arXiv:1306.3359v1 [q-fin.CP] 14 Jun 2013

Making Mean-Variance Hedging Implementable in a Partially Observable Market *

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First version: June 14, 2013

Abstract

The mean-variance hedging (MVH) problem is studied in a partially observable market where the drift processes can only be inferred through the observation of asset or index processes. Although most of the literatures treat the MVH problem by the duality method, here we study a system consisting of three BSDEs derived by Mania and Tevzadze (2003) and Mania et.al.(2008) and try to provide more explicit expressions directly implementable by practitioners. Under the Bayesian and Kalman-Bucy frameworks, we find that a relevant BSDE yields a semi-closed solution via a simple set of ODEs which allow a quick numerical evaluation. This renders remaining problems equivalent to solving European contingent claims under a new forward measure, and it is straightforward to obtain a forward looking non-sequential Monte Carlo simulation scheme. We also give a special example where the hedging position is available in a semi-closed form. For more generic setups, we provide explicit expressions of approximate hedging portfolio by an asymptotic expansion. These analytic expressions not only allow the hedgers to update the hedging positions in real time but also make a direct analysis of the terminal distribution of the hedged portfolio feasible by standard Monte Carlo simulation.

Keywords : Mean-variance hedging, BSDE, Bayesian analysis, Kalman-Bucy filter, asymptotic expansion, particle method

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1 Introduction

Since the last financial crisis, there are market-wide efforts for standardization of financial products so that they can be traded through security exchanges or central counterparties. This is expected to make them more liquid, transparent, remote from counterparty credit risks, and in particular significantly reduces the regulatory cost for financial firms. For these products, an idealistic situation for electronic trading is emerging and many financial firms are heavily investing to setup sophisticated e-trading systems to maintain their profitability for coming years. At first sight, it might appear that it leads the financial market closer to the ideal "complete" environment. However, on the other hand, remaining uncleared OTC contracts are going to be severely penalized in terms of regulatory cost so that it gives financial firms a strong incentive to walk away from them. This inevitably makes a part of security universe less liquid and costlier to trade, and can make practitioners reluctant to use them even if they were the most efficient hedging instruments before the crisis. The last crisis also created another complication by pushing all the practitioners into a new pricing regime for the collateralized contracts. Growing recognition of the critical importance of the choice of collateral and its funding cost makes it impossible to perfectly hedge even a very simple cash flow unless one has an easy access to the relevant collateral assets or there exist very liquid basis markets.

Considering the above situation, we naturally expect that there is a growing need of systematic hedging method allowing investors to flexibly choose the hedging instruments based on their own regulatory and accessibility conditions. Mean-variance hedging (MVH) is a one possible approach to this problem. MVH has been studied by many authors and there exist vast literatures on the related issues. See, as some of recent works, Laurent & Pham (1999) [8], Pham (2001) [14], Pham & Quenez (2001) [15] and references therein. Although the mathematical understanding of the MVH problem has been greatly progressed by those adopting the duality method, more practical issues related to the actual implementation of a hedging program have not attracted much attention so far and there exist only a few special examples reported with explicit expressions. In this paper, we try to make a progress in that direction by studying the system of equations derived by Mania, Tevzadza and their co-authors.

In Mania & Tevzadze (2003) [9], the authors studied a minimizing problem of a convex cost function and showed that the optimal value function follows a backward stochastic partial differential equation (BSPDE). They have used the flow dynamics of the value function derived from the Itô-Ventzell formula combined with a martingale property of the optimal value function to obtain a BSPDE as a sufficient condition for the optimality. For the MVH problem, they showed that the BSPDE can be decomposed into three backward stochastic differential equations (BSDEs). The technique is extended for a partial information setup by Mania et.al.(2008) [11], for utility maximization by Mania & Santacroce (2010) [12], and for MVH problem with general semimartingales by Jeanblanc et.al. (2012) [6].

In the following, we consider the MVH problem in a partially observable market where the drift processes can only be inferred through the observation of stock or any index processes driven by Brownian motions possibly with stochastic volatilities. Under the Bayesian and Kalman-Bucy frameworks, we find that a relevant BSDE yields an semi-closed solution via a simple set of ODEs allowing a quick numerical evaluation. This renders remaining problems equivalent to solving European contingent claims, and it is straightforward to obtain a forward looking Monte Carlo simulation scheme using a simple particle method [4]. As far as the optimal hedging positions are concerned, it is also pointed out that one only needs the standard simulations for the terminal liability and its Delta sensitivities against the state processes under a certain forward measure. We also provide explicit expressions for a solvable case and approximate hedging portfolio for more generic setups by an asymptotic expansion method. These explicit forms allow the hedgers to update the hedging positions in real time, and also make the direct analysis of the terminal distribution of the hedged portfolio feasible by standard Monte Carlo simulation. We also provide several numerical examples to demonstrate our procedures.

2 The Market Setup

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$, where T is a fixed time horizon. We consider a financial market with a risk-less asset, d tradable stocks or indexes $S = \{S_i\}_{1\leq i\leq d}$, and $m := (n-d)$ non-tradable indexes or otherwise state processes relevant for stochastic volatilities $Y = \{Y_j\}_{d+1\leq j\leq n}$. For simplicity, we assume that the interest rate is zero by focusing on a relatively short time period. Using a vector notation of S and Y, we write the dynamics of the underlyings as

$$dS_t = \sigma(t, S_t, Y_t)\Big(dW_t + \theta_t dt\Big) \tag{2.1}$$

$$dY_t = \bar{\sigma}(t, S_t, Y_t)\Big(dW_t + \theta_t dt\Big) + \rho(t, S_t, Y_t)\Big(dB_t + \alpha_t dt\Big) \tag{2.2}$$

Here, (W, B) are independent $(\mathbf{P}, \mathcal{F})$-Brownian motions with dimension d and m. θ and α are $\{\mathcal{F}_t\}$-adapted market price-of-risk (MPR) processes for W and B. $\sigma(t, s, y)$, $\bar{\sigma}(t, s, y)$ and $\rho(t, s, y)$ are assumed to be known smooth functions taking values in $\mathbb{R}^{d\times d}$, $\mathbb{R}^{m\times d}$ and $\mathbb{R}^{m\times m}$. We assume all of them satisfy the technical conditions to allow unique strong solutions for S and Y.

We denote the available information set for the investor by a sub-σ-field $\mathcal{G}_t \subset \mathcal{F}_t$. We assume that $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ is the $\mathbf{P}$-augmentation of filtration generated by the processes of all the stocks S and a subset $\{Y\}^{obs} \subset \{Y_j\}_{d+1\leq j\leq n}$ which are continuously observable assets or indexes but not tradable by the investor by regulatory or some other reasons. Although S and $\{Y\}^{obs}$ can be observed continuously, we assume that the investor cannot identify their drifts and Brownian shocks independently, which is most likely the case in the real financial market. Thus, neither θ nor α is $\{\mathcal{G}_t\}$-adapted. Through the observation of quadratic covariation of S and $\{Y\}^{obs}$, we can recover the values of $\sigma_t\sigma_t^\top$, $\bar{\sigma}_t^{obs}\sigma_t^\top$ and $(\bar{\sigma}_t\bar{\sigma}_t^\top + \rho_t\rho_t^\top)^{obs}$ at each time. We assume the maps $(\sigma, \bar{\sigma}, \rho)$ are constructed in such a way that they allow to fix the values of all the remaining $Y_k \in \{Y\}_{d+1\leq j\leq n}\backslash\{Y\}^{obs}$ uniquely from the values of $\left\{S_t, Y_t^{obs}, \sigma_t\sigma_t^\top, \bar{\sigma}_t^{obf}\sigma_t^\top, (\bar{\sigma}_t\bar{\sigma}_t^\top + \rho_t\rho_t^\top)^{obs}\right\}$ at every time t. Thus, under the above construction, the whole elements of $\{Y\}_{d+1\leq j\leq n}$ are in fact $\{\mathcal{G}_t\}$-adapted. Let us further assume σ and ρ are always nonsingular and thus

$$\begin{aligned} \widetilde{W}_t &:= \int_0^t \sigma^{-1}(u, S_u, Y_u)dS_u \\ &= W_t + \int_0^t \theta_u du \end{aligned} \tag{2.3}$$

$$\begin{aligned}\widetilde{B}_t &:= \int_0^t \rho^{-1}(u,S_u,Y_u)\Big(dY_u - \bar{\sigma}(u,S_u,Y_u)\sigma^{-1}(u,S_u,Y_u)dS_u\Big) \\ &= B_t + \int_0^t \alpha_u du \end{aligned} \tag{2.4}$$

are actually $\{\mathcal{G}_t\}$-adapted processes.

3 Linear Filtering

From the expressions (2.3) and (2.4) and the fact that both of $(\widetilde{W}, \widetilde{B})$ are observable, we have a linear observation system for the MPR processes. If we further assume that the MPRs are either constants or linear Gaussian processes in $(\mathbf{P}, \mathcal{F})$, then the system becomes a well-known Bayesian or Kalman-Bucy filtering model. See, a textbook written by Bain & Crisan (2008) [1] for the details of stochastic filtering.

Let us denote

$$z_t := \begin{pmatrix}\theta_t \\ \alpha_t\end{pmatrix}, \quad \omega_t = \begin{pmatrix} W_t \\ B_t \end{pmatrix} \tag{3.1}$$

for notational simplicity, and then we put

$$\Lambda_t = \exp\Big(-\int_0^t z_s^\top d\omega_s - \frac{1}{2}\int_0^t ||z_s||^2 ds\Big). \tag{3.2}$$

For linear filtering models we discuss below, Λ is actually shown to be a true $(\mathbf{P}, \mathcal{F})$-martingale. We can then define a new measure $\widetilde{P}$ by

$$\left.\frac{d\widetilde{\mathbf{P}}}{d\mathbf{P}}\right|_{\mathcal{F}_t} = \Lambda_t \tag{3.3}$$

then, it is easy to check

$$\widetilde{\omega}_t := \begin{pmatrix} \widetilde{W}_t \\ \widetilde{B}_t \end{pmatrix} \tag{3.4}$$

is a n-dimensional $(\widetilde{\mathbf{P}}, \mathcal{F})$-Brownian motion. By (2.1) and (2.2), one can see that $\mathbb{G}$ is actually the augmented filtration generated by $(\widetilde{W}, \widetilde{B})$ (See Ref. [15] for details.). $(\widetilde{\mathbf{P}}, \mathcal{F})$-martingale $\widetilde{\Lambda}_t = 1/\Lambda_t$ gives the inverse relation between the measures

$$\left.\frac{d\mathbf{P}}{d\widetilde{\mathbf{P}}}\right|_{\mathcal{F}_t} = \widetilde{\Lambda}_t\ . \tag{3.5}$$

We denote the expectation of the MPRs conditional on $\mathcal{G}_t$ by

$$\hat{z}_t := \begin{pmatrix} \hat{\theta}_t \\ \hat{\alpha}_t \end{pmatrix} := \begin{pmatrix} \mathbb{E}[\theta_t|\mathcal{G}_t] \\ \mathbb{E}[\alpha_t|\mathcal{G}_t] \end{pmatrix}. \tag{3.6}$$

By Kallianpur-Striebel formula, it is given by

$$\hat{z}_t = \frac{\widetilde{\mathbb{E}}[z_t \widetilde{\Lambda} | \mathcal{G}_t]}{\widetilde{\mathbb{E}}[\widetilde{\Lambda}_t | \mathcal{G}_t]} \ . \tag{3.7}$$

where $\widetilde{\mathbb{E}}[\ ]$ is the expectation under $\widetilde{\mathbf{P}}$ measure. This equation can be explicitly solvable for a Bayesian and also for a linear Gaussian model. Note that the processes defined by

$$N_t = \widetilde{W}_t - \int_0^t \hat{\theta}_s ds \tag{3.8}$$

$$M_t = \widetilde{B}_t - \int_0^t \hat{\alpha}_s ds \tag{3.9}$$

are called innovation processes and they are $(\mathbf{P}, \mathcal{G})$-Brownian motions.

3.1 A Bayesian model

In this section, we consider a Bayesian model in which the MPR is assumed to be $\mathcal{F}_0$-measurable with a known prior distribution. The constant vector

$$z = \begin{pmatrix} \theta \\ \alpha \end{pmatrix} \tag{3.10}$$

denotes a value of the MPR. For a concrete calculation, let us assume that z has a prior Gaussian distribution with the mean z_0 and its covariance denoted by a positive definite symmetric matrix Σ_0. Let us denote the corresponding density function by $\varsigma(z)$.

In this setup, one has

$$\widetilde{\Lambda}_t = \exp\left(z^\top \widetilde{\omega}_t - \frac{t}{2} ||z||^2 \right) \tag{3.11}$$

and hence

$$\begin{aligned} F(t, \widetilde{\omega}_t) \ &:= \ \widetilde{\mathbb{E}}[\widetilde{\Lambda}_t | \mathcal{G}_t] \\ &= \int_{\mathbb{R}^n} \exp\left(\widetilde{\omega}_t^\top z - \frac{t}{2} ||z||^2 \right) \varsigma(z) d^n z \ . \end{aligned} \tag{3.12}$$

This yields

$$\hat{z}_t = \frac{\partial_w F(t, \widetilde{\omega}_t)}{F(t, \widetilde{\omega}_t)}. \tag{3.13}$$

For a Gaussian prior distribution, $\hat{z}$ can be evaluated explicitly. One can show that

$$F(t, \widetilde{\omega}_t) = \frac{1}{(2\pi)^{n/2} |\Sigma_0|^{1/2}} \int \exp\left(\widetilde{\omega}_t^\top z - \frac{t}{2} ||z||^2 - \frac{1}{2} (z - z_0)^\top \Sigma_0^{-1} (z - z_0) \right) d^n z. \tag{3.14}$$

Using a new positive definite symmetric matrix $\Sigma(t)$ defined by

$$\Sigma(t) := [\Sigma_0^{-1} + t\mathbb{I}]^{-1} \tag{3.15}$$

and $x := z - z_0$, one obtains

$$F(t,\widetilde{\omega}_t) = \frac{\exp\Big(\widetilde{\omega}_t^\top z_0 - \frac{t}{2}||z_0||^2\Big)}{(2\pi)^{n/2}|\Sigma_0|^{1/2}} \int \exp\Big([\widetilde{\omega}_t - tz_0]^\top x - \frac{1}{2}x^\top \Sigma(t)^{-1}x\Big) d^n x. \qquad (3.16)$$

Then, simple calculation gives

$$F(t,\widetilde{\omega}_t) = \sqrt{\frac{|\Sigma(t)|}{|\Sigma_0|}} \exp\left(-\frac{t}{2}||z_0||^2 + \widetilde{\omega}_t^\top z_0 + \frac{1}{2}[\widetilde{\omega}_t - tz_0]^\top \Sigma(t)[\widetilde{\omega}_t - tz_0]\right). \qquad (3.17)$$

As a result, the conditional expectation of the MPR is given by

$$\hat{z}_t = z_0 + \Sigma(t)[\widetilde{\omega}_t - tz_0] \qquad (3.18)$$

Using a simple fact

$$\frac{d}{dt}(\Sigma(t)\Sigma(t)^{-1}) = 0 \qquad (3.19)$$

one can easily confirm that

$$\frac{d}{dt}\Sigma(t) = -\Sigma(t)^2. \qquad (3.20)$$

Thus, the dynamics of $\hat{z}$ can be written as

$$d\hat{z}_t = -\Sigma(t)\hat{z}_t dt + \Sigma(t)d\widetilde{\omega}_t \qquad (3.21)$$

i.e.,

$$d\hat{z}_t = \Sigma(t)dn_t \ . \qquad (3.22)$$

where we have used a shorthand notation

$$n_t := \begin{pmatrix} N_t \\ M_t \end{pmatrix} \ . \qquad (3.23)$$

Thus, we can see that $\hat{z}$ is a Gaussian martingale process in $(\widetilde{\mathbf{P}}, \mathcal{G})$.

3.2 A Kalman-Bucy model

In this model, we assume z_t (or, “signal”) follows a linear Gaussian process in $(\mathbf{P}, \mathcal{F})$:

$$dz_t = [\mu - Fz_t]dt + \delta dV_t \qquad (3.24)$$

where $\mu \in \mathbb{R}^n$ and $\delta \in \mathbb{R}^{n\times p}, F \in \mathbb{R}^{n\times n}$ are constants. V denotes p-dimensional $(\mathbf{P}, \mathcal{F})$-Brownian motions independent of (W, B). The MPR is assumed to have a prior Gaussian distribution with mean z_0 and covariance matrix Σ_0.

The observation is made through

$$d\widetilde{\omega}_t = z_t dt + d\omega_t \ . \qquad (3.25)$$

In this case, we have a well-known result that

$$d\hat{z}_t = [\mu - F\hat{z}_t]dt + \Sigma(t)dn_t, \quad \hat{z}_0 = z_0 \tag{3.26}$$

where $\Sigma(t) \in \mathbb{R}^{n\times n}$ is a deterministic function given as a solution of the following ODE:

$$\frac{d\Sigma(t)}{dt} = \delta\delta^\top - F\Sigma(t) - \Sigma(t)F^\top - \Sigma(t)^2 \tag{3.27}$$

with the initial condition $\Sigma(0) = \Sigma_0$. In the remainder of the paper, we provide the detailed calculations only for this Kalman-Buycy model. For Bayesian case, one can get the equivalent results by simply putting $\mu = F = 0$ and using the relevant $\Sigma(t)$ given in (3.15) in the corresponding formulas.

4 A System of BSDEs for Mean-Variance Hedging

Since we are assuming that the interest rate is zero [1], the dynamics of wealth with the initial capital w at $s < t$ is given by

$$\mathcal{W}_t^\pi(s,w) = w + \int_s^t \pi_u^\top dS_u \tag{4.1}$$

where $\pi \in \Pi$ is a portfolio strategy. Here, Π denotes a set of d-dimensional $\mathbb{G}$-predictable processes satisfying appropriate integrability conditions. Our problem is to solve

$$V(t,w) = \operatorname*{ess\,inf}_{\pi\in\Pi} \mathbb{E}\left[\left(\mathcal{W}_T^\pi(t,w) - H\right)^2 \Big| \mathcal{G}_t\right] . \tag{4.2}$$

In this paper, we suppose H is some $\mathcal{G}_T$-measurable (and hence the investor can exactly know the terminal liability) square integrable random variable.

Mania & Tevzadze [9, 11] proved (using more general setup) that a solution of the above problem is given by

$$V(t,w) = w^2 V_2(t) - 2wV_1(t) + V_0(t) \tag{4.3}$$

where V_2, V_1 and V_0 are the solutions of the following BSDEs:

$$V_2(t) = 1 - \int_t^T \frac{||Z_2(s) + V_2(s)\hat{\theta}_s||^2}{V_2(s)} ds - \int_t^T Z_2(s)^\top dN_s - \int_t^T \Gamma_2(s)^\top dM_s \tag{4.4}$$

$$V_1(t) = H - \int_t^T \frac{[Z_2(s) + V_2(s)\hat{\theta}_s]^\top [Z_1(s) + V_1(s)\hat{\theta}_s]}{V_2(s)} ds - \int_t^T Z_1(s)^\top dN_s - \int_t^T \Gamma_1(s)^\top dM_s \tag{4.5}$$

$$V_0(t) = H^2 - \int_t^T \frac{||Z_1(s) + V_1(s)\hat{\theta}_s||^2}{V_2(s)} ds - \int_t^T Z_0(s)^\top dN_s - \int_t^T \Gamma_0(s)^\top dM_s \tag{4.6}$$

[1] Practically, one can still include bonds in his/her portfolio by modeling their dynamics directly just as equities or commodities.

with some positive constant c such that $c < V_2$ under the existence of equivalent martingale measures and with some mild conditions. Here, all the $\{Z_i, \Gamma_i\}$ are $\{\mathcal{G}_t\}$-adapted processes with appropriate dimensionality.

The corresponding optimal wealth process is given by

$$\mathcal{W}_T^{\pi^*}(t,w) = w + \int_t^T \frac{[Z_1(s) + V_1(s)\hat{\theta}_s]^\top}{V_2(s)}[dN_s + \hat{\theta}_s ds] - \int_t^T \mathcal{W}_s^{\pi^*}(t,w)\frac{[Z_2(s) + V_2(s)\hat{\theta}_s]^\top}{V_2(s)}[dN_s + \hat{\theta}_s ds]. \tag{4.7}$$

Using the relationship

$$dN_s + \hat{\theta}_s ds = \sigma^{-1}(s, S_s, Y_s)dS_s \tag{4.8}$$

one can easily read off the optimal hedging position from (4.7) as

$$\pi_s^* = (\sigma^{-1})^\top(s, S_s, Y_s)\frac{1}{V_2(s)}\Big\{[Z_1(s) + V_1(s)\hat{\theta}_s] - \mathcal{W}_s^{\pi^*}[Z_s(s) + V_2(s)\hat{\theta}_s]\Big\} . \tag{4.9}$$

In our setup with Brownian motions, derivation of the above BSDEs is quite straightforward by using Itô-Ventzell formula and the martingale property of $V(t, \mathcal{W}_t^{\pi^*})$ for the optimal strategy. The main ideas are briefly explained in Appendix A. The detailed explanation on Itô-Ventzell formula is available, for example, in the section 3.3 of a textbook [7] as a *generalized* Itô formula. It is quite interesting to see there exists a direct link between the BSPDE and the usual HJB equation. See discussions given in Mania & Tevzadze (2008) [10] for this point.

5 Solving V_2 by ODEs

We now try to solve V_2 for our Kalman-Bucy filtering model. Firstly, using the fact that $0 < c < V_2$, we transform V_2, Z_2 and Γ_2 as follows:

$$V_L(t) = \log V_2(t), \quad Z_L(t) = Z_2(t)/V_2(t), \quad \Gamma_L(t) = \Gamma_2(t)/V_2(t) . \tag{5.1}$$

Simple calculation gives a quadratic growth BSDE

$$V_L(t) = -\int_t^T \left\{\frac{1}{2}(||Z_L(s)||^2 - ||\Gamma_L(s)||^2) + 2\hat{\theta}_s^\top Z_L(s) + ||\hat{\theta}_s||^2\right\} ds - \int_t^T Z_L(s)^\top dN_s - \int_t^T \Gamma_L(s)^\top dM_s . \tag{5.2}$$

The only ingredient of the BSDE is $\hat{z}$ and it has a linear Gaussian form.

Now, let us suppose that the solution has the following form:

$$V_L(t) = \frac{1}{2}\hat{z}_t^\top a^{[2]}(t)\hat{z}_t + a^{[1]}(t)^\top \hat{z}_t + a^{[0]}(t) \tag{5.3}$$

where $\{a^{[i]}\}$ are deterministic functions taking values in $a^{[2]}(t) \in \mathbb{R}^{n\times n}$, $a^{[1]}(t) \in \mathbb{R}^n$ and $a^{[0]}(t) \in \mathbb{R}$. We can take $a^{[2]}$ as a symmetric form. Then, simple application of Itô formula gives

$$\begin{pmatrix} Z_L(t) \\ \Gamma_L(t) \end{pmatrix} = \Sigma(t)[a^{[1]}(t) + a^{[2]}(t)\hat{z}_t] . \tag{5.4}$$

Substituting this result into (5.2), one obtains

$$\begin{aligned}dV_L(t) &= \Big\{\frac{1}{2}a^{[1]}(t)^\top\Xi(t)a^{[1]}(t) + \Big[\Big(a^{[2]}(t)\Xi(t) + 2\mathbf{1}_{(d,0)}\Sigma(t)\Big)a^{[1]}(t)\Big]^\top \hat{z}_t \\ &+\frac{1}{2}\hat{z}_t^\top\Big[2\mathbf{1}_{(d,0)} + a^{[2]}(t)\Xi(t)a^{[2]}(t) + 2\mathbf{1}_{(d,0)}\Sigma(t)a^{[2]}(t) + 2a^{[2]}(t)\Sigma(t)\mathbf{1}_{(d,0)}\Big]\hat{z}_t\Big\}dt \\ &+Z_L(t)^\top dN_t + \Gamma_L(t)^\top dM_t \ . \end{aligned} \tag{5.5}$$

Here, we have defined

$$\Xi(t) := (\Sigma_d^\top\Sigma_d)(t) - (\Sigma_m^\top\Sigma_m)(t) \tag{5.6}$$

and Σ_d (Σ_m) are $d\times n$ ($m\times n$) matrices obtained by restricting to the first d (last m) rows of $\Sigma(t)$, and $\mathbf{1}_{(d,0)}$ is the diagonal matrix which has 1 for the first d elements and 0 for all the others.

On the other hand, the dynamics of $\hat{z}$ in (3.26) and Itô formula yield

$$\begin{aligned}dV_L(t) &= \Big\{\dot{a}^{[0]}(t) + \mu^\top a^{[1]}(t) + \frac{1}{2}\mathrm{tr}(a^{[2]}(t)\Sigma^2(t)) \\ &+\Big[\dot{a}^{[1]}(t) - F^\top a^{[1]}(t) + a^{[2]}(t)\mu\Big]^\top\hat{z}_t \\ &+\frac{1}{2}\hat{z}_t^\top\Big[\dot{a}^{[2]}(t) - F^\top a^{[2]}(t) - a^{[2]}(t)F\Big]\hat{z}_t\Big\}dt \\ &+Z_L(t)^\top dN_t + \Gamma_L(t)^\top dM_t \ . \end{aligned} \tag{5.7}$$

Matching the coefficients of $(\hat{z}\hat{z},\hat{z})$ and a remaining constant term respectively, and using the fact that $V_L(T)=0$, one obtains the following ODEs [2]:

$$\begin{aligned}\dot{a}^{[2]}(t) &= 2\mathbf{1}_{(d,0)} + a^{[2]}(t)\Xi(t)a^{[2]}(t) \\ &\quad +F^\top a^{[2]}(t) + a^{[2]}(t)F + 2\Big(\mathbf{1}_{(d,0)}\Sigma(t)a^{[2]}(t) + a^{[2]}(t)\Sigma(t)\mathbf{1}_{(d,0)}\Big) \end{aligned} \tag{5.8}$$

$$\dot{a}^{[1]}(t) = -a^{[2]}(t)\mu + \Big[F^\top + a^{[2]}(t)\Xi(t) + 2\mathbf{1}_{(d,0)}\Sigma(t)\Big]a^{[1]}(t) \tag{5.9}$$

$$\dot{a}^{[0]}(t) = -\mu^\top a^{[1]}(t) - \frac{1}{2}\mathrm{tr}(a^{[2]}(t)\Sigma^2(t)) + \frac{1}{2}a^{[1]}(t)^\top\Xi(t)a^{[1]}(t) \tag{5.10}$$

with terminal conditions $a^{[2]}(T) = a^{[1]}(T) = a^{[0]}(T) = 0$.

The ODEs can be solved sequentially in ($a^{[2]}\to a^{[1]}\to a^{[0]}$) order. Due to the quadratic form, the existence of $a^{[2]}$ is not guaranteed and the detailed conditions are difficult to obtain due to its multi-dimensionality. However, it is clear that $a^{[2]}$ stays finite in $[0,T]$ unless the maturity is too long or the size of Ξ is very large. In any case, the behavior of $a^{[2]}$ can be easily checked by numerically solving the ODE. Once we assume its existence, it is clearly seen that (5.3) actually satisfies the BSDE by a standard application of Itô formula. In fact, this technique for a quadratic BSDE was already discussed in Schroder & Skiadas (1999) [16] in the application to a recursive utility, but to the best of our knowledge, it is the first time as the application to the MVH problem in Mania & Tevzadze approach.

[2]Put $\mu = F = 0$ and use the corresponding $\Sigma(t)$ for our first Bayesian model.

Remark:

It is instructive to apply the perturbative solution technique of FBSDEs proposed by Fujii & Takahashi (2012) [3] to (5.2). One can confirm that the V_L has a quadratic form of $\hat{z}$ and (Z_L, Γ_L) have a linear form of $\hat{z}$ at an arbitrary order of the perturbative expansion. This is actually how we have noticed the existence of a quadratic-form solution.

6 V_1 as a simple forward expectation of H

Since the BSDE for V_1 is linear

$$dV_1(t) = [Z_L(t) + \hat{\theta}_t]^\top [Z_1(t) + V_1(t)\hat{\theta}_t]dt + Z_1(t)^\top dN_t + \Gamma_1(t)^\top dM_t \tag{6.1}$$

with $V_1(T) = H$, it is clear that we have

$$V_1(t) = \mathbb{E}^{\mathcal{A}}\left[H \exp\Bigl(-\int_t^T \bigl[||\hat{\theta}_s||^2 + \hat{\theta}_s^\top Z_L(s) \bigr] ds \Bigr) \Big| \mathcal{G}_t \right] . \tag{6.2}$$

Here, the measure $\mathbf{P}^{\mathcal{A}}$ is defined by

$$\left. \frac{d\mathbf{P}^{\mathcal{A}}}{d\mathbf{P}} \right|_{\mathcal{G}_t} = \eta_t \tag{6.3}$$

where

$$\eta_t = \exp\Bigl(-\int_0^t [Z_L(s) + \hat{\theta}_s]^\top dN_s - \frac{1}{2}\int_0^t ||Z_L(s) + \hat{\theta}_s||^2 ds \Bigr) . \tag{6.4}$$

By the result of the previous section, $Z_L + \hat{\theta}$ is a linear Gaussian process and hence the above measure change can be justified, for example, by Lemma 3.9 in [1].

Now, let us evaluate

$$A(t,T) := \mathbb{E}^{\mathcal{A}}\left[\exp\Bigl(-\int_t^T \bigl[||\hat{\theta}_s||^2 + \hat{\theta}_s^\top Z_L(s) \bigr] ds \Bigr) \Big| \mathcal{G}_t \right] . \tag{6.5}$$

The argument of exp() has a quadratic Gaussian form and is given by

$$A(t,T) = \mathbb{E}^{\mathcal{A}}\left[\exp\Bigl(-\int_t^T \Bigl\{ \frac{1}{2}\hat{z}_s^\top b^{[2]}(s)\hat{z}_s + b^{[1]}(s)^\top \hat{z}_s \Bigr\} ds \Bigr) \Big| \mathcal{G}_t \right] \tag{6.6}$$

where $b^{[2]}(t) \in \mathbb{R}^{n\times n}$ and $b^{[1]}(t) \in \mathbb{R}^n$ are deterministic functions defined as

$$b^{[2]}(t) := 2\mathbf{1}_{(d,0)} + \mathbf{1}_{(d,0)}\Sigma(t)a^{[2]}(t) + a^{[2]}(t)\Sigma(t)\mathbf{1}_{(d,0)} \tag{6.7}$$

$$b^{[1]}(t) := \mathbf{1}_{(d,0)}\Sigma(t)a^{[1]}(t) . \tag{6.8}$$

One may notice that the problem is equivalent to the pricing of the zero-coupon bond in a quadratic Gaussian short rate model, and we in fact borrow the same technique below.

Let us focus on the Kalman-Bucy model. The result for the Bayesian model can be obtained by the simple parameter replacement as before. In the measure $\mathbf{P}^{\mathcal{A}}$, the MPR follows

$$d\hat{z}_t = [\varphi(t) + \kappa(t)\hat{z}_t]dt + \Sigma(t)dn_t^{\mathcal{A}} \tag{6.9}$$

where

$$\begin{aligned}\varphi(t) &:= \mu - (\Sigma_d^\top \Sigma_d)(t)a^{[1]}(t) \\ \kappa(t) &:= -\Big[F + (\Sigma_d^\top \Sigma_d)(t)a^{[2]}(t) + \Sigma(t)\mathbf{1}_{(d,0)}\Big]\end{aligned} \tag{6.10}$$

and $n_t^{\mathcal{A}}$ is the $(\mathbf{P}^{\mathcal{A}}, \mathcal{G})$-Brownian motion which is related to n_t by Girsanov's theorem as

$$n_t^{\mathcal{A}} = n_t + \int_0^t \mathbf{1}_{(d,0)}\Big[\Sigma(s)[a^{[1]}(s) + a^{[2]}(s)\hat{z}_s] + \hat{z}_s\Big]ds\ . \tag{6.11}$$

Let us suppose A is given in the following form [3]:

$$A(t,T) = \exp\left(\frac{1}{2}\hat{z}_t^\top c^{[2]}(t)\hat{z}_t + c^{[1]}(t)^\top \hat{z}_t + c^{[0]}(t)\right)\ . \tag{6.12}$$

with deterministic functions $\{c^{[i]}\}$ taking values in $c^{[2]}(t) \in \mathbb{R}^{n\times n}$, $c^{[1]}(t) \in \mathbb{R}^n$, $c^{[0]}(t) \in \mathbb{R}$. From (6.6), one sees that the dynamics of A is given by

$$dA(t,T) = A(t,T)\Big\{\frac{1}{2}\hat{z}_t^\top b^{[2]}(t)\hat{z}_t + b^{[1]}(t)^\top \hat{z}_t\Big\}dt + (\cdots)dn_t^{\mathcal{A}}\ , \tag{6.13}$$

but from (6.12) and the dynamics of $\hat{z}$ tell us that

$$\begin{aligned}&dA(t,T) \\ &= A(t,T)\Big\{\frac{1}{2}\hat{z}_t^\top\big[\dot{c}^{[2]}(t) + c^{[2]}(t)\kappa(t) + \kappa(t)^\top c^{[2]}(t) + c^{[2]}(t)\Sigma^2(t)c^{[2]}(t)\big]\hat{z}_t \\ &\quad + \big[\dot{c}^{[1]}(t) + \kappa(t)^\top c^{[1]}(t) + c^{[2]}(t)\varphi(t) + c^{[2]}(t)\Sigma^2(t)c^{[1]}(t)\big]^\top \hat{z}_t \\ &\quad + \big[\dot{c}^{[0]}(t) + \varphi(t)^\top c^{[1]}(t) + \frac{1}{2}\mathrm{tr}\big(c^{[2]}(t)\Sigma^2(t)\big) + \frac{1}{2}c^{[1]}(t)^\top\Sigma^2(t)c^{[1]}(t)\big]\Big\}dt + (\cdots)dn_t^{\mathcal{A}}\ .\end{aligned} \tag{6.14}$$

Therefore, one can see that the solution of A is given by the form (6.12) if and only if $\{c^{[i]}\}$ solve the following ODEs:

$$\dot{c}^{[2]}(t) = b^{[2]}(t) - c^{[2]}(t)\kappa(t) - \kappa(t)^\top c^{[2]}(t) - c^{[2]}(t)\Sigma^2(t)c^{[2]}(t) \tag{6.15}$$

$$\dot{c}^{[1]}(t) = b^{[1]}(t) - \kappa(t)^\top c^{[1]}(t) - c^{[2]}(t)\varphi(t) - c^{[2]}(t)\Sigma^2(t)c^{[1]}(t) \tag{6.16}$$

$$\dot{c}^{[0]}(t) = -\varphi(t)^\top c^{[1]}(t) - \frac{1}{2}\mathrm{tr}\big(c^{[2]}(t)\Sigma^2(t)\big) - \frac{1}{2}c^{[1]}(t)^\top\Sigma^2(t)c^{[1]}(t) \tag{6.17}$$

with the terminal conditions $c^{[2]}(T) = c^{[1]}(T) = c^{[0]}(T) = 0$. Numerical evaluation can be easily performed in ($c^{[2]} \to c^{[1]} \to c^{[0]}$) order. The solutions of the ODEs have the same

[3] The argument T is omitted in $\{c^{[i]}\}$ for notational simplicity.

problem for their existence due to the quadratic term of $c^{[2]}$ as in the case for $a^{[2]}$. In the remainder, let us suppose that there exists a finite solution for $(c^{[2]}, c^{[1]}, c^{[0]})$ in $[0,T]$ for a given parameter set, which can be checked numerically in any case.

If there exists a solution for $\{c^{[i]}\}$, we can define a very useful forward measure $\mathbf{P}^{\mathcal{A}_T}$ by

$$\left.\frac{d\mathbf{P}^{\mathcal{A}_T}}{d\mathbf{P}^{\mathcal{A}}}\right|_{\mathcal{G}_t} = \frac{A(t,T)}{A(0,T)\exp\Bigl(\int_0^t \bigl[||\hat{\theta}_s||^2 + \hat{\theta}_s^\top Z_L(s)\bigr]\,ds\Bigr)} \tag{6.18}$$

under which standard Brownian motion is given by the relation

$$\begin{aligned} n_t^{\mathcal{A}_T} &= n_t + \int_0^t \mathbf{1}_{(d,0)}\Bigl\{\Sigma(s)[a^{[1]}(s) + a^{[2]}(s)\hat{z}_s] + \hat{z}_s\Bigr\}ds \\ &\quad - \int_0^t \Sigma(s)[c^{[1]}(s) + c^{[2]}(s)\hat{z}_s]ds \end{aligned} \tag{6.19}$$

by Girsanov's theorem. Using this measure, one can now express V_1 in a very simple fashion:

$$V_1(t) = A(t,T)\mathbb{E}^{\mathcal{A}_T}\Bigl[H\Bigl|\mathcal{G}_t\Bigr] \ . \tag{6.20}$$

7 Monte Carlo Method

In this section, we consider how to evaluate (V_1, Z_1) and V_0 by Mote Carlo simulation. Although V_0 is not necessary for the specification of the optimal hedging position, it is needed to obtain the optimal value function $V(t,w)$ as well as the *mean-variance price* of H that is the w minimizing $V(t,w)$ for a given terminal liability H.

For notational simplicity, let us put

$$X_t := \begin{pmatrix} S_t \\ Y_t \end{pmatrix} \tag{7.1}$$

$$\gamma(t,X_t) := \begin{pmatrix} \sigma(t,X_t) & 0 \\ \bar{\sigma}(t,X_t) & \rho(t,X_t) \end{pmatrix}, \tag{7.2}$$

then, the relevant dynamics under $(\mathbf{P},\mathcal{G})$ can be written as

$$dX_t = \gamma(t,X_t)[dn_t + \hat{z}_t dt] \ . \tag{7.3}$$

In the forward measure $(\mathbf{P}^{\mathcal{A}_T},\mathcal{G})$, it becomes

$$dX_t = \gamma(t,X_t)\Bigl\{dn_t^{\mathcal{A}_T} + \bigl[\psi(t) + \Psi(t)\hat{z}_t\bigr]dt\Bigr\} \tag{7.4}$$

where ψ and Ψ are deterministic functions given below:

$$\psi(t) := \Sigma(t)c^{[1]}(t) - \mathbf{1}_{(d,0)}\Sigma(t)a^{[1]}(t) \tag{7.5}$$

$$\Psi(t) := \mathbf{1}_{(0,m)} + \Sigma(t)c^{[2]}(t) - \mathbf{1}_{(d,0)}\Sigma(t)a^{[2]}(t) \ . \tag{7.6}$$

Similarly, the dynamics of $\hat{z}$ in $(\mathbf{P}^{\mathcal{A}_T},\mathcal{G})$ is given by

$$d\hat{z}_t=\Big[\phi(t)-\Phi(t)\hat{z}_t\Big]dt+\Sigma(t)dn_t^{\mathcal{A}_T} \tag{7.7}$$

with deterministic functions (ϕ,Φ):

$$\begin{aligned}\phi(t) &:= \mu-(\Sigma_d^\top\Sigma_d)(t)a^{[1]}(t)+\Sigma^2(t)c^{[1]}(t) &(7.8)\\ \Phi(t) &:= F+(\Sigma_d^\top\Sigma_d)(t)a^{[2]}(t)+\Sigma(t)\mathbf{1}_{(d,0)}-\Sigma^2(t)c^{[2]}(t)\ . &(7.9)\end{aligned}$$

In the remainder, we consider a situation where the terminal liability H is given by some function of X_T, i.e.,

$$H=H(X_T)\ . \tag{7.10}$$

7.1 Evaluation of (V_1,Z_1)

Of course, the evaluation of

$$V_1(t)=A(t,T)\mathbb{E}^{\mathcal{A}_T}\Big[H(X_T)\Big|\mathcal{G}_t\Big] \tag{7.11}$$

can be performed by simply running $(X_t,\hat{z}_t)$ under $(\mathbf{P}^{\mathcal{A}_T},\mathcal{G})$ in standard simulation.

For the evaluation of Z_1, we need to introduce the three *stochastic flows*, $(\xi_{t,u},\chi_{t,u},\widetilde{\chi}_{t,u})$. They are associated with the sensitivity of the values $\hat{z}_u$ and X_u at certain future time $u\ (>t)$ against the small changes of their initial values at time t. The first one is defined as, for $1\leq i,j\leq n$,

$$(\xi_{t,u})_{i,j}:=\frac{\partial\hat{z}_u^j(t,\hat{z})}{\partial\hat{z}^i} \tag{7.12}$$

and is actually given as the solution of the following ODE:

$$\frac{d\xi_{t,u}}{du}=-\xi_{t,u}\Phi^\top(u),\qquad(\xi_{t,t})_{i,j}=\delta_{i,j}\ . \tag{7.13}$$

Here, the notation $\hat{z}_u(t,\hat{z})$ emphasizes that $\hat{z}_u$ started from the value $\hat{z}$ at time t.

The next two quantities are similarly defined as

$$(\chi_{t,u})_{i,j}:=\frac{\partial X_u^j(t,x,z)}{\partial x^i},\qquad(\widetilde{\chi}_{t,u})_{i,j}:=\frac{\partial X_u^j(t,x,z)}{\partial\hat{z}^i}\ . \tag{7.14}$$

The three arguments $(t,x,\hat{z})$ indicate that X stated from x at time t but its future value X_u also depends on the value of $\hat{z}$ at time t. One can show that they follow the SDEs

$$\begin{aligned}d(\chi_{t,u})_{i,j}&=(\chi_{t,u})_{i,k}\partial_k\gamma_j(u,X_u)\Big\{dn_u^{\mathcal{A}_T}+[\psi(u)+\Psi(u)\hat{z}_u]du\Big\} &(7.15)\\ d(\widetilde{\chi}_{t,u})_{i,j}&=(\widetilde{\chi}_{t,u})_{i,k}\partial_k\gamma_j(u,X_u)\Big\{dn_u^{\mathcal{A}_T}+[\psi(u)+\Psi(u)\hat{z}_u]du\Big\}\\ &\quad+(\gamma(u,X_u)\Psi(u))_{j,k}(\xi_{t,u}^\top)_{k,i}du &(7.16)\end{aligned}$$

with initial conditions $(\chi_{t,t})_{i,j}=\delta_{i,j}$ and $\widetilde{\chi}_{t,t}=0$, respectively. In the above equations, and also in the reminder of the paper, we will often use the so-called *Einstein convention* which assumes the summation of the duplicated indexes. For example, (7.15) should be understood to involve $\sum_{k=1}^{n}$.

Using the above stochastic flows, one obtains

$$\begin{pmatrix} Z_1(t) \\ \Gamma_1(t) \end{pmatrix} = V_1(t)\Big(\Sigma(t)[c^{[1]}(t)+c^{[2]}(t)\hat{z}_t]\Big)$$
$$+A(t,T)\Big\{\mathbb{E}^{\mathcal{A}_T}\left[(\chi_{t,T})_{i,j}\partial_j H(X_T)|\mathcal{G}_t\right]\gamma_i(t,X_t)+\mathbb{E}^{\mathcal{A}_T}\left[(\widetilde{\chi}_{t,T})_{i,j}\partial_j H(X_T)|\mathcal{G}_t\right]\Sigma_i(t)\Big\}. \tag{7.17}$$

Thus, the simulation of those stochastic flows alongside of the original underlyings $(X,\hat{z})$ provides us the wanted quantity.

Remark: Calculation of Z_1 from Delta sensitivity

In the previous formulation, we have introduced the stochastic flows. This complication is not avoidable in order to make a *one-shot* Monte Carlo simulation possible for the evaluation of V_0 which will be explained in the next section. However, if one only needs the hedging position at time t (and if the dimension n is not too large), we can take a much simpler approach. As one can imagine from the definitions of the stochastic flows, the second line of (7.17) can also be estimated by the usual "Delta" sensitivity of the terminal liability:

$$\mathbb{E}^{\mathcal{A}_T}[(\chi_{t,T})_{i,j}\partial_j H(X_T)|\mathcal{G}_t] = \frac{\partial}{\partial x^i}\mathbb{E}^{\mathcal{A}_T}[H(X_T)|\mathcal{G}_t]$$
$$\mathbb{E}^{\mathcal{A}_T}[(\widetilde{\chi}_{t,T})_{i,j}\partial_j H(X_T)|\mathcal{G}_t] = \frac{\partial}{\partial \hat{z}^i}\mathbb{E}^{\mathcal{A}_T}[H(X_T)|\mathcal{G}_t]\ . \tag{7.18}$$

Thus, the required simulations to obtain (V_1,Z_1) are only those for the estimations of the terminal liability $H(X_T)$ and its *Delta* sensitivities against the underlyings $(X,\hat{z})$ in $(\mathbf{P}^{\mathcal{A}_T},\mathcal{G})$ measure.

7.2 Evaluation of V_0

Let us define, for $(t<s<T)$ and $(1\le r\le n)$,

$$\mathcal{Z}_s(X_T,\chi_{s,T},\widetilde{\chi}_{s,T}) := A(s,T)H(X_T)\Big\{\Sigma(s)[c^{[1]}(s)+c^{[2]}(s)\hat{z}_s]+\hat{z}_s\Big\}$$
$$+A(s,T)\Big\{(\chi_{s,T})_{i,j}\partial_j H(X_T)\gamma_i(s,X_s)+(\widetilde{\chi}_{s,T})_{i,j}\partial_j H(X_T)\Sigma_i(s)\ \Big\}. \tag{7.19}$$

We also put

$$\zeta_1(s) := \begin{pmatrix} Z_1(s) \\ \Gamma_1(s) \end{pmatrix} \tag{7.20}$$

for a lighter notation. Then, it is easy to confirm that

$$\zeta_1(s)+V_1(s)\hat{z}_s = \mathbb{E}^{\mathcal{A}_T}\left[\mathcal{Z}_s(X_T,\chi_{s,T},\widetilde{\chi}_{s,T})\Big|\mathcal{G}_s\right]\ . \tag{7.21}$$

Note that the Radon-Nikodym derivative between $\mathbf{P}^{\mathcal{A}_T}$ and $\mathbf{P}$ conditional on $\mathcal{G}_t$ is given by

$$\begin{aligned} L_t &:= \left.\frac{d\mathbf{P}^{\mathcal{A}_T}}{d\mathbf{P}}\right|_{\mathcal{G}_t} \\ &= \exp\left(\int_0^t [G(s)+K(s)\hat{z}_s]dn_s - \frac{1}{2}\int_0^t ||G(s)+K(s)\hat{z}_s||^2 ds\right) \end{aligned} \tag{7.22}$$

where G and K are the deterministic functions defined as

$$G(t) := \Sigma(t)c^{[1]}(t) - \mathbf{1}_{(d,0)}\Sigma(t)a^{[1]}(t) \tag{7.23}$$

$$K(t) := \Sigma(t)c^{[2]}(t) - \mathbf{1}_{(d,0)}\Sigma(t)a^{[2]}(t) - \mathbf{1}_{(d,0)} \ . \tag{7.24}$$

Then the inverse relation is given by

$$\begin{aligned} L_t^{-1} &= \left.\frac{d\mathbf{P}}{d\mathbf{P}^{\mathcal{A}_T}}\right|_{\mathcal{G}_t} \\ &= \exp\left(-\int_0^t [G(s)+K(s)\hat{z}_s]dn_s^{\mathcal{A}_T} - \frac{1}{2}\int_0^t ||G(s)+K(s)\hat{z}_s||^2 ds\right) \ . \end{aligned} \tag{7.25}$$

Since V_0 follows a linear BSDE, it is easy to see that V_0 satisfies

$$V_0(t) = \mathbb{E}\left[H^2(X_T) - \int_t^T e^{-V_L(s)}[\zeta_1(s)+V_1(s)\hat{z}_s]^\top \mathbf{1}_{(d,0)}[\zeta_1(s)+V_1(s)\hat{z}_s]ds\middle|\mathcal{G}_t\right] \ . \tag{7.26}$$

Changing the measure to $\mathbf{P}^{\mathcal{A}_T}$, one can express it as

$$\begin{aligned} V_0(t) = L_t\mathbb{E}^{\mathcal{A}_T}\Big[L_T^{-1}H^2(X_T) - \int_t^T L_s^{-1}e^{-V_L(s)}\mathbb{E}^{\mathcal{A}_T}\left[\mathcal{Z}_s(X_T,\chi_{s,T},\widetilde{\chi}_{s,T})|\mathcal{G}_s\right]^\top \\ \times\mathbf{1}_{(d,0)}\mathbb{E}^{\mathcal{A}_T}\left[\mathcal{Z}_s(X_T,\chi_{s,T},\widetilde{\chi}_{s,T})|\mathcal{G}_s\right]ds\Big|\mathcal{G}_t\Big]. \end{aligned} \tag{7.27}$$

Unfortunately, the naive evaluation of the above expression requires sequential Monte Carlo simulations and seems numerically too burdensome to be useful in practice.

However, there is a nice way called a *particle method* to compress convoluted expectations. The method describes a physical system where multiple copies of particles are created at random interaction times following Poisson law. After the creation, the particles belonging to a common specie follow the same probability law but are driven by independent Brownian motions. This idea was introduced by McKean (1975) [13] to solve a certain type of semilinear PDE and has been applied to various research areas since then.

For the current problem (7.27), let us introduce a deterministic intensity λ_t and denote the corresponding random interaction time by τ. Then, $V_0(t)$ can be represented by

$$\begin{aligned} &V_0(t) = L_t\mathbb{E}^{\mathcal{A}_T}\left[L_T^{-1}H^2(X_T)\middle|\mathcal{G}_t\right] \\ &-\mathbf{1}_{\{\tau>t\}}L_t\mathbb{E}^{\mathcal{A}_T}\Big[\mathbf{1}_{\{t<\tau<T\}}L_\tau^{-1}e^{-V_L(\tau)+\int_t^\tau \lambda_u du} \\ &\times\frac{1}{\lambda_\tau}\Big(\mathcal{Z}_\tau(X_T,\chi_{\tau,T},\widetilde{\chi}_{\tau,T})\Big)^{p=1}\mathbf{1}_{(d,0)}\Big(\mathcal{Z}_\tau(X_T,\chi_{\tau,T},\widetilde{\chi}_{\tau,T})\Big)^{p=2}\Big|\mathcal{G}_t\Big] \ . \end{aligned} \tag{7.28}$$

Here, the underlyings (or "particles") $(X, \hat{z}, \chi, \widetilde{\chi})$ belong to either the group $(p = 1)$ or $(p = 2)$, and they follow the SDEs having the same form (7.4), (7.7), (7.15) and (7.16) respectively, but driven by two independent n-dimensional Brownian motions $n^{\mathcal{A}_T}(p = 1)$ and $n^{\mathcal{A}_T}(p = 2)$. This particle representation allows a one-shot non-sequential Monte Carlo simulation. See Fujii & Takahashi (2012) [4] for the details of the particle method as a solution technique for BSDEs, and also Fujii et.al.(2012) [5] as a concrete application to the pricing of American options.

As long as there exist solutions for $\{a^{[i]}\}$ and $\{c^{[i]}\}$, the explained procedures allow us to obtain the solutions for the three BSDEs given in Sec. 4 under a quite general setup. However, it may be tough to update the hedging positions in timely manner in a volatile market, and in addition, it seems almost impossible to analyze the terminal distribution of the hedged portfolio, which may be important for financial firms from a risk-management perspective, by simulating (4.7) in the current approach. In the remainder of the paper, we give an explicitly solvable example and then an asymptotic expansion method to answer this issue.

8 A simple solvable example

In this section, we consider a solvable case where the terminal liability depends only on a non-tradable index $Y^I \in \{Y\}^{obs}$

$$H(X_T) = Y^I_T \ . \tag{8.1}$$

Let us suppose that $\gamma^I(t, X_t) = Y^I_t \sigma_y^\top$ where $\sigma_y \in \mathbb{R}^n$ is a n-dimensional constant vector. Then from (7.4), the index's dynamics under $(\mathbf{P}^{\mathcal{A}_T}, \mathcal{G})$ can be written as

$$dY^I_s = Y^I_s \sigma_y^\top \left[\psi(s) + \Psi(s)\hat{z}_s\right] ds + Y^I_s \sigma_y^\top dn^{\mathcal{A}_T}_s \ . \tag{8.2}$$

In order to get V_1, it is enough to evaluate

$$\mathbb{E}^{\mathcal{A}_T}\left[Y^I_T | \mathcal{G}_t\right] = Y^I_t \mathbb{E}^{\mathcal{A}_T}\left[\exp\Bigl(\int_t^T \sigma_y^\top \left[\psi(s) + \Psi(s)\hat{z}_s\right] ds\Bigr)\Big|\mathcal{G}_t\right] \ . \tag{8.3}$$

Since it has an affine structure, one can evaluate the above expectation by the same method used for the evaluation of $A(t, T)$. One can show that

$$P(t, T) := \mathbb{E}^{\mathcal{A}_T}\left[\exp\Bigl(\int_t^T \sigma_y^\top \left[\psi(s) + \Psi(s)\hat{z}_s\right] ds\Bigr)\Big|\mathcal{G}_t\right] \tag{8.4}$$

can be written by the deterministic functions $(\beta^{[1]}(t) \in \mathbb{R}^n, \beta^{[0]}(t) \in \mathbb{R})$ and $\hat{z}_t$ as

$$P(t, T) = \exp\Bigl(\beta^{[1]}(t)^\top \hat{z}_t + \beta^{[0]}(t)\Bigr) \tag{8.5}$$

where $\{\beta^{[i]}\}$ solve the following ODEs:

$$\dot{\beta}^{[1]}(t) = \Phi(t)^\top \beta^{[1]}(t) - \Psi(t)^\top \sigma_y \tag{8.6}$$

$$\dot{\beta}^{[0]}(t) = -\phi(t)^\top \beta^{[1]}(t) - \frac{1}{2}\beta^{[1]}(t)^\top \Sigma^2(t)\beta^{[1]}(t) - \psi(t)^\top \sigma_y \tag{8.7}$$

with terminal conditions $\beta^{[1]}(T) = \beta^{[0]}(T) = 0$.

Now, from the above arguments, one obtains

$$V_1(t) = Y_t^I A(t,T) P(t,T) \; . \tag{8.8}$$

A simple application of Itô formula gives

$$\begin{pmatrix} Z_1(t) \\ \Gamma_1(t) \end{pmatrix} = V_1(t) \Big\{ \sigma_y + \Sigma(t) \big[c^{[1]}(t) + \beta^{[1]}(t) + c^{[2]}(t) \hat{z}_t \big] \Big\} \; . \tag{8.9}$$

Once we calculate and store all the relevant deterministic functions, it is straightforward to evaluate V_0 from

$$V_0(t) = \mathbb{E}\left[(Y_T^I)^2 - \int_t^T \frac{||Z_1(s) + V_1(s)\hat{\theta}_s||^2}{V_2(s)} ds \Big| \mathcal{G}_t \right] \tag{8.10}$$

by standard Monte Carlo simulation.

8.1 A numerical test using the solvable example

Let us provide an interesting numerical example which tests the consistency of our procedures. In this solvable example, we can directly run the optimal wealth process $\mathcal{W}_t^{\pi^*}$ given in (4.7). Thus, it is possible to compare $V(0,w) = w^2 V_2(0) - 2wV_1(0) + V_0(0)$, which is obtained by the ODEs and a standard Monte Carlo simulation for (8.10), with $\mathbb{E}[(Y_T^I - \mathcal{W}_T^{\pi^*})^2]$ directly obtained by running the simulation for Y^I and $\mathcal{W}^{\pi^*}$.

Let us use the following parameters with $(n = 3, \; d = 2)$: [4]

$$z_0 = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.1 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0.06 \\ 0.06 \\ 0.02 \end{pmatrix}, \quad F = \begin{pmatrix} 0.2 & 0.07 & 0.05 \\ 0.07 & 0.2 & 0.03 \\ 0.05 & 0.03 & 0.2 \end{pmatrix}$$
$$\delta = \begin{pmatrix} 0.3 & 0.15 & -0.1 \\ 0.15 & 0.3 & -0.08 \\ -0.03 & -0.07 & 0.3 \end{pmatrix}, \quad \Sigma_0 = \begin{pmatrix} 0.2 & 0.1 & -0.01 \\ 0.1 & 0.2 & -0.05 \\ -0.01 & -0.05 & 0.2 \end{pmatrix} \tag{8.11}$$

and

$$\sigma_y^\top = (-0.07, -0.12, 0.27) \tag{8.12}$$

with initial value $Y_0^I = 1$.

For $T = 0.5$, we have obtained $V_2(0) = 0.9263, V_1(0) = 0.9399$ by numerically solving ODEs, and $V_0(0) = 0.9974$ after $(100{,}000 + 100{,}000$ antipathetic$)$ paths with step size $dt = 2 \times 10^{-3}$. The standard error for V_0 simulation is about 4×10^{-4}. In Fig. 1, we have compared the quadratic form of $V(0,w)$ to the results of direct simulation of hedged portfolio with various initial capitals with the same number of paths and step size for the evaluation of V_0. The standard error for the portfolio simulation is less than 4×10^{-4}.

[4] Here, we put $p = 3$ in (3.24). But the choice is free and what only matters for the dynamics of $\hat{z}$ is $(\delta\delta^\top) \in \mathbb{R}^{n \times n}$.

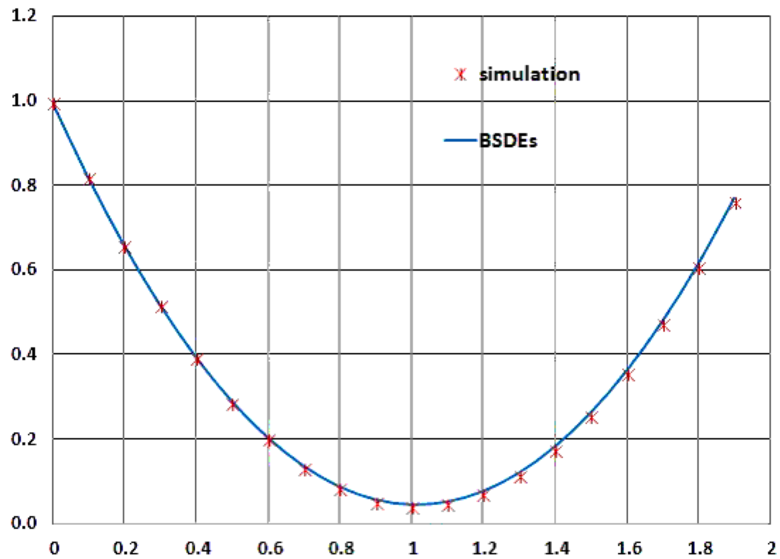

Figure 1: Comparison of $V(0, w) = w^2 V_2(0) - 2wV_1(0) + V_0(0)$ and direct simulation of $\mathbb{E}(Y_T^I - \mathcal{W}_T^{\pi^*})^2$. The solid line is based on the quadratic form of $V(0, w)$ and $\{*\}$ marks are those obtained from the direct simulation of wealth. The horizontal axis denotes the size of initial capital w.

One can see that the prediction of the BSDEs matches very well with the result of the direct simulation of the hedged portfolio.

One can also study the terminal distribution of the hedged portfolio: $(Y_T^I - \mathcal{W}_T^{\pi^*})$. In Fig. 2, we have plotted the terminal distribution of $(Y_T^I - \mathcal{W}_T^{\pi^*})$ for the five choices of the initial capital $w = \{0,\ 0.5,\ 1,\ 1.5,\ 2\}$. The graphs are obtained by connecting the histograms after sampling $400,000$ scenarios with the same parameters used to obtain Fig. 1. One can see distributions of the hedged portfolios change consistently with the result of Fig. 1 and achieves the smallest variance at $w = 1$ scenario among the five choices.

9 An asymptotic expansion method

Although it is impossible to obtain a closed-form solution for

$$V_1(t) = A(t, T)\mathbb{E}^{\mathcal{A}_T}\big[H(X_T)|\mathcal{G}_t\big] \tag{9.1}$$

in general, its evaluation is clearly equivalent to solving a European contingent claim. Thus, one can borrow various techniques developed for the pricing of financial derivatives from the vast existing literatures. Here, we adopt an asymptotic expansion method to obtain explicit approximate expressions. See, for example, [18, 17, 19] and references therein for the details of the method. In those works, the terminal probability distribution of the underlying process is estimated, which is then applied to a generic payoff function to price an interested contingent claim. In this article, however, we adopt a slightly simplified approach in which the asymptotic expansion is directly applied to the terminal

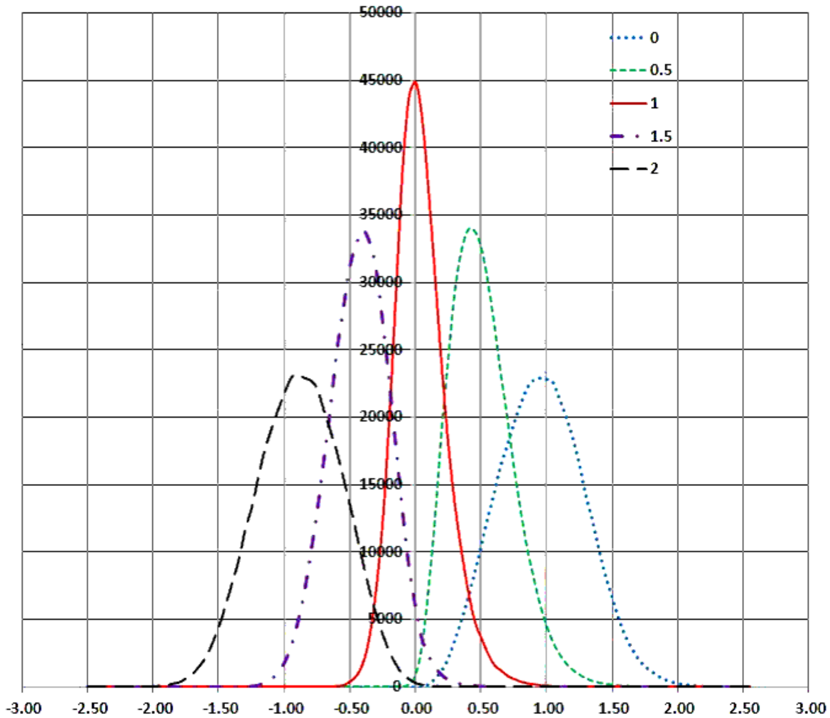

Figure 2: The terminal distribution of $(Y_T^I - \mathcal{W}_T^{\pi^*})$ for the five choices of the initial capital $w = \{0,\ 0.5,\ 1,\ 1.5,\ 2\}$. The graphs are obtained by connecting the histograms after sampling $400,000$ paths.

payoff by assuming $H(x)$ is a smooth function of x. If necessary, we can also apply the original method in [18, 17, 19] to the current problem, but the resultant formula and required calculation would be more involved. We also assume time-homogeneous volatility structure $\gamma(X_t)$ without explicit dependence on t for simplicity.

9.1 Approximation scheme

Firstly, let us introduce an auxiliary parameter ϵ and ϵ-dependent processes

$$\begin{aligned} dX_s^\epsilon &= \epsilon\gamma(X_s^\epsilon)\mathbf{1}_{(0,m)}\hat{z}_s^\epsilon ds + \epsilon\gamma(X_s^\epsilon)dn_s^{\mathcal{A}_T} \\ &\quad +\epsilon^2\gamma(X_s^\epsilon)[\psi(s) + \widetilde{\Psi}(s)\hat{z}_s^\epsilon]ds \qquad (9.2) \\ d\hat{z}_s^\epsilon &= \epsilon\big[\phi(s) - \Phi(s)\hat{z}_s^\epsilon\big]ds + \epsilon\Sigma(s)dn_s^{\mathcal{A}_T}\ , \qquad (9.3) \end{aligned}$$

where

$$\widetilde{\Psi}(s) := \Sigma(s)c^{[2]}(s) - \mathbf{1}_{(d,0)}\Sigma(s)a^{[2]}(s) \qquad (9.4)$$

is a deterministic function [5]. The idea behind this setup is to assume

$$\Sigma(s),\ \gamma(x),\ \mu,\ F \tag{9.5}$$

have small enough sizes relative to $\mathbf{1}$. Then, the auxiliary parameter ϵ is introduced to count the order of those small quantities appearing in the expansion. Since Ψ contains $\mathbf{1}_{(0,m)}$, the remaining small term is extracted as $\widetilde{\Psi}$ in (9.4).

Suppose, we have expanded the ϵ-dependent process X^ϵ as a power series of ϵ in the following form:

$$X_s^\epsilon = X_s^{(0)} + \epsilon X_s^{(1)} + \epsilon^2 X_s^{(2)} + \cdots \tag{9.6}$$

where

$$X_s^{(k)} := \frac{1}{k!}\frac{\partial^k X_s^\epsilon}{\partial \epsilon^k}\bigg|_{\epsilon=0} \tag{9.7}$$

Since the term $X^{(k)}$ contains the k-th order products of small quantities in (9.5), higher order terms in (9.6) can be naturally neglected for the approximation purpose. Putting $\epsilon = 1$ at the end of calculation provides an approximate valuation for the original process X. In the current work, we will provide the formula for (V_1, Z_1) up to the third order contribution. The accuracy of approximation is, of course, determined by the size of the quantities given in (9.5) [6]. As we can see in the numerical examples provided later in the paper, the scheme seems to work well with realistic parameters, at least for relatively short maturities.

9.2 Asymptotic expansions of the underlying processes

Let us consider the expansion of $(X_s^\epsilon, \hat{z}_s^\epsilon)$ for $(s > t)$ under the given condition at t. Obviously, we have

$$X_s^{(0)} \equiv x \tag{9.8}$$
$$\hat{z}_s^{(0)} \equiv \hat{z} \tag{9.9}$$

with the conventions that $x := X_t^\epsilon$ and $\hat{z} := \hat{z}_t^\epsilon$.

Assuming $\gamma(x)$ is smooth enough, one can easily derive

$$dX_s^{(1)} = \gamma(x)\mathbf{1}_{(0,m)}\hat{z}ds + \gamma(x)dn_s^{\mathcal{A}_T} \tag{9.10}$$

$$\begin{aligned} dX_s^{(2)} = &\Big\{X_s^{i,(1)}\partial_i\gamma(x)\mathbf{1}_{(0,m)}\hat{z} + \gamma(x)\mathbf{1}_{(0,m)}\hat{z}_s^{(1)} + \gamma(x)[\psi(s) + \widetilde{\Psi}(s)\hat{z}]\Big\}ds \\ &+X_s^{i,(1)}\partial_i\gamma(x)dn_s^{\mathcal{A}_T} \end{aligned} \tag{9.11}$$

$$\begin{aligned} dX_s^{(3)} = &\Big\{\big[X_s^{i,(2)}\partial_i\gamma(x) + \frac{1}{2}X_s^{i,(1)}X_s^{j,(1)}\partial_{i,j}\gamma(x)\big]\mathbf{1}_{(0,m)}\hat{z} \qquad (9.12) \\ &+X_s^{i,(1)}\partial_i\gamma(x)\mathbf{1}_{(0,m)}\hat{z}_s^{(1)} + \gamma(x)\mathbf{1}_{(0,m)}\hat{z}_s^{(2)} \\ &+X_s^{i,(1)}\partial_i\gamma(x)[\psi(s) + \widetilde{\Psi}(s)\hat{z}] + \gamma(x)\widetilde{\Psi}(s)\hat{z}_s^{(1)}\Big\}ds \\ &+\big[X_s^{i,(2)}\partial_i\gamma(x) + \frac{1}{2}\partial_{i,j}\gamma(x)\big]dn_s^{\mathcal{A}_T} \end{aligned} \tag{9.13}$$

[5] In theory, there is no need to expand $\hat{z}$ by introducing ϵ since it already has a linear dynamics. However, if one treats $\hat{z}$ exactly, the calculations associated with X become hugely involved due to the presence of $\hat{z}$ in its drift process mostly likely with only a minor improvement of accuracy.

[6] More precisely speaking, we need to consider the effect of time-integration together.

with initial conditions $X_t^{(i)} = 0$ for $i \in \{1,2,3\}$. Similarly, for $\hat{z}_s^{(i)}$, one obtains

$$d\hat{z}_s^{(1)} = [\phi(s) - \Phi(s)\hat{z}]ds + \Sigma(s)dn_s^{\mathcal{A}_T} \tag{9.14}$$
$$d\hat{z}_s^{(2)} = -\Phi(s)\hat{z}_s^{(1)}ds \tag{9.15}$$
$$d\hat{z}_s^{(3)} = -\Phi(s)\hat{z}_s^{(2)}ds \tag{9.16}$$

with $\hat{z}_t^{(i)} = 0$ for $i \in \{1,2,3\}$.

9.2.1 Approximation of V_1

Under the assumption that $H(x)$ is smooth enough, one can expand it as

$$\mathbb{E}^{\mathcal{A}_T}\left[H(X_T^\epsilon)|\mathcal{G}_t\right]$$
$$= H(x) + \epsilon\partial_i H(x)\mathbb{E}^{\mathcal{A}_T}\left[X_T^{i,(1)}|\mathcal{G}_t\right] \tag{9.17}$$
$$+\epsilon^2\Big\{\partial_i H(x)\mathbb{E}^{\mathcal{A}_T}\left[X_T^{i,(2)}|\mathcal{G}_t\right] + \frac{1}{2}\partial_{i,j}H(x)\mathbb{E}^{\mathcal{A}_T}\left[X_T^{i,(1)}X_T^{j,(1)}|\mathcal{G}_t\right]\Big\} \tag{9.18}$$
$$+\epsilon^3\Big\{\partial_i H(x)\mathbb{E}^{\mathcal{A}_T}\left[X_T^{i,(3)}|\mathcal{G}_t\right] + \partial_{i,j}H(x)\mathbb{E}^{\mathcal{A}_T}\left[X_T^{i,(2)}X_T^{i,(1)}|\mathcal{G}_t\right]$$
$$+\frac{1}{6}\partial_{i,j,k}H(x)\mathbb{E}^{\mathcal{A}_T}\left[X_T^{i,(1)}X_T^{j,(1)}X_T^{k,(1)}|\mathcal{G}_t\right]\Big\} + \mathcal{O}(\epsilon^4). \tag{9.19}$$

Since $A(t,T)$ is already available as a solution of the ODEs, one only needs the expectations of $\{X_T^{(i)}\}$ and their cross products to obtain an analytic expression of $V_1(t)$. This is actually calculable because all the $\{X^{(i)}\}$ have linear dynamics thanks to the way we have introduced ϵ in (9.2) and (9.3). Once this is done, $Z_1(t)$ can be easily derived by the simple application of Itô formula.

Let us put

$$g(x,\hat{z}) := \gamma(x)\mathbf{1}_{(0,m)}\hat{z} \quad \in \mathbb{R}^n \tag{9.20}$$

and a shorthand notation of a time integration, such as

$$[f]_t^T := \int_t^T f(s)ds$$
$$\left[[f]_t^s\right]_t^T := \int_t^T\Big(\int_t^s f(u)du\Big)ds$$
$$\cdots \tag{9.21}$$

to lighten the expressions. From the application of Itô formula, we can obtain all the necessary expectations as follows:

$$\mathbb{E}^{\mathcal{A}_T}\left[X_T^{(1)}|\mathcal{G}_t\right] = (T-t)g(x,\hat{z})$$
$$\mathbb{E}^{\mathcal{A}_T}\left[X_T^{(2)}|\mathcal{G}_t\right] = \frac{1}{2}(T-t)^2\partial_i g(x,\hat{z})g^i(x,\hat{z}) + \gamma(x)\Big([\psi]_t^T + \mathbf{1}_{(0,m)}\left[[\phi]_t^s\right]_t^T\Big)$$
$$+\gamma(x)\Big([\widetilde{\Psi}]_t^T - \mathbf{1}_{(0,m)}\left[[\Phi]_t^s\right]_t^T\Big)\hat{z}$$
$$\mathbb{E}^{\mathcal{A}_T}\left[X_T^{i,(1)}X_T^{j,(1)}|\mathcal{G}_t\right] = (T-t)^2 g^i(x,\hat{z})g^j(x,\hat{z}) + (T-t)(\gamma\gamma^\top)_{i,j}(x) \tag{9.22}$$

and

$$\mathbb{E}^{\mathcal{A}_T}\big[X_T^{(3)}|\mathcal{G}_t\big] = \frac{1}{6}(T-t)^3\Big\{\partial_i g(x,\hat{z})\partial_j g^i(x,\hat{z})g^j(x,\hat{z}) + \partial_{i,j}g(x,\hat{z})g^i(x,\hat{z})g^j(x,\hat{z})\Big\}$$
$$+\frac{1}{4}(T-t)^2\partial_{i,j}g(x,\hat{z})(\gamma\gamma^\top)_{i,j}(x) + \Big(\partial_i\gamma(x)\mathbf{1}_{(0,m)}\big[[\Sigma]_t^s\big]_t^T\gamma^\top(x)\Big)_i$$
$$+\partial_i g(x,\hat{z})\gamma_i(x)\Big(\big[[\psi]_t^s\big]_t^T + \mathbf{1}_{(0,m)}\big[\big[[\phi]_t^u\big]_t^s\big]_t^T\Big)$$
$$+\partial_i g(x,\hat{z})\gamma_i(x)\Big(\big[[\widetilde{\Psi}]_t^s\big]_t^T - \mathbf{1}_{(0,m)}\big[\big[[\Phi]_t^u\big]_t^s\big]_t^T\Big)\hat{z}$$
$$+g^i(x,\hat{z})\partial_i\gamma(x)\Big\{[(s-t)\psi]_t^T + \mathbf{1}_{(0,m)}\Big(\big[\big[[\phi]_t^u\big]_t^s\big]_t^T + \big[[(u-t)\phi]_t^s\big]_t^T\Big)\Big\}$$
$$+g^i(x,\hat{z})\partial_i\gamma(x)\Big\{[(s-t)\widetilde{\Psi}]_t^T - \mathbf{1}_{(0,m)}\Big(\big[\big[[\Phi]_t^u\big]_t^s\big]_t^T + \big[[(u-t)\Phi]_t^s\big]_t^T\Big)\Big\}\hat{z}$$
$$+\gamma(x)\Big(\big[\widetilde{\Psi}[\phi]_t^s\big]_t^T - \mathbf{1}_{(0,m)}\big[\big[\Phi[\phi]_t^u\big]_t^s\big]_t^T\Big)$$
$$+\gamma(x)\Big(-\big[\widetilde{\Psi}[\Phi]_t^s\big]_t^T + \mathbf{1}_{(0,m)}\big[\big[\Phi[\Phi]_t^u\big]_t^s\big]_t^T\Big)\hat{z}$$
$$\mathbb{E}^{\mathcal{A}_T}\big[X_T^{i,(2)}X_T^{j,(1)}|\mathcal{G}_t\big] = \frac{1}{2}(T-t)^3 g^k(x,\hat{z})\partial_k g^i(x,\hat{z})g^j(x,\hat{z}) + \Big(\gamma(x)\mathbf{1}_{(0,m)}\big[[\Sigma]_t^s\big]_t^T\gamma^\top(x)\Big)_{i,j}$$
$$+\frac{1}{2}(T-t)^2\Big\{g^k(x,\hat{z})((\partial_k\gamma)\gamma^\top)_{i,j}(x) + \partial_k g^i(x,\hat{z})(\gamma\gamma^\top)_{k,j}\Big\}$$
$$+g^j(x,\hat{z})\gamma_i(x)\Big\{\big[[\psi]_t^s\big]_t^T + [(s-t)\psi]_t^T + \mathbf{1}_{(0,m)}\Big(2\big[\big[[\phi]_t^u\big]_t^s\big]_t^T + \big[[(u-t)\phi]_t^s\big]_t^T\Big)\Big\}$$
$$+g^j(x,\hat{z})\gamma_i(x)\Big\{\big[[\widetilde{\Psi}]_t^s\big]_t^T + [(s-t)\widetilde{\Psi}]_t^T - \mathbf{1}_{(0,m)}\Big(2\big[\big[[\Phi]_t^u\big]_t^s\big]_t^T + \big[[(u-t)\Phi]_t^s\big]_t^T\Big)\Big\}\hat{z}$$
$$\mathbb{E}^{\mathcal{A}_T}\big[X_T^{i,(1)}X_T^{j,(1)}X_T^{k,(1)}|\mathcal{G}_t\big]$$
$$= (T-t)^2\Big\{g^i(x,\hat{z})(\gamma\gamma^\top)_{j,k}(x) + g^j(x,\hat{z})(\gamma\gamma^\top)_{k,i}(x) + g^k(x,\hat{z})(\gamma\gamma^\top)_{i,j}(x)\Big\}$$
$$+(T-t)^3 g^i(x,\hat{z})g^j(x,\hat{z})g^k(x,\hat{z}) \tag{9.23}$$

Although the expressions are rather lengthy for higher order corrections, there is an important feature making our method useful. As one can see from the above result, the stochastic variable ($x = X_t^\epsilon, \hat{z} = \hat{z}_t^\epsilon$) are separated from all the necessary time integrations. Thus, one can carry out the required integrations beforehand and store them in the memory, which then makes possible to use $V_1(t)$ in the simulation with only the usual update of underlying state processes $(X_t^\epsilon, \hat{z}_t^\epsilon)$. As we shall see next, this property continues to hold for $Z_1(t)$.

9.2.2 Approximation of (Z_1, Γ_1)

We now try to expand

$$\zeta_1^\epsilon(t) := \begin{pmatrix} Z_1^\epsilon(t) \\ \Gamma_1^\epsilon(t) \end{pmatrix} \tag{9.24}$$

as

$$\zeta_1^\epsilon(t) = \epsilon\zeta_1^{(1)}(t) + \epsilon^2\zeta_1^{(2)}(t) + \epsilon^3\zeta_1^{(3)}(t) + \cdots \tag{9.25}$$

up to the ϵ-third order corrections. Since the expansion for

$$V_1^\epsilon(t) = A(t,T)\mathbb{E}^{\mathcal{A}_T}[H(X_T^\epsilon)|\mathcal{G}_t] \tag{9.26}$$

is already obtained, one only needs a simple application of Itô formula. Since it increases ϵ-order by $\mathbf{1}$, we only need up to the 2nd order corrections of V_1^ϵ, and also there is no 0-th order contribution to ζ_1.

By extracting the coefficients (as *row* vector) of the n-dimensional Brownian motion from the SDEs of the following conditional expectations,

$$\begin{aligned}
\overline{X}_{t,T}^{i,(1)}(x,\hat{z}) &:= \mathbb{E}^{\mathcal{A}_T}[X_T^{i,(1)}|\mathcal{G}_t] \\
\overline{X}_{t,T}^{i,(2)}(x,\hat{z}) &:= \mathbb{E}^{\mathcal{A}_T}[X_T^{i,(2)}|\mathcal{G}_t] \\
\overline{X}_{t,T}^{(i,j),(1,1)}(x,\hat{z}) &:= \mathbb{E}^{\mathcal{A}_T}[X_T^{i,(1)}X_T^{j,(1)}|\mathcal{G}_t]
\end{aligned} \tag{9.27}$$

one obtains

$$\bar{\sigma}_{t,T}^{i,(1)}(x,\hat{z}) := (T-t)\Big\{\partial_j g^i(x,\hat{z})\gamma_j(x) + \gamma_i(x)\mathbf{1}_{(0,m)}\Sigma(t)\Big\} \tag{9.28}$$

$$\begin{aligned}
\bar{\sigma}_{t,T}^{i,(2)}(x,\hat{z}) &:= \frac{1}{2}(T-t)^2\Big[\partial_{j,k}g^i(x,\hat{z})g^j(x,\hat{z}) + \partial_j g^i(x,\hat{z})\partial_k g^j(x,\hat{z})\Big]\gamma_k(x) \\
&+\frac{1}{2}(T-t)^2\Big[g^j(x,\hat{z})\partial_j\gamma_i(x) + \partial_j g^i(x,\hat{z})\gamma_j(x)\Big]\mathbf{1}_{(0,m)}\Sigma(t) \\
&+\partial_j\gamma_i(x)\Big\{\Big([\psi]_t^T + \mathbf{1}_{(0,m)}\big[[\phi]_t^s\big]_t^T\Big) + \Big([\widetilde{\Psi}]_t^T - \mathbf{1}_{(0,m)}\big[[\Phi]_t^s\big]_t^T\Big)\hat{z}\Big\}\gamma_j(x) \\
&+\gamma_i(x)\Big([\widetilde{\Psi}]_t^T - \mathbf{1}_{(0,m)}\big[[\Phi]_t^s\big]_t^T\Big)\Sigma(t)
\end{aligned} \tag{9.29}$$

$$\begin{aligned}
\bar{\sigma}_{t,T}^{(i,j),(1,1)}(x,\hat{z}) &= (T-t)^2\Big[\partial_k g^i(x,\hat{z})g^j(x,\hat{z}) + g^i(x,\hat{z})\partial_k g^j(x,\hat{z})\Big]\gamma_k(x) \\
&+(T-t)\Big[(\partial_k\gamma\gamma^\top)_{i,j}(x) + (\partial_k\gamma\gamma^\top)_{j,i}(x)\Big]\gamma_k(x) \\
&+(T-t)^2\Big[g^j(x,\hat{z})\gamma_i(x) + g^i(x,\hat{z})\gamma_j(x)\Big]\mathbf{1}_{(0,m)}\Sigma(t)\ ,
\end{aligned} \tag{9.30}$$

respectively. Using this result, one can show that the expansion is finally given by

$$\begin{aligned}
\zeta_1^{(1)}(t)^\top &= A(t,T)H(x)\Big[c^{[1]}(t)^\top + \hat{z}^\top c^{[2]}(t)\Big]\Sigma(t) + A(t,T)\partial_i H(x)\gamma_i(x) \\
\zeta_1^{(2)}(t)^\top &= A(t,T)\Big(\partial_i H(x)\overline{X}_{t,T}^{i,(1)}(x,\hat{z})\Big)\Big[c^{[1]}(t)^\top + \hat{z}^\top c^{[2]}(t)\Big]\Sigma(t) \\
&+A(t,T)\Big[\partial_{i,j}H(x)\overline{X}_{t,T}^{i,(1)}(x,\hat{z})\gamma_j(x) + \partial_i H(x)\bar{\sigma}_{t,T}^{i,(1)}(x,\hat{z})\Big] \\
\zeta_1^{(3)}(t)^\top &= A(t,T)\big[\partial_i H(x)\overline{X}_{t,T}^{i,(2)}(x,\hat{z}) + \frac{1}{2}\partial_{i,j}H(x)\overline{X}_{t,T}^{(i,j),(1,1)}(x,\hat{z})\big][c^{[1]}(t)^\top + \hat{z}^\top c^{[2]}(t)]\Sigma(t) \\
&+A(t,T)\Big[\partial_{i,j}H(x)\overline{X}_{t,T}^{i,(2)}(x,\hat{z})\gamma_j(x) + \partial_i H(x)\bar{\sigma}_{t,T}^{i,(2)}(x,\hat{z})\Big] \\
&+\frac{A(t,T)}{2}\Big[\partial_{i,j,k}H(x)\overline{X}_{t,T}^{(i,j),(1,1)}(x,\hat{z})\gamma_k(x) + \partial_{i,j}H(x)\bar{\sigma}_{t,T}^{(i,j),(1,1)}(x,\hat{z})\Big].
\end{aligned} \tag{9.31}$$

9.3 Numerical Examples

As a simple application of the asymptotic expansion, let us consider

$$H(X_T) = Y_T^I \tag{9.32}$$

as in Sec. 8, but now

$$\gamma^I(X_t) = (Y_t^I)^\beta \sigma_y^\top \tag{9.33}$$

for its volatility term. Here, $\beta \in [0,1]$ is some constant, and $\sigma_y \in \mathbb{R}^n$ is a constant vector. In this case, many cross terms vanish in the asymptotic expansion and one obtains rather simple formulas. The results of the asymptotic expansion for this model are summarized in Appendix B.

	$V_1^{(0)}$	$V_1^{(1)}$	$V_1^{(2)}$	$V_1^{(3)}$	$V_0^{(0)}$	$V_0^{(1)}$	$V_0^{(2)}$	$V_0^{(3)}$
$\beta = 0.25$	0.87206	0.89560	0.90216	0.90409	0.9052	1.0095	1.0116	1.0088
$\beta = 0.5$	0.87206	0.89560	0.90224	0.90596	0.9106	1.0142	1.0164	1.0160

Table 1: The numerical results for $V_1^{(i)}, V_0^{(i)}$ for $\beta = 0.25$ and $\beta = 0.5$ models. $V_1^{(i)}$ is calculated based on the asymptotic expansion *including all the contribution up to* the i-th order. $V_0^{(i)}$ is obtained by running simulation for (8.10) with the corresponding order of approximation for (V_1, Z_1).

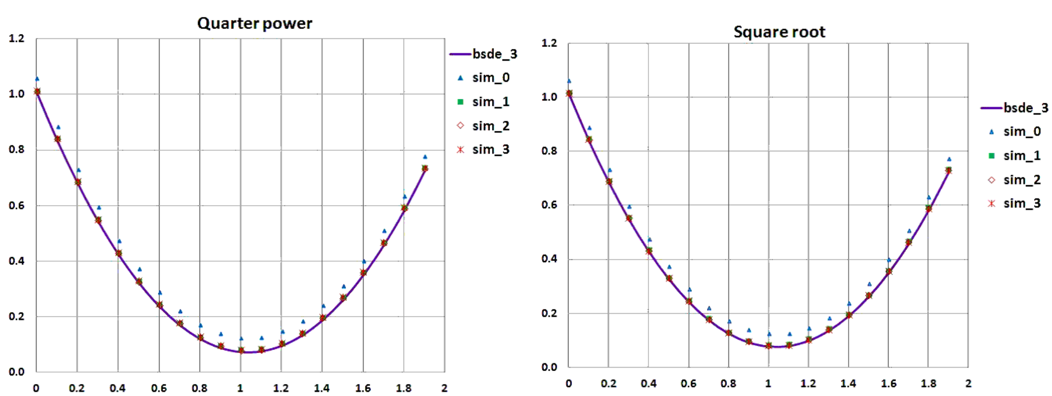

Figure 3: Comparison of $V(0,w) \simeq w^2 V_2(0) - 2wV_1^{(3)}(0) + V_0^{(3)}(0)$ and direct simulation of $\mathbb{E}(Y_T^I - \mathcal{W}_T^{\pi^*})^2$ with each approximation order of (V_1, Z_1). The solid line is based on the quadratic form of $V(0,w)$ and the other symbols are those obtained from the direct simulation of wealth with each approximation order. The horizontal axis denotes the size of initial capital w.

We have studied $\beta = 0.25$ and $\beta = 0.5$ cases for $T = 1$yr maturity. For the remaining parameters $(z_0,\ \mu,\ F,\ \delta,\ \Sigma_0)$ and also σ_y are those we have used in Sec. 8.1. We have also set $Y_0^I = 1$ for both of the models. $V_2(0)$ is independent from the model of Y^I and we have obtained $V_2(0) = 0.8721$ by numerically solving the ODEs. In Table 1, we have listed the numerical results for $V_1(0)$ and $V_0(0)$.

There, the results of $\{V_1^{(i)}\}$ are based on the asymptotic expansion *including all the contribution up to* the i-th order, and $\{V_0^{(i)}\}$ are calculated by simulating (8.10) with the corresponding order of approximation for (V_1, Z_1). The number of simulation paths and step size are the same as those used in Sec. 8.1. The standard error of V_0 simulation is around 7×10^{-4} for both of the models.

In Fig. 3, we have done the same consistency test as in Sec. 8.1, where we have compared the quadratic form of $V(0, w)$ and direct simulation of $\mathbb{E}(Y_T^I - \mathcal{W}_T^{\pi^*})^2$. The solid line corresponds to the prediction of $V(0, w)$ using the 3rd order approximation, and the other symbols denote the results of direct simulation of $\mathbb{E}(Y_T^I - \mathcal{W}_T^{\pi^*})^2$ using each order of approximation of (V_1, Z_1). One can confirm the consistency of our approximation and also that even the 1st order approximation realizes the most part of the hedging benefit of the variance reduction.

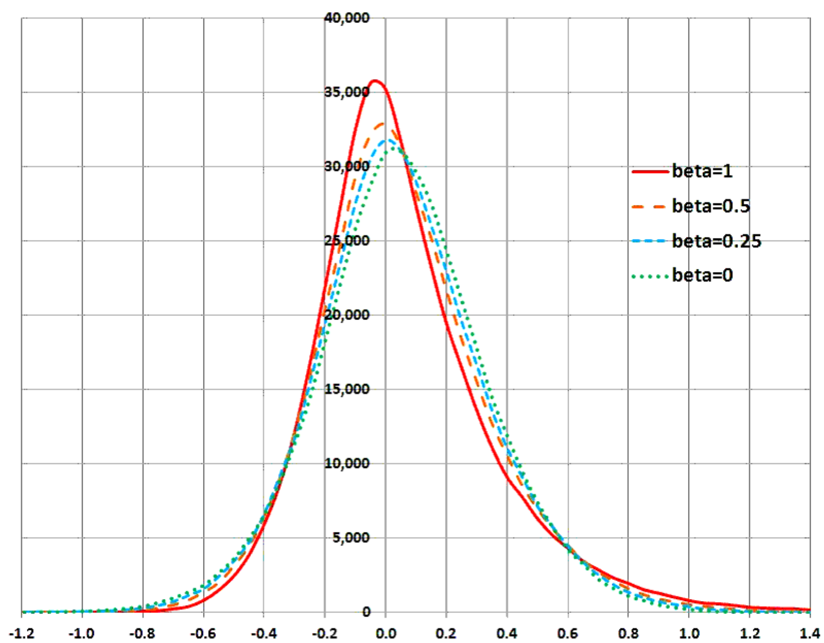

Figure 4: The comparison of the terminal distribution $(Y_T^I - \mathcal{W}_T^{\pi^*})$ with the initial capital $w = 1$ using the third order asymptotic expansion. The graphs are obtained by connecting the histograms after sampling 400,000 paths.

It might be surprising that these results are very close between the two choices of β, but in fact, this result is naturally expected. Actually, one can easily confirm that $(V_1^{(0)}(0),\ V_1^{(1)}(0))$ should have exactly the same value with arbitrary $\beta \in [0, 1]$ in the current setup. Furthermore, the common σ_y and the initial value of $Y_0^I = 1$ indicate that every model with $\beta \in [0, 1]$ has almost the same variance for relatively short maturities, which naturally leads to the similar variance for the hedging error. However, as can be seen from Fig. 4, there appears a difference in the distribution of the hedged portfolio.

There, we have compared the terminal distribution $(Y_T^I - \mathcal{W}_T^{\pi^*})$ with the initial capital $w = 1$ for four models of $\beta = \{1,\ 0.5,\ 0.25,\ 0\}$. The graphs of distribution were obtained by connecting the histogram after sampling 400,000 paths. These difference may become important for financial firms from a risk-management perspective.

10 Conclusions

In this article, we have studied the mean-variance hedging (MVH) problem in a partially observable market by studying a set of three BSDEs derived by Mania & Tevzadze [9]. Under the Bayesian and Kalman-Bucy frameworks, we have found that one of these BSDEs yields a semi-closed solution via a simple set of ODEs which allow a quick numerical evaluation. We have proposed a Monte Carlo scheme using a particle method to solve the remaining two BSDEs without nested simulations. As far as the optimal hedging positions are concerned, it is also pointed out that one only needs the standard simulations for the terminal liability and its Delta sensitivities against the state processes under a new measure $(\mathbf{P}^{\mathcal{A}_T}, \mathcal{G})$.

We gave a special example where the hedging position is available in a semi-closed form and presented an interesting consistency test by directly simulating the optimal portfolio. For more general situations, we have provided explicit expressions of the approximate hedging portfolio by an asymptotic expansion method and demonstrated the procedures by several numerical examples. It would be interesting future works to apply the obtained asymptotic expansion formula to more involved situations where the payoff function H is non-linear or dependent on both S and Y.

Although the simplifying assumptions on the MPR dynamics in $(\mathbf{P}, \mathcal{F})$ are very restrictive, generalization to a non-linear dynamics remains as a very challenging issue of the non-linear filtering problem with infinite degrees of freedom. It may be worth considering to use a similar asymptotic expansion technique (see, for example, Fujii (2013) [2].) for this problem. If the MPR process is perfectly observable, then, in principle, we can take its non-linear effects into account perturbatively by the method proposed in [3].

A Derivation of BSDEs

In this section, for interested readers, we briefly explain the main ideas of Mania & Tevzadze leading to the system of BSDEs. Since $V(t,w)$ defined by (4.2) is a $\{\mathcal{G}_t\}$-adapted semimartingale in general, using the *"representation theorem"* (see, Lemma 4.1 of [15]), one can decompose it as

$$V(t,w) = V(s,w) + \int_s^t a(u,w)du + \int_s^t Z(u,w)^\top dN_u + \int_s^t \Gamma(u,w)^\top dM_u \qquad \text{(A.1)}$$

with an appropriate $\{\mathcal{G}_t\}$-adapted triple (a, Z, Γ). Assuming appropriate conditions for the use of Itô-Ventzell formula [7], one obtains

$$\begin{aligned}
V(t,\mathcal{W}_t^\pi) &= V(s,w) + \int_s^t a(u,\mathcal{W}_u^\pi)ds + \int_s^t Z(u,\mathcal{W}_u^\pi)^\top dN_u + \int_s^t \Gamma(u,\mathcal{W}_u^\pi)^\top dM_u \\
&+ \int_s^t V_w(u,\mathcal{W}_u^\pi)\pi_u^\top \sigma_u [dN_u + \hat{\theta}_u du] + \int_s^t \pi_u^\top \sigma_u Z_w(u,\mathcal{W}_u^\pi)du \\
&+ \int_s^t \frac{1}{2} V_{ww}(u,\mathcal{W}_u^\pi)\pi_u^\top (\sigma\sigma^\top)(u)\pi_u du \qquad \text{(A.2)}
\end{aligned}$$

Here, we have written $\mathcal{W}_t^\pi(s,w)$ as $\mathcal{W}_t^\pi$ for simplicity. It is easy to see $V(t,\mathcal{W}_t^\pi)$ should be a $(\mathbf{P},\mathcal{G})$-martingale for the optimal strategy π^* (and submartingale otherwise). Then, one obtains

$$\begin{aligned}
a(s,w) &= -\inf_{\pi\in\Pi}\left\{ \frac{1}{2} V_{ww}(s,w) \Big|\Big| \sigma^\top(s)\pi_s + \frac{Z_w(s,w)+V_w(s,w)\hat{\theta}_s}{V_{ww}(s,w)} \Big|\Big|^2 \right\} \\
&\quad + \frac{||Z_w(s,w)+V_w(s,w)\hat{\theta}_s||^2}{2V_{ww}(s,w)} \qquad \text{(A.3)}
\end{aligned}$$

as a drift condition.

Assuming the π which makes the first term zero is admissible and hence corresponding to π^*, one obtains

$$a(s,w) = \frac{||Z_w(s,w)+V_w(s,w)\hat{\theta}_s||^2}{2V_{ww}(s,w)} \ . \qquad \text{(A.4)}$$

Substituting the above result into (A.1) yields a BSPDE

$$\begin{aligned}
V(t,w) &= \Big| H - w \Big|^2 - \frac{1}{2}\int_t^T \frac{||Z_w(s,w)+V_w(s,w)\hat{\theta}_s||^2}{V_{ww}(s,w)} ds \\
&- \int_t^T Z(s,w)^\top dN_s - \int_t^T \Gamma(s,w)^\top dM_s \ . \qquad \text{(A.5)}
\end{aligned}$$

The optimal wealth dynamics can also be read as

$$\mathcal{W}_T^{\pi^*}(t,w) = w - \int_t^T \frac{[Z_w(s,\mathcal{W}_s^{\pi^*}) + V_w(s,\mathcal{W}_s^{\pi^*})\hat{\theta}_s]^\top}{V_{ww}(s,\mathcal{W}_s^{\pi^*})}[dN_s + \hat{\theta}_s ds] \ . \qquad \text{(A.6)}$$

Since (A.5) needs to hold for arbitrary w, one may suppose the following decomposition holds:

$$V(t,w) = w^2 V_2(t) - 2wV_1(t) + V_0(t) \qquad \text{(A.7)}$$

where $\{V_i\}$ do not depend on w. Then, inserting back to (A.5) leads to the desired set of BSDEs. Economic meanings of V_i are explained in [9].

B Asymptotic expansion formulas for the model in Sec. 9.3

Firstly, let us put

$$\begin{aligned}
\overline{Y}^{I,(1)}_{t,T}(y,\hat{z}) &:= \mathbb{E}^{\mathcal{A}_T}[Y^{I,(1)}_T|\mathcal{G}_t] \\
\overline{Y}^{I,(2)}_{t,T}(y,\hat{z}) &:= \mathbb{E}^{\mathcal{A}_T}[Y^{I,(2)}_T|\mathcal{G}_t] \\
\overline{Y}^{I,(3)}_{t,T}(y,\hat{z}) &:= \mathbb{E}^{\mathcal{A}_T}[Y^{I,(3)}_T|\mathcal{G}_t] \ ,
\end{aligned} \tag{B.1}$$

with the convention that

$$y := Y^{I,\epsilon}_t \ . \tag{B.2}$$

From the results in Sec. 9.2.1 and 9.2.2, one obtains

$$\begin{aligned}
&\overline{Y}^{I,(1)}_{t,T}(y,\hat{z}) = (T-t)y^\beta(\sigma_y^\top \mathbf{1}_{(0,m)}\hat{z}) \\
&\overline{Y}^{I,(2)}_{t,T}(y,\hat{z}) = \frac{1}{2}(T-t)^2\beta y^{2\beta-1}(\sigma_y^\top \mathbf{1}_{(0,m)}\hat{z})^2 + y^\beta\sigma_y^\top\Big([\psi]_t^T + \mathbf{1}_{(0,m)}\big[[\phi]_t^s\big]_t^T\Big) \\
&\qquad +y^\beta\sigma_y^\top\Big([\widetilde{\Psi}]_t^T - \mathbf{1}_{(0,m)}\big[[\Phi]_t^s\big]_t^T\Big)\hat{z} \\
&\overline{Y}^{I,(3)}_{t,T}(y,\hat{z}) \\
&\quad = \frac{1}{6}(T-t)^3(2\beta^2-\beta)y^{3\beta-2}(\sigma_y^\top\mathbf{1}_{(0,m)}\hat{z})^3 + \frac{1}{4}(T-t)^2(\beta^2-\beta)y^{3\beta-2}(\sigma_y^\top\mathbf{1}_{(0,m)}\hat{z})||\sigma_y||^2 \\
&\quad +\beta y^{2\beta-1}(\sigma_y^\top\mathbf{1}_{(0,m)}\hat{z})\sigma_y^\top\Big\{\big[[\psi]_t^s\big]_t^T + [(s-t)\psi]_t^T + \mathbf{1}_{(0,m)}\Big(2\big[\big[[\phi]_t^u\big]_t^s\big]_t^T + \big[[(u-t)\phi]_t^s\big]_t^T\Big)\Big\} \\
&\quad +\beta y^{2\beta-1}(\sigma_y^\top\mathbf{1}_{(0,m)}\hat{z})\sigma_y^\top\Big\{\big[[\widetilde{\Psi}]_t^s\big]_t^T + [(s-t)\widetilde{\Psi}]_t^T - \mathbf{1}_{(0,m)}\Big(2\big[\big[[\Phi]_t^u\big]_t^s\big]_t^T + \big[[(u-t)\Phi]_t^s\big]_t^T\Big)\Big\}\hat{z} \\
&\quad +y^\beta\sigma_y^\top\Big(\big[\widetilde{\Psi}[\phi]_t^s\big]_t^T - \mathbf{1}_{(0,m)}\big[\big[\Phi[\phi]_t^u\big]_t^s\big]_t^T\Big) + y^\beta\sigma_y^\top\Big(-\big[\widetilde{\Psi}[\Phi]_t^s\big]_t^T + \mathbf{1}_{(0,m)}\big[\big[\Phi[\Phi]_t^u\big]_t^s\big]_t^T\Big)\hat{z} \\
&\quad +\beta y^{2\beta-1}\big(\sigma_y^\top\mathbf{1}_{(0,m)}\big[[\Sigma]_t^s\big]_t^T\sigma_y\big) \ .
\end{aligned} \tag{B.3}$$

Using the above results, one can show $V_1^\epsilon(t)$ can be expanded as

$$V_1^\epsilon(t) = A(t,T)\Big\{y + \epsilon\overline{Y}^{I,(1)}_{t,T}(y,\hat{z}) + \epsilon^2\overline{Y}^{I,(2)}_{t,T}(y,\hat{z}) + \epsilon^3\overline{Y}^{I,(3)}_{t,T}(y,\hat{z}) + o(\epsilon^3)\Big\} \ . \tag{B.4}$$

It is also straightforward to obtain

$$\begin{aligned}
\zeta_1^{(1)}(t)^\top &= A(t,T)\Big\{y[c^{[1]}(t)^\top + \hat{z}^\top c^{[2]}(t)]\Sigma(t) + y^\beta\sigma_y^\top\Big\} \\
\zeta_1^{(2)}(t)^\top &= A(t,T)\Big\{\overline{Y}^{I,(1)}_{t,T}(y,\hat{z})[c^{[1]}(t)^\top + \hat{z}^\top c^{[2]}(t)]\Sigma(t) + \bar{\sigma}^{I,(1)}_{t,T}(y,\hat{z})\Big\} \\
\zeta_1^{(3)}(t)^\top &= A(t,T)\Big\{\overline{Y}^{I,(2)}_{t,T}(y,\hat{z})[c^{[1]}(t)^\top + \hat{z}^\top c^{[2]}(t)]\Sigma(t) + \bar{\sigma}^{I,(2)}_{t,T}(y,\hat{z})\Big\} \ ,
\end{aligned} \tag{B.5}$$

with the definitions of

$$
\begin{aligned}
\bar{\sigma}^{I,(1)}_{t,T}(y,\hat{z}) &:= (T-t)\Big\{\beta y^{2\beta-1}(\sigma_y^\top \mathbf{1}_{(0,m)}\hat{z})\sigma_y^\top + y^\beta(\sigma_y^\top \mathbf{1}_{(0,m)}\Sigma(t))\Big\} \\
\bar{\sigma}^{I,(2)}_{t,T}(y,\hat{z}) &= \frac{1}{2}(T-t)^2(2\beta^2-\beta)y^{3\beta-2}(\sigma_y^\top \mathbf{1}_{(0,m)}\hat{z})^2\sigma_y^\top \\
&\quad +(T-t)^2\beta y^{2\beta-1}(\sigma_y^\top \mathbf{1}_{(0,m)}\hat{z})(\sigma_y^\top \mathbf{1}_{(0,m)}\Sigma(t)) \\
&\quad +\beta y^{2\beta-1}\sigma_y^\top\Big[\Big([\psi]_t^T + \mathbf{1}_{(0,m)}\big[[\phi]_t^s\big]_t^T\Big) + \Big([\widetilde{\Psi}]_t^T - \mathbf{1}_{(0,m)}\big[[\Phi]_t^s\big]_t^T\Big)\hat{z}\Big]\sigma_y^\top \\
&\quad +y^\beta\sigma_y^\top\Big([\widetilde{\Psi}]_t^T - \mathbf{1}_{(0,m)}\big[[\Phi]_t^s\big]_t^T\Big)\Sigma(t).
\end{aligned}
\tag{B.6}
$$

Acknowledgement

This research is partially supported by Center for Advanced Research in Finance (CARF).

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