrow \hat{\eta}^{-1}_{\mathbf{X}}"] & & & SX_s \arrow[d, "\hat{S}\mathbf{X}", "\shortmid"{marking}] \\
X_t \arrow[l, phantom, "\overset{f}{\Leftarrow}"] \arrow[rrr, "\hat{\eta}_{X_t}"', "\shortmid"{marking}] \arrow[d, "(f_t)_*", "\shortmid"{marking}] \arrow[rrrd, phantom, "\Downarrow \hat{\eta}^{-1}_{(f_t)_*}"] & & & SX_t \arrow[d, "(Sf_t)_*", "\shortmid"{marking}] \\
Y_t \arrow[rrr, "\hat{\eta}_{Y_t}"', "\shortmid"{marking}] & & & SY_t.
\end{tikzcd}$$

We argue similarly for $\hat{\epsilon}$. Finally, this data is required to satisfy (HTA2) and (HTA3); but these follow immediately from the strong transformation axioms satisfied by $\hat{\eta}$ and $\hat{\epsilon}$. $\square$

And now we lift this data to $\mathcal{B}\big(\mathbb{C}oll(S)\big)$:

Proposition 79. *There are horizontal arrows*

$$\begin{tikzcd}
\mathrm{id}_{\mathbb{C}at} \arrow[r, Rightarrow, "\boldsymbol{\eta}", "|" marking] \arrow[d, Rightarrow, "\eta"'] \arrow[dr, phantom, "\Downarrow\tilde{\boldsymbol{\eta}}"] & S \arrow[d, Rightarrow, "\mathrm{id}_S"] & , & SS \arrow[r, Rightarrow, "\boldsymbol{\mu}", "|" marking] \arrow[d, Rightarrow, "\mu"'] \arrow[dr, phantom, "\Downarrow\tilde{\boldsymbol{\mu}}"] & S \arrow[d, Rightarrow, "\mathrm{id}_S"] & , & T \arrow[r, Rightarrow, "\boldsymbol{\epsilon}", "|" marking] \arrow[d, Rightarrow, "\mathrm{id}_S"'] \arrow[dr, phantom, "\Downarrow\tilde{\boldsymbol{\epsilon}}"] & \mathrm{id}_{\mathbb{C}at} \arrow[d, Rightarrow, "\eta"] & , & T \arrow[r, Rightarrow, "\boldsymbol{\Delta}", "|" marking] \arrow[d, Rightarrow, "\mathrm{id}_S"'] \arrow[dr, phantom, "\Downarrow\tilde{\boldsymbol{\Delta}}"] & TT \arrow[d, Rightarrow, "\mu"] \\
S \arrow[r, Rightarrow, "S\mathbf{I}"', "|" marking] & S & & S \arrow[r, Rightarrow, "S\mathbf{I}"', "|" marking] & S & & S \arrow[r, Rightarrow, "S\mathbf{I}"', "|" marking] & S & & S \arrow[r, Rightarrow, "S\mathbf{I}"', "|" marking] & S
\end{tikzcd}$$

of $\mathbb{C}oll(S)$*, lifting* $\boldsymbol{\eta}$*,* $\boldsymbol{\mu}$*,* $\boldsymbol{\epsilon}$ *and* $\boldsymbol{\Delta}$*.*

Proof. We illustrate the case of $\boldsymbol{\eta}$. Consider the following cartesian lifting:

$$\begin{tikzcd}
X \arrow[r, "\langle\eta_X, \mathrm{id}_{SX}\rangle_* \mathbf{I}_{SX}", "|" marking] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "\Downarrow\langle\eta_X, \mathrm{id}_{SX}\rangle"] & SX \arrow[d, "\mathrm{id}_{SX}"] \\
SX \arrow[r, "\mathbf{I}_{SX}"', "|" marking] & SX.
\end{tikzcd}$$

From the proof of Proposition 50 that we have $\langle\eta_X, \mathrm{id}_{SX}\rangle_* \mathbf{I}_{SX} = (\eta_X)_* = \boldsymbol{\eta}X$; so we shall take the central natural transformation of $\tilde{\boldsymbol{\eta}}$ to have component at X given by

$$\tilde{\boldsymbol{\eta}}_X = \boldsymbol{\eta}X \xrightarrow{\langle\eta_X, \mathrm{id}_{SX}\rangle} \mathbf{I}_{SX} \xrightarrow{\mathfrak{e}_X} S\mathbf{I}_X.$$

We must check two things: firstly, that this defines a cartesian natural transformation $\eta_c \Rightarrow S\mathbf{I}_{(-)} \colon \mathbb{C}at_0 \to \mathbb{C}at_1$, and secondly that it satisfies the coherence condition for a modification.

For the first of these, we must check that diagrams of the following form commute and are pullbacks:

$$\begin{tikzcd}
\boldsymbol{\eta}X \arrow[r, "\boldsymbol{\eta}f"] \arrow[d, "\tilde{\boldsymbol{\eta}}_X"'] & \boldsymbol{\eta}Y \arrow[d, "\tilde{\boldsymbol{\eta}}_Y"] \\
S\mathbf{I}_X \arrow[r, "S\mathbf{I}_f"'] & S\mathbf{I}_Y.
\end{tikzcd}$$

Now, we can see that the 'source' and 'target' squares for this, namely

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] & \text{and} & SX \arrow[r, "Sf"] \arrow[d, "\mathrm{id}_{SX}"'] & SY \arrow[d, "\mathrm{id}_{SY}"] \\
SX \arrow[r, "Sf"'] & SY & & SX \arrow[r, "Sf"'] & SY
\end{tikzcd}$$

commute and are pullbacks, so it suffices to check that the diagrams

$$\begin{tikzcd}
\boldsymbol{\eta}X(\langle y_i\rangle; x) \arrow[r, "\boldsymbol{\eta}f"] \arrow[d, "\tilde{\eta}_X"'] & \boldsymbol{\eta}Y(\langle fy_i\rangle; x) \arrow[d, "\tilde{\eta}_Y"] \\
SX(\langle y_i\rangle; x) \arrow[r, "Sf"'] & SY(\langle fy_i\rangle; \langle fx\rangle)
\end{tikzcd}$$

commute and are pullbacks: that they commute follows from unrolling the definitions, and they are pullbacks since both vertical arrows are isomorphisms.

Secondly, we must check that the requisite squares commute making this into a modification; thus given $\mathbf{X}\colon X_s \to\!\!\!\!\!+ X_t$ in $\mathbb{C}at_1$, we need the following diagram to commute:

$$\begin{tikzcd}
S\mathbf{X} \otimes \boldsymbol{\eta}X_s \arrow[r, "\mathrm{id}_{S\mathbf{X}} \otimes \tilde{\boldsymbol{\eta}}_{X_s}"] \arrow[d, "\eta_{\mathbf{X}}"'] & S\mathbf{X} \otimes S\mathbf{I}_{X_s} \arrow[d, "\mathfrak{m}^{-1}_{\mathbf{I}_{X_t},\mathbf{X}} \circ \mathfrak{l}_{\mathbf{X}} \circ \mathfrak{r}^{-1}_{\mathbf{X}} \circ \mathfrak{m}_{\mathbf{X},\mathbf{I}_{X_s}}"] \\
\boldsymbol{\eta}X_t \otimes \mathbf{X} \arrow[r, "\tilde{\boldsymbol{\eta}}_{X_t} \otimes \eta_{\mathbf{X}}"'] & S\mathbf{I}_{X_t} \otimes S\mathbf{X};
\end{tikzcd}$$

again, it's easy to check that the 'source' and 'target' squares for this commute, so it suffices to check that the following diagrams commute in **Set**:

$$\begin{tikzcd}
(SX \otimes \boldsymbol{\eta}X_s)(\langle y_i\rangle; x) \arrow[r, "\mathrm{id}_{SX} \otimes \tilde{\boldsymbol{\eta}}_{X_s}"] \arrow[dd, "\eta_{\mathbf{X}}"'] & (SX \otimes S\mathbf{I}_{X_s})(\langle y_i\rangle; \langle x\rangle) \arrow[dr, "\mathfrak{r}^{-1}_{\mathbf{X}} \circ \mathfrak{m}_{\mathbf{X},\mathbf{I}_{X_s}}"] & \\
 & & SX(\langle y_i\rangle; \langle x\rangle) \\
(\boldsymbol{\eta}X_t \otimes X)(\langle y_i\rangle; x) \arrow[r, "\tilde{\boldsymbol{\eta}}_{X_t} \otimes \eta_X"'] & (S\mathbf{I}_{X_t} \otimes SX)(\langle y_i\rangle; \langle x\rangle) \arrow[ur, "\mathfrak{l}^{-1}_{\mathbf{X}} \circ \mathfrak{m}_{\mathbf{X},\mathbf{I}_{X_s}}"'] &
\end{tikzcd}$$

and again, unrolling the definitions shows that they indeed do. We proceed in the same fashion to construct $\tilde{\boldsymbol{\mu}}$, $\tilde{\boldsymbol{\epsilon}}$ and $\tilde{\boldsymbol{\Delta}}$. $\square$

So it remains to lift (PMD3) and (PCD3). Again, we start by lifting to $\mathcal{B}\big([\mathbb{C}at, \mathbb{C}at]_\psi\big)$:

Proposition 80. *The modifications λ, ρ and τ for the pseudomonad $\hat{S}$ lift to* *special modifications*

$$\boldsymbol{\lambda}\colon \mathbf{I}_S \Rrightarrow \boldsymbol{\mu} \otimes S\boldsymbol{\eta}\colon S \nRightarrow S$$

$$\boldsymbol{\rho}\colon \mathbf{I}_S \Rrightarrow \boldsymbol{\mu} \otimes \boldsymbol{\eta}S\colon S \nRightarrow S$$

$$\textit{and } \boldsymbol{\tau}\colon \boldsymbol{\mu} \otimes \boldsymbol{\mu}S \Rrightarrow \boldsymbol{\mu} \otimes S\boldsymbol{\mu}\colon SSS \nRightarrow S.$$

Similarly, the modifications λ', ρ' and τ' for the pseudocomonad $\hat{T}$ lift to special modifications

$$\boldsymbol{\lambda}'\colon T\boldsymbol{\epsilon} \otimes \boldsymbol{\Delta} \Rrightarrow \mathbf{I}_T\colon T \nRightarrow T$$

$$\boldsymbol{\rho}'\colon \boldsymbol{\epsilon}T \otimes \boldsymbol{\Delta} \Rrightarrow \mathbf{I}_T\colon T \nRightarrow T$$

$$\textit{and } \boldsymbol{\tau}'\colon T\boldsymbol{\Delta} \otimes \boldsymbol{\Delta} \Rrightarrow \boldsymbol{\Delta}T \otimes \boldsymbol{\Delta}\colon T \nRightarrow TTT.$$

Proof. For λ, we give special maps

$$\lambda_X\colon \mathbf{I}_{SX} \Rightarrow (\mu_X)_* \otimes \hat{S}(\eta_X)_*$$

by taking this to be the component of the modification $\hat{\lambda}$ at X. We must check that these maps are natural in X, which amounts to checking that the following two composites agree:

$$\begin{tikzcd}
SX \arrow[rr, "\mathbf{I}_{SX}", "\shortmid" marking] \arrow[d, "Sf"'] & {} \arrow[d, phantom, "\Downarrow \mathbf{I}_{Sf}"] & SX \arrow[d, "Sf"] & & SX \arrow[rr, "\mathbf{I}_{SX}", "\shortmid" marking] \arrow[d, equal] & {} \arrow[d, phantom, "\Downarrow \lambda_X"] & SX \arrow[d, equal] \\
SY \arrow[rr, "\mathbf{I}_{SY}", "\shortmid" marking] \arrow[d, equal] & {} \arrow[d, phantom, "\Downarrow \lambda_Y"] & SY \arrow[d, equal] & = & SX \arrow[r, "S\boldsymbol{\eta}X", "\shortmid" marking] \arrow[d, "Sf"'] \arrow[dr, phantom, "\Downarrow S\boldsymbol{\eta}f"] & SSX \arrow[r, "\boldsymbol{\mu}X", "\shortmid" marking] \arrow[d, "SSf"] \arrow[dr, phantom, "\Downarrow \boldsymbol{\mu}f"] & SX \arrow[d, "Sf"] \\
SY \arrow[r, "S\boldsymbol{\eta}Y"', "\shortmid" marking] & SSY \arrow[r, "\mathbf{m}uY"', "\shortmid" marking] & SY & & SY \arrow[r, "S\boldsymbol{\eta}Y"', "\shortmid" marking] & SSY \arrow[r, "\boldsymbol{\mu}Y"', "\shortmid" marking] & SY,
\end{tikzcd}$$

but this follows the equality of pastings:

$$\begin{tikzcd}
SX \arrow[rr, bend left=40, "\mathbf{I}_{SX}", "\shortmid"{marking}] \arrow[drr, bend left=20, "\hat{S}(f_*)", "\shortmid"{marking}] \arrow[d, "\hat{S}(f_*)"', "\shortmid"{marking}] \arrow[d, bend right=70, "(Sf)_*"', "\shortmid"{marking}] & & SX \arrow[d, "\hat{S}(f_*)", "\shortmid"{marking}] \arrow[d, bend left=70, "(Sf)_*", "\shortmid"{marking}] \\
SY \arrow[r, "\hat{S}((\eta_Y)_*)"', "\shortmid"{marking}] \arrow[rr, bend left=50, "\mathbf{I}_{SY}", "\shortmid"{marking}] & SSY \arrow[r, "(\mu_Y)_*"', "\shortmid"{marking}] \arrow[u, phantom, "\Downarrow\hat{\lambda}_Y"] & SY
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
SX \arrow[rr, bend left=50, "\mathbf{I}_{SX}", "\shortmid"{marking}] \arrow[r, "\hat{S}((\eta_X)_*)", "\shortmid"{marking}] \arrow[d, "\hat{S}(f_*)"', "\shortmid"{marking}] \arrow[d, bend right=70, "(Sf)_*"', "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow\hat{S}\hat{\eta}^{-1}_{f_*}"] & SSX \arrow[r, "(\mu_X)_*", "\shortmid"{marking}] \arrow[d, "\hat{S}\hat{S}(f_*)", "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow\hat{\mu}^{-1}_{f_*}"] \arrow[u, phantom, "\Downarrow\hat{\lambda}_X"] & SX \arrow[d, "\hat{S}(f_*)", "\shortmid"{marking}] \arrow[d, bend left=70, "(Sf)_*", "\shortmid"{marking}] \\
SY \arrow[r, "\hat{S}((\eta_Y)_*)"', "\shortmid"{marking}] & SSY \arrow[r, "(\mu_Y)_*"', "\shortmid"{marking}] & SY
\end{tikzcd}$$

exhibiting $\hat{\lambda}$ as a modification $\mathrm{id}_{\hat{S}} \Rrightarrow \hat{\mu} \circ \hat{S}\hat{\eta}$. We proceed identically to construct ρ, τ, λ', ρ' and τ'. ☐

And now we lift to $\mathcal{B}\big(\mathbb{C}oll(S)\big)$:

Proposition 81. *The invertible special modifications $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$, $\boldsymbol{\tau}$, $\boldsymbol{\lambda}'$, $\boldsymbol{\rho}'$ and $\boldsymbol{\tau}'$ lift to invertible special cells of $\mathbb{C}oll(S)$.*

Proof. We must check, for instance, that $\boldsymbol{\lambda}$ lifts to a cell

$$\boldsymbol{\lambda} \colon (\boldsymbol{\mu}, \tilde{\boldsymbol{\mu}}) \otimes (S, \mathrm{id}_S)(\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}) \Rrightarrow \mathbf{I}_{(S, \mathrm{id}_S)}.$$

All that is required for this is to check that $\boldsymbol{\lambda}$ is compatible in $\mathbb{C}oll(S)_1$ with the projections of the 'source' and 'target' objects down to $S\mathbf{I}$. This is a somewhat long but unenlightening diagram chase which we therefore omit. ☐

10.3.2 Lifting δ

We have seen that we can lift $\hat{S}$ and $\hat{T}$ to $\mathcal{B}\big(\mathbb{C}oll(S)\big)$, and thus it makes sense to ask for data for a pseudo-distributive law between them. We now wish to see how we can use the theory of double clubs to reduce this to a collection of data in $\mathcal{B}\big(\mathbb{C}at/S\mathbf{I}_1\big)$. We begin with (PDA1), for which we must produce a horizontal arrow

$$(\boldsymbol{\delta}, \tilde{\boldsymbol{\delta}})\colon (TS, \mu) \relbar\joinrel\mapstochar\joinrel\rightarrow (ST, \mu)$$

of $\mathbb{C}oll(S)$, i.e., a horizontal transformation and a cartesian modification as follows:

$$\begin{tikzcd}
TS \arrow[r, Rightarrow, "\boldsymbol{\delta}", "\shortmid" marking] \arrow[d, Rightarrow, "\mu"'] \arrow[dr, phantom, "\Downarrow \tilde{\boldsymbol{\delta}}"] & ST \arrow[d, Rightarrow, "\mu"] \\
S \arrow[r, Rightarrow, "S\mathbf{I}"', "\shortmid" marking] & S.
\end{tikzcd}$$

Now, suppose we have a horizontal arrow

$$\begin{tikzcd}
TS1 \arrow[r, "\mathbf{d}", "\shortmid" marking] \arrow[d, "\mu_1"'] \arrow[dr, phantom, "\Downarrow \tilde{\mathbf{d}}"] & ST1 \arrow[d, "\mu_1"] \\
S1 \arrow[r, "S\mathbf{I}_1"', "\shortmid" marking] & S1
\end{tikzcd}$$

of $\mathbb{C}at/S\mathbf{I}_1$. We should like to say that $(\mathbf{d}, \tilde{\mathbf{d}})$ is the component at 1 of some horizontal arrow $(\boldsymbol{\delta}, \tilde{\boldsymbol{\delta}})$ of $\mathbb{C}oll(S)$, which amounts to asking for the double homomorphism $F\colon \mathbb{C}oll(S) \to \mathbb{C}at/S\mathbf{I}_1$ to be 'horizontally full', in the following sense:

Proposition 82. *Let (A_s, α_s) and (A_t, α_t) be objects of $\mathbb{C}oll(S)$, and suppose that we have a horizontal arrow*

$$\begin{tikzcd}
A_s1 \arrow[r, "\mathbf{a}", "\shortmid" marking] \arrow[d, "(\alpha_s)_1"'] \arrow[dr, phantom, "\Downarrow \boldsymbol{\theta}"] & A_t1 \arrow[d, "(\alpha_t)_1"] \\
S1 \arrow[r, "S\mathbf{I}_1"', "\shortmid" marking] & S1
\end{tikzcd}$$

of $\mathbb{C}at/S\mathbf{I}_1$. *Then there is a horizontal arrow* $(\mathbf{A}, \boldsymbol{\alpha})$ *of* $\mathbb{C}oll(S)$*:-*

$$\begin{tikzcd}
A_s \arrow[r, Rightarrow, "\mathbf{A}", "\shortmid"{marking}] \arrow[d, Rightarrow, "\alpha_s"'] \arrow[dr, phantom, "\Downarrow\boldsymbol{\alpha}"] & A_t \arrow[d, Rightarrow, "\alpha_t"] \\
S \arrow[r, Rightarrow, "S\mathbf{I}"', "\shortmid"{marking}] & S
\end{tikzcd}$$

such that $F(\mathbf{A}, \boldsymbol{\alpha}) = (\mathbf{a}, \boldsymbol{\theta})$.

To prove this, we shall need the following result:

Proposition 83. *Let* $\mathbb{K}$ *and* $\mathbb{L}$ *be pseudo double categories, and suppose that the functor* $[s,t]\colon L_1 \to L_0 \times L_0$ *admits cartesian liftings of isomorphisms. Then the functor* $[s,t]\colon [\mathbb{K}, \mathbb{L}]_{h\psi} \to [\mathbb{K}, \mathbb{L}]_{v\psi} \times [\mathbb{K}, \mathbb{L}]_{v\psi}$ *also admits cartesian liftings of isomorphisms.*

Proof. Suppose we are given a horizontal transformation $\mathbf{B}\colon B_s \Rrightarrow B_t \colon \mathbb{K} \to \mathbb{L}$ and vertical isomorphisms $f_s\colon A_s \Rightarrow B_s$ and $f_t\colon A_t \Rightarrow B_t$; then we must construct a modification $\langle f_s, f_t\rangle\colon \langle f_s, f_t\rangle_*(\mathbf{B}) \Rrightarrow \mathbf{B}$ as follows:-

$$\begin{tikzcd}
A_s \arrow[r, Rightarrow, "\langle f_s, f_t\rangle_*(\mathbf{B})", "\shortmid"{marking}] \arrow[d, Rightarrow, "f_s"'] \arrow[dr, phantom, "\Downarrow\langle f_s, f_t\rangle"] & A_t \arrow[d, Rightarrow, "f_t"] \\
B_s \arrow[r, Rightarrow, "\mathbf{B}"', "\shortmid"{marking}] & B_t.
\end{tikzcd}$$

We start by giving $\mathbf{A} = \langle f_s, f_t\rangle_*(\mathbf{B})$. Observe that given $X \in K_0$, we have the following cartesian liftings:

$$\begin{tikzcd}
A_sX \arrow[r, "\langle (f_s)_X, (f_t)_X\rangle_*(\mathbf{B}X)", "\shortmid"{marking}] \arrow[d, "(f_s)_X"'] \arrow[dr, phantom, "\Downarrow\langle (f_s)_X, (f_t)_X\rangle"] & A_tX \arrow[d, "(f_t)_X"] \\
B_sX \arrow[r, "\mathbf{B}X"', "\shortmid"{marking}] & B_tX.
\end{tikzcd}$$

Let us write f_X for $\langle (f_s)_X, (f_t)_X\rangle$ and $(f_X)_*(\mathbf{B}X)$ for $\langle (f_s)_X, (f_t)_X\rangle_*(\mathbf{B}X)$. Observe that since $(f_s)_X$ and $(f_t)_X$ are invertible maps, so also will f_X be, and furthermore, we have

$$f_X^{-1} = \big\langle (f_s)_X^{-1}, (f_t)_X^{-1} \big\rangle \colon \mathbf{B}X \to \langle (f_s)_X, (f_t)_X \rangle_* (\mathbf{B}X).$$

So we give the components functor A_c on objects by setting $\mathbf{A}X = (f_X)_*(\mathbf{B}X)$, and on maps $g\colon X \to Y$ by setting $\mathbf{A}g\colon \mathbf{A}X \to \mathbf{A}Y$ be the map $(f_X)_*(\mathbf{B}g)$ induced by the universal property of cartesian liftings and satisfying

$$\mathbf{B}g \circ f_X = f_Y \circ \mathbf{A}g.$$

Observe that this makes the maps f_X into the components of a natural transformation $f\colon A_c \Rightarrow B_c$. To give the pseudonaturality invertible special transformation for $\mathbf{A}$, we take the component at $\mathbf{X}$ to be given by the composite

$$A_t\mathbf{X} \otimes \mathbf{A}X_s \xrightarrow{(f_t)_{\mathbf{X}} \otimes f_{X_s}} B_t\mathbf{X} \otimes \mathbf{B}X_s \xrightarrow{B_{\mathbf{X}}} \mathbf{B}X_t \otimes B_s\mathbf{X} \xrightarrow{f_{X_t}^{-1} \otimes (f_s)_{\mathbf{X}}^{-1}} \mathbf{A}X_t \otimes A_s\mathbf{X}.$$

Observe that this is indeed a special map in L_1, since its source and target are the maps

$$A_sX_s \xrightarrow{(f_s)_{X_s}} B_sX_s \xrightarrow{\mathrm{id}} B_sX_s \xrightarrow{(f_s)_{X_s}^{-1}} A_sX_s$$

and

$$A_tX_t \xrightarrow{(f_t)_{X_t}} B_tX_t \xrightarrow{\mathrm{id}} B_tX_t \xrightarrow{(f_t)_{X_t}^{-1}} A_tX_t.$$

The naturality of these maps in $\mathbf{X}$ follows from the naturality of $B_{(-)}$, f, f_s and f_t. That the required coherence diagrams commute follows straightforwardly, as we are just conjugating by f. It remains to give the modification $\langle f_s, f_t \rangle \colon \mathbf{A} \Rrightarrow \mathbf{B}$. We take its central natural transformation to be $f\colon A \Rightarrow B\colon K_0 \to L_1$; easily (MA1) is satisfied, whilst for (MA2) we require diagrams of the following form to commute:

$$\begin{tikzcd}
A_t\mathbf{X} \otimes AX_s \arrow[r, "A_{\mathbf{X}}"] \arrow[d, "(f_t)_{\mathbf{X}} \otimes f_{X_s}"'] & AX_t \otimes A_s\mathbf{X} \arrow[d, "f_{X_t} \otimes (f_s)_{\mathbf{X}}"] \\
B_t\mathbf{X} \otimes BX_s \arrow[r, "B_{\mathbf{X}}"'] & BX_t \otimes B_s\mathbf{X},
\end{tikzcd}$$

which they do by definition of $A_{\mathbf{X}}$. It remains to check that this lifting is cartesian; but any lifting of an isomorphism is automatically cartesian, so we are done. □

Now we are ready to prove Proposition 82:

Proof. Let us write $(\hat{\mathbf{A}}, \hat{\boldsymbol{\alpha}})$ for $G(\mathbf{a}, \boldsymbol{\theta})$; so we have

$$\begin{tikzcd} \hat{A}_s \arrow[r, Rightarrow, "\hat{\mathbf{A}}"] \arrow[d, Rightarrow, "\hat{\alpha}_s"'] \arrow[dr, phantom, "\Downarrow\hat{\boldsymbol{\alpha}}"] & \hat{A}_t \arrow[d, Rightarrow, "\hat{\alpha}_t"] \\ S \arrow[r, Rightarrow, "S\mathbf{I}"'] & S. \end{tikzcd}$$

Furthermore, we have invertible vertical transformations

$$\eta_s := \eta_{(A_s,\alpha_s)} \colon A_s \Rightarrow \hat{A}_s \quad \text{and} \quad \eta_t := \eta_{(A_t,\alpha_t)} \colon A_t \Rightarrow \hat{A}_t$$

such that $\hat{\alpha}_s\eta_s = \alpha_s$ and $\hat{\alpha}_t\eta_t = \alpha_t$. By Proposition 50, the functor $[s,t] \colon \mathbb{C}at_1 \to \mathbb{C}at_0 \times \mathbb{C}at_0$ is a fibration, and so certainly admits cartesian liftings of isomorphisms. Thus, by the previous proposition, the functor $[s,t] \colon [\mathbb{C}at, \mathbb{C}at]_{h\psi} \to [\mathbb{C}at, \mathbb{C}at]_{v\psi} \times [\mathbb{C}at, \mathbb{C}at]_{v\psi}$ also admits cartesian liftings of isomorphisms. Thus we may form the cartesian lifting

$$\begin{tikzcd} A_s \arrow[r, Rightarrow, "\langle\eta_s,\eta_t\rangle_*(\hat{\mathbf{A}})"] \arrow[d, Rightarrow, "\eta_s"'] \arrow[dr, phantom, "\Downarrow\langle\eta_s,\eta_t\rangle"] & A_t \arrow[d, Rightarrow, "\eta_t"] \\ \hat{A}_s \arrow[r, Rightarrow, "\hat{\mathbf{A}}"'] & \hat{A}_t; \end{tikzcd}$$

so now we take $\mathbf{A} = \langle\eta_s, \eta_t\rangle_*(\hat{\mathbf{A}})$ and

$$\boldsymbol{\alpha} = \langle\eta_s, \eta_t\rangle_*(\hat{\mathbf{A}}) \xrightarrow{\langle\eta_s,\eta_t\rangle} \hat{\mathbf{A}} \xrightarrow{\hat{\boldsymbol{\alpha}}} S\mathbf{I}.$$

Observe that $\langle\eta_s, \eta_t\rangle$ is invertible and hence certainly a cartesian modification, so that $\boldsymbol{\alpha}$ is itself a cartesian modification as required; and since $\hat{\alpha}_s\eta_s = \alpha_s$ and $\hat{\alpha}_t\eta_t = \alpha_t$, $\boldsymbol{\alpha}$ has the correct source and target.

It remains to check that $F(\mathbf{A}, \boldsymbol{\alpha}) = (\mathbf{a}, \boldsymbol{\theta})$. Now $F(\mathbf{A}, \boldsymbol{\alpha})$ is given by

$$\langle(\eta_s)_1, (\eta_t)_1\rangle_*(\hat{\mathbf{A}}1) \xrightarrow{\langle(\eta_s)_1,(\eta_t)_1\rangle} \hat{\mathbf{A}}1 \xrightarrow{\hat{\boldsymbol{\alpha}}_1} S\mathbf{I}_1;$$

but by definition, we have that

$$\hat{\mathbf{A}}1 = \mathbf{a}, \quad \hat{\boldsymbol{\alpha}}_1 = \boldsymbol{\theta}, \quad (\eta_s)_1 = \mathrm{id}_{A_s 1} \quad \text{and} \quad (\eta_t)_1 = \mathrm{id}_{A_t 1}.$$

Therefore $\langle (\eta_s)_1, (\eta_t)_1 \rangle = \mathrm{id}_{\hat{\mathbf{A}}1} = \mathrm{id}_{\mathbf{a}}$; so the above composite is indeed equal to $\mathbf{a} \xrightarrow{\boldsymbol{\theta}} S\mathbf{I}$ as required. □

Thus, if we can find a horizontal arrow

$$(\mathbf{d}, \tilde{\mathbf{d}})\colon (TS1, \mu_1) \twoheadrightarrow (ST1, \mu_1)$$

of $\mathbb{C}at/S\mathbf{I}_1$, then by Proposition 82, we obtain a horizontal arrow

$$(\boldsymbol{\delta}, \tilde{\boldsymbol{\delta}})\colon (TS, \mu) \twoheadrightarrow (ST, \mu)$$

of $\mathbb{C}oll(S)$ as required, whose image under F is precisely $(\mathbf{d}, \tilde{\mathbf{d}})$. Once we have this, deriving the remaining data (PDD2) and (PDD3) is straightforward. For instance, considering $\overline{\boldsymbol{\eta}}$, we must find a special invertible cell

$$\overline{\boldsymbol{\eta}}\colon (\boldsymbol{\delta}, \tilde{\boldsymbol{\delta}}) \otimes (T, \mathrm{id}_S)(\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}) \Rrightarrow (\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}})(T, \mathrm{id}_S)$$

of $\mathbb{C}oll(S)$. Considering the double homomorphism $F\colon \mathbb{C}oll(S) \to \mathbb{C}at/S\mathbf{I}_1$, we know that both F_0 and F_1 form one side of an equivalence of categories. In particular, the functor $F_1\colon \mathbb{C}oll(S)_1 \to \mathbb{C}at_1/S\mathbf{I}_1$ is full and faithful, and thus it suffices to find a special invertible cell

$$\overline{\boldsymbol{\eta}}_1\colon (\mathbf{d}, \tilde{\mathbf{d}}) \otimes \big((T, \mathrm{id}_S)(\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}})\big)1 \Rrightarrow \big((\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}})(T, \mathrm{id}_S)\big)1$$

of $\mathbb{C}at/S\mathbf{I}_1$. We proceed similarly for the remaining data.

Finally, we must ensure that (PDA1)–(PDA10) are satisfied, which amounts to checking certain equalities of pastings in $\mathcal{B}\big(\mathbb{C}oll(S)\big)$, which amounts to checking certain equalities of maps in $\mathbb{C}oll(S)_1$; but since the functor $F_1\colon \mathbb{C}oll(S)_1 \to \mathbb{C}at_1/S\mathbf{I}_1$ is faithful, it suffices to check that these equalities hold in $\mathbb{C}at/S\mathbf{I}_1$.

Chapter 11

Constructing the pseudo-distributive law at 1

In the last chapter, we showed how to reduce the prospect of constructing the desired pseudo-distributive law in $[\mathbf{Mod}, \mathbf{Mod}]_\psi$ to the prospect of constructing a pseudo-distributive law 'at 1' in the bicategory $\mathcal{B}\big(\mathbb{C}at/S\mathbf{I}_1\big)$. In this chapter we duly construct such a pseudo-distributive law.

11.1 Spans

Before we begin, we shall need a few preliminaries about acyclic and connected graphs. We seek to capture their combinatorial essence in a categorical manner, allowing a smooth presentation of the somewhat involved proof which follows.

The objects of our attention are spans in **FinCard**, i.e., diagrams $n \leftarrow k \rightarrow m$ in the category of finite cardinals and all maps. When we write 'span' in future, it should be read as 'span in **FinCard**' unless otherwise stated. We also make use without comment of the evident inclusions **FinOrd** → **FinCard** and $S1 \rightarrow$ **FinCard**.

Now, each span $n \leftarrow k \rightarrow m$ determines a (categorist's) graph $k \rightrightarrows n + m$; if we forget the orientation of the edges of this graph, we get a (combinatorialist's) undirected multigraph. We say that a span $n \leftarrow k \rightarrow m$ is **acyclic** or **connected** if the associated multigraph is so. Note that the *acyclic* condition includes the assertion that there are no multiple edges.

Proposition 84. *Given a span* $n \xleftarrow{\theta_1} k \xrightarrow{\theta_2} m$*, the number of connected components of the graph induced by the span is given by the cardinality of* r *in the pushout* *diagram*

$$\begin{tikzcd}
k \arrow[r, "\theta_2"] \arrow[d, "\theta_1"'] & m \arrow[d, "\tau_2"] \\
n \arrow[r, "\tau_1"'] & r
\end{tikzcd}$$

in **FinCard**.

Proof. Given the above pushout diagram, set $n_i = \tau_1^{-1}(i)$ and $m_i = \tau_2^{-1}(i)$ (for $i = 1, \ldots, r$). Now we observe that, for $i \neq j$, we have

$$\theta_1^{-1}(n_i) \cap \theta_2^{-1}(m_j) = \theta_1^{-1}(n_i) \cap \theta_1^{-1}(n_j) = \emptyset,$$

so that induced graph of the span has at least r unconnected parts (with respective vertex sets $n_i + m_i$). On the other hand, if the induced graph G had strictly more than r connected components, we could find vertex sets $v_1, \ldots, v_{r+1}$ which partition $v(G)$, and for which

$$x \in v_i,\ y \in v_j \text{ (for } i \neq j) \qquad \text{implies} \qquad x \text{ is not adjacent to } y. \qquad (\dagger)$$

But now define maps $\tau_1 \colon n \to r+1$ and $\tau_2 \colon m \to r+1$ by letting $\tau_i(x)$ be the p for which $x \in v_p$. Then by condition ($\dagger$), we have $\tau_1(\theta_1(a)) = \tau_2(\theta_2(a))$ for all $a \in k$, and so we have a commuting diagram

$$\begin{tikzcd}
k \arrow[r, "\theta_2"] \arrow[d, "\theta_1"'] & m \arrow[d, "\tau_2"] \\
n \arrow[r, "\tau_1"'] & r+1
\end{tikzcd}$$

for which the bottom right vertex does not factor through r, contradicting the assumption that r was a pushout. Hence G has precisely r connected components. □

Corollary 85. *A span* $n \xleftarrow{\theta_1} k \xrightarrow{\theta_2} m$ *is connected if and only if the diagram*

$$\begin{tikzcd}
k \arrow[r, "\theta_2"] \arrow[d, "\theta_1"'] & m \arrow[d] \\
n \arrow[r] & 1
\end{tikzcd}$$

is a pushout in **FinCard**.

Proposition 86. *A span* $n \xleftarrow{\theta_1} k \xrightarrow{\theta_2} m$ *is acyclic if and only, for every monomorphism* $\iota\colon k' \hookrightarrow k$*,*

$$\begin{tikzcd} k \arrow[r, "\theta_2"] \arrow[d, "\theta_1"'] & m \arrow[d] \\ n \arrow[r] & r \end{tikzcd} \quad \textit{a pushout implies} \quad \begin{tikzcd} k' \arrow[r, "\theta_2\iota"] \arrow[d, "\theta_1\iota"'] & m \arrow[d] \\ n \arrow[r] & r \end{tikzcd} \quad \textit{not a pushout.}$$

Proof. Suppose the left hand diagram is a pushout; then the associated graph G of the span has r connected components.

Suppose first that G is acyclic, and $\iota\colon k' \hookrightarrow k$. Then the graph G' associated to the span $n \xleftarrow{\theta_1\iota} k' \xrightarrow{\theta_2\iota} m$ has the same vertices as G but strictly fewer edges; and since G is acyclic, G' must have strictly more than r connected components, and hence r cannot be a pushout for the right-hand diagram.

Conversely, if G has a cycle, then we can remove some edge of G without changing the number of connected components; and thus we obtain some monomorphism $\iota\colon k' \hookrightarrow k$ making the right-hand diagram a pushout. □

Proposition 87. *Suppose we have a commuting diagram*

$$\begin{tikzcd} k \arrow[r, "\theta_2"] \arrow[d, "\theta_1"'] & m \arrow[d, "\phi_2"] \\ n \arrow[r, "\phi_1"'] & r. \end{tikzcd} \tag{$*$}$$

Then the spans $m^{(i)} \leftarrow k^{(i)} \rightarrow n^{(i)}$ *(for* $i = 1, \ldots, r$*) induced by pulling back along elements* $i\colon 1 \rightarrow r$ *are all connected if and only if* $(*)$ *is a pushout.*

Proof. Suppose all the induced spans are connected; then each diagram

$$\begin{tikzcd} k^{(i)} \arrow[r, "\theta_2^{(i)}"] \arrow[d, "\theta_1^{(i)}"'] & m^{(i)} \arrow[d] \\ n^{(i)} \arrow[r] & 1 \end{tikzcd}$$

is a pushout; hence the diagram

$$\begin{tikzcd}
\sum_i k^{(i)} \arrow[r, "\sum_i \theta_2^{(i)}"] \arrow[d, "\sum_i \theta_1^{(i)}"'] & \sum_i m^{(i)} \arrow[d] \\
\sum_i n^{(i)} \arrow[r] & r
\end{tikzcd}$$

is also a pushout, whence it follows that $(*)$ is itself a pushout.

Conversely, if $(*)$ is a pushout, then pulling this back along the map $i\colon 1 \to r$ yields another pushout in **FinCard**, so that each induced span is connected. $\square$

Proposition 88. *Let G be a graph with finite edge and vertex sets. Any two of the following conditions implies the third:*

- *G is acyclic;*
- *G is connected;*
- $|v(G)| = |e(G)| + 1$.

Proof.

- If G is acyclic and connected, then it is a tree, and so $|v(G)| = |e(G)| + 1$;
- if G is connected with $|v(G)| = |e(G)| + 1$, then it is minimally connected, hence a tree, and so acyclic;
- if G is acyclic with $|v(G)| = |e(G)| + 1$, then it is maximally acyclic, hence a tree, and so connected.

$\square$

Corollary 89. *A span $n \xleftarrow{\theta_1} k \xrightarrow{\theta_2} m$ is acyclic and connected if and only if the diagram*

$$\begin{tikzcd}
k \arrow[r, "\theta_2"] \arrow[d, "\theta_1"'] & m \arrow[d] \\
n \arrow[r] & 1
\end{tikzcd}$$

is a pushout in **FinCard**, *and $n + m = k + 1$.*

Corollary 90. *Suppose we have a commuting diagram*

$$\begin{tikzcd} k \arrow[r, "\theta_2"] \arrow[d, "\theta_1"'] & m \arrow[d, "\phi_2"] \\ n \arrow[r, "\phi_1"'] & r. \end{tikzcd} \tag{$*$}$$

then the induced spans $m^{(i)} \leftarrow k^{(i)} \rightarrow n^{(i)}$ *(for* $i = 1, \dots, r$*) are acyclic and connected if and only if* $(*)$ *is a pushout and* $m + n = k + r$*.*

11.2 (PDD1)

We are now ready to give our pseudo-distributive law at 1, and we begin with (PDD1):

Definition 91. We give the horizontal arrow

$$\begin{tikzcd} TS1 \arrow[r, "\mathbf{d}", "\shortmid"{marking}] \arrow[d, "\mu_1"'] \arrow[dr, phantom, "\Downarrow\tilde{\mathbf{d}}"] & ST1 \arrow[d, "\mu_1"] \\ S1 \arrow[r, "S\mathbf{I}_1"', "\shortmid"{marking}] & S1 \end{tikzcd}$$

of $\mathbb{C}at/S\mathbf{I}_1$ as follows. The profunctor $d\colon \hat{T}\hat{S}1 \to \hat{S}\hat{T}1$ is the following functor $d\colon (SS1)^{\mathrm{op}} \times SS1 \to \mathbf{Set}$:

- **On objects**: elements $f \in d(\phi; \psi)$ are bijections f_n fitting into the diagram

  $$\begin{tikzcd} n_\phi \arrow[r, "f_n"] \arrow[d, "\phi"'] & n_\psi \arrow[d, "\psi"] \\ m_\phi & m_\psi \end{tikzcd}$$

  such that the span $m_\phi \xleftarrow{\phi} n_\phi \xrightarrow{\psi \circ f_n} m_\psi$ is acyclic and connected.

- **On maps**: Let $g\colon \psi \to \rho$ in $TS1$ and let $f \in d(\phi; \psi)$. Then we give $g \bullet f \in d(\phi; \rho)$ by

  $$\begin{tikzcd} n_\phi \arrow[r, "g_n \circ f_n"] \arrow[d, "\phi"'] & n_\rho \arrow[d, "\rho"] \\ m_\phi & m_\rho \end{tikzcd}$$

> This action is evidently functorial, but we still need to check that it really does yield an element of $d(\phi; \rho)$; that is, we need the associated span to be acyclic and connected. But this span is the top path of the diagram
>
> $$\begin{tikzcd} & & n_\phi \arrow[ddll, "\phi"'] \arrow[dr, "f_n"] & & & \\ & & & n_\psi \arrow[r, "g_n"] \arrow[dr, "\psi"'] & n_\rho \arrow[dr, "\rho"] & \\ m_\phi & & & & m_\psi \arrow[r, "g_m"'] & m_\rho; \end{tikzcd}$$
>
> and therefore also the bottom path, since the right-hand square commutes. But since g_m is an isomorphism, the graph induced by the span $m_\phi \xleftarrow{\phi} n_\phi \xrightarrow{\psi f_n} m_\psi$ is isomorphic to the graph induced by the span $m_\phi \xleftarrow{\phi} n_\phi \xrightarrow{g_m \psi f_n} m_\rho$, and hence the latter is acyclic and connected since the former is. So we have a well-defined left action of $TS1$ on d; proceeding similarly we give a well-defined right action of $ST1$ on d.

This completes the definition of d; we now give the 2-cell $\tilde{d}$, for which we must give natural maps

$$\tilde{d}_{\phi,\psi} \colon d(\phi; \psi) \to S1(n_\phi, n_\psi).$$

But this is straightforward: we simply send

$$\begin{tikzcd} n_\phi \arrow[r, "f_n"] \arrow[d, "\phi"'] & n_\psi \arrow[d, "\psi"] \\ m_\phi & m_\psi \end{tikzcd}$$

in $d(\phi; \psi)$ to f_n in $S1(n_\phi; n_\psi)$. It's visibly the case that this satisfies the required naturality conditions.

Now, consider the pseudo-natural transformation $\delta \colon \hat{T}\hat{S} \Rightarrow \hat{S}\hat{T}$ induced by this $(\mathbf{d}, \tilde{\mathbf{d}})$; its component at a discrete category X has $\delta_X(\{\Sigma_m\}_{1 \leqslant m \leqslant j}; \{\Gamma_n\}_{1 \leqslant n \leqslant k})$ given by the set of admissible matchings of $\{\Sigma_m\}$ with $\{\Gamma_n\}$, which is precisely what we sought in Chapter 9.

11.3 (PDD2)

For (PDD2) we must produce the component of the invertible special modifications $\overline{\boldsymbol{\eta}}$ and $\overline{\boldsymbol{\epsilon}}$ at 1:

Proposition 92. *There is an invertible special cell*

$$\begin{tikzcd}
T1 \arrow[d, "(T\boldsymbol{\eta})1"', "\shortmid" marking] \arrow[dr, "(\boldsymbol{\eta}T)1", "\shortmid" marking] \arrow[dr, phantom, "\overset{\overline{\boldsymbol{\eta}}1}{\Rightarrow}"' near start, shift right=3] & \\
TS1 \arrow[r, "\mathbf{d}"', "\shortmid" marking] & ST1
\end{tikzcd}$$

mediating the centre of this diagram in $\mathbb{C}oll(S)$ (where we omit the projections to $S\mathbf{I}$).

Proof. With respect to the descriptions of $S1$ and S^21 given above, we observe that that the functors $T\eta_1 \colon T1 \to TS1$ and $\eta_{T1} \colon T1 \to ST1$ are given by

$$T\eta_1 \colon n \mapsto (n \xrightarrow{\text{id}} n) \qquad\qquad \eta_{T1} \colon n \mapsto (n \xrightarrow{!} 1)$$
$$f \mapsto (f, f) \qquad\qquad f \mapsto (f, !)$$

and hence $(\boldsymbol{\eta}T)1 \colon TS1^{\text{op}} \times T1 \to \mathbf{Set}$ and $(T\boldsymbol{\eta})1 \colon ST1^{\text{op}} \times T1 \to \mathbf{Set}$ are given by:

$$\begin{aligned}
(\boldsymbol{\eta}T)1(\phi; n) &= (\eta_{T1})_*(\phi; n) = ST1(\phi, (n \xrightarrow{\text{id}} n)) \\
(T\boldsymbol{\eta})1(\phi; n) &= \hat{T}(\eta_1)_*(\phi; n) \cong (T\eta_1)_*(\phi; n) = TS1(\phi, (n \xrightarrow{!} 1))
\end{aligned}$$

Thus the composite along the upper side of this diagram is given by

$$(\boldsymbol{\eta}T)1(\phi; n) = ST1(\phi, (n \xrightarrow{!} 1)) \cong \begin{cases} S1(n_\phi, n) & \text{if } m_\phi = 1; \\ \emptyset & \text{otherwise,} \end{cases} \tag{1}$$

where the isomorphism is natural in ϕ and n; and with respect to this isomorphism, the projection down to $S\mathbf{I}$ is given simply by the inclusion

$$(\boldsymbol{\eta}T)1(\phi; n) \hookrightarrow S1(n_\phi, n).$$

Now, the lower side is given by

$$(\mathbf{d} \otimes (T\boldsymbol{\eta})1)(\phi; n) = \int^{\psi \in TS1} TS1(\psi, (n \xrightarrow{\mathrm{id}} n)) \times d(\phi; \psi),$$

which is isomorphic to $d(\phi; (n \xrightarrow{\mathrm{id}} n))$, naturally in ϕ and n. Now, any element f of $d(\phi; (n \xrightarrow{\mathrm{id}} n))$, given by

$$\begin{tikzcd}
n_\phi \arrow[r, "f_n"] \arrow[d, "\phi"'] & n \arrow[d, "\mathrm{id}"] \\
m_\phi & n
\end{tikzcd}$$

say, must satisfy $m_\phi + n = n_\phi + 1$; but since $n = n_\phi$, this can only happen if $m_\phi = 1$; and in this case, the diagram

$$\begin{tikzcd}
n_\phi \arrow[r, "f_n"] \arrow[d, "\phi"'] & n \arrow[d, "!"] \\
m_\phi \arrow[r, "!"'] & 1
\end{tikzcd}$$

is necessarily a pushout. Hence

$$(\mathbf{d} \otimes (T\boldsymbol{\eta})1)(\phi; n) \cong \begin{cases} S1(n_\phi, n) & \text{if } m_\phi = 1; \\ \emptyset & \text{otherwise,} \end{cases} \tag{2}$$

naturally in ϕ and n; and once again, the projection down to $S\mathbf{I}$ is given simply by inclusion. So, composing the isomorphisms (1) and (2), we get a special invertible cell $\overline{\boldsymbol{\eta}}1$ which is compatible with the projections down to $S\mathbf{I}$, as required. □

Proposition 93. *There is an isomorphic 2-cell*

$$\begin{tikzcd}
TS1 \arrow[r, "\boldsymbol{\delta}1", "|" marking] \arrow[d, "(\boldsymbol{\epsilon}S)1"', "|" marking] \arrow[r, phantom, shift right=4, "\overset{\bar{\boldsymbol{\epsilon}}1}{\Rightarrow}"] & ST1 \arrow[dl, "(S\boldsymbol{\epsilon})1", "|" marking] \\
S1 &
\end{tikzcd}$$

mediating the centre of this diagram in $\mathbb{C}\mathit{oll}(S)$ (where we omit the projections to $S\mathbf{I}$).

Proof. Dual to the above. ☐

11.4 (PDD3)

For (PDD3) we must produce the component of the invertible special modifications $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\Delta}}$ at 1:

Proposition 94. *There is an isomorphic 2-cell*

$$\begin{tikzcd}
TS1 \arrow[rr, "\mathbf{d}", "\shortmid"{marking}] \arrow[d, "(\boldsymbol{\Delta}S)1"', "\shortmid"{marking}] & \Downarrow \overline{\boldsymbol{\Delta}}1 & ST1 \arrow[d, "(S\boldsymbol{\Delta})1", "\shortmid"{marking}] \\
TTS1 \arrow[r, "(T\boldsymbol{\delta})1"', "\shortmid"{marking}] & TST1 \arrow[r, "(\boldsymbol{\delta}T)1"', "\shortmid"{marking}] & STT1
\end{tikzcd}$$

mediating the centre of this diagram in $\mathbb{C}oll(S)$ *(where we omit the projections to* $S\mathbf{I}$*).*

Proof. Let us describe explicitly the horizontal arrows involved in the above diagram. The functors $\mu_{S1} \colon TTS1 \to TS1$ and $S\mu_1 \colon STT1 \to ST1$ in **Cat** are given by

$$\begin{aligned}
\mu_{S1} \colon (n_\phi \xrightarrow{\phi_1} m_\phi \xrightarrow{\phi_2} r_\phi) &\mapsto (n_\phi \xrightarrow{\phi_1} m_\phi) & S\mu_1 \colon (n_\phi \xrightarrow{\phi_1} m_\phi \xrightarrow{\phi_2} r_\phi) &\mapsto (n_\phi \xrightarrow{\phi_2\phi_1} r_\phi) \\
(f_n, f_m, f_r) &\mapsto (f_n, f_m) & (f_n, f_m, f_r) &\mapsto (f_n, f_r)
\end{aligned}$$

and hence $(\boldsymbol{\Delta}S)1 \colon TTS1^{\mathrm{op}} \times TS1 \to \mathbf{Set}$ and $(S\boldsymbol{\Delta})1 \colon STT1^{\mathrm{op}} \times ST1 \to \mathbf{Set}$ are given by:

$$\begin{aligned}
(\boldsymbol{\Delta}S)1(\phi; \psi) &= (\mu_{S1})^*(\phi; \psi) = TS1((n_\phi \xrightarrow{\phi_1} m_\phi), \psi) \\
(S\boldsymbol{\Delta})1(\phi; \psi) &= \hat{S}(\mu_1)^*(\phi; \psi) \cong (S\mu_1)^*(\phi; \psi) = ST1((n_\phi \xrightarrow{\phi_2\phi_1} r_\phi), \psi).
\end{aligned}$$

We now wish to describe $(\boldsymbol{\delta}T)1$ and $(T\boldsymbol{\delta})1$; let us abbreviate these as dT and Td respectively. It's a straightforward calculation to see that $dT \colon STT1^{\mathrm{op}} \times TST1 \to \mathbf{Set}$ is given as follows:

- **On objects**: elements $f \in dT(\phi; \psi)$ are pairs of bijections f_n and f_m fitting in the diagram

$$\begin{tikzcd}
n_\phi \arrow[r, "f_n"] \arrow[d, "\phi_1"'] & n_\psi \arrow[d, "\psi_1"] \\
m_\phi \arrow[r, "f_m"'] \arrow[d, "\phi_2"'] & m_\psi \arrow[d, "\psi_2"] \\
r_\phi & r_\psi
\end{tikzcd}$$

such that the span $r_\phi \xleftarrow{\phi_2} m_\phi \xrightarrow{\psi_2 \circ f_m} r_\psi$ is acyclic and connected.

- **On maps**: Let $g\colon \psi \to \rho$ in $TST1$ and let $f \in dT(\phi;\psi)$. Then we give an element $g \circ f \in dT(\phi;\rho)$ by

$$\begin{tikzcd}
n_\phi \arrow[r, "g_n \circ f_n"] \arrow[d, "\phi_1"'] & n_\rho \arrow[d, "\rho_1"] \\
m_\phi \arrow[r, "g_m \circ f_m"'] \arrow[d, "\phi_2"'] & m_\rho \arrow[d, "\rho_2"] \\
r_\phi & r_\rho;
\end{tikzcd}$$

  and we give in a similar way the right action of $STT1$.

Similarly, it's easy to calculate that $Td\colon TST1^{\mathrm{op}} \times TTS1 \to \mathbf{Set}$ is given by:

- **On objects**: elements $f \in Td(\phi;\psi)$ are pairs of bijections $f_n\colon n_\phi \to n_\psi$ and $f_r\colon r_\phi \to r_\psi$ fitting in the diagram

$$\begin{tikzcd}
n_\phi \arrow[r, "f_n"] \arrow[d, "\phi_1"'] & n_\psi \arrow[d, "\psi_1"] \\
m_\phi \arrow[d, "\phi_2"'] & m_\psi \arrow[d, "\psi_2"] \\
r_\phi \arrow[r, "f_r"'] & r_\psi
\end{tikzcd}$$

such that for each $i = 1, \ldots, r_\psi$, the induced spans

$$\begin{tikzcd}
 & n_\phi^{(i)} \arrow[ddl, "\phi_1^{(i)}"'] \arrow[dr, "f_n^{(i)}"] & & \\
 & & n_\psi^{(i)} \arrow[dr, "\psi_1^{(i)}"] & \\
m_\phi^{(i)} & & & m_\psi^{(i)}
\end{tikzcd}$$

are acyclic and connected.

[Let us clarify what the induced spans referred to above actually are. We have the commuting diagram

$$\begin{tikzcd}
n_\phi \arrow[r, "f_n"] \arrow[d, "\phi_1"'] & n_\psi \arrow[r, "\psi_1"] & m_\psi \arrow[d, "\psi_2"] \\
m_\phi \arrow[r, "\phi_2"'] & r_\phi \arrow[r, "f_r"'] & r_\psi
\end{tikzcd} \tag{$*$}$$

and the induced spans are the result of pulling this diagram back along elements $i \colon 1 \to r_\psi$. By the results of the first section of this chapter, these spans are all acyclic and connected if and only if $(*)$ is a pushout and $r_\psi + n_\phi = m_\phi + m_\psi$.]

- **On maps**: Let $g \colon \psi \to \rho$ in $TTS1$ and let $f \in Td(\phi; \psi)$. Then we give an element $g \circ f \in Td(\phi; \rho)$ by

$$\begin{tikzcd}
n_\phi \arrow[r, "g_n \circ f_n"] \arrow[d, "\phi_1"'] & n_\rho \arrow[d, "\rho_1"] \\
m_\phi \arrow[d, "\phi_2"'] & m_\rho \arrow[d, "\rho_2"] \\
r_\phi \arrow[r, "g_r \circ f_r"'] & r_\rho;
\end{tikzcd}$$

again, we give a right action of $TST1$ similarly.

So, returning to the diagram in question, the upper side is given by

$$((S\mathbf{\Delta})1 \otimes \mathbf{d})(\phi; \rho) = \int^{\psi \in ST1} d(\psi; \rho) \times ST1\big((n_\phi \xrightarrow{\phi_2\phi_1} r_\phi), \psi\big),$$

which is isomorphic to $d\big((n_\phi \xrightarrow{\phi_2\phi_1} r_\phi);\rho\big)$, naturally in ϕ and ρ. With respect to this isomorphism, the projection onto $S\mathbf{I}$ has components

$$d\big((n_\phi \xrightarrow{\phi_2\phi_1} r_\phi);\rho\big) \to S1(n_\phi;n_\rho)$$

which send

$$\begin{tikzcd}
n_\phi \arrow[r, "f_n"] \arrow[d, "\phi_2\phi_1"'] & n_\rho \arrow[d, "\rho"] \\
r_\phi & m_\rho
\end{tikzcd}$$

to f_n. The lower side of this diagram, which we denote by K, is given by

$$\begin{aligned}
K(\phi;\rho) &= ((\boldsymbol{\delta}T)1 \otimes (T\boldsymbol{\delta})1 \otimes (\boldsymbol{\Delta}S)1)(\phi;\rho) \\
&= \int^{\substack{\psi\in TST1,\\ \xi\in TTS1}} TS1((n_\xi \xrightarrow{\xi_1} m_\xi),\rho) \times Td(\psi;\xi) \times dT(\phi;\psi).
\end{aligned}$$

We may represent a typical element $x \in K(\phi;\rho)$ as $x = f \otimes g \otimes h$, where $f \in dT(\phi;\psi)$, $g \in Td(\psi;\xi)$, and $h \in TS1((n_\xi \xrightarrow{\xi_1} m_\xi),\rho)$:

$$\begin{tikzcd}
n_\phi \arrow[r, "f_n"] \arrow[d, "\phi_1"'] & n_\psi \arrow[r, "g_n"] \arrow[d, "\psi_1"'] & n_\xi \arrow[r, "h_n"] \arrow[d, "\xi_1"'] & n_\rho \arrow[d, "\rho"] \\
m_\phi \arrow[r, "f_m"'] \arrow[d, "\phi_2"'] & m_\psi \arrow[d, "\psi_2"'] & m_\xi \arrow[r, "h_m"'] \arrow[d, "\xi_2"'] & m_\rho \\
r_\phi & r_\psi \arrow[r, "g_r"'] & r_\xi. &
\end{tikzcd}$$

Then the projection onto $S\mathbf{I}$ has components

$$\begin{aligned}
K(\phi;\rho) &\to S1(n_\phi,n_\rho) \\
f \otimes g \otimes h &\mapsto h_n \circ g_n \circ f_n.
\end{aligned}$$

So, we need to set up an isomorphism between $K(\phi;\rho)$ and $d((n_\phi \xrightarrow{\phi_2\phi_1} r_\phi);\rho)$ which is natural in ϕ and ρ and compatible with the projection onto $S\mathbf{I}$. In one direction, we send the element $x \in K(\phi;\rho)$:

$$\begin{tikzcd}
n_\phi \arrow[r, "f_n"] \arrow[d, "\phi_1"'] & n_\psi \arrow[r, "g_n"] \arrow[d, "\psi_1"'] & n_\xi \arrow[r, "h_n"] \arrow[d, "\xi_1"'] & n_\rho \arrow[d, "\rho"] \\
m_\phi \arrow[r, "f_m"'] \arrow[d, "\phi_2"'] & m_\psi \arrow[d, "\psi_2"'] & m_\xi \arrow[r, "h_m"'] \arrow[d, "\xi_2"'] & m_\rho \\
r_\phi & r_\psi \arrow[r, "g_r"'] & r_\xi &
\end{tikzcd}$$

to the element $\hat{x}$ of $d_1\big((n_\phi \xrightarrow{\phi_2\phi_1} r_\phi);\rho\big)$ given by

$$\begin{tikzcd}
n_\phi \arrow[r, "h_n g_n f_n"] \arrow[d, "\phi_2\phi_1"'] & n_\rho \arrow[d, "\rho"] \\
r_\phi & m_\rho .
\end{tikzcd}$$

Note that this element is independent of the representation of x that we chose, that this assignation is natural in ϕ and ρ, and is compatible with the projection down to $S\mathbf{I}$; but for it to be well-defined, we need still to check that the span $r_\phi \xleftarrow{\phi_2\phi_1} n_\phi \xrightarrow{\rho h_n g_n f_n} m_\rho$ is acyclic and connected. For this, we observe first that in the following diagram

$$\begin{tikzcd}
n_\phi \arrow[r, "f_n"] \arrow[d, "\phi_1"'] & n_\psi \arrow[r, "g_n"] \arrow[d, "\psi_1"'] & n_\xi \arrow[r, "\xi_1"] & m_\xi \arrow[r, "h_n"] \arrow[d, "\xi_2"] & m_\rho \arrow[dd] \\
m_\phi \arrow[r, "f_m"'] \arrow[d, "\phi_2"'] & m_\psi \arrow[r, "\psi_2"'] & r_\psi \arrow[r, "g_r"'] \arrow[d] & r_\xi \arrow[d] & \\
r_\phi \arrow[rr] & & 1 \arrow[r] & 1 \arrow[r] & 1
\end{tikzcd}$$

each of the smaller squares is a pushout; and hence the outer square is also a pushout. But the top edge is $h_n\xi_1 g_n f_n = \rho h_n g_n f_n$, so that the square

$$\begin{tikzcd}
n_\phi \arrow[r, "\rho h_n g_n f_n"] \arrow[d, "\phi_2\phi 1"'] & n_\rho \arrow[d] \\
r_\psi \arrow[r] & 1
\end{tikzcd}$$

is a pushout as required. Furthermore, the following equalities hold:

$$\begin{aligned} r_\phi + r_\psi &= m_\phi + 1 \\ m_\psi + m_\xi &= n_\psi + r_\xi \\ m_\psi &= m_\phi \\ m_\rho &= m_\xi \\ r_\psi &= r_\xi \\ n_\psi &= n_\phi, \end{aligned}$$

whence we have $m_\rho + r_\phi = n_\phi + 1$. So the span $r_\phi \xleftarrow{\phi_2\phi_1} n_\phi \xrightarrow{\rho h_n g_n f_n} m_\rho$ is acyclic and connected as required.

Conversely, suppose we are given an element k of $d_1((n_\phi \xrightarrow{\phi_2\phi_1} r_\phi);\rho)$:

$$\begin{tikzcd} n_\phi \arrow[r, "k_n"] \arrow[d, "\phi_2\phi_1"'] & n_\rho \arrow[d, "\rho"] \\ r_\phi & m_\rho; \end{tikzcd}$$

then we take the following pushout:

$$\begin{tikzcd} n_\phi \arrow[r, "\rho k_n"] \arrow[d, "\phi_1"'] & m_\rho \arrow[d, "i_2"] \\ m_\phi \arrow[r, "i_1"'] & r. \end{tikzcd}$$

Now, the map i_1 in this pushout square need not be order-preserving; but it has a (non-unique) factorisation as $m_\phi \xrightarrow{\alpha_1} r_1 \xrightarrow{\sigma_1} r$, where α_1 is order-preserving and σ_1 a bijection. Similarly, we can factorise i_2 as $m_\rho \xrightarrow{\alpha_2} r_2 \xrightarrow{\sigma_2} r$ with α_2 is order-preserving and σ_2 a bijection. [Note that it follows that each of the diagrams

$$\begin{tikzcd} n_\phi \arrow[r, "\rho k_n"] \arrow[d, "\phi_1"'] & m_\rho \arrow[d, "\sigma_1^{-1} i_2"] \\ m_\phi \arrow[r, "\alpha_1"'] & r_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} n_\phi \arrow[r, "\rho k_n"] \arrow[d, "\phi_1"'] & m_\rho \arrow[d, "\alpha_2"] \\ m_\phi \arrow[r, "\sigma_2^{-1} i_1"'] & r_2 \end{tikzcd}$$

is also a pushout.] Now we send k to the element $\hat{k}$ of $K(\phi;\rho)$ represented by the following:

$$\begin{tikzcd}
n_\phi \arrow[r, "\mathrm{id}"] \arrow[d, "\phi_1"'] & n_\phi \arrow[r, "k_n"] \arrow[d, "\phi_1"'] & n_\rho \arrow[r, "\mathrm{id}"] \arrow[d, "\rho"'] & n_\rho \arrow[d, "\rho"] \\
m_\phi \arrow[r, "\mathrm{id}"'] \arrow[d, "\phi_2"'] & m_\phi \arrow[d, "\alpha_1"'] & m_\rho \arrow[r, "\mathrm{id}"'] \arrow[d, "\alpha_2"'] & m_\rho. \\
r_\phi & r_1 \arrow[r, "\sigma_2^{-1}\sigma_1"'] & r_2 &
\end{tikzcd}$$

This is visibly compatible with the projection down onto $S\mathbf{I}$, but we need to check that it is in fact a valid element of $K(\phi;\rho)$. Clearly all squares commute in the diagram above, so we need only check the acyclic and connected conditions. We start with connectedness; for the middle map, the diagram

$$\begin{tikzcd}
n_\phi \arrow[r, "k_n"] \arrow[d, "\phi_1"'] & n_\rho \arrow[r, "\rho"] & m_\rho \arrow[d, "\alpha_2"] \\
m_\phi \arrow[r, "\alpha_1"'] & r_1 \arrow[r, "\sigma_2^{-1}\sigma_1"'] & r_2
\end{tikzcd}
=
\begin{tikzcd}
n_\phi \arrow[r, "\rho k_n"] \arrow[d, "\phi_1"'] & m_\rho \arrow[d, "\alpha_2"] \\
m_\phi \arrow[r, "\sigma_2^{-1}i_1"'] & r_2
\end{tikzcd}$$

is indeed a pushout, so the induced spans for the middle map are connected. For the left-hand map, consider the diagram

$$\begin{tikzcd}
n_\phi \arrow[r, "\rho k_n"] \arrow[d, "\phi_1"'] & m_\rho \arrow[d, "\sigma_1^{-1}i_2"] \\
m_\phi \arrow[r, "\alpha_1"'] \arrow[d, "\phi_2"'] & r_1 \arrow[d] \\
r_\phi \arrow[r] & 1;
\end{tikzcd}$$

the outer square and the upper square are both pushouts, and hence so is the lower square; so the left-hand span is connected.

And now acyclicity. For the middle map, we need that, given any monomorphism

$\iota \colon n'_\phi \hookrightarrow n_\phi$, the diagram

$$\begin{tikzcd}
n'_\phi \arrow[r, "\rho k_n \iota"] \arrow[d, "\phi_1 \iota"'] & m_\rho \arrow[d, "\alpha_2"] \\
m_\phi \arrow[r, "\sigma_2^{-1} i_1"'] & r_2
\end{tikzcd}$$

is no longer a pushout. But suppose it were; then in the diagram

$$\begin{tikzcd}
n'_\phi \arrow[r, "\rho k_n \iota"] \arrow[d, "\phi_1 \iota"'] & m_\rho \arrow[d, "\sigma_1^{-1} i_2"] \\
m_\phi \arrow[r, "\alpha_1"'] \arrow[d, "\phi_2"'] & r_1 \arrow[d] \\
r_\phi \arrow[r] & 1
\end{tikzcd}$$

the upper and lower squares would be pushouts, hence making the outer edge a pushout; but this contradicts the acyclicity of the span $r_\phi \leftarrow n_\phi \rightarrow m_\rho$. So the induced spans for the middle map are acyclic. Thus we now know that the following equations hold:

$$m_\phi + m_\rho = n_\phi + r_2$$
$$r_\phi + m_\rho = n_\phi + 1$$
$$r_1 = r_2,$$

and so can deduce that $r_1 + r_\phi = m_\phi + 1$, as required for the left-hand span to be acyclic.

It remains to check that these two assignations are mutually inverse. It is evident, given $k \in d_1((n_\phi \xrightarrow{\phi_2 \phi_1} r_\phi); \rho)$, that $\hat{\hat{k}} = k$. For the other direction, we send

$$x = \begin{tikzcd}
n_\phi \arrow[r, "f_n"] \arrow[d, "\phi_1"'] & n_\psi \arrow[r, "g_n"] \arrow[d, "\psi_1"'] & n_\xi \arrow[r, "h_n"] \arrow[d, "\xi_1"'] & n_\rho \arrow[d, "\rho"] \\
m_\phi \arrow[r, "f_m"'] \arrow[d, "\phi_2"'] & m_\psi \arrow[d, "\psi_2"'] & m_\xi \arrow[r, "h_m"'] \arrow[d, "\xi_2"'] & m_\rho \\
r_\phi & r_\psi \arrow[r, "g_r"'] & r_\xi .
\end{tikzcd} \quad \text{to} \quad \hat{\hat{x}} = \begin{tikzcd}
n_\phi \arrow[r, "\mathrm{id}"] \arrow[d, "\phi_1"'] & n_\phi \arrow[r, "k_n"] \arrow[d, "\phi_1"'] & n_\rho \arrow[r, "\mathrm{id}"] \arrow[d, "\rho"'] & n_\rho \arrow[d, "\rho"] \\
m_\phi \arrow[r, "\mathrm{id}"'] \arrow[d, "\phi_2"'] & m_\phi \arrow[d, "\alpha_1"'] & m_\rho \arrow[r, "\mathrm{id}"'] \arrow[d, "\alpha_2"'] & m_\rho \\
r_\phi & r_1 \arrow[r, "\sigma_2^{-1} \sigma_1"'] & r_2 .
\end{tikzcd}$$

We claim that these two diagrams represent the same element of $K(\phi;\rho)$. Indeed, note that in the diagram

$$\begin{tikzcd}
n_\phi \arrow[r, "f_n"] \arrow[d, "\phi_1"'] & n_\psi \arrow[r, "g_n"] \arrow[d, "\psi_1"'] & n_\xi \arrow[r, "\xi_1"] & m_\xi \arrow[r, "h_m"] \arrow[d, "\xi_2"'] & m_\rho \arrow[d, "g_r^{-1}\xi_2 h_m^{-1}"] \\
m_\phi \arrow[r, "f_m"'] & m_\psi \arrow[r, "\psi_2"'] & r_\psi \arrow[r, "g_r"'] & r_\xi \arrow[r, "g_r^{-1}"'] & r_\psi
\end{tikzcd}$$

each of the smaller squares is a pushout, and hence the outer edge is. But the upper edge is $h_m\xi_1 g_n f_n = \rho h_n g_n f_n = \rho k_n$, so that the diagram

$$\begin{tikzcd}
n_\phi \arrow[r, "\rho k_n"] \arrow[d, "\phi_1"'] & m_\rho \arrow[d, "g_r^{-1}\xi_2 h_m^{-1}"] \\
m_\phi \arrow[r, "\psi_2 f_m"'] & r_\psi
\end{tikzcd}$$

is a pushout. Since r_1 is also a pushout for this diagram, it follows that there is an isomorphism $\beta_1\colon r_1 \to r_\psi$ such that $\beta_1\alpha_1 = \psi_2 f_m$; hence the following diagram commutes:

$$\begin{tikzcd}
n_\phi \arrow[r, "f_n"] \arrow[d, "\phi_1"'] & n_\psi \arrow[d, "\psi_1"] \\
m_\phi \arrow[r, "f_m"] \arrow[d, "\alpha_1"'] & m_\psi \arrow[d, "\psi_2"] \\
r_1 \arrow[r, "\beta_1"'] & r_\psi
\end{tikzcd}$$

Similarly, we see that

$$\begin{tikzcd}
n_\phi \arrow[r, "\rho k_n"] \arrow[d, "\phi_1"'] & m_\rho \arrow[d, "\xi_2 h_m^{-1}"] \\
m_\phi \arrow[r, "g_r\psi_2 f_m"'] & r_\xi
\end{tikzcd}$$

is a pushout, and so there is an isomorphism $\beta_2\colon r_\xi \to r_2$ such that $\beta_2\xi_2 h_m^{-1} = \alpha_2$, i.e., $\beta_2\xi_2 = \alpha_2 h_m$. Hence the following diagram commutes:

$$\begin{tikzcd}
n_\xi \arrow[r, "h_n"] \arrow[d, "\xi_1"'] & n_\rho \arrow[d, "\rho"] \\
m_\xi \arrow[r, "h_m"] \arrow[d, "\xi_2"'] & m_\rho \arrow[d, "\alpha_2"] \\
r_\xi \arrow[r, "\beta_2"'] & r_2.
\end{tikzcd}$$

Furthermore, we have $r_1 \xrightarrow{\beta_1} r_\psi \xrightarrow{g_r} r_\xi \xrightarrow{\beta_2} r_2 = r_1 \xrightarrow{\sigma_1} r \xrightarrow{\sigma_2^{-1}} r_2$, since each of these objects is a pushout of the same span, and the isomorphisms between them are isomorphisms of pushouts. Thus, using an evident notation for the internal actions, we have

$$x = \begin{tikzcd}
n_\phi \arrow[r, "f_n"] \arrow[d, "\phi_1"'] & n_\psi \arrow[r, "g_n"] \arrow[d, "\psi_1"'] & n_\xi \arrow[r, "h_n"] \arrow[d, "\xi_1"'] & n_\rho \arrow[d, "\rho"] \\
m_\phi \arrow[r, "f_m"'] \arrow[d, "\phi_2"'] & m_\psi \arrow[d, "\psi_2"'] & m_\xi \arrow[r, "h_m"'] \arrow[d, "\xi_2"'] & m_\rho \\
r_\phi & r_\psi \arrow[r, "g_r"'] & r_\xi. &
\end{tikzcd} \equiv \begin{tikzcd}
n_\phi \arrow[r, "\mathrm{id}"] \arrow[d, "\phi_1"'] & n_\phi \arrow[r, "f_n"] \arrow[d, "\phi_1"'] & n_\psi \arrow[r, "g_n"] \arrow[d, "\psi_1"] & n_\xi \arrow[r, "h_n"] \arrow[d, "\xi_1"'] & n_\rho \arrow[r, "\mathrm{id}"] \arrow[d, "\rho"'] & n_\rho \arrow[d, "\rho"] \\
m_\phi \arrow[r, "\mathrm{id}"'] \arrow[d, "\phi_2"'] & m_\phi \arrow[r, "f_m"] \arrow[d, "\alpha_1"'] & m_\psi \arrow[d, "\psi_2"] & m_\xi \arrow[r, "h_m"] \arrow[d, "\xi_2"'] & m_\rho \arrow[r, "\mathrm{id}"'] \arrow[d, "\alpha_2"'] & m_\rho \\
r_\phi & r_1 \arrow[r, "\beta_1"'] & r_\psi \arrow[r, "g_r"'] & r_\xi \arrow[r, "\beta_2"'] & r_2. &
\end{tikzcd}$$

$$\equiv \begin{tikzcd}
n_\phi \arrow[r, "\mathrm{id}"] \arrow[d, "\phi_1"'] & n_\phi \arrow[r, "k_n"] \arrow[d, "\phi_1"'] & n_\rho \arrow[r, "\mathrm{id}"] \arrow[d, "\rho"'] & n_\rho \arrow[d, "\rho"] \\
m_\phi \arrow[r, "\mathrm{id}"'] \arrow[d, "\phi_2"'] & m_\phi \arrow[d, "\alpha_1"'] & m_\rho \arrow[r, "\mathrm{id}"'] \arrow[d, "\alpha_2"'] & m_\rho \\
r_\phi & r_1 \arrow[r, "\sigma_2^{-1}\sigma_1"'] & r_2. &
\end{tikzcd} = \hat{\hat{x}}.$$

So the assignations $x \mapsto \hat{x}$ and $k \mapsto \hat{k}$ are mutually inverse as required. It now follows that the assignation $d_1((n_\phi \xrightarrow{\phi_2\phi_1} r_\phi); \rho) \to K(\phi; \rho)$ is natural in ϕ and ρ, since its inverse is. This completes the proof. □

Proposition 95. *There is an isomorphic 2-cell*

$$\begin{tikzcd}
TSS1 \arrow[r, "(\boldsymbol{\delta}S)1", "\shortmid"{marking}] \arrow[d, "(T\boldsymbol{\mu})1"', "\shortmid"{marking}] & STS1 \arrow[r, "(S\boldsymbol{\delta})1", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\overline{\boldsymbol{\mu}}"] & SST1 \arrow[d, "(\boldsymbol{\mu}T)1", "\shortmid"{marking}] \\
TS1 \arrow[rr, "\mathbf{d}"', "\shortmid"{marking}] & & ST1
\end{tikzcd}$$

mediating the centre of this diagram in $\mathbb{C}oll(S)$ *(where we omit the projections to* $S\mathbf{I}$*).*

Proof. Dual to the above. ☐

11.5 (PDA1)–(PDA10)

It remains only to show that the data produced above satisfies the ten coherence axioms (PDA1)–(PDA10). At first this may appear somewhat forbidding, but our job is made rather simple by the following argument.

Definition 96. We say that a cell

$$\begin{tikzcd}
X_s \arrow[r, "\mathbf{X}", "\shortmid"{marking}] \arrow[d, "f_s"'] \arrow[dr, phantom, "\Downarrow\mathbf{f}"] & X_t \arrow[d, "f_t"] \\
Y_s \arrow[r, "\mathbf{Y}"', "\shortmid"{marking}] & Y_t
\end{tikzcd}$$

of $\mathbb{C}at$ is **locally monomorphic** if it is a monomorphism when viewed as a map of $[X_t^{\mathrm{op}} \times X_s, \mathbf{Set}]$:

$$\begin{tikzcd}
X_t^{\mathrm{op}} \times X_s \arrow[rr, "f_t^{\mathrm{op}} \times f_s"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{f}{\Rightarrow}"] & Y_t^{\mathrm{op}} \times Y_s \arrow[dl, "Y"] \\
& \mathbf{Set}. &
\end{tikzcd}$$

In terms of its components, this happens if and only if each of the maps

$$f_{x_t,x_s} \colon X(x_t; x_s) \to Y(f_t x_t; f_s x_s)$$

is a monomorphism.

Now, local monomorphisms admit a limited form of 'left cancellation'. Indeed, suppose we are given objects $\mathbf{X} = X \colon X_s \nrightarrow X_t$ and $\mathbf{X}' = X' \colon X_s \nrightarrow X_t$ of $\mathbb{C}at_1$, and *special* maps $\mathbf{g}_1$ and $\mathbf{g}_2 \colon \mathbf{X}' \to \mathbf{X}$; then given a local monomorphism $\mathbf{f} \colon \mathbf{X} \to \mathbf{Y}$, we have that

$$\mathbf{f} \circ \mathbf{g}_1 = \mathbf{f} \circ \mathbf{g}_1 \quad \text{implies} \quad \mathbf{g}_1 = \mathbf{g}_2,$$

since to give a special map $g_i \colon \mathbf{X}' \to \mathbf{X}$ is equivalently to give a natural transformation $g_i \colon X' \Rightarrow X$; therefore the result follows from the fact that $f \colon X \Rightarrow (Y \circ f_t^{\mathrm{op}} \times f_s)$ is a monomorphism in $[X_t^{\mathrm{op}} \times X_s, \mathbf{Set}]$.

Also, given a special isomorphism $\mathbf{g} \colon \mathbf{X}' \to \mathbf{X}$ and a local monomorphism $\mathbf{f} \colon \mathbf{X} \to \mathbf{Y}$, we observe that $\mathbf{f} \circ \mathbf{g}$ is also a local monomorphism.

Proposition 97. *Consider each of the pasting diagrams in the axioms (PDA1)–(PDA10) as a diagram in* $\mathbb{C}at/S\mathbf{I}_1$*. Then the projection map from each 'source' and 'target' face down onto* $S\mathbf{I}_1$ *is a local monomorphism.*

Proof. Observe that every special cell in the pasting diagrams for (PDA1)–(PDA10) is invertible, and therefore, for each pasting diagram it suffices to show for *any one* path through it that the projection onto $S\mathbf{I}_1$ is a local monomorphism; it then follows, by the discussion preceding this proposition, that the same is true for all other paths. We now work our way through the ten axioms:

- (AX 1): Let us write K for the composite $T1 \xrightarrow{\epsilon 1} \mathrm{id} \xrightarrow{\eta 1} S1$; then we have

  $$K(m; n) = \begin{cases} \{\,*\,\} & \text{if } m = n = 1; \\ \emptyset & \text{otherwise.} \end{cases}$$

  and the projection down onto $S\mathbf{I}_1$ simply sends the unique element of $K(1; 1)$ to the unique element of $S1(1; 1)$, and thus is a local monomorphism as required.

- (AX 2)–(AX 5): For each of these we look at the path $\mathbf{d} \colon TS1 \to ST1$, and from the definitions, the projection onto $S\mathbf{I}_1$ is visibly a local monomorphism.

- (AX 6): Let us write K for the composite

  $$TSSS1 \xrightarrow{(TS\boldsymbol{\mu})1} TSS1 \xrightarrow{(T\boldsymbol{\mu})1} TS1 \xrightarrow{\boldsymbol{\delta}1} ST1.$$

  Then we have an isomorphism

  $$K(\phi;\psi) \cong d\big(\phi; (n_\psi \xrightarrow{\psi_3\psi_2\psi_1} s_\psi)\big)$$

  natural in ϕ and ψ, where we are writing a typical element of $TSSS1$ as $\psi = n_\psi \xrightarrow{\psi_1} m_\psi \xrightarrow{\psi_2} r_\psi \xrightarrow{\psi_3} s_\psi$ in the evident way. With respect to this isomorphism, the projection down onto $S\mathbf{I}_1$ is given simply by the value of $\tilde{d}$ there, which is a monomorphism as required.

- (AX 7): Dually to (AX 6).

- (AX 8): Let us write K for the composite $TSS1 \xrightarrow{(T\boldsymbol{\mu})1} TS1 \xrightarrow{\mathbf{d}} ST1 \xrightarrow{(S\boldsymbol{\epsilon})1} S1$; then we have

  $$K(m;\phi) = d\big((m \xrightarrow{\mathrm{id}} m); (n_\phi \xrightarrow{\phi_2\phi_1} r_\phi)\big)$$

  and again the projection down onto $S\mathbf{I}_1$ is simply given by the value of $\tilde{d}$ there, and thus is a local monomorphism.

- (AX 9): Dually to (AX 8).

- (AX 10): Let us write K for the composite $TSS1 \xrightarrow{(T\boldsymbol{\mu})1} TS1 \xrightarrow{\mathbf{d}} ST1 \xrightarrow{(S\boldsymbol{\Delta})1} STT1$; then we have

  $$K(\psi;\phi) = d\big((n_\psi \xrightarrow{\psi_2\psi_1} r_\psi); (n_\phi \xrightarrow{\phi_2\phi_1} r_\phi)\big)$$

  and again the projection down onto $S\mathbf{I}_1$ is simply given by the value of $\tilde{d}$ there, and thus is a local monomorphism. □

Corollary 98. *The pasting equalities (PDA1)–(PDA10), when viewed as diagrams in $\mathbb{C}\mathrm{at}/S\mathbf{I}_1$, hold for the data (PDD1)–(PDD5) constructed above.*

Proof. Consider (PDA1) for example. The two pasting diagrams under consideration pick out two arrows **f** and **g** of $\mathbb{C}at_1/S\mathbf{I}_1$:

$$\begin{tikzcd}
(\boldsymbol{\epsilon}S)1 \otimes (T\boldsymbol{\eta})1 \arrow[rr, "\mathbf{f}"] \arrow[dr, "\boldsymbol{\pi}_1"'] & & (S\boldsymbol{\epsilon})1 \otimes (\boldsymbol{\eta}T)1 \arrow[dl, "\boldsymbol{\pi}_2"] \\
& S\mathbf{I}_1 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\boldsymbol{\epsilon}S)1 \otimes (T\boldsymbol{\eta})1 \arrow[rr, "\mathbf{g}"] \arrow[dr, "\boldsymbol{\pi}_1"'] & & (S\boldsymbol{\epsilon})1 \otimes (\boldsymbol{\eta}T)1 \arrow[dl, "\boldsymbol{\pi}_2"] \\
& S\mathbf{I}_1, &
\end{tikzcd}$$

where both the above diagrams commute. But by the previous proposition, the projections $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are local monomorphisms, and since **f** and **g** are special maps, we have

$$\boldsymbol{\pi}_2 \circ \mathbf{f} = \boldsymbol{\pi}_1 = \boldsymbol{\pi}_2 \circ \mathbf{g} \quad \text{implying} \quad \mathbf{f} = \mathbf{g}.$$

We argue similarly for the other nine diagrams. ☐

This completes the definition of our pseudo-distributive law in $\mathcal{B}(\mathbb{C}at/S\mathbf{I}_1)$; so now, by the arguments of the previous Chapter, we can produce from this a pseudo-distributive law in $\mathcal{B}\big(\mathbb{C}oll(S)\big)$, and thence, via the strict homomorphism

$$V := \mathcal{B}\big(\mathbb{C}oll(S)\big) \xrightarrow{\mathcal{B}U} \mathcal{B}\big([\mathbb{C}at, \mathbb{C}at]_\psi\big) \xrightarrow{\mathcal{B}(-)} [\mathbf{Mod}, \mathbf{Mod}]_\psi,$$

our desired pseudo-distributive law $\delta\colon \hat{T}\hat{S} \Rightarrow \hat{S}\hat{T}$ in **Mod**. So finally, we can honestly state our preferred definition of polycategory:

Definition 99. A polycategory is a monad on a discrete object X in the bicategory $Kl(\delta)$.

Chapter 12

Closing Remarks

Let us take stock of what we have achieved. We set out to establish an abstract formulation of the theory of *polycategories*, a formulation based on the theory of *pseudo-distributive laws*. By setting up a suitable such pseudo-distributive law $\delta\colon \hat{T}\hat{S} \Rightarrow \hat{S}\hat{T}$ of the free symmetric strict monoidal category pseudocomonad over itself qua pseudomonad, we are able to view polycategories as *monads* in the 'two-sided Kleisli bicategory' of this pseudo-distributive law.

In order to set up such a pseudo-distributive law, we first developed the theory of *pseudo double categories* and *double clubs*. In particular, we established the existence of a monoidal double category $\mathbb{C}oll(S)$ and an equivalence of pseudo double categories

$$\mathbb{C}oll(S) \simeq \mathbb{C}at/S\mathbf{I}_1.$$

We were then able to 'lift' the pseudomonad $\hat{S}$ and pseudocomonad $\hat{T}$ from $[\mathbf{Mod}, \mathbf{Mod}]_\psi$ to $\mathbb{C}oll(S)$, and, using the above equivalence, to consider them as data in $\mathbb{C}at/S\mathbf{I}_1$. With this in place, we were able to construct our pseudo-distributive law δ by reducing to the construction of a pseudo-distributive law in $\mathbb{C}at/S\mathbf{I}_1$.

Several directions for further research suggest themselves at this point. Most straightforward is to ascertain the natural higher-dimensional structure into which polycategories form themselves. This should itself be a pseudo double category, of 'polycategories, polyfunctors, polymodules and polytransformations'. To explore this, we would extend the bicategory $Kl(\delta)$ itself to a pseudo double category and utilise the 'monad' construction detailed by Leinster [Lei04a].

Also of interest would be an investigation into the higher dimensional structure of pseudo double categories themselves. As mentioned in passing above, the 2-category $\mathbf{DblCat}_\psi$ is a monoidal bicategory, and hence a suitable base for enriched bicategory theory in the sense of [Car95, Lac95]. A bicategory enriched in $\mathbf{DblCat}_\psi$ is an interesting structure: it is genuinely three-dimensional, but significantly less unwieldy than a tricategory, since we have one less dimension of coherence to deal with, its associativity and unitality being given up to *isomorphism* rather than *equivalence*. A leading example of such a structure would be $\mathbf{DblCat}_\psi$ itself; indeed, as we observed, $\mathbf{DblCat}_\psi$ is a *biclosed* monoidal bicategory, and hence canonically enriched over itself (see [Lac95]).

One might also hope that the theory of this thesis can be modified to deal with the 'multi-bicategories' and 'poly-bicategories' of [CKS03]. Essentially, we can view these as 'many-object' versions of multicategory or polycategory respectively; therefore to describe a multi-bicategory or poly-bicategory with object set X, we would replace our base monoidal 2-category **Cat** with the monoidal 2-category $\mathbf{Cat}/X \times X$ and develop the rest of the theory from there.

Leaving higher-dimensional structures aside, several other directions suggest themselves. Firstly, can we extend this result from **Mod** to $\mathcal{V}$-**Mod**, where $\mathcal{V}$ is a suitable base for enriched category theory, thereby giving a description of $\mathcal{V}$-polycategories? It is not an immediately straightforward task, since in this thesis we have leaned heavily on cartesian notions which do not translate well to the enriched setting.

Secondly, are there other pseudo-distributive laws $\delta \colon \hat{T}\hat{S} \Rightarrow \hat{S}\hat{T}$ which would yield different flavours of generalised categorical structure? For instance, is there a pseudo-distributive law for a polycategory-like structure where we may now plug *several* outputs of one map into the inputs of another?

Thirdly, are there different choices for pseudomonad and pseudocomonad that we could take? For example, there is a pseudomonad on **Mod** which freely adds products and dually, a pseudocomonad which freely adds coproducts: is there a natural choice of pseudo-distributive law mediating between these two, yielding a further different generalised categorical structure?

More interestingly, we may seek motivation from the field of linear logic. Polycategories provide a model for a certain fragment of the system of linear logic, namely the *multiplicative* fragment. Can we find a suitable analogue of the polycategory which models the larger *multiplicative-additive* fragment?

For this, we would like to consider the pseudomonad P for the 'free symmetric strict monoidal category with products' on a category; and dual to this, the pseudocomonad C for the 'free symmetric strict monoidal category with coproducts' on a category. A suitable distributive law of the latter over the former should give rise to a generalised categorical structure modelling the multiplicative-additive fragment of linear logic. At present the technology is not in place to describe the pseudomonad P, or even show that it exists, so this last direction remains a rather distant prospect; but an enticing one nonetheless.

Appendix A

Whiskering and double clubs

We have defined the concept of double club in terms of closure under the structure of monoidal double category. However, we may also ask about closure under the 'whiskering' operations of Chapter 1. *Prima facie*, this may appear to be a strictly stronger requirement, but in fact it follows from our definition of double club.

We begin with a preliminary general result on endohom double categories. We saw how to construct the monoidal structure on $[\mathbb{K}, \mathbb{K}]_\psi$ using the whiskering operations $G(–)$ and $(–)G$. We can also to a certain extent go in the other direction, and derive something like the whiskering homomorphisms from the monoidal structure on $[\mathbb{K}, \mathbb{K}]_\psi$. Indeed, given a homomorphism $G\colon \mathbb{K} \to \mathbb{K}$, we obtain homomorphisms

$$(–) \bullet \mathbf{I}_G \colon [\mathbb{K}, \mathbb{K}]_\psi \xrightarrow{\cong} [\mathbb{K}, \mathbb{K}]_\psi \times 1 \xrightarrow{\mathrm{id} \times \ulcorner \mathbf{I}_G \urcorner} [\mathbb{K}, \mathbb{K}]_\psi \times [\mathbb{K}, \mathbb{K}]_\psi \xrightarrow{\bullet} [\mathbb{K}, \mathbb{K}]_\psi$$

and

$$\mathbf{I}_G \bullet (–) \colon [\mathbb{K}, \mathbb{K}]_\psi \xrightarrow{\cong} 1 \times [\mathbb{K}, \mathbb{K}]_\psi \xrightarrow{\ulcorner \mathbf{I}_G \urcorner \times \mathrm{id}} [\mathbb{K}, \mathbb{K}]_\psi \times [\mathbb{K}, \mathbb{K}]_\psi \xrightarrow{\bullet} [\mathbb{K}, \mathbb{K}]_\psi.$$

And these homomorphisms approximate the operation of whiskering by G in the following sense:

Proposition 100. *There are canonical invertible vertical transformations*

$$l_G \colon G(–) \Rightarrow \mathbf{I}_G \bullet (–) \quad \textit{and} \quad r_G \colon (–)G \Rightarrow (–) \bullet \mathbf{I}_G$$

which are natural in G.

Proof. We have $\big(G(–)\big)_0 = \big(\mathbf{I}_G \bullet (–)\big)_0$ and $\big((–)G\big)_0 = \big((–) \bullet \mathbf{I}_G\big)_0$, so we can take

$(l_G)_0$ and $(r_G)_0$ to be identity natural transformations. For $(l_G)_1$ and $(r_G)_1$, observe that we have

$$\begin{aligned}\big(\mathbf{I}_G \bullet (-)\big)_1 &= \mathbf{I}_G(-)_t \otimes G(-) = \mathbf{I}_{G(-)_t} \otimes G(-)\\ \text{and } \big((-) \bullet \mathbf{I}_G\big)_1 &= (-)G \otimes (-)_s\mathbf{I}_G.\end{aligned}$$

Therefore we take $(l_G)_1$ to be the natural transformation

$$(l_G)_1 = G(-) \xRightarrow{\mathfrak{l}_{G(-)}} \mathbf{I}_{G(-)_t} \otimes G(-)$$

and $(r_G)_1$ to be the natural transformation

$$(r_G)_1 = (-)G \xRightarrow{\mathfrak{r}_{(-)G}} (-)G \otimes \mathbf{I}_{(-)_s G} \xRightarrow{\mathrm{id} \otimes \mathfrak{e}_G} (-)G \otimes (-)_s\mathbf{I}_G\,.$$

It's now routine diagram chasing to check that l and r satisfy all the required axioms for a vertical transformation, and that they are natural in G as required. □

Proposition 101. *Let S be a double club, and let (A, α) be an object of $\mathbb{C}\mathit{oll}(S)$. Then the whiskering homomorphisms*

$$(-)A \colon [\mathbb{K}, \mathbb{K}]_\psi \to [\mathbb{K}, \mathbb{K}]_\psi \quad \textit{and} \quad A(-) \colon [\mathbb{K}, \mathbb{K}]_\psi \to [\mathbb{K}, \mathbb{K}]_\psi$$

lift to homomorphisms

$$(-)(A, \alpha) \colon \mathbb{C}\mathit{oll}(S) \to \mathbb{C}\mathit{oll}(S) \quad \textit{and} \quad (A, \alpha)(-) \colon \mathbb{C}\mathit{oll}(S) \to \mathbb{C}\mathit{oll}(S).$$

Proof. We give the details for $(A, \alpha)(-)$, since $(-)(A, \alpha)$ follows similarly. Following Proposition 100, we have the homomorphism $\mathbf{I}_{(A,\alpha)} \bullet (-) \colon \mathbb{C}\mathit{oll}(S) \to \mathbb{C}\mathit{oll}(S)$; further we have the invertible special vertical transformation

$$l_A \colon A(-) \Rightarrow \mathbf{I}_A \bullet (-) \colon \mathbb{K} \to \mathbb{K}$$

So we give $(A, \alpha)(-)$ as follows. Its component $\big((A, \alpha)(-)\big)_0 \mathbb{C}\mathit{oll}(S)_0 \to \mathbb{C}\mathit{oll}(S)_0$ is simply $\big(\mathbf{I}_{(A,\alpha)} \bullet (-)\big)_0 = (A, \alpha) \bullet (-)$, whilst $\big((A, \alpha)(-)\big)_1 \colon \mathbb{C}\mathit{oll}(S)_1 \to \mathbb{C}\mathit{oll}(S)_1$ is given as follows:

- **On objects**: given $(\mathbf{B}, \boldsymbol{\beta})$ in $\mathbb{C}oll(S)_1$, we take $(A, \alpha)(\mathbf{B}, \boldsymbol{\beta})$ to be the modification

$$A\mathbf{B} \xRightarrow{(l_A)_{\mathbf{B}}} \mathbf{I}_A \bullet \mathbf{B} \xRightarrow{\mathbf{I}_\alpha \bullet \boldsymbol{\beta}} \mathbf{I}_S \bullet S\mathbf{I} \xRightarrow{\mathfrak{e} \bullet S\mathbf{I}} S\mathbf{I} \bullet S\mathbf{I} \xRightarrow{\mathfrak{m}} S\mathbf{I}.$$

  The first modification above is cartesian since it is invertible, whilst the remaining composite is $\mathbf{I}_{(A,\alpha)} \bullet (\mathbf{B}, \boldsymbol{\beta})$, and hence cartesian since S is a double club; thus the entire composite is cartesian as required.

- **On maps**: given $\boldsymbol{\delta}\colon (\mathbf{B}, \boldsymbol{\beta}) \to (\mathbf{C}, \boldsymbol{\gamma})$, we take $(A, \alpha)(\boldsymbol{\delta})$ to be given by

$$A\boldsymbol{\delta}\colon (A, \alpha)(\mathbf{B}, \boldsymbol{\beta}) \to (A, \alpha)(\mathbf{C}, \boldsymbol{\gamma}).$$

  That this map is compatible with the projections down to $S\mathbf{I}$ is an easy diagram chase.

It's immediate that these definitions are compatible with source and target; it remains to give the comparison maps $\mathfrak{m}$ and $\mathfrak{e}$, for which we simply take

$$\begin{aligned} \mathfrak{e}_{(B,\beta)} &= \mathfrak{e}_B\colon \mathbf{I}_{AB} \Rrightarrow A\mathbf{I}_B \\ \text{and } \mathfrak{m}_{(\mathbf{B},\boldsymbol{\beta}),(\mathbf{B}',\boldsymbol{\beta}')} &= \mathfrak{m}_{\mathbf{B},\mathbf{B}'}\colon A\mathbf{B} \otimes A\mathbf{B}' \to A(\mathbf{B} \otimes \mathbf{B}'). \end{aligned}$$

That these maps are compatible with the projections down to $S\mathbf{I}$ is another straightforward diagram chase, whilst the coherence axioms for $\mathfrak{m}$ and $\mathfrak{e}$ follows from those for $A(–)$ on $[\mathbb{K}, \mathbb{K}]_\psi$. □

For completeness, we also observe the following:

Proposition 102. *Let S be a double club, and let $\gamma\colon (A, \alpha) \to (B, \beta)$ be a vertical arrow of $\mathbb{C}oll(S)$. Then the whiskering vertical transformations*

$$(–)\gamma\colon (–)A \Rightarrow (–)B \quad \textit{and} \quad \gamma(–)\colon A(–) \Rightarrow B(–)$$

lift to vertical transformations

$$(–)\gamma\colon (–)(A, \alpha) \Rightarrow (–)(B, \beta) \quad \textit{and} \quad \gamma(–)\colon (A, \alpha)(–) \Rightarrow (B, \beta)(–).$$

The proof is straightforward: one must simply show that the components of $\gamma(-)$ and $(-)\gamma$ are compatible with the projections down to $S\mathbf{I}$.

Appendix B

Pseudomonads and pseudocomonads

We recall here the definition of a pseudomonad on a bicategory (as found in, for example, [Mar99, Lac00]). This is a specialisation of the notion of pseudomonad *in* a tricategory (see [Lac00]), where the tricategory in question is taken to be the tricategory of all bicategories.

Definition 103. A **pseudomonad** on a bicategory $\mathcal{B}$ consists of the following data:

(PMD1) A homomorphism $S\colon \mathcal{B} \to \mathcal{B}$;

(PMD2) Pseudonatural transformations $\eta\colon \mathrm{id}_{\mathcal{B}} \Rightarrow S$ and $\mu\colon SS \Rightarrow S$;

(PMD3) Invertible modifications

$$\begin{tikzcd}
S \arrow[d, "S\eta"'] \arrow[dr, "\mathrm{id}_S"] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"', near start] & \\
SS \arrow[r, "\mu"'] & S,
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[d, "\eta S"'] \arrow[dr, "\mathrm{id}_S"] \arrow[dr, phantom, "\overset{\rho}{\Rightarrow}"', near start] & \\
SS \arrow[r, "\mu"'] & S,
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
SSS \arrow[r, "S\mu"] \arrow[d, "\mu S"'] \arrow[dr, phantom, "\overset{\tau}{\Rightarrow}"] & SS \arrow[d, "\mu"] \\
SS \arrow[r, "\mu"'] & S.
\end{tikzcd}$$

All subject to the following two axioms:

(PMA1) The following pastings agree:

$$\begin{tikzcd}
S^4 \arrow[rr, "SS\mu"] \arrow[dr, "S\mu S"] \arrow[dd, "\mu SS"'] & & S^3 \arrow[dr, "S\mu"] \arrow[dl, phantom, "\overset{S\tau}{\Rightarrow}"] & \\
\arrow[dr, phantom, "\overset{\tau S}{\Rightarrow}"] & S^3 \arrow[rr, "S\mu"'] \arrow[dd, "\mu S"'] \arrow[ddrr, phantom, "\overset{\tau}{\Rightarrow}"] & & S^2 \arrow[dd, "\mu"] \\
S^3 \arrow[dr, "\mu S"'] & & & \\
& S^2 \arrow[rr, "\mu"'] & & S
\end{tikzcd}
=
\begin{tikzcd}
S^4 \arrow[rr, "SS\mu"] \arrow[dd, "\mu SS"'] \arrow[ddrr, phantom, "\overset{\cong}{\Rightarrow}"] & & S^3 \arrow[dr, "S\mu"] \arrow[dd, "\mu S"] \arrow[ddr, phantom, "\overset{\tau}{\Rightarrow}"] & \\
& & & S^2 \arrow[dd, "\mu"] \\
S^3 \arrow[rr, "S\mu"'] \arrow[dr, "\mu S"'] \arrow[drr, phantom, "\overset{\tau}{\Rightarrow}"] & & S^2 \arrow[dr, "\mu"'] & \\
& S^2 \arrow[rr, "\mu"'] & & S;
\end{tikzcd}$$

(PMA2) The following pastings agree:

$$\begin{tikzcd}
& S^3 \arrow[rr, "\mu S"] \arrow[dr, "S\mu"] \arrow[d, phantom, "\Downarrow S\rho"] & & S^2 \arrow[dr, "\mu"] \arrow[d, phantom, "\Downarrow \tau"] & \\
S^2 \arrow[ur, "S\eta S"] \arrow[rr, "\mathrm{id}"'] & {} & S^2 \arrow[rr, "\mu"'] & {} & S
\end{tikzcd}
=
\begin{tikzcd}
& S^3 \arrow[dr, "\mu S"] \arrow[d, phantom, "\Downarrow \lambda S"] & & \\
S^2 \arrow[ur, "S\eta S"] \arrow[rr, "\mathrm{id}"'] & {} & S^2 \arrow[r, "\mu"'] & S.
\end{tikzcd}$$

Dually, we have the notion of a pseudocomonad on a bicategory:

Definition 104. A **pseudocomonad** on a bicategory $\mathcal{B}$ consists of the following data:

(PCD1) A homomorphism $T\colon \mathcal{B} \to \mathcal{B}$;

(PCD2) Pseudonatural transformations $\epsilon\colon T \Rightarrow \mathrm{id}_{\mathcal{B}}$ and $\Delta\colon T \Rightarrow TT$;

(PCD3) Invertible modifications

$$\begin{tikzcd}
T \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}_T"'] & T^2, \arrow[d, "T\epsilon"] \arrow[dl, phantom, "\overset{\lambda'}{\Rightarrow}"] \\
& T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}_T"'] & T^2 \arrow[d, "\epsilon T"] \arrow[dl, phantom, "\overset{\rho'}{\Rightarrow}"] \\
& T
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
T \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\overset{\tau'}{\Rightarrow}"] & T^2 \arrow[d, "\Delta T"] \\
T^2 \arrow[r, "T\Delta"'] & T^3.
\end{tikzcd}$$

Subject to the two axioms:

(PCA1) The following pastings agree:

$$\begin{tikzcd}
T \arrow[rr, "\Delta"] \arrow[dr, "\Delta"] \arrow[dd, "\Delta"'] & & T^2 \arrow[dr, "\Delta T"] \arrow[dl, phantom, "\overset{\tau'}{\Rightarrow}"] & \\
 & T^2 \arrow[rr, "T\Delta"'] \arrow[dd, "\Delta T"] \arrow[dl, phantom, "\overset{\tau'}{\Rightarrow}"] \arrow[ddrr, phantom, "\overset{\cong}{\Rightarrow}"] & & T^3 \arrow[dd, "\Delta TT"] \\
T^2 \arrow[dr, "T\Delta"'] & & & \\
 & T^3 \arrow[rr, "TT\Delta"'] & & T^4
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[rr, "\Delta"] \arrow[dd, "\Delta"'] \arrow[ddrr, phantom, "\overset{\tau'}{\Rightarrow}"] & & T^2 \arrow[dr, "\Delta T"] \arrow[dd, "\Delta T"] \arrow[ddr, phantom, "\overset{\tau' T}{\Rightarrow}"] & \\
 & & & T^3 \arrow[dd, "\Delta TT"] \\
T^2 \arrow[rr, "T\Delta"'] \arrow[dr, "T\Delta"'] & & T^3 \arrow[dr, "T\Delta T"] \arrow[dl, phantom, "\overset{T\tau'}{\Rightarrow}"] & \\
 & T^3 \arrow[rr, "TT\Delta"'] & & T^4;
\end{tikzcd}$$

(PCA2) The following pastings agree:

$$\begin{tikzcd}
T \arrow[rr, "\Delta"] \arrow[dr, "\Delta"'] & & T^2 \arrow[rr, "\mathrm{id}_{T^2}"] \arrow[dr, "T\Delta"'] \arrow[dl, phantom, "\Downarrow\tau'"] & & T^2 \\
 & T^2 \arrow[rr, "\Delta T"'] & & T^3 \arrow[ur, "T\epsilon T"'] \arrow[u, phantom, "\Downarrow T\rho'"] &
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[r, "\Delta"] & T^2 \arrow[rr, "\mathrm{id}_{T^2}"] \arrow[dr, "\Delta T"'] & & T^2. \\
 & & T^3 \arrow[ur, "T\epsilon T"'] \arrow[u, phantom, "\Downarrow\lambda' T"] &
\end{tikzcd}$$

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