JOURNAL OF FINANCIAL AND QUANTITATIVE ANALYSIS VOL. 31, NO. 4, DECEMBER 1996

Multifactor Portfolio Efficiency and Multifactor Asset Pricing

Eugene F. Fama*

Abstract

The concept of multifactor portfolio efficiency plays a role in Merton's intertemporal CAPM (the ICAPM), like that of mean-variance efficiency in the Sharpe-Lintner CAPM. In the CAPM, the relation between the expected return on a security and its risk is just the condition on security weights that holds in any mean-variance-efficient portfolio, applied to the market portfolio M. The risk-return relation of the ICAPM is likewise just the application to M of the condition on security weights that produces ICAPM multifactor-efficient portfolios. The main testable implication of the CAPM is that equilibrium security prices require that M is mean-variance-efficient. The main testable implication of the ICAPM is that securities must be priced so that M is multifactor-efficient. As in the CAPM, building the ICAPM on multifactor efficiency exposes its simplicity and allows easy economic insights.

I. Introduction

In applications that require estimates of expected returns, the capital-asset-pricing model (CAPM) of Sharpe (1964) and Lintner (1965) is the popular choice. For example, textbooks in corporate finance typically recommend CAPM expected returns for estimating a firm's cost of capital. Studies of the performance of portfolio managers commonly use the CAPM to estimate benchmark expected returns.

The CAPM says that the expected return on a security depends only on the sensitivity of its return to the market return—its market β. There is, however, evidence that market β does not suffice to describe expected return. Variables that seem to help explain expected return include a firm's market capitalization, earnings/price ratio, leverage, and book-to-market-equity ratio (Banz (1981), Basu (1983), Bhandari (1988), Rosenberg, Reid, and Lanstein (1985), Chan, Hamao, and Lakonishok (1991), Fama and French (1992)). Moreover, Chen, Roll, and Ross (1986) and Fama and French (1996) find that the CAPM fares poorly in competition with multifactor alternatives. This evidence suggests that multifactor

*Graduate School of Business, University of Chicago, 1101 East 58th Street, Chicago, IL 60637. The comments of John Cochrane, George Constantinides, Shmuel Kandel, Francis Longstaff, Robert Merton, and René Stulz are gratefully acknowledged. Kenneth French, Anthony Lynch, and Wayne Ferson (the referee) get special thanks.

models should be considered in research applications that require estimates of expected returns.

One popular multifactor model is the arbitrage-pricing-theory (APT) of Ross (1976). The pure-arbitrage version of the APT has a well-known shortcoming. It provides exact predictions of expected returns only for perfectly diversified portfolios, that is, portfolios whose returns are completely captured by the common risk factors in returns. Connor's (1984) version of the APT produces exact statements about the expected returns on all securities. But his results come with two costs: i) the market portfolio must be perfectly diversified, and ii) the simple arbitrage argument that makes the APT so attractive is abandoned in favor of arguments based on utility maximization.

In a classic paper, Merton (1973) develops an intertemporal model (the ICAPM) that uses utility maximization to get exact multifactor predictions of expected security returns. He gets exact results without assuming the market portfolio is perfectly diversified. The drawback of Merton's approach is degree of difficulty. His continuous-time methods do not yield easy insights. Long (1974) provides a discrete-time version of the ICAPM and an excellent discussion of the model's economics, but his formal story is again difficult. Although Merton and Long show that the CAPM is a special case of the ICAPM, their formal treatments of the ICAPM lack the simple intuition that makes the CAPM so attractive.

The powerful intuition of the CAPM centers on Markowitz' (1959) concept of mean-variance-efficiency. The CAPM starts with assumptions that imply that investors hold mean-variance-efficient (MVE) portfolios. When there is a risk-free security, f, MVE portfolios combine f with one MVE portfolio of risky securities, the tangency portfolio T. Since T is the risky component of all MVE portfolios, market-clearing security prices require that T is the value-weight market portfolio M. The familiar CAPM relation between the expected return on any security i, $E(r_i)$, and its market risk, β_{iM} (the slope in the regression of r_i on r_M),

$$(1) \qquad E(r_i) - r_f \quad = \quad \beta_{iM}\left[E(r_M) - r_f\right],$$

is then just the application to M of the condition on security weights that holds in any MVE portfolio (Fama (1976), ch. 8, Roll (1977)).

My goal is to show that Merton's ICAPM can be built on similar intuition. ICAPM investors hold multifactor-efficient portfolios that generalize the notion of portfolio efficiency. Like CAPM investors, ICAPM investors dislike wealth uncertainty. But ICAPM investors are also concerned with hedging more specific aspects of future consumption-investment opportunities, such as the relative prices of consumption goods and the risk-return tradeoffs they will face in capital markets. As a result, the typical ICAPM multifactor-efficient portfolio combines one of Markowitz' MVE portfolios with hedging portfolios that mimic uncertainty about the S future consumption-investment state variables of concern to investors.

The ICAPM risk-return relation is then a natural generalization of (1). One simply adds risk premiums for the sensitivities of r_i to the returns r_s, $s = 1, \ldots, S$, on the state-variable mimicking portfolios,

$$(2) \qquad E(r_i) - r_f \quad = \quad \beta_{iM}\left[E(r_M) - r_f\right] + \sum_{s=1}^{S} \beta_{is}\left[E(r_s) - r_f\right],$$

where β_{iM} and β_{is}, $s = 1, \ldots, S$, are the slopes from the multiple regression of r_i on r_M and r_s, $s = 1, \ldots, S$.

As in the CAPM, the ICAPM relation (2) between expected return and multifactor risks is the condition on the weights for securities that holds in any multifactor-efficient portfolio, applied to the market portfolio M. And just as market equilibrium in the CAPM requires that M is mean-variance-efficient, in the ICAPM market-clearing prices imply that M is multifactor-efficient.

The development of this story begins with a portfolio model for ICAPM investors (Section II), which leads to the characterization of optimal portfolios as multifactor-efficient (Section III). Sections IV to VI detail the properties of multifactor-efficient portfolios that, combined with market-clearing conditions in Section VII, deliver the main results of the ICAPM. The final step (Sections VIII and IX) is to explore special insights obtained from the multifactor-efficiency approach to the ICAPM.

II. The Investor's Decision Problem

The consumption-investment problem facing an ICAPM investor is that posed by Fama (1970). Time is discrete and, at any time $t-1$, the investor divides total wealth w_{t-1} between consumption c_{t-1} and a portfolio p that generates wealth $w_t = (w_{t-1} - c_{t-1})(1 + r_{pt})$, which, in turn, must be divided between consumption and a portfolio at time t. In making this sequence of decisions, the investor is assumed to be risk-averse and to maximize the expected utility of lifetime consumption.

Fama (1970) shows that the dynamic program for the lifetime consumption-investment problem produces a derived utility function to be used in the decision at time $t-1$. The derived utility function, $U(C_{t-1}, w_t|K_t)$, depends on i) $C_{t-1} = (\ldots, c_{t-2}, c_{t-1})$, consumption up to and including time $t-1$, ii) wealth at the next point in time, w_t, and iii) S state variables, $K_t = (k_{1t}, \ldots, k_{St})$, to be observed at t. (Upper-case symbols denote vectors and matrices.) The investor's risk aversion with respect to lifetime consumption carries over to the derived utility function: $U(C_{t-1}, w_t|K_t)$ is increasing and strictly concave in (C_{t-1}, w_t).

The state variables enter $U(C_{t-1}, w_t|K_t)$ because the utility of wealth at t depends on the way it can be used in the markets for consumption goods and securities at t and periods after t. The state variables K_t include the prices of consumption goods at t, the risk-return tradeoffs that will be available in security markets at t, and variables observed at t that are informative about consumption-investment opportunities for periods after t. Other state variables, for example, employment opportunities, are also important. For concreteness, and because the analysis does not cover labor-supply decisions, I refer to K_t as consumption-investment opportunities. But it is understood that K_t includes *all* state variables of concern to *any* investor.

By way of contrast, the Sharpe-Lintner CAPM ignores incentives to use investments at $t-1$ to hedge future consumption-investment opportunities, implied by the dependence of utility on the state variables in $U(C_{t-1}, w_t|K_t)$. Instead, the CAPM uses the simpler portfolio model of Markowitz (1959), which assumes that utility $U(C_{t-1}, w_t)$ depends only on the history of consumption and the wealth generated at t by the portfolio chosen at $t-1$. If portfolio return distributions are normal, optimal CAPM portfolios are easily described. Normality implies that, given a choice of c_{t-1}, the portfolio decision reduces to choosing a combination of mean and variance of return. Since investors are risk-averse, optimal portfolios are *minimum-variance* (MV): they have the smallest return variances, given their expected returns. Since investors prefer more wealth to less, optimal portfolios are *mean-variance-efficient* (MVE): they are the subset of MV portfolios that maximize expected return, given their variances.

Unlike the CAPM, the ICAPM works with the general utility function $U(C_{t-1}, w_t|K_t)$, and a key idea is that covariance between security returns and the state variables allows investors to use their portfolio choices to hedge uncertainty about future consumption-investment opportunities. These hedging demands differentiate the ICAPM portfolio problem from the simpler mean-variance model of the CAPM. Merton (1973) and Long (1974) emphasize that because ICAPM investors use their portfolios to hedge consumption-investment opportunities, optimal portfolios are not typically mean-variance-efficient.

My main point is that a generalized concept of portfolio efficiency, multifactor efficiency, drives the ICAPM. Building the ICAPM on multifactor efficiency leads to simple stories for the model's main results on i) spanning portfolios that can generate all the portfolios relevant for ICAPM investors, and ii) the relation between expected return and the multifactor risks of securities and portfolios.

III. The Multifactor Efficiency of Optimal Portfolios

Consider the consumption-investment decision at time $t-1$. Given a choice of c_{t-1}, the utility function $U(C_{t-1}, w_t|K_t)$ says that the investor is concerned with w_t, the wealth generated at t by the portfolio chosen at $t-1$, and with future consumption-investment opportunities, summarized by the state variables K_t. The properties of any portfolio p as a source of wealth at t and a hedge against state-variable uncertainty are summarized by the joint distribution of r_{pt} and K_t. Like Merton and Long, I assume i) that there is complete agreement among investors about the joint distribution of security returns, $R_t = (r_{1t}, \ldots, r_{Nt})'$, and the state variables, K_t, and ii) the distribution is multivariate normal. Though not explicit in the notation, it is also understood that the joint distribution of R_t and K_t is conditional on the information available at $t-1$.

Multivariate normality of R_t and K_t implies that the joint distribution of the return on any portfolio, r_{pt}, and the state variables K_t is multivariate normal. The joint distribution of r_{pt} and K_t is thus described by i) the mean and variance of the portfolio's return, $E(r_p)$ and $\sigma^2(r_p)$, ii) the covariances between r_{pt} and the state variables, $\text{cov}(r_p, k_s)$, $s = 1, \ldots, S$, and iii) the joint distribution of the state variables. The distribution of the state variables is the same for all portfolios. Thus, the portfolio decision reduces to a choice of $E(r_p)$, $\sigma^2(r_p)$, and $\text{cov}(r_p, k_s)$, $s = 1, \ldots, S$. The covariances, $\text{cov}(r_p, k_s)$, $s = 1, \ldots, S$, capture a portfolio's properties as a hedge against future consumption-investment opportunities.

There is another way to state this result that leads to the concept of multifactor efficiency. The joint normality of returns and the state variables implies that the relation between the return on any security or portfolio p and the state variables is described by the linear regression,

$$(3) \qquad r_{pt} = E\left(r_{pt}\right) + \sum_{s=1}^{S} b_{ps} k_{st} + \epsilon_{pt}, \quad E\left(\epsilon_{pt}\right) = 0,$$

$$\mathrm{cov}\left(\epsilon_{pt}, K_{st}\right) = 0, \quad s = 1, \ldots, S,$$

where, without loss of generality, the state variables are scaled to have means equal to 0.0. The vector of slopes $B_p = (b_{p1}, \ldots, b_{pS})'$ is the vector of covariances, $\mathrm{cov}(r_p, k_s)$, $s = 1, \ldots, S$, multiplied by the inverse of the covariance matrix of the state variables. This inverse is the same for all p. Thus, a portfolio's properties as a hedge against state-variable uncertainty are also captured by its regression slopes or loadings on the state variables. The joint normality of returns and the state variables then implies that, given a choice of c_{t-1}, the optimal portfolio for an investor depends only on $E(r_p)$, $\sigma^2(r_p)$, and B_p. More specifically,

Proposition 1. Since ICAPM investors are risk-averse with respect to wealth uncertainty, their optimal portfolios are *multifactor-minimum-variance* (MMV): they minimize $\sigma^2(r_p)$, given their $E(r_p)$ and B_p. Since ICAPM investors like wealth, they choose from the subset of MMV portfolios that are *multifactor-efficient* (ME): they maximize $E(r_p)$, given their $\sigma^2(r_p)$ and B_p.

The rest of the paper develops the implications of the multifactor efficiency of optimal portfolios. My claim is that multifactor efficiency provides a simple key to understanding the multifactor ICAPM, in the same way that mean-variance efficiency is the key to the CAPM.

IV. Weights for Securities in Multifactor-Minimum-Variance (MMV) Portfolios

An MMV portfolio minimizes $\sigma^2(r_p)$, given its $E(r_p)$ and B_p. Assume, for the moment, that there is no risk-free security. Suppose $B_e = (b_{e1}, \ldots, b_{eS})'$ are the target loadings on the state variables for an MMV portfolio e, $E(r_e)$ is the target expected return, and σ_{ij} is the covariance between the returns on securities i and j. The MMV portfolio e is defined by the weights $X_e = (x_{1e}, \ldots, x_{Ne})'$ for securities that

$$(4a) \qquad \min \sigma^2(r_e) = \sum_{i=1}^{N} \sum_{j=1}^{N} x_{ie} x_{je} \sigma_{ij}, \quad \text{subject to,}$$

$$(4b) \qquad \sum_{i=1}^{N} x_{ie} b_{is} = b_{es}, \quad s = 1, \ldots, S,$$

$$(4c) \qquad \sum_{i=1}^{N} x_{ie} E\left(r_i\right) = E\left(r_e\right),$$

$$(4d) \qquad \sum_{i=1}^{N} x_{ie} = 1.$$

The set of MMV portfolios is given by the solutions to (4) for all feasible combinations of $E(r_e)$ and B_e. With unrestricted short-selling (assumed in (4)) and N much greater than S, it is reasonable to assume that all combinations of loadings on the state variables are feasible. I also assume that all combinations of $E(r_e)$ and B_e are feasible, and this warrants comment. In the ICAPM, the number of securities is finite, and multifactor-efficient (ME) portfolios typically are not perfectly diversified, that is, they have return variation in (3) that is unexplained by the state variables. The residual variance of an ME portfolio is undiversifiable uncertainty about wealth that must be compensated in expected returns. Expected returns thus vary independently of loadings on the state variables. This key point distinguishes Merton's ICAPM from Ross' APT. It is discussed in more detail later.

With unrestricted short-selling, the solution to (4) is obtained by forming the Lagrangean,

$$(5) \quad G = \sum_{i=1}^{N}\sum_{j=1}^{N} x_{ie}x_{je}\sigma_{ij} + \sum_{s=1}^{S} 2\lambda_{se}\left[b_{es} - \sum_{i=1}^{N} x_{ie}b_{is}\right] + 2\lambda_{S+1,e}\left[E(r_e) - \sum_{i=1}^{N} x_{ie}E(r_i)\right] + 2\lambda_{S+2,e}\left[1 - \sum_{i=1}^{N} x_{ie}\right],$$

where $2\lambda_{se}$, $s = 1,\ldots,S+2$, are Lagrange multipliers for the constraints in (4). If the covariance matrix of security returns is positive definite, one solves (4) by differentiating G with respect to each of the x_{ie} and the $2\lambda_{se}$ and setting the derivatives equal to 0.0. Differentiating G with respect to the Lagrange multipliers just says that the weights x_{ie} for securities in the MMV portfolio e must satisfy (4b) to (4d). The interesting conditions are the N equations obtained by differentiating G with respect to the x_{ie},

$$(6) \quad \sum_{j=1}^{N} x_{je}\sigma_{ij} - \sum_{s=1}^{S} \lambda_{se}b_{is} - \lambda_{S+1,e}E(r_i) - \lambda_{S+2,e} = 0,$$

$$\mathrm{cov}(r_i, r_e) - \sum_{s=1}^{S} \lambda_{se}b_{is} - \lambda_{S+1,e}E(r_i) - \lambda_{S+2,e} = 0, \quad i = 1,\ldots,N.$$

Equation (6), the condition on security weights in an MMV portfolio, drives my analysis of the ICAPM. Much of what follows involves manipulating (6), first to characterize MMV portfolios, and then to show that the ICAPM risk-return relation (2) is implied by (6) when the MMV portfolio e is the market portfolio M.

Since Markowitz' minimum-variance (MV) portfolios play a similar central role in the CAPM, it is interesting to compare (6) with the condition on security weights in an MV portfolio. Since CAPM investors do not use their portfolios to hedge uncertainty about consumption-investment state variables, the MV portfolio with expected return $E(r_e)$ is just the solution to (4) without the constraints of (4b) on state-variable loadings. Dropping (4b) implies dropping the terms involving state-variable loadings in (6). MV portfolios are thus special cases of (6). Specifically, the weights for securities in the MV portfolio with expected return $E(r_e)$

must be chosen to produce a linear relation between the expected return on any security and the covariance of its return with the return on e,

$$\text{cov}(r_i, r_e) - \lambda_{S+1,e}E(r_i) - \lambda_{S+2,e} = 0, \quad i = 1, \ldots, N.$$

Thus, $\text{cov}(r_i, r_e)$, the contribution of security i to $\sigma^2(r_e)$, is the risk of i in the minimum-variance portfolio e. In contrast, (6) says that a security's risks in the multifactor-minimum-variance portfolio e of the ICAPM also include $b_{i1}, \ldots, b_{iS}$, the loadings of its return on the state variables of concern to investors.

V. Basis or Spanning Portfolios for MMV Portfolios

Characterizing minimum-variance (MV) and mean-variance efficient (MVE) portfolios is important for understanding the Sharpe-Lintner CAPM. With a risk-free security, f, all MV and MVE portfolios combine f and the MVE tangency portfolio T. When there is no risk-free security, all MV and MVE portfolios are spanned by (they can be generated from) any two MV portfolios (Black (1972)).

Characterizing multifactor-minimum-variance (MMV) and multifactor-efficient (ME) portfolios is likewise important in Merton's ICAPM. This section develops the properties of MMV portfolios implied by (6). The exercise produces two main results: i) all portfolios of MMV portfolios are MMV; ii) with S state variables, any $S+2$ linearly-independent MMV portfolios span all MMV and ME portfolios. Thus, whereas two minimum-variance portfolios span the portfolios relevant to investors in the CAPM, $S+2$ multifactor-minimum-variance portfolios are needed in the ICAPM. The additional S portfolios cover the demands of investors to hedge uncertainty about future consumption-investment opportunities.

The first step is to show that all MMV portfolios can be expressed as portfolios of a particular set of $S+2$ portfolios. Given the Lagrange multipliers for the MMV portfolio e, (6) is a set of N linear equations that can be solved for the security weights in e. A little matrix algebra (see Appendix) shows that the weights are

$$(7) \qquad x_{ie} = \sum_{s=1}^{S} \lambda_{se} \left(\sum_{j=1}^{N} d_{ij} b_{js} \right) + \lambda_{S+1,e} \left(\sum_{j=1}^{N} d_{ij} E(r_j) \right) + \lambda_{S+2,e} \left(\sum_{j=1}^{N} d_{ij} \right), \quad i = 1, \ldots, N,$$

where the d_{ij} are the elements of D, the inverse of the covariance matrix of security returns.

Only the Lagrange multipliers in (7) differ from one MMV portfolio to another. The bracketed terms in (7) are the same for all MMV portfolios. To show that this implies that all MMV portfolios are portfolios of $S+2$ portfolios, rescale the $S+2$ bracketed terms in (7) so that each becomes a set of weights for securities that sums to 1.0,

$$\text{(8a)} \quad x_{is} = \sum_{j=1}^{N} d_{ij} b_{js} / \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} b_{js}, \quad i = 1, \ldots, N, \quad s = 1, \ldots, S,$$

$$\text{(8b)} \quad x_{i,S+1} = \sum_{j=1}^{N} d_{ij} E\left(r_j\right) / \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} E\left(r_j\right), \quad i = 1, \ldots, N,$$

$$\text{(8c)} \quad x_{i,S+2} = \sum_{j=1}^{N} d_{ij} / \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij}, \quad i = 1, \ldots, N.$$

To reproduce the weights for securities in the MMV portfolio e given by (7), one must then scale the Lagrange multipliers in (7) with the denominators of the portfolio weights in (8),

$$\text{(9)} \quad q_{se} = \lambda_{se} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} b_{js} \right), \quad s = 1, \ldots, S,$$

$$q_{S+1,e} = \lambda_{S+1,e} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} E(r_j) \right),$$

$$q_{S+2,e} = \lambda_{S+2,e} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} \right).$$

With (8) and (9), (7) becomes

$$\text{(10)} \quad x_{ie} = \sum_{s=1}^{S+2} x_{is} q_{se}, \quad i = 1, \ldots, N.$$

With (10), the return on the MMV portfolio e is

$$\text{(11)} \quad r_e = \sum_{i=1}^{N} x_{ie} r_i = \sum_{i=1}^{N} \left(\sum_{s=1}^{S+2} x_{is} q_{se} \right) r_i = \sum_{s=1}^{S+2} q_{se} \left(\sum_{i=1}^{N} x_{is} r_i \right) = \sum_{s=1}^{S+2} q_{se} r_s,$$

where r_s, $s = 1, \ldots, S+2$, are the portfolio returns defined by the security weights in (8), and q_{se}, $s = 1, \ldots, S+2$, are the weights on these portfolios of (9). Since (4) says that e is a portfolio (the sum over i of x_{ie} is 1.0), and since (8) says that each r_s is a portfolio return (the sum over i of x_{is} is 1.0), one can infer that the solution to (4) requires that the sum over s of the q_{se} in (11) is 1.0.

Since (11) applies to any MMV portfolio e, (11) implies that any MMV portfolio is indeed a portfolio of the $S+2$ portfolios of (8). Moreover, since all combinations of the weights q_{se}, $s = 1, \ldots, S+2$, in (11) that sum to 1.0 are feasible:

Proposition 2. The $S+2$ portfolios of (8) are a spanning set that can generate all MMV portfolios, and any portfolio of the $S+2$ portfolios of (8) is MMV.

The $S+2$ portfolios of (8) are MMV since they are trivial cases of portfolios of the portfolios of (8). More important, Proposition 2 is next shown to imply:

Proposition 3. Any portfolio of MMV portfolios is MMV.

Proposition 3 holds if any portfolio of MMV portfolios satisfies Proposition 2, that is, it reduces to a portfolio of the $S+2$ portfolios of (8). Consider any portfolio p formed by investing x_{ep}, $e = 1, \ldots, A$, in any number A of MMV portfolios e,

$$r_p = \sum_{e=1}^{A} x_{ep} r_e, \quad \sum_{e=1}^{A} x_{ep} = 1.0. \tag{12}$$

Using (11), r_p can be expressed as a combination of the $S+2$ portfolios of (8),

$$r_p = \sum_{e=1}^{A} x_{ep} r_e = \sum_{e=1}^{A} x_{ep} \sum_{s=1}^{S+2} q_{se} r_s. \tag{13}$$

Proposition 2 says that p is an MMV portfolio as long as the sum of the weights on the r_s in (13) is 1.0. For each of the MMV portfolios e used to form p, the sum over s of q_{se}, $\Sigma_s q_{se}$, is 1.0. Since the sum over e of x_{ep} is 1.0, p is indeed MMV, and (Proposition 3) any portfolio of MMV portfolios is MMV.

The $S+2$ MMV portfolios of (8) are a basis set that spans all MMV portfolios. But this basis set is hardly unique. The Appendix shows that Propositions 2 and 3 imply:

Proposition 4. Any MMV portfolio can be generated from any $S+2$ linearly-independent MMV portfolios. Thus, any $S+2$ linearly-independent MMV portfolios are a basis or spanning set.

Proposition 3 is central when I later argue that in the ICAPM, a market equilibrium requires that the market portfolio is multifactor-efficient. Proposition 4, on the other hand, is the ICAPM analogy to Black's (1972) result that when there is no risk-free security, all the minimum-variance and mean-variance-efficient portfolios of the CAPM can be generated from any two minimum-variance portfolios. In the ICAPM, $S+2$ multifactor-minimum-variance portfolios are needed to span multifactor-minimum-variance and multifactor-efficient portfolios. The additional S portfolios cover the demands of investors to hedge uncertainty about consumption-investment state variables.

Finally, much of the preceding is old stuff. The lifetime consumption-investment problem in Section I follows Fama (1970). Multifactor efficiency is similar to Grinblatt and Titman's (1987) local efficiency. The analysis of the optimality of multifactor-efficient portfolios in Section III and the setup of the variance-minimization problem (4) in Section IV are similar to Elton and Gruber (1992), which derives results like Propositions 2 to 4. The algebra in Propositions 2 to 4 appears in many places, e.g., Jobson and Korkie (1985), Grinblatt and Titman (1987), and Huberman, Kandel, and Stambaugh (1987). In bringing this material together, and extending it (mostly in the sections that follow), what I hope to show is that i) the role of multifactor efficiency in Merton's ICAPM is as central as the role of mean-variance-efficiency in the Sharpe-Lintner CAPM, and ii) building the ICAPM on multifactor efficiency exposes its simplicity and allows easy insights into its economics.

VI. Multifactor-Efficient Portfolios

Propositions 2 to 4 describe multifactor-minimum-variance (MMV) portfolios. Since ICAPM investors hold multifactor-efficient (ME) portfolios, I next determine which MMV portfolios are ME.

The characterization of ME portfolios is striking. Markowitz' mean-variance-efficient (MVE) portfolios are also ME. Other ME portfolios combine MVE portfolios with mimicking portfolios for the state variables. In economic terms, ICAPM investors use Markowitz' MVE portfolios to optimize the tradeoff of expected return for non-state-variable return variance, but they add in mimicking portfolios for the state variables to hedge more specific aspects of future consumption-investment opportunities.

A. The Spanning Portfolios of Proposition 2

Characterizing multifactor-efficient portfolios largely involves developing the properties of the spanning portfolios of (8). I show that these portfolios are solutions to the variance-minimization problem (4), always imposing the constraint (4d) that the sum of the weights for securities is 1.0, but imposing at most one of the constraints of (4b) and (4c) on state-variable loadings and the target expected return.

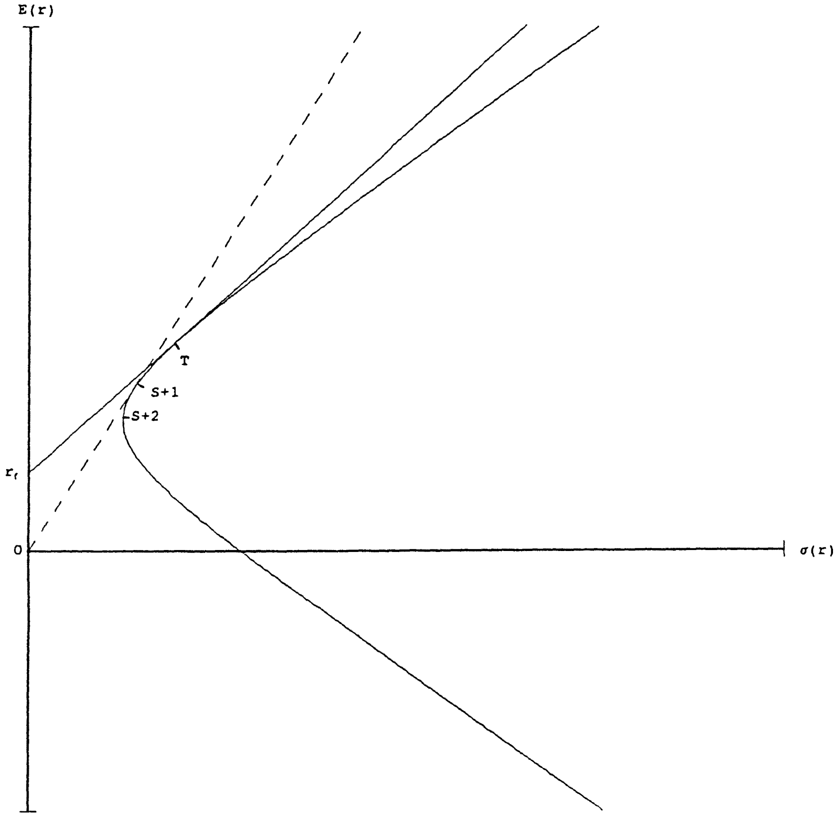
First, suppose the constraints of (4b) and (4c) on expected return and state-variable loadings are dropped. The resulting portfolio has the smallest possible return variance: it is the global-minimum-variance (GMV) portfolio. Dropping (4b) and (4c) from the variance-minimization problem (4) is equivalent to setting the Lagrange multipliers λ_{se}, $s = 1, \ldots, S+1$, in (5) to (7) equal to 0.0. Equations (8) to (11) then imply that the GMV portfolio is the portfolio $S+2$, defined by the security weights of (8c).

Next, consider the portfolio obtained by solving (4) without the constraints of (4b) on state-variable loadings. This portfolio has the smallest return variance given its expected return: it is one of Markowitz' minimum-variance (MV) portfolios. Dropping (4b) from (4) amounts to setting the Lagrange multipliers λ_{se}, $s = 1, \ldots, S$, in (5) to (7) equal to 0.0. Equations (8) to (11) then say that the portfolio is a combination $[q_{S+1,e} r_{S+1} + (1 - q_{S+1,e}) r_{S+2}]$ of the portfolios $S+1$ and $S+2$ defined by the security weights of (8b) and (8c). The set of MV portfolios is the set of all values of $q_{S+1,e}$ in $[q_{S+1,e} r_{S+1} + (1 - q_{S+1,e}) r_{S+2}]$. Thus, portfolio $S+1$, like $S+2$, is one of Markowitz' MV portfolios. In fact, Roll ((1977), p. 165) shows that if $E(r_{S+2}) > 0.0$, $S+1$ is mean-variance-efficient (MVE). Specifically, portfolio $S+1$ is the tangency of the line from the origin to the MVE boundary in Figure 1. In this case, $E(r_{S+1}) > E(r_{S+2})$ and MVE portfolios are the subset of $[q_{S+1,e} r_{S+1} + (1 - q_{S+1,e}) r_{S+2}]$ with $q_{S+1,e} \geq 0.0$. Since portfolios $S+1$ and $S+2$ are also multifactor-minimum-variance (MMV), Proposition 3 says that Markowitz' MV and MVE portfolios are MMV.

FIGURE 1

Markowitz' Mean-Variance-Efficient (MVE) Portfolios

With risk-free borrowing and lending, MVE portfolios are all combinations $[q_{Te}r_T + (1 - q_{Te})r_f]$, $q_{Te} \geq 0.0$, where T is the tangency of a line from r_f to the curved boundary of MVE portfolios of risky securities. If $r_f = 0.0$, the tangency point moves down the curved boundary to portfolio $S+1$. When there is no risk-free security, $S+1$ is the portfolio defined by the weights for securities in (8b). MVE portfolios are then combinations $[q_{S+1,e}r_{S+1} + (1 - q_{S+1,e})r_{S+2}]$, $q_{S+1,e} \geq 0.0$, where $S+2$ is the global-minimum-variance portfolio of risky securities, defined by the security weights in (8c). Multifactor-efficient portfolios combine an MVE portfolio with the S mimicking portfolios for the state variables, defined by the security weights in (8a). Technical point: the curved minimum-variance boundary for risky securities is a hyperbola (Merton (1972)). Thus, the tangency portfolio T is well-defined only if $E(r_{S+2}) \neq r_f$, and portfolio $S+1$ of (8b) is well-defined only if $E(r_{S+2}) \neq 0.0$.



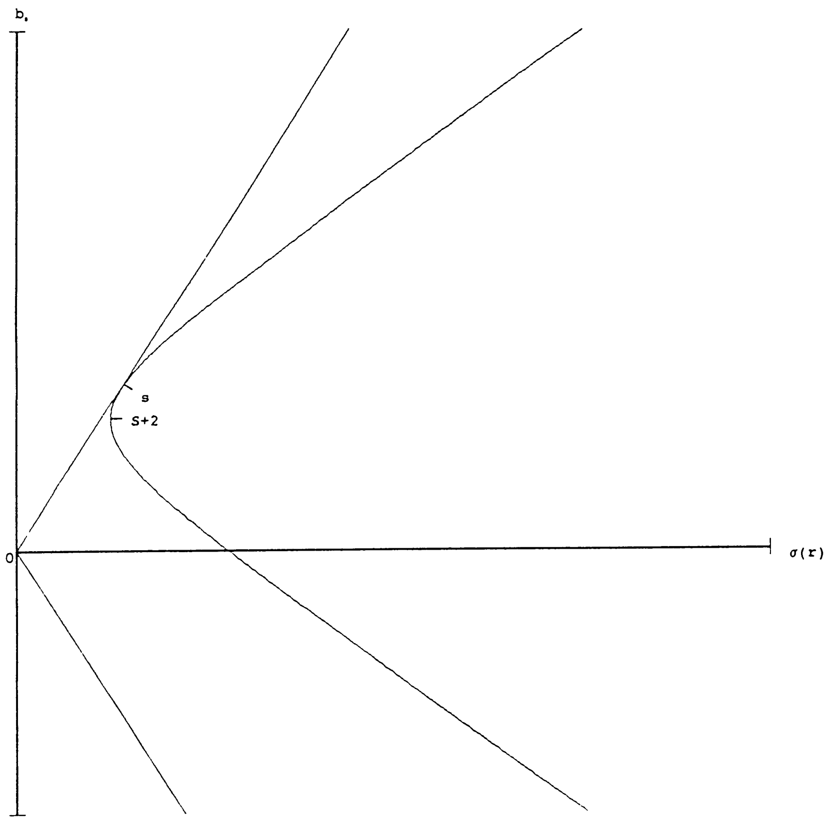
Similar arguments imply that a portfolio that minimizes return variance subject to a target loading on state variable s, but with no constraint on expected return or loadings on other state variables, is a combination $[q_{se}r_s + (1 - q_{se})r_{S+2}]$ of the GMV portfolio $S+2$ and the portfolio s defined by the security weights of (8a). The portfolios s, $s = 1, \ldots, S$, of (8a) can then be viewed as mimicking portfolios for the state variables. Specifically, each r_s, $s = 1, \ldots, S$, obtained from (8a) is a portfolio return with the smallest variance, given a particular target loading on state variable

s (that implied by $q_{se} = 1.0$ in $[q_{se}r_s + (1 - q_{se})r_{S+2}]$), but with no constraints on the portfolio's expected return or its loadings on other state variables. Geometrically, portfolio $S + 2$ is the global-minimum-variance portfolio in the $[b_s, \sigma(r)]$ plane (Figure 2), and a variant of Roll's ((1977), p. 165) analysis implies that portfolio s is the tangency of the line from the origin to the minimum-variance boundary in the $[b_s, \sigma(r)]$ plane.

FIGURE 2

Portfolios that Minimize Variance Subject to Having a Particular Loading on State Variable *s*, but with No Constraint on Expected Return or Loadings on Other State Variables

When there is no risk-free security, these MMV portfolios are the curved boundary defined by the combinations $[q_{se}r_S + (1 - q_{se})r_{S+2}]$, where $S + 2$ is the global-minimum-variance portfolio of (8c). With a risk-free security, the portfolios are on the two line segments given by $[q_{se}r_S + (1 - q_{se})r_f]$, where s is the tangency of the line from the origin $[b_{fS} = \sigma(r_f) = 0.0]$ to the curved boundary. (If b_{S+2} were less than 0.0, s would be the tangency portfolio of a line from the origin to the negatively sloped portion of the curved boundary.) With or without a risk-free security, r_S is the return on the mimicking portfolio for state variable s of (8a). Technical point: the analysis in Merton (1972) implies that the curved boundary for risky securities is a hyperbola. Thus, the mimicking portfolio s of (8a) is well-defined only if $b_{S+2,s} \neq 0.0$.



B. The Set of Multifactor-Efficient Portfolios

Equation (11) says that any MMV portfolio is a portfolio of the r_s, $s = 1, \ldots, S+2$, of (8),

$$r_e = \sum_{s=1}^{S+2} q_{se} r_s. \tag{14}$$

Any combination of q_{se}, $s = 1, \ldots, S+2$, in (14) that sums to 1.0 is a multifactor-minimum-variance (MMV) portfolio. Which MMV portfolios are multifactor-efficient (ME)? Merton (1973) and Long (1974) argue that without further restrictions on investor tastes, one cannot restrict the loadings on the state variables in optimal portfolios. There is an ME portfolio for any set of b_{es}, $s = 1, \ldots, S$, and, thus, for any set of q_{se}, $s = 1, \ldots, S$, in (14). Intuitively, given the premiums for state-variable risks, some investors may take long positions in a state-variable risk, while others take short positions.

With respect to residual variance in (3), that is, return variance not explained by the state variables, the issue is more clear-cut. For risk-averse investors, only MMV portfolios that offer a positive tradeoff of expected return for residual variance are multifactor-efficient. To identify these portfolios, restate (14) to acknowledge that the role of the GMV portfolio $S+2$ is to ensure that in any MMV portfolio e, the weights on the $S+2$ portfolios of (8) sum to 1.0,

$$r_e = \sum_{s=1}^{S} q_{se} r_s + q_{S+1,e} r_{S+1} + \left[1 - \sum_{s=1}^{S} q_{se} - q_{S+1,e}\right] r_{S+2}. \tag{15}$$

Roughly speaking, (15) says that the weights q_{se}, $s = 1, \ldots, S$, can control the loadings of MMV portfolios on the state variables. To ensure that the portfolio weights sum to 1.0, the sum of $-q_{se}$, $s = 1, \ldots, S$, is invested in r_{S+2}. To control the tradeoff of expected return for residual variance in MMV portfolios, one can then vary the investments $q_{S+1,e}$ in r_{S+1} and $1 - q_{S+1,e}$ in r_{S+2}.

Varying $q_{S+1,e}$ in $[q_{S+1,e} r_{S+1} + (1 - q_{S+1,e}) r_{S+2}]$ generates the set of Markowitz' minimum-variance (MV) portfolios, which includes mean-variance-efficient (MVE) portfolios. An MVE portfolio maximizes expected return, given its return variance. But this implies that it maximizes expected return, given its return variance and whatever happen to be its state-variable loadings. MVE portfolios are thus multifactor-efficient. Conversely, an MV portfolio that is not MVE has minimum expected return, given its return variances, which means it has minimum expected return, given its return variance and state-variable loadings. It is thus multifactor-inefficient as well as mean-variance-inefficient.

In short, in terms of (15), a multifactor-efficient portfolio combines one of Markowitz' MVE portfolios (an MVE combination $[q_{S+1,e} r_{S+1} + (1 - q_{S+1,e}) r_{S+2}]$) with the S state-variable mimicking portfolios, with the positions in the mimicking portfolios financed by offsetting positions in the GMV portfolio $S+2$. All such combinations of an MVE portfolio with the state-variable mimicking portfolios are ME. In economic terms, Merton's ICAPM investors use Markowitz' MVE portfolios to optimize the tradeoff of expected return for non-state-variable return variance, and they combine MVE portfolios with state-variable mimicking portfolios to hedge uncertainty about future consumption-investment opportunities.

It is worth noting that since MVE portfolios are also multifactor-efficient, the investors of the Sharpe-Lintner CAPM (who do not differentiate among sources of risk) are covered by the ICAPM. Investors concerned with some but not all state-variable risks are also covered. They choose portfolios that maximize expected return, given their return variances and loadings on the state variables of interest. But such portfolios maximize expected return, given their return variances and loadings on all state variables. Like Markowitz's MVE portfolios, they are special cases of multifactor-efficient portfolios.

Finally, it is easy to show that any portfolio of multifactor-efficient portfolios where the weights on the component ME portfolios are positive, reduces to a combination of an MVE portfolio with the state-variable mimicking portfolios. Thus,

Proposition 5. All portfolios of positively-weighted ME portfolios are ME.

C. Risk-Free Borrowing and Lending

Like the Sharpe-Lintner CAPM, most treatments of the ICAPM assume that there is a risk-free security. Adding a risk-free security to the opportunity set simply involves adding a security with sure return r_f to the variance-minimization problem (4). Skipping the details, i) the risk-free security replaces the GMV portfolio $S+2$ in the basis set of (15). ii) Adding a risk-free security has no effect on the state-variable mimicking portfolios in (15); they are still defined by the security weights of (8b). iii) With a risk-free security, Markowitz's MV portfolios are combinations $[q_{Te}r_T + (1 - q_{Te})r_f]$, where portfolio T is the tangency of a line from r_f to the MVE boundary for risky securities in Figure 1. T thus replaces portfolio $S+1$ (the tangency of a line from the origin to the MVE boundary) in the basis set of (15).

In short, when there is a risk-free security, r_f replaces r_{S+2} in (15), r_T replaces r_{S+1}, and MMV portfolios are all combinations of r_f, r_T, and r_s, $s = 1, \ldots, S$, in

$$(16) \qquad r_e = \sum_{s=1}^{S} q_{se}r_s + q_{Te}r_T + \left[1 - \sum_{s=1}^{S} q_{se} - q_{Te}\right] r_f.$$

A multifactor-efficient (ME) portfolio again combines one of Markowitz' mean-variance-efficient portfolios ($[q_{Te}r_T + (1 - q_{Te})r_f]$, $q_{Te} \geq 0$ in Figure 1) with the S state-variable mimicking portfolios, with the positions in the mimicking portfolios financed by offsetting positions in the risk-free security. All such combinations of an MVE portfolio with the state-variable mimicking portfolios are ME.

Adding a risk-free security does not affect Proposition 5: any portfolio of positively-weighted multifactor-efficient portfolios reduces to a combination of an MVE portfolio with the state-variable mimicking portfolios, so all portfolios of positively-weighted ME portfolios are ME.

VII. Market Equilibrium and the Market Portfolio

This section discusses the main results of the ICAPM on spanning portfolios and the relation between multifactor risks and expected return. I show that when there is no risk-free security, the market portfolio M can replace portfolio $S + 1$ in the spanning set of (15). Or, when there is risk-free borrowing and lending, M can replace the tangency portfolio T in the spanning set of (16). I then show that the risk-return relation (2) of the ICAPM is just the first-order condition on the weights for securities in any MMV portfolio, applied to M. All these results use the fact, established first, that in a market equilibrium, the market portfolio must be multifactor-efficient.

A. Market Equilibrium

Market-clearing prices imply that supply equals demand for each security. Equivalently, a market equilibrium requires that when one combines the portfolios chosen by investors, weighting each by that investor's nonnegative share of invested wealth, one gets the market portfolio M. Since investors choose ME portfolios, M is a portfolio of positively-weighted ME portfolios, so, from Proposition 5,

Proposition 6. In a market equilibrium, the value-weight market portfolio M is multifactor-efficient.

Suppose there is no risk-free security and the market portfolio has a nonzero weight $q_{S+1,e}$ in (15). In other words, M is not just a combination of the GMV portfolio $S + 2$ and the S state-variable mimicking portfolios. Then, since M is multifactor-efficient, Proposition 4 says that replacing r_{S+1} with r_M in (15) produces another basis set that spans MMV and ME portfolios. Thus,

Proposition 7. When there is no risk-free security, MMV and ME portfolios are spanned by $S + 2$ portfolios that include M, the S state-variable mimicking portfolios, and the GMV portfolio.

Alternatively, suppose there is a risk-free security and M has a nonzero weight on portfolio T in (16). Since M is multifactor-efficient, Proposition 4 says that replacing r_T with r_M in (16) produces another basis set for MMV and ME portfolios. Thus,

Proposition 8. When there is a risk-free security f, MMV and ME portfolios are spanned by $S + 2$ portfolios that include f, M, and the S state-variable mimicking portfolios.

Proposition 8 is commonly taken to be the central portfolio-spanning result of the ICAPM.

A caveat is in order. The fact that the market portfolio can replace portfolio $S + 1$ in (15), or the tangency portfolio T in (16), is implied by Proposition 4: any $S + 2$ linearly-independent MMV portfolios are a basis set. In particular, replacing $S + 1$ (or T) with M does not mean that M is mean-variance-efficient (MVE). In the ICAPM, M is multifactor-efficient, but it almost surely is not MVE.

Since it goes to the heart of the difference between the ICAPM and the CAPM, this point warrants emphasis. ICAPM investors use Markowitz' MVE portfolios to optimize the tradeoff of expected return for generalized return variance. But ICAPM investors are also concerned with state-variable risks, and optimal (multifactor-efficient) portfolios typically combine an MVE portfolio with some or all of the state-variable mimicking portfolios. This means that the market portfolio (which aggregates the portfolios chosen by investors) almost surely combines an MVE portfolio with some or all of the state-variable mimicking portfolios. Thus, M is multifactor-efficient, but it is not mean-variance-efficient.

B. The Risk-Return Relation (2)

The characterizations of MMV spanning portfolios in (15) and (16) derive from the first-order condition on security weights in any MMV portfolio. The multifactor efficiency of the market portfolio then allows one to transform (15) and (16) into the ICAPM spanning Propositions 7 and 8.

The ICAPM risk-return relation (2) is also just a manipulated version of the first-order condition on security weights in an MMV portfolio, applied to M. The Appendix shows that when there is a risk-free security, the expression for the weights of securities in any MMV portfolio e implies

$$E(r_i) - r_f = \beta_{ie}\left[E(r_e) - r_f\right] + \sum_{s=1}^{S} \beta_{is}\left[E(r_s) - r_f\right], \quad i = 1, \ldots, N, \tag{17}$$

where β_{ie} and β_{is}, $s = 1, \ldots, S$, are the slopes in the multiple regression of the return on security i on the returns on the MMV portfolio e and the S state-variable mimicking portfolios. The tedious algebra in the Appendix just says that the security weights that satisfy the first-order condition for the MMV portfolio e also produce regression slopes (multifactor risk measures) that satisfy the risk-return relation (17).

The risk-return equation (2) of the ICAPM is (17) with the market portfolio M as the MMV portfolio e. A market equilibrium indeed implies that M is multifactor-efficient. Thus, M can be the MMV portfolio e in (17), and (2) holds.

My main point is now clear. The multifactor risk-return relation (2) is just the fact that in the ICAPM, securities must be priced so that their weights in the market portfolio satisfy the first-order condition on the weights in any multifactor-efficient portfolio. This generalizes the result that the risk-return relation (1) of the CAPM is just the fact that in a market equilibrium, M must be mean-variance-efficient (Fama (1976), Roll (1977)). Moreover, the difference between the ICAPM risk-return relation (2) and the CAPM risk-return relation (1) is that in the ICAPM, the market portfolio is multifactor-efficient, but it is not mean-variance-efficient. Thus, market β does not suffice to explain expected return.

VIII. Odds and Ends

The spanning Proposition 8 and the risk-return relation (2) are the main results of the ICAPM. I turn now to peripheral issues neatly tackled with the multifactor-efficiency story. I consider i) the risk premiums in (2), ii) the special role of the market portfolio in tests of the ICAPM, and iii) the links between Merton's ICAPM and Ross' APT.

A. Expected Risk Premiums

The ICAPM provides a ready-made open system for testing hypotheses about which state variables are of concern to investors and so affect expected returns. But this flexibility has a cost. The signs of the premiums for state-variable risks, $E(r_s) - r_f$, $s = 1, \ldots, S$ in (2), are indeterminate. They can be positive or negative, depending on investor tastes for different aspects of future consumption-investment opportunities.

The risk aversion of investors does imply a positive premium for the residual variance of an ME portfolio, that is, return variance left unexplained by the state variables in (3). Does this imply a positive expected premium, $E(r_M) - r_f$, in the market return? Since M is multifactor-efficient, the premium for the residual variance of M is positive. But, at least in principle, this positive component of $E(r_M) - r_f$ can be offset by negative premiums due to the properties of M as a hedge against state-variable uncertainty. Thus, the sign of $E(r_M) - r_f$ is also indeterminate!

In truth, even the positive premium $E(r_T) - r_f$ for the MVE tangency portfolio in Figure 1 requires that the expected return $E(r_{S+2})$ on the GMV portfolio is greater than r_f. In the CAPM, market clearing requires that $E(r_{S+2}) > r_f$, but I see no similar argument for the ICAPM. If $E(r_{S+2}) < r_f$, then (as in Figure 3) T is the tangency of a line from r_f to the negatively sloped portion of the minimum-variance boundary, and $E(r_T) - r_f < 0.0$. MVE portfolios are then combinations $[q_{Te} r_T + (1 - q_{Te}) r_f]$ with $q_{Te} \leq 0.0$, and multifactor-efficient portfolios also involve short positions in T. Note, though, that Propositions 5, 6, and 8 and the risk-return relation (2) hold whether or not T is MVE.

B. The Special Role of the Market Portfolio

Huberman and Kandel (1987) shows that if a set of Y portfolios spans Markowitz' minimum-variance (MV) portfolios, then in the regression of the return on any security on the returns on the Y portfolios, the intercept is 0.0 and the slopes sum to 1.0. Since MV portfolios are a subset of the set of multifactor-minimum-variance (MMV) portfolios, Proposition 4 implies that any basis set of MMV portfolios spans MV portfolios. The return on any security i can thus be expressed as

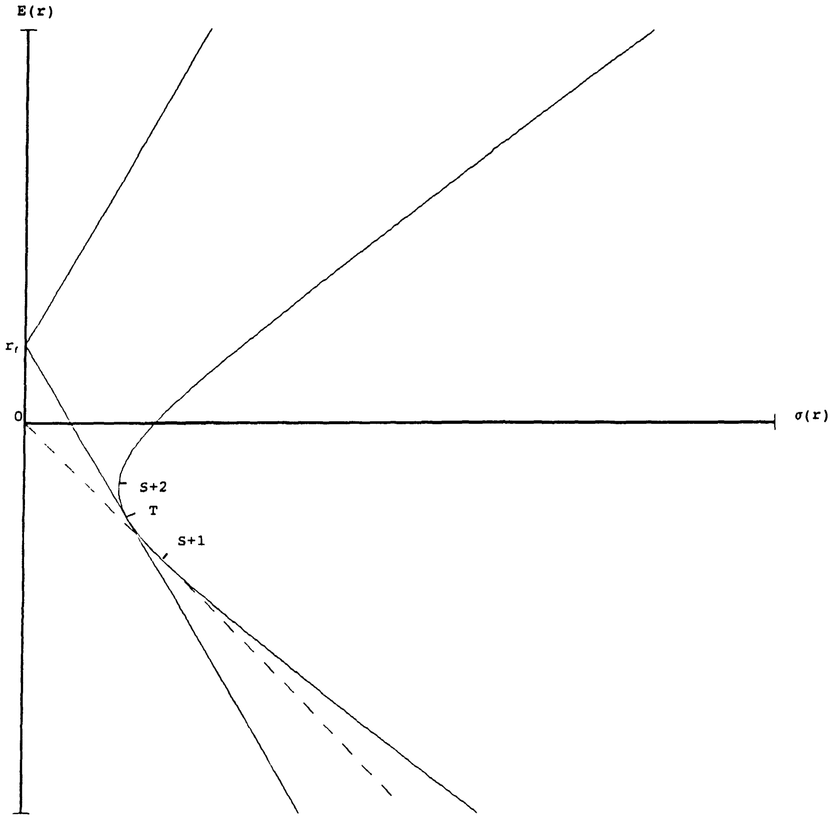
$$(18) \qquad r_i - r_e = \sum_{v=1}^{S+1} \beta_{iv} (r_v - r_e) + \epsilon_i,$$

where r_e and r_v, $v = 1, \ldots, S + 1$, are the returns on any basis set of $S + 2$ MMV portfolios, e is any portfolio from this set, and I use HK's result that the slope for

FIGURE 3

Markowitz' Mean-Variance-Efficient (MVE) Portfolios when the Expected Return on the GMV Portfolio of Risky Securities is Less than the Risk-Free Rate [$E(r_{S+2}) < r_f$]

With risk-free borrowing and lending, MVE portfolios are combinations [$q_{Te}r_T + (1 - q_{Te})r_f$], $q_{Te} \leq 0.0$, where T is the tangency of a line from r_f to the negatively sloped portion of Markowitz' minimum-variance boundary for risky securities. When there is no risk-free security, the position of the portfolio $S+1$, defined by the weights for securities in (8b), depends on whether the expected return on the GMV portfolio $S+2$ is greater or less than 0.0. When $E(r_{S+2}) > 0.0$, portfolio $S+1$ is the tangency of a line from the origin to the MVE boundary (Figure 1). When, as shown here, $E(r_{S+2}) < 0.0$, portfolio $S+1$ is the tangency of a line from the origin to the negatively sloped portion of the minimum-variance boundary. MVE portfolios are then combinations [$q_{S+1,e}r_{S+1} + (1 - q_{S+1,e})r_{S+2}$], with $q_{S+1,e} \leq 0.0$. Multifactor-efficient portfolios always combine an MVE portfolio with the S mimicking portfolios for the state variables, defined by the weights in (8a).



any portfolio e in a basis set is 1.0 minus the sum of the slopes on the other $S+1$ portfolios. Taking expected values in (18) yields

$$E(r_i - r_e) = \sum_{v=1}^{S+1} \beta_{iv} E(r_v - r_e). \quad (19)$$

Thus, there is a risk-return relation (19), like (2), for any basis set of MMV portfolios. And excess returns can be measured relative to any portfolio e in a basis set. Note, though, that these results are just implications of the algebra of MMV and MV portfolios. They say nothing about asset pricing.

The point merits discussion. Suppose asset pricing is governed by an ICAPM in which there are S state variables that carry special premiums. It is nevertheless true that any two of Markowitz' minimum-variance portfolios can describe the expected returns on all securities and portfolios. As Fama (1976) and Roll (1977) emphasize, this result is implied by the algebra of MV portfolios. It does not imply CAPM pricing. The testable equilibrium condition of the CAPM is that the market portfolio M is priced to be mean-variance efficient, so M can be one of the two MV portfolios used to describe expected returns.

Similarly, consider the set of MMV portfolios that minimize variance subject to constraints on expected return and loadings on a given subset $S' < S$ of the state variables. The algebra of MMV portfolios implies that any $S' + 2$ portfolios from this set can describe the expected returns on all securities and portfolios; that is, there are equations like (18) and (19) for these $S' + 2$ portfolios. But this does not mean expected returns are governed by an ICAPM in which only the $S' < S$ state variables carry special premiums. The equilibrium condition of an S' state-variable ICAPM is that the market portfolio M is one of the $S' + 2$ MMV portfolios that can used to describe expected returns.

In short, the market portfolio has a special role in Merton's ICAPM, much like its role in the Sharpe-Lintner CAPM. The main testable implication of the CAPM (Fama (1976), Roll (1977)) is that equilibrium security prices require that M is mean-variance-efficient. The main testable implication of the ICAPM is that securities must be priced so that M is multifactor-efficient.

IX. The Market Return and the APT

A. Is the Market Return a State Variable?

The market portfolio and the state-variable mimicking portfolios seem to have similar roles in the risk-return relation (2). This suggests that r_M should be treated as just another state variable. This approach, however, is likely to obscure the economic role of M in (2). In the ICAPM, the number of securities is finite and individual securities can be nontrivial in value relative to total wealth. Thus, multifactor-efficient (ME) portfolios generally have positive residual variances in (3). With risk-averse investors, the residual variances of ME portfolios are undiversifiable risks that must be compensated in expected returns. With S state variables, the residual variances of ME portfolios are the reason $S + 2$ (rather than $S + 1$) portfolios are needed to span MMV portfolios, and $S + 1$ (rather than S) risk premiums are needed to describe expected returns.

In the basis set of (16), the tangency portfolio T captures variation in expected returns that is independent of loadings on the S state variables—the variation in expected returns due to the residual variances of ME portfolios. But this role can be played by any ME (or MMV) portfolio that is not just a combination of the state-variable mimicking portfolios. In the basis set of (2), the market portfolio fills in for T. Put a bit differently, M appears in (2), along with the mimicking portfolios for the state variables, because in the ICAPM, M is multifactor-efficient and has a positive residual variance in (3). Thus, the part of r_M legitimately treated as a separate state variable is the variation in r_M—the uncertainty about total invested wealth—not explained by the other S state variables.

B. The ICAPM and the APT

The argument that the market portfolio appears in the ICAPM risk-return relation (2) because r_M contains undiversifiable residual variance becomes clear if Merton's ICAPM is contrasted with Ross' APT, especially Connor's (1984) version of the APT.

First some housekeeping. Since a state variable may affect the return on only one security, Merton's state variables need not be Ross' common factors in returns. If many securities have private state variables, however, Merton's ICAPM is empty; its restrictions on expected returns are too loose. Conversely, all the common factors in Ross' APT need not be state variables that give rise to special hedging demands by investors. The comparison of the two models is simplified, however, if one assumes that Merton's state variables are the common factors of the APT.

The pure-arbitrage version of the APT is easily summarized. By definition, a perfectly diversified portfolio has no residual variance in (3). Suppose there are S perfectly diversified state-variable mimicking portfolios that, along with a risk-free security, generate perfectly diversified portfolios with any loadings on the state variables. The APT then says that the ICAPM risk-return relation (2) holds for these perfectly diversified portfolios, but without the term, $\beta_{pM}[E(r_M - r_f)]$, for the market portfolio,

$$E\left(r_p\right) - r_f \quad = \quad \sum_{s=1}^{S} \beta_{ps} \left[E(r_s) - r_f\right]. \tag{20}$$

Intuitively, perfectly diversified portfolios have no residual variances in (3), so the absence of arbitrage opportunities implies that their expected returns do not contain the compensation for residual variance captured by the market term in (2). The expected returns on perfectly diversified portfolios depend only on the sensitivities of their returns to the returns on the state-variable mimicking portfolios.

The pure-arbitrage version of the APT delivers (20) only for perfectly diversified portfolios. Connor (1984) develops an APT in which (20) holds for all securities. (See also Chen and Ingersoll (1983).) Like the pure-arbitrage APT, Connor assumes that the state-variable mimicking portfolios are perfectly diversified. Like Merton, however, Connor uses utility maximization to develop an exact expression for expected security returns. Unlike Merton, Connor assumes that the market portfolio is perfectly diversified. My multifactor-efficiency story provides a simple ICAPM interpretation of the effect of this key assumption.

In the ICAPM, when there is a risk-free security f, all MMV portfolios are portfolios of f, the market portfolio M, and the S state-variable mimicking portfolios. But if M and the mimicking portfolios are perfectly diversified (Connor's assumptions), they have no residual variances in (3). This means there is a portfolio of the risk-free security and the state-variable mimicking portfolios that has the same loadings on the state variables as M and is perfectly correlated with M. The absence of arbitrage then implies that this portfolio and M must have the same expected return. M is thus redundant in the basis set of (2). All MMV portfolios are combinations of the risk-free security and the S state-variable mimicking portfolios. Proposition 3 in Huberman and Kandel (1987) then says that M is also redundant for describing expected returns; expected returns on all securities are given by (20).

In short, Connor's APT can be viewed as a special case of Merton's ICAPM in which all MMV and ME portfolios are perfectly diversified so there is no variation in expected returns independent of loadings on the state variables. $S + 1$ linearly-independent MMV portfolios span MMV portfolios and describe expected security returns. In contrast, in the general version of the ICAPM, the market portfolio is not perfectly diversified. The residual variances of M and other ME portfolios are then undiversifiable risks that must be compensated in expected returns. (Dybvig (1983) and Grinblatt and Titman (1983) can be interpreted as making a similar point.) Because of these residual risks (and the risks of the S state variables), $S + 2$ portfolios are needed to span MMV and ME portfolios and to describe expected returns.

X. Conclusions

The Sharpe-Lintner CAPM starts with assumptions that imply that investors hold mean-variance-efficient (MVE) portfolios. Assumptions are added to guarantee that the market portfolio M is MVE. The risk-return relation of the CAPM is then just the application to M of the condition on security weights that holds in any MVE portfolio (Fama (1976), ch. 8, Roll (1977)).

There is a similar story for Merton's intertemporal CAPM. ICAPM investors hold multifactor-efficient portfolios that generalize the notion of portfolio efficiency. Like CAPM investors, ICAPM investors dislike wealth uncertainty, and they use Markowitz' MVE portfolios to optimize the tradeoff of expected return for general sources of return variance. But ICAPM investors are also concerned with hedging more specific aspects of future consumption-investment opportunities, such as the relative prices of consumption goods and the risk-return tradeoffs they will face in capital markets. As a result, the typical multifactor-efficient portfolio of the ICAPM combines an MVE portfolio with hedging portfolios that mimic uncertainty about consumption-investment state variables.

As in the CAPM, the relation between expected return and multifactor risks in the ICAPM is the condition on the weights for securities that holds in any multifactor-efficient portfolio, applied to the market portfolio M. And just as market equilibrium in the CAPM requires that M is mean-variance-efficient, in the ICAPM, market-clearing prices imply that M is multifactor-efficient.

Appendix

This appendix derives i) equation (7) for the security weights in any MMV portfolio, ii) Proposition 4, and iii) the risk-return relation (2) of the ICAPM.

A. The Weights of (7) for Securities in an MMV Portfolio

Given the Lagrange multipliers for the MMV portfolio e, (6) is a set of N linear equations that can be solved for the security weights $X_e = (x_{ie}, \ldots, x_{Ne})'$. Define:

V = $N \times N$ nonsingular matrix of covariances σ_{ij} between security returns, with $D = V^{-1}$;
B = $N \times S$ matrix of loadings b_{is} of security returns on the S state variables;
$E(R)$ = $N \times 1$ vector of expected security returns, $E(r_i)$, $i = 1, \ldots, N$;
L_e = $S \times 1$ vector of λ_{se}, $s = 1, \ldots, S$;
1_N = $N \times 1$ vector of 1s.

In matrix form, the N equations of (6) are then

$$VX_e = BL_e + \lambda_{S+1,e}E(R) + \lambda_{S+2,e}1_N, \tag{21}$$

and the security weights that solve the variance-minimization problem (4) are

$$X_e = DBL_e + \lambda_{S+1,e}[DE(R)] + \lambda_{S+2,e}[D1_N]. \tag{22}$$

Equation (22) is the matrix version of (7).

B. Proposition 4

Equation (10) for the security weights in any MMV portfolio e can be written as

$$X_e = X_{S+2}Q_e = X_{S+2}ZZ^{-1}Q_e, \tag{23}$$

where X_{S+2} is the $N \times (S+2)$ matrix of security weights for the $S+2$ portfolios of (8), Q_e is the $(S+2) \times 1$ vector of weights for these portfolios of (9), and Z is any nonsingular $(S+2) \times (S+2)$ matrix with $Z'1_{S+2} = 1_{S+2}$.

Proposition 2 implies that $X_{S+2}Z$ is a set of weights for S+2 linearly-independent MMV portfolios. Equation (23) then says that $Z^{-1}Q_e$ is the $(S+2) \times 1$ vector of weights for these portfolios that produces the weights for securities in the MMV portfolio e. Since e can be any MMV portfolio, (23) says that the $S+2$ MMV portfolios $X_{S+2}Z$ can generate all MMV portfolios. Moreover, Proposition 2 implies that there is a Z such that $X_{S+2}Z$ generates any particular $S+2$ linearly-independent MMV portfolios. One can infer that (Proposition 4) any $S+2$ linearly-independent MMV portfolios are a basis set for MMV portfolios.

C. The Risk-Return Relation (2) of Theorem 2

I next show that the risk-return relation (2) is a manipulated version of the first-order condition on the weights for securities in an MMV portfolio. The analysis fleshes out Breeden ((1979), fn. 7).

When there is a risk-free security, the matrix version of the first-order condition (6) on the weights for securities in the target MMV portfolio e is

$$\begin{aligned} E(R) - R_f &= \left(1/\lambda_{S+1,e}\right) VX_e - B\left(L_e/\lambda_{S+1,e}\right) \\ &= \left(1/\lambda_{S+1,e}\right) V_{ie} - VDB\left(L_e/\lambda_{S+1,e}\right), \end{aligned} \tag{24}$$

where, $E(R) - R_f$ is the $N \times 1$ vector of expected excess security returns, and $V_{ie} = VX_e$ is the vector of $\text{cov}(r_i, r_e)$, $i = 1, \ldots, N$.

Let X_s be the $N \times S$ matrix of weights for securities in the mimicking portfolios for the S state variables in (8a). Since the columns of DB are proportional to the columns of X_s, no harm is done (I simply redefine the Lagrange multipliers L_e), if I replace DB in (24) with X_s. Then,

$$\begin{aligned} E(R) - R_f &= \left(1/\lambda_{S+1,e}\right) V_{ie} - VX_s\left(L_e/\lambda_{S+1,e}\right) \\ &= \left(1/\lambda_{S+1,e}\right) V_{ie} - V_{is}\left(L_e/\lambda_{S+1,e}\right), \end{aligned} \tag{25}$$

where $V_{is} = VX_s$ is the $N \times S$ matrix of covariances between security returns and the mimicking portfolio returns. If I define the $N \times (S+1)$ matrix $V_{i,es} = [V_{ie}, V_{is}]$, (25) becomes

$$E(R) - R_f = V_{i,es} \begin{pmatrix} 1/\lambda_{S+1,e} \\ -L_e/\lambda_{S+1,e} \end{pmatrix}. \tag{26}$$

The final step to the risk-return relation (2) involves expressing the Lagrange multipliers in (26) in terms of the expected excess returns on the target MMV portfolio e and the S mimicking portfolios for the state variables. Let $E(R_s) - R_f$ be the $S \times 1$ vector of expected excess returns on the mimicking portfolios, and let $X_{es} = [X_e, X_s]$ be the $N \times (S+1)$ matrix of weights for securities in e and the S mimicking portfolios. The vector of expected excess returns on e and the S mimicking portfolios can then be expressed in terms of the expected excess returns on securities given by (26) as

$$\begin{aligned} \begin{bmatrix} E(r_e) - r_f \\ E(R_s) - R_f \end{bmatrix} &= X'_{es} V_{i,es} \begin{pmatrix} 1/\lambda_{S+1,e} \\ -L_e/\lambda_{S+1,e} \end{pmatrix} \\ &= V_{es,es} \begin{pmatrix} 1/\lambda_{S+1,e} \\ -L_e/\lambda_{S+1,e} \end{pmatrix}, \end{aligned} \tag{27}$$

where $V_{es,es}$ is the $(S+1) \times (S+1)$ covariance matrix for the returns on e and the S mimicking portfolios. If $D_{es,es}$ is the inverse of $V_{es,es}$, then (26) says that the vector of Lagrange multipliers is

$$D_{es,es} \begin{bmatrix} E(r_e) - r_f \\ E(R_s) - R_f \end{bmatrix} = \begin{pmatrix} 1/\lambda_{S+1,e} \\ -L_e/\lambda_{S+1,e} \end{pmatrix}. \tag{28}$$

Substituting (28) into (26) gives,

$$E(R) - R_f = V_{i,es} D_{es,es} \begin{bmatrix} E(r_e) - r_f \\ E(R_s) - R_f \end{bmatrix}. \tag{29}$$

Finally, the $N \times (S + 1)$ matrix of slopes in the (with intercept) regressions of security returns on the returns on the MMV portfolio e and the S mimicking portfolios for the state variables is

$$\beta_{i,es} = V_{i,es} D_{es,es}, \tag{30}$$

and (29) becomes

$$E(R) - R_f = \beta_{i,es} \begin{bmatrix} E(r_e) - r_f \\ E(R_s) - R_f \end{bmatrix}. \tag{31}$$

Equation (31) is a manipulated version of (24), the condition on the weights for securities in any MMV portfolio e. The tedious algebra above simply says that the security weights that solve (24) also deliver regression slopes that satisfy (31). The risk-return relation (2) is (31) with the market portfolio M as the MMV portfolio e. Market-clearing prices for securities indeed imply that M is multifactor-efficient. Thus, M can be the MMV portfolio e in (31).

There is a simpler path to (2). Like any basis set, the set of MMV portfolios that includes the risk-free security, the market portfolio, and the S state-variable mimicking portfolios can generate all of Markowitz' minimum-variance portfolios. Proposition 3 of Huberman and Kandel (1987) then implies that the relation between the excess return on any security and the excess returns on the market portfolio and the mimicking portfolios for the state variables is the linear regression,

$$r_i - r_f = \beta_{iM} \left(r_M - r_f \right) + \sum_{s=1}^{S} \beta_{is} \left(r_s - r_f \right) + \epsilon_i, \tag{32}$$

Taking expected values in (32) yields (2).

But the elegant conciseness of this proof conceals the main point of equations (24) to (31), indeed the main point of this paper: the risk-return relation (2) of Merton's ICAPM is just the first-order condition on the weights for securities in any MMV portfolio, applied to the market portfolio M.

References

Banz, R. W. "The Relationship between Return and Market Value of Common Stocks." *Journal of Financial Economics,* 9 (1981), 3–18.

Basu, S. "The Relationship between Earnings Yield, Market Value, and Return for NYSE Common Stocks: Further Evidence." *Journal of Financial Economics,* 12 (1983), 129–156.

Bhandari, L. C. "Debt/Equity Ratio and Expected Common Stock Returns: Empirical Evidence." *Journal of Finance,* 43 (1988), 507–528.

Black, F. "Capital Market Equilibrium with Restricted Borrowing, *Journal of Business,* 45 (1972), 444–455.

Breeden, D. T. "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities" *Journal of Financial Economics,* 7 (1979), 265–296.

Chan, L. K.; Y. Hamao; and J. Lakonishok. "Fundamentals and Stock Returns in Japan." *Journal of Finance*, 46 (1991), 1739–1764.

Chen, N.-F., and J. Ingersoll. "Exact Pricing in Linear Factor Models with Infinitely Many Assets: A Note." *Journal of Finance,* 38 (1983), 985–988.

Chen, N.-F.; R. Roll; and S. A. Ross. "Economic Forces and the Stock Market." *Journal of Business,* 59 (1986), 383–403.

Connor, G. "A Unified Beta Pricing Theory." *Journal of Economic Theory*, 34 (1984), 13–31.

Dybvig, P. H. "An Explicit Bound on Individual Assets' Deviations from APT Pricing in a Finite Economy." *Journal of Financial Economics,* 12 (1983), 493–496.

Elton, E. J., and M. J. Gruber. "Portfolio Analysis with a Nonnormal Multi-Index Return-Generating Process." *Review of Quantitative Finance and Accounting,* 2 (1992), 5–16.

Fama, E. F. "Multiperiod Consumption-Investment Decisions." *American Economic Review,* 60 (1970), 163–174.

________________. *Foundations of Finance.* New York, NY: Basic Books (1976).

Fama, E. F., and K. R. French. "The Cross-Section of Expected Stock Returns." *Journal of Finance,* 47 (1992), 427–465.

________________. "Common Risk Factors in the Returns on Stocks and Bonds." *Journal of Financial Economics,* 33 (1993), 3–56.

________________. "Multifactor Explanations of Asset-Pricing Anomalies." *Journal of Finance,* 51 (1996), 55–84.

Grinblatt, M., and S. Titman. "Factor Pricing in a Finite Economy." *Journal of Financial Economics,* 12 (1983), 497–507.

________________. "The Relation between Mean-Variance Efficiency and Arbitrage Pricing." *Journal of Business,* 60 (1987), 97–112.

Huberman, G., and S. Kandel. "Mean-Variance Spanning." *Journal of Finance,* 42 (1987), 873–888.

Huberman, G.; S. Kandel; and R. F. Stambaugh. "Mimicking Portfolios and Exact Arbitrage Pricing." *Journal of Finance,* 42 (1987), 1–9.

Jobson, J. D., and B. Korkie. "Some Tests of Linear Asset Pricing with Multivariate Normality." *Canadian Journal of Administrative Sciences,* 2 (1985), 114–138.

Lintner, J. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." *Review of Economics and Statistics,* 47 (1965), 13–37.

Long, J. B., Jr. "Stock Prices, Inflation and the Term Structure of Interest Rates." *Journal of Financial Economics,* 1 (1974), 131–170.

Markowitz, H. *Portfolio Selection: Efficient Diversification of Investments.* New York, NY: Wiley (1959).

Merton, R. C. "An Analytic Derivation of the Efficient Portfolio Frontier." *Journal of Financial and Quantitative Analysis,* 7 (1972), 1151–1172.

________________. "An Intertemporal Capital Asset Pricing Model." *Econometrica,* 41 (1973), 867–887.

Roll, R. "A Critique of the Asset Pricing Theory's Tests' Part I: On Past and Potential Testability of the Theory." *Journal of Financial Economics,* 4 (1977), 129–176.

Rosenberg, B.; K. Reid; and R. Lanstein. "Persuasive Evidence of Market Inefficiency." *Journal of Portfolio Management,* 11 (Spring 1985), 9–17.

Ross, S. A. "The Arbitrage Theory of Capital Asset Pricing." *Journal of Economic Theory,* 13 (1976), 341–360.

Sharpe, W. F. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." *Journal of Finance,* 19 (1964), 425–442.