EXPLICIT SOLUTION TO AN INVERSE FIRST-PASSAGE TIME PROBLEM FOR LÉVY PROCESSES. APPLICATION TO COUNTERPARTY CREDIT RISK

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Abstract. For a given Markov process X and survival function $\overline{H}$ on $\mathbb{R}_+$, the *inverse first-passage time problem* (IFPT) is to find a barrier function $b:\mathbb{R}_+\to[-\infty,+\infty]$ such that the survival function of the first-passage time $\tau_b=\inf\{t\geq 0: X(t)\leq b(t)\}$ is given by $\overline{H}$. In this paper we consider a version of the IFPT problem where the barrier is *fixed at zero* and the problem is to find an entrance law μ and a time-change I such that for the time-changed process $X\circ I$ the IFPT problem is solved by a constant barrier at the level zero. For any Lévy process X satisfying a Cramér assumption, we identify explicitly the solution of this problem, which is given in terms of a quasi-invariant distribution of the process X killed at the epoch of first entrance into the negative half-axis. For a given multi-variate survival function $\overline{H}$ of generalised frailty type we construct subsequently an explicit solution to the corresponding IFPT with the barrier level fixed at zero. We apply these results to the valuation of financial contracts that are subject to counterparty credit risk.

1. Introduction

Financial models incorporating the idea that a firm defaults on its debt when the value of the debt exceeds the value of the firm were originally introduced by Merton [25]. Because 'firm value' cannot be directly measured, later contributors such as Longstaff & Schwartz [24] and Hull & White [17] have moved to stylized models in which default occurs when some process $Y(t)$ – interpreted as 'distance to default' – crosses a given, generally time-varying, barrier $b(t)$. The risk-neutral distribution of the default time can be inferred from the firm's credit default swap spreads, and Hull & White [17] provide a numerical algorithm to determine $b(t)$ such that the first hitting time distribution H is equal to this market-implied default time distribution.

As we will show, these calculations are greatly simplified if, instead of starting at a fixed point $Y(0)=x>0$ and calibrating the barrier $b(t)$ we fix the barrier at $b(t)\equiv 0$ and start Y at a random point $Y(0)=Y_0$, where Y_0 has a distribution function F on $\mathbb{R}^+$, to be chosen. If we combine this with a deterministic time change then it turns out that essentially any continuous distribution H can be realized in this way, often with closed-form expressions for F.

In precise terms, the *inverse first-passage time* (IFPT) problem may be described as follows. Let (Y,P^μ) be a real-valued Markov process with càdlàg[1] paths that has entrance law μ on $\mathbb{R}_+\backslash\{0\}$ (*i.e.*, $P^\mu(Y_0\in \mathrm{d}x)=\mu(\mathrm{d}x)$). Given a CDF H on $\mathbb{R}_+$, the IFPT for the process (Y,P^μ) is to find a barrier function $b:\mathbb{R}_+\to[-\infty,+\infty]$ such that the first-passage time τ_b^Y of the process Y below the barrier b has CDF H:

$$P^\mu(\tau_b^Y\leq t)=H(t),\qquad t\in\mathbb{R}_+, \tag{1.1}$$
$$\text{with } \tau_b^Y=\inf\{t\geq 0: Y_t\in(-\infty,b(t))\}.$$

Recently there has been a renewed interest in the IFPT problem, in good part motivated by the above questions of credit risk modeling. Chen *et al.* [11] proved existence and uniqueness of the IFPT of an arbitrary continuous CDF on $\mathbb{R}_+$ for a diffusion with smooth bounded coefficients and strictly positive volatility function. In [1, 16, 17, 29, 30] a number of methods have been developed to compute this boundary, which is in general non-linear. Zucca & Sacerdote [30] analyse a Monte Carlo approximation method and a method based on the discretization

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[1] càdlàg = right-continuous with left-limits

of the Volterra integral equation satisfied by the boundary, which was derived in Peskir [27], while related integral equations are studied in Jaimungal *et al.* [20]. Avellaneda & Zhu [1] derive a free boundary problem for the density of a diffusion killed upon first hitting the boundary, where the free boundary is the solution to the IFPT, and Cheng *et al.* [12] established the existence and uniqueness of a solution to this free-boundary problem. A related "smoothed" version of the IFPT problem is considered in Ettinger *et al.* [15]: for any prescribed life-time it is shown that there exists a unique continuously differentiable boundary for which a standard Brownian motion killed at a rate that is a given function of this boundary exactly has the prescribed life-time.

In this paper we consider a related inverse problem where the barrier is *fixed* to be equal to zero, and the problem is to identify in a given family a stochastic process of which the first-passage time below the level zero has the given probability distribution. For a given Markov process X, the class of stochastic processes that we consider consists of the collection $(P^\mu, X\circ I)$ that is obtained by time-changing X by a continuous increasing function I and by varying the entrance law μ of X over the set of all probability measures on the positive half-line. Here $I:\mathbb{R}_+\to[0,\infty]$ is a function that is continuous and increasing on its domain, i.e. at all t for which $I(t)$ is finite, and the time-changed process $X\circ I=\{(X\circ I)(t), t\in\mathbb{R}_+\}$ is defined by $X\circ I(t)=X(I(t))$ if $I(t)$ is finite, and by $\limsup_{t\to\infty}X(t)$ otherwise.

Definition 1.1. For a continuous CDF H, the *randomised and time-changed inverse first-passage problem* (RIFPT) is to find a probability measure μ on $(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+))$ and an increasing continuous function $I:\mathbb{R}_+\to[0,\infty]$ such that for the time-changed process $Y=X\circ I$ the first-passage time of Y into the negative half-line $(-\infty,0)$ has CDF H:

$$P^\mu(\tau_0^Y\le t)=H(t),\qquad t\in\mathbb{R}_+,\tag{1.2}$$
$$\text{with } \tau_0^Y=\inf\{t\in\mathbb{R}_+: Y_t\in(-\infty,0)\}.$$

The fact that the boundary is constant and known is helpful for practical implementation of the model, *e.g.* in subsequent counterparty risk valuation computations and for the matching of model- and market prices.

In this paper we concentrate on the case where X is a Lévy process satisfying a Cramér condition. The class of Lévy processes has been extensively deployed in financial modeling, see the monograph Cont & Tankov [13]. For the general theory of Lévy processes we refer to the monographs Applebaum [2], Bertoin [4], Kyprianou [21] and Sato [28].

The key step is to determine, for some $\lambda\in\mathbb{R}_+$, a λ-invariant distribution for the process X killed at the first hitting time of 0; see Definition 2.4 below. If μ is λ-invariant then under P^μ the first passage time τ_0^X is exponentially distributed with parameter λ, so (μ,I) with $I(t)=t$ solves the RIFPT problem when H is $\text{Exp}(\lambda)$. The solution for other continuous distribution functions H is then obtained by an obvious deterministic time change.

The paper is structured as follows. In Section 2 we formulate the problem and state the main results for the RIFPT problem, Theorems 2.2 and 2.6. The proof of Theorem 2.2 is also given, together with an illustrative example where the Lévy process is drifting Brownian motion. In Section 3 a multi-dimensional version of the RIFPT theorem is stated for a specific class of multivariate default-time distributions; its proof follows quite easily given the results of Section 2. The proof of Theorem 2.6 involves the relationship between first-passage times and the so-called Wiener-Hopf factors; these matters are discussed in Section 4. In Section 5 the results of Theorem 2.6 are established for the special case of *mixed-exponential Lévy processes.* The proof is then completed in Section 6, exploiting the fact that mixed-exponential Lévy processes are dense in the class of Lévy processes. The concluding Section 7 demonstrates the application of our results to a problem of counterparty risk valuation.

2. IFPT Problem formulation and main results

Let $(\Omega,\mathcal{F},\mathbf{F},P)$ be a filtered probability space with completed filtration $\mathbf{F}=\{\mathcal{F}_t\}_{t\ge0}$, and X be an $\mathbf{F}$-Lévy process, i.e., an $\mathbf{F}$-adapted stochastic process with càdlàg paths that has stationary independent increments, with $X_0 = 0$ and the property that for each $s \leq t < u$ the increment $X_u - X_t$ is independent of $\mathcal{F}_s$. Let $\{P_x, x \in \mathbb{R}\}$ be the family of probability measures corresponding to shifts of the Lévy process X by x and, more generally, denote by P^μ the family of measures with entrance law (the distribution of X_0) equal to μ; thus $P_x = P^{\delta_x}$ where δ_x is the Dirac measure at x. To avoid degeneracies we exclude the case that $|X|$ is a subordinator. As standing notation we denote $X_*(t) = \inf_{s \leq t} X(s)$ and $X^*(t) = \sup_{s \leq t} X(s)$. Furthermore we make throughout the following assumption.

Assumption 2.1. The Gaussian coefficient σ^2 and Lévy measure ν of X satisfy at least one of the following conditions:

$$\text{(i) } \sigma^2 > 0, \qquad \text{(ii) } \nu(-1,1) = +\infty, \qquad \text{(iii) } \nu \text{ has no atoms and } S^\nu \cap (-\infty, 0) \neq \emptyset,$$

where S^ν denotes the support of ν.

When neither of Assumptions 2.1(i) and (ii) hold, the process X is of the form $X_t = \mathtt{d}t + \sum_{s \in (0,t]} \Delta X_s$, where $\Delta X_s = X_s - X_{s-}$ denotes the jump-size of X at time s, for some constant $\mathtt{d}$, which is called the infinitesimal drift of X.

The first observation is that for any entrance law there exists a unique time-change that solves the RIFPT problem. For a given probability measure μ on the positive real line, define the function $I_\mu : \mathbb{R}_+ \to [0, \infty]$ by

$$I_\mu(t) = \overline{F}_\mu^{-1}(\overline{H}(t)), \qquad t \in \mathbb{R}_+, \tag{2.1}$$

$$\text{with } \overline{F}_\mu^{-1}(x) = \inf\{t \in \mathbb{R}_+ : \overline{F}_\mu(t) < x\}, \tag{2.2}$$

where $\overline{H} = 1 - H$ and $\overline{F}_\mu$ denote the survival functions corresponding to the CDF H and to the CDF of the first-passage time τ_0^X of X into the negative half-line $(-\infty, 0)$ under the probability measure P^μ. Here and throughout this paper, we use the convention $\inf \emptyset = +\infty$.

Theorem 2.2. *Let H be a given CDF on $\mathbb{R}_+ \backslash \{0\}$, and let μ be a probability measure on $(\mathbb{R}_+ \backslash \{0\}, \mathcal{B}(\mathbb{R}_+ \backslash \{0\}))$. Assume that μ is continuous if Assumption 2.1(i) and (ii) are not satisfied. If H is continuous, then the function I_μ defined in Eqn.* (2.1) *is the unique time-change such that (μ, I_μ) is a solution of the RIFPT problem.*

For the proof, we need some properties of the distribution of the running infimum.

Lemma 2.3. (i) *If X satisfies Assumption 2.1(i) or (ii), the CDF of $X_*(t)$ is continuous, for any strictly positive t.*
(ii) *If $\nu(-1,1)$ is finite and Assumption 2.1(iii) holds, then the CDF of $X_*(t)$ is continuous on the set $\mathbb{R}_- \setminus \min\{\mathtt{d}t, 0\}$, with $\mathbb{R}_- = (-\infty, 0]$.*

The proof of Lemma 2.3(i) can be found in Sato [28, Lemma 49.3] and Pecherskii & Rogozin [26, Lemma 1], while Lemma 2.3(ii) follows by conditioning on the first jump of the process X.

Proof of Theorem 2.2. Denote by $\mathtt{c}$ the value 0 or $\max\{-\mathtt{d}, 0\}$ according to whether or not X satisfies at least one of the Assumptions 2.1(i) and (ii). The key observation in the proof is that, for any $x \in \mathbb{R}_+ \backslash \{\mathtt{c}t\}$, the map $t \mapsto P_x(\tau_0^X > t)$ is (a) continuous and (b) strictly decreasing. To verify claim (a) it suffices to show that $P_x(\tau_0^X = t)$ is zero for any non-negative t and strictly positive $x \neq \mathtt{c}t$. The latter follows as consequence of the bound $P_x(\tau_0^X = t) \leq P_0(X_*(t) = -x)$ that holds for any strictly positive x and t, and the fact (from Lemma 2.3) that the CDF of $X_*(t)$ is continuous on $(-\infty, 0] \backslash \{-\mathtt{c}t\}$ for any strictly positive t. To see that claim (b) is true, observe that, by the Markov property, we have for strictly positive x, t and s

$$\begin{aligned} P_x\left(\tau_0^X > t\right) - P_x\left(\tau_0^X > t + s\right) &= P_x\left(\tau_0^X > t, \tau_0^X \leq t + s\right) \\ &\geq E\left(\mathbf{1}_{\{X_*(t) > -x\}} P(X_*(s) < -x - z)|_{z = X(t)}\right). \end{aligned} \tag{2.3}$$

Since for any strictly positive epoch s the random variable X_s has an infinitely divisible distribution and any infinitely divisible distribution (not equal to a point mass or subordinator) has unbounded support that contains

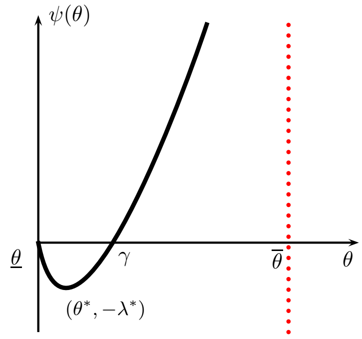

FIGURE 1. The graph of the Laplace exponent ψ on the interval $[\underline{\theta}, \overline{\theta})$ of a Lévy process satisfying Assumption 2.5, with γ denoting the largest root of the Cramér-Lundberg equation $\psi(\theta) = 0$, θ^* the solution of Petrov's equation $\psi'(\theta) = 0$ and $-\lambda^* = \psi(\theta^*)$. The left- and right-inverses of ψ are denoted by ϕ and $\overline{\phi}$.

the positive half-axis (*e.g.*, [28, Corollary 24.4]), it follows that we have

$$P(X_*(s) < -x) \geq P(X(s) < -x) > 0, \qquad s > 0, x \geq 0. \tag{2.4}$$

By combining Eqns. (2.3) and (2.4) we have for any strictly positive x, t and s,

$$P_x\left(\tau_0^X > t\right) > P_x\left(\tau_0^X > t + s\right),$$

so that it follows that claim (b) holds true.

The above key observation in conjunction with Lebesgue's Dominated Convergence Theorem and the assumption that μ is continuous if X does not satisfy Assumption 2.1(i) and (ii) imply that the map $t \mapsto \overline{F}_\mu(t)$ is continuous and strictly decreasing. Denote by Y^μ the time-changed process $X \circ I_\mu$. Since I_μ is monotone increasing and continuous, we have

$$P^\mu\left(\tau_0^{Y^\mu} \geq t\right) = P^\mu\left(\tau_0^X \geq I_\mu(t)\right) = \overline{F}_\mu\left(\overline{F}_\mu^{-1}(\overline{H}(t))\right) = \overline{H}(t) \tag{2.5}$$

for $t \in \mathbb{R}_+$, where we used in the final equality that $\overline{F}_\mu$ is continuous. □

We next turn to the specification of the second degree of freedom, the entrance law μ. By an appropriate choice of the randomisation μ the form of the function $\overline{F}_\mu$ in the specification of the time-change I_μ in Eqn. (2.1) can be considerably simplified. In particular, the function $\overline{F}_\mu$ is equal to an exponential if μ is taken to be equal to any quasi-invariant distribution of the process X killed at the epoch of first-passage below the level 0, the definition of which, we recall, is as follows:

Definition 2.4. For given $\lambda \in \mathbb{R}_+$, the probability measure μ on the measurable space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ is a λ-*invariant distribution* for the process X killed at the epoch of first entrance into the negative half-axis $(-\infty, 0)$ if it holds

$$P^\mu\left(X_t \in A, t < \tau_0^X\right) = \mu(A)\mathrm{e}^{-\lambda t} \qquad \text{for all } A \in \mathcal{B}(\mathbb{R}_+). \tag{2.6}$$

To guarantee existence of such distributions we restrict ourselves in the subsequent analysis to Lévy processes X satisfying the following assumption:

Assumption 2.5 (Cramér Assumption)**.** The distribution of X_1 satisfies *Cramér's condition*

$$E_0[\mathrm{e}^{\gamma X_1}] = 1 \qquad \text{for some } \gamma \in (0, \infty),$$

where $E_0[\cdot]$ denotes the expectation under the probability measure P_0.

Under the Cramér Assumption, there exists a continuum of quasi-invariant distributions of the process X killed upon the first moment of entrance into the negative half-axis, which are given in terms of the Laplace exponent and the positive Wiener-Hopf factor of X. The Laplace exponent $\psi : \mathbb{R} \to (-\infty, \infty]$ of X, given by $\psi(\theta) = \log E[\mathrm{e}^{\theta X_1}]$ for real θ, is finite valued and convex when restricted to the interior $(\underline{\theta}, \overline{\theta})$ of its maximal domain, where $\overline{\theta} = \sup\{\theta \in \mathbb{R} : E[\exp\{\theta X_1\}] < \infty\}$ and $\underline{\theta} = \inf\{\theta \in \mathbb{R} : E[\exp\{\theta X_1\}] < \infty\}$ (In Figure 1 the Laplace exponent of a Lévy process satisfying Assumption 2.5 is plotted.) Since ψ is a convex function that is zero at zero and, under Assumption 2.5, also takes the value zero at a strictly positive point γ, it follows that the function ψ attains a strictly negative minimum

$$-\lambda_* := \min_{\theta \in [\underline{\theta}, \overline{\theta}]} \psi(\theta) = \psi(\theta_*),$$

$$\text{where } \theta_* \in (0, \overline{\theta}) \text{ solves } \psi'(\theta^*) = 0. \tag{2.7}$$

We refer to the relation in Eqn. (2.7) satisfied by θ_* as the *Petrov equation*, and to the constants θ_* and $\lambda_* = \psi(\theta_*)$ as the *Petrov shift* and the *Petrov-coefficient*. On the intervals $[\theta_*, \overline{\theta})$ and $(\underline{\theta}, \theta_*]$ the function ψ is continuous and strictly monotone with inverses denoted by

$$\phi : \left[-\lambda_*, \psi(\overline{\theta})\right) \to \left[\theta_*, \bar{\theta}\right), \qquad \bar{\phi} : \left[-\lambda_*, \psi(\underline{\theta})\right) \to \left(\underline{\theta}, \theta_*\right]. \tag{2.8}$$

The positive Wiener-Hopf factor is the map $\Psi^+ : (0, \infty) \times \mathbb{R} \to \mathbb{C}$ given by

$$\Psi^+(q, \theta) = E[\exp(\mathbf{i}\theta X^*_{e(q)})], \qquad q > 0, \theta \in \mathbb{R}, \tag{2.9}$$

with $e(q)$ an $\mathrm{Exp}(q)$ random time that is independent of X. It is shown in Lemma 4.4 below that, under the Cramér Assumption 2.5, the definition of the map Ψ^+ can be uniquely extended to the set $D^{\mathrm{cl}} = \{(q, \theta) : \Re(q) \geq -\lambda^*, \Im(\theta) \geq 0\}$ (by analytical and continuous extension).

Consider for any $\lambda \in (0, \lambda^*]$ the probability measure μ_λ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ that is characterised by its Laplace transform $\widehat{\mu}_\lambda$ that is given by

$$\widehat{\mu}_\lambda(\theta) = \frac{\bar{\phi}(-\lambda)}{\bar{\phi}(-\lambda) + \theta} \cdot \Psi^+(-\lambda, \mathbf{i}\theta), \qquad \lambda \in (0, \lambda^*]. \tag{2.10}$$

The members of the family $\{\mu_\lambda, \lambda \in (0, \lambda^*]\}$ are quasi-invariant distributions for the Lévy process killed upon first entrance into the negative half-line $(-\infty, 0)$:

Theorem 2.6. *For any $\lambda \in (0, \lambda^*]$, μ_λ is the unique probability measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ that is a λ-invariant distribution of $\{X_t, t < \tau_0^X\}$.*

The proof of Theorem 2.6 is provided in Section 6 below. Under any of the entrance laws μ_λ given in Theorem 2.6 the distribution of the first-passage time τ_0^X is exponential and thus the corresponding survival function $\overline{F}_{\mu_\lambda}$ and time change I_{μ_λ} defined in Eqn. (2.1) take an explicit form:

$$\overline{F}_{\mu_\lambda}(t) = \exp(-\lambda t), \qquad t \in \mathbb{R}_+, \lambda \in (0, \lambda^*],$$
$$I_{\mu_\lambda} = -\frac{1}{\lambda} \log \overline{H}(t), \qquad t \in \mathbb{R}_+.$$

The combination of Theorems 2.2 and 2.6 immediately yields the following result:

Corollary 2.7. *For any given continuous survival function $\overline{H}$ and $\lambda \in (0, \lambda^*]$, the RIFPT problem is solved by the pair $(\mu_\lambda, I_{\mu_\lambda})$:*

$$P^{\mu_\lambda}\left(\tau_0^{Y^{\mu_\lambda}} > t\right) = \overline{H}(t), \qquad t \in \mathbb{R}_+.$$

2.1. **Example.** As a simple example, let us consider the case where X_t is Brownian motion with drift, with entrance law μ, or equivalently $X_t = X_0 + W_t + \eta t$ where W_t is a standard Brownian motion, $\eta \in \mathbb{R}$ and $X_0 \sim \mu$ is a random variable independent of $\{W_t, t \in \mathbb{R}^+\}$. In this case

$$\psi(\theta) = \log E[e^{\theta X_1}] = \eta\theta + \frac{1}{2}\theta^2$$

and $\underline{\theta} = -\infty, \overline{\theta} = +\infty$, so the Petrov coefficients are $\theta^* = -\eta$, $\lambda^* = \frac{1}{2}\eta^2$ and the inverse of ϕ to the left of θ^* is

$$\overline{\phi}(y) = -\eta - \sqrt{\eta^2 + 2y}.$$

The positive Wiener-Hopf factor is

$$\Psi^+(q,\theta) = \frac{-\mathbf{i}(\eta - \sqrt{\eta^2+2q})}{\theta - \mathbf{i}(\eta - \sqrt{\eta^2+2q})}$$

The Laplace transform of the λ-invariant distribution is therefore given by

$$\begin{aligned} \hat{\mu}_\lambda(\theta) &= \left(\frac{-\eta - \sqrt{\eta^2-2\lambda}}{\theta - (\eta + \sqrt{\eta^2-2\lambda})}\right)\left(\frac{-\eta + \sqrt{\eta^2-2\lambda}}{\theta - (\eta - \sqrt{\eta^2-2\lambda})}\right) \\ &= \frac{2\lambda}{\theta_+ - \theta_-}\left(\frac{1}{\theta-\theta_+} - \frac{1}{\theta-\theta_-}\right), \end{aligned} \tag{2.11}$$

where $\theta_\pm = \eta \pm \sqrt{\eta^2 - 2\lambda}$. The condition $\eta \le \sqrt{2\lambda}$ is necessary and sufficient for the expression at (2.11) to be the Laplace transform of a probability measure on $\mathbb{R}^+$, and we note that this is the same as the condition $\lambda \in (0, \lambda^*]$ of Theorem 2.6. Under this condition μ_λ is a mixture of exponentials (or a gamma distribution if $\eta = \sqrt{2\lambda}$). This special case was presented in our earlier paper [14].

3. Multi-dimensional RIFPT

Given a joint survival function $\overline{H} : (\mathbb{R}_+)^d \to [0,1]$ and a d-dimensional Lévy process, a d-dimensional version of the RIFPT problem is phrased as the problem to find a probability measure on $\mathbb{R}^d$ and a collection of increasing continuous functions $I^1, \ldots, I^d$ such that the following identity holds:

$$P^\mu\left(\tau^{Y^1} > t_1, \ldots, \tau^{Y^d} > t_d\right) = \overline{H}(t_1,\ldots,t_d), \qquad \text{for all } t_1,\ldots,t_d \in [0,1], \tag{3.1}$$

$$Y^i := X \circ I^i \text{ for } i = 1,\ldots,d. \tag{3.2}$$

In order to present a solution we will impose some structure on the joint survival function $\overline{H}$, assuming that it is from the class of multivariate *generalised frailty survival functions* that is defined as follows:

Definition. A joint survival function $\overline{H} : \mathbb{R}^d_+ \to [0,1]$ is called a (d-dimensional) *generalised frailty distribution* if there exists a random vector $\Upsilon = (\Upsilon_1, \ldots, \Upsilon_m)$ for some $m \in \mathbb{N}$ such that we have

$$\overline{H}(t_1,\ldots,t_d) = E\left[\prod_{i=1}^d \overline{H}_i(t_i|\Upsilon)\right], \qquad t_1,\ldots,t_d \in [0,1],$$

where $\overline{H}_i(\cdot|u) : \mathbb{R}_+ \to [0,1]$, $i = 1,\ldots,d$, $u \in \mathbb{U}^m$ denotes a collection of survival functions, where $\mathbb{U}^m$ denotes the image of the random vector Υ.

When we denote by $(T_1,\ldots,T_d)$ a random vector with joint survival function $\overline{H}$, the condition in the definition can be phrased as the requirement that there exists a finite-dimensional random vector Υ such that, conditional on Υ, the random variables $T_1,\ldots,T_d$ are mutually independent. In the context of credit risk modeling, for example, one may interpret the vector Υ as the common factors driving the solvency of a collection of d companies (such as economic environment, as opposed to idiosyncratic factors).

We remark that the terminology of generalised frailty is derived from the theory of (multi-variate) survival modeling, in which *frailty* refers to a common factor driving the survival probabilities of the individual entities. One of the commonly studied models is that of *multiplicative frailty* where the frailty appears as a multiplicative factor in the individual hazard functions, in which case the conditional individual survival functions $\overline{H}_i(\cdot|u)$ take the form $\overline{H}_i(\cdot)^u$ for $u \in \mathbb{R}_+$.

Assume henceforth that $\overline{H}$ is a d-dimensional generalised frailty survival function, and denote the corresponding collection of conditional survival functions by $\{\overline{H}_i(\cdot|u), i = 1,\ldots,d, u \in \mathbb{U}^m\}$ for some $m \in \mathbb{N}$. A solution to the multi-dimensional IFPT of the survival function $\overline{H}$ can be constructed by application of the construction that was used in Corollary 2.7 to the conditional survival functions $\overline{H}_i(\cdot|u)$. To formulate this result, let $\{X^{i|u}, i \in \{1,\ldots,d\}, u \in \mathbb{U}^m\}$ be a collection of independent Lévy processes, each satisfying the Cramèr

Assumption, Assumption 2.5, and denote by $\{\mu_i(\cdot|u), i \in \{1,\dots,d\}, u \in \mathbb{U}^m\}$ the probability distributions that have Laplace transforms $\widehat{\mu}_i(\cdot|u)$ given by

$$\widehat{\mu}_i(\theta|u) = \frac{\bar{\phi}_{i|u}(-\lambda_{i|u})}{\bar{\phi}_{i|u}(-\lambda_{i|u}) + \theta} \cdot \Psi^+_{i|u}(-\lambda_{i|u}, \mathbf{i}\theta), \qquad \text{for some } \lambda_{i|u} \in (0, \lambda^*_{i|u}],$$

where $\bar{\phi}_{i|u}$, $\Psi^+_{i|u}$, $\lambda^*_{i|u}$ are the corresponding left-inverse of the Laplace exponent, positive Wiener-Hopf factor and Petrov coefficient of the Lévy process $X^{i|u}$, respectively. Finally, let $\{I_i(\cdot|u), i \in \{1,\dots,m\}, u \in \mathbb{U}^m\}$ denote the collection of time-changes given by

$$I_i(t|u) = -\frac{1}{\lambda_{i|u}} \log \overline{H}_i(t|u), \qquad t \in \mathbb{R}_+.$$

The solution of the multi-dimensional IFPT is given as follows:

Theorem 3.1. *It holds*

$$P\left(\tau_0^{Y^1} > t_1, \dots, \tau_0^{Y^d} > t_d\right) = \overline{H}(t_1,\dots,t_d), \qquad t_1,\dots,t_d \in [0,1],$$
$$\text{with } Y^i(t) = Y_0^{i|\Upsilon} + X^{i|\Upsilon}\left(I_i(t|\Upsilon)\right), \qquad i = 1,\dots,d,$$

where, conditional on $\Upsilon = u \in \mathbb{U}^m$, the random variable $Y_0^{i|u}$ follows the probability distribution $\mu_i(\cdot|u)$ and is independent of the vector $(X^{1|u},\dots,X^{d|u})$ of Lévy processes.

Proof. By the tower-property of conditional expectations and the fact that, conditional on the random variable Υ, the set $\{Y^{i|\Upsilon}, i = 1,\dots,d\}$ forms a collection of independent random variables, we have, for any vector $(t_1,\dots,t_d) \in [0,1]^d$),

$$P\left(\tau_0^{Y^1} > t_1, \dots, \tau_0^{Y^d} > t_d\right) = E\left[\prod_{i=1}^d P\left(\tau_0^{Y^i} > t_i \middle| \Upsilon\right)\right]$$
$$= E\left[\prod_{i=1}^d P^{\mu_i(\cdot|\Upsilon)}\left(\tau_0^{X^{i|\Upsilon}} > I_i(t_i|\Upsilon)\right)\right] = E\left[\prod_{i=1}^d \overline{H}_i(t_i|\Upsilon)\right] = \overline{H}(t_1,\dots t_d)$$

where in the second line we used Corollary 2.7. □

4. Wiener-Hopf factorisation and first-passage times

This section is devoted to a number of auxiliary results concerning the Wiener-Hopf factorisation of X. Denote by Ψ the characteristic exponent of X, *i.e.*, the unique map $\Psi : \mathbb{R} \to \mathbb{C}$ that satisfies $E[\exp\{\mathbf{i}\theta X_t\}] = \exp\{t\Psi(\theta)\}$ for any $t \in \mathbb{R}_+$. According to the Lévy-Khintchine formula, the characteristic exponent is given by

$$\Psi(\theta) = \mathbf{i}c\theta - \frac{\sigma^2}{2}\theta^2 + \int_{\mathbb{R}} [\mathrm{e}^{\mathbf{i}\theta z} - 1 - \mathbf{i}\theta z]\nu(\mathrm{d}z), \qquad \theta \in \mathbb{R}, \tag{4.1}$$

where c and σ^2 are the instantaneous drift and variance of the continuous martingale part of X, and ν denotes the Lévy measure of X. Under Assumption 2.5 the random variable X_1 has negative mean and the Lévy measure ν of X satisfies the condition (*e.g.*, Sato [28, Theorem 25.3])

$$\int_{(1,\infty)} \mathrm{e}^{\gamma x}\nu(\mathrm{d}x) < \infty. \tag{4.2}$$

Under the exponential moment condition in Eqn. (4.2) the function Ψ can be analytically extended to the strip

$$\mathcal{S} = \{\theta \in \mathbb{C} : \Im(\theta) \in (-\mathbf{i}\overline{\theta}, -\mathbf{i}\underline{\theta}) \cup \{0\}\}$$
$$\text{with} \qquad \overline{\theta} = \sup\{\theta \in \mathbb{R} : E[\exp\{\theta X_1\}] < \infty\}, \qquad \underline{\theta} = \inf\{\theta \in \mathbb{R} : E[\exp\{\theta X_1\}] < \infty\},$$

with $\Im(\theta)$ denoting the imaginary part of θ and where we have $\underline{\theta} \le 0 < \overline{\theta}$ given the fact $\Psi(0) = 0$ and Assumption 2.5. This analytical extension of Ψ will also be denoted by Ψ. The characteristic exponent Ψ is related to the Laplace exponent $\psi : \mathbb{R} \to (-\infty, \infty]$ of X by $\psi(\theta) = \Psi(-\mathbf{i}\theta)$.

The probability distributions of the running supremum X^*_t and the running infimum $X_*(t)$ of X up to time t, are related to the characteristic exponent Ψ by the Wiener-Hopf factorization of X, which represents Ψ as the product of the Wiener-Hopf factors Ψ^+ and $\Psi^- : (0,\infty)\times\mathbb{R}\to\mathbb{C}$, as follows:

$$\frac{q}{q-\Psi(\theta)} = \Psi^+(q,\theta)\Psi^-(q,\theta), \qquad \theta\in\mathbb{R},\ q>0, \tag{4.3}$$

with $\Psi^+(q,\theta)$ given in Eqn. (2.9) and $\Psi^-(q,\theta) = E[\exp\{\mathbf{i}\theta X_*(e(q))\}]$, where, as before, $e(q)$ denotes an independent exponential random variable with mean q^{-1} that is independent of X. By analytical extension the Wiener-Hopf factorization in (4.3) continues to hold for all θ in the strip $\mathcal{S}$, when we denote the analytical extension of the two Wiener-Hopf factors to the strip $\mathcal{S}$ also by $\Psi^+(q,\theta)$ and $\Psi^-(q,\theta)$.

The probabilistic form of the Wiener-Hopf factorization of X, states that, under the probability measure $P = P_0$, (i) the random variables $X(e(q)) - X_*(e(q))$ and $X_*(e(q))$ are independent, and (ii) the random variables $X(e(q)) - X_*(e(q))$ and $X^*(e(q))$ have the same probability distribution. Using this form of the Wiener-Hopf factorisation the Laplace transforms in x of the functions $K_{\theta,q}, L_{\theta,q}(x) : \mathbb{R}_+\to\mathbb{R}$ given by

$$K_{\theta,q}(x) = E_x[\mathrm{e}^{-\theta X(e(q))}\mathbf{1}_{\{\tau_0^X > e(q)\}}], \qquad L_{\theta,q}(x) = E_x[\mathrm{e}^{-q\tau_0^X+\theta X_{\tau_0^X}}], \qquad x\in\mathbb{R}_+,$$

for non-negative q,θ, can be expressed in terms of the Wiener-Hopf factors Ψ^+ and Ψ^-.

Lemma 4.1. *For $\theta, q\in\mathbb{R}_+\backslash\{0\}$ the Laplace transforms $\widehat{K}_{\theta,q}, \widehat{L}_{\theta,q}$ of the functions $K_{\theta,q}$ and $L_{\theta,q}$ are finite and given by*

$$\widehat{K}_{\theta,q}(u) = \frac{\Psi^+(q,\mathbf{i}\theta)\Psi^-(q,-\mathbf{i}u)}{\theta+u}, \qquad \widehat{L}_{\theta,q}(u) = \frac{1}{u-\theta}\left(1-\frac{\Psi^-(q,-\mathbf{i}u)}{\Psi^-(q,-\mathbf{i}\theta)}\right), \qquad u\in\mathbb{R}_+. \tag{4.4}$$

Proof. The probabilistic form of the Wiener-Hopf factorisation and the fact that the events $\{\tau_0^X > e(q)\}$ and $\{X_*(e(q))\geq 0\}$ are equal P_x-a.s. for any nonnegative x (*i.e.*, the probability $P_x(\Delta)$ of the difference Δ of these two sets is 0) imply that we have the identities

$$\begin{aligned} K_{\theta,q}(x) &= E_x[\mathrm{e}^{-\theta X(e(q))}\mathbf{1}_{\{\tau_0^X > e(q)\}}] = E_x[\mathrm{e}^{-\theta(X-X_*)(e(q)+X_*(e(q)))}\mathbf{1}_{\{X_*(e(q))\geq 0\}}] \\ &= \mathrm{e}^{-\theta x}E_0[\mathrm{e}^{-\theta X^*(e(q))}]E_0[\mathrm{e}^{-\theta X_*(e(q))}\mathbf{1}_{\{X_*(e(q))\geq -x\}}] \end{aligned} \tag{4.5}$$

for any nonnegative real x. Hence the Laplace transform $\widehat{K}_{\theta,q}$ of the function $K_{\theta,q}$ is equal to

$$\widehat{K}_{\theta,q}(u) = \Psi^+(q,\mathbf{i}\theta)\frac{1}{\theta+u}E_0[\mathrm{e}^{uX_*(e(q))}], \qquad u\in\mathbb{R}_+, \tag{4.6}$$

which yields the identity in Eqn. (4.4) in view of the definition of the Wiener-Hopf factor Ψ^-. The form of the Laplace transform $\widehat{L}_{\theta,q}$ follows by combining Eqns. (4.5)–(4.6) with the following identities:

$$\begin{aligned} E_x[\mathrm{e}^{\theta X_*(e(q))}\mathbf{1}_{\{\tau_0^X\geq e(q)\}}] &= E_x[\mathrm{e}^{\theta X_*(e(q))}] - E_x[\mathrm{e}^{\theta X_*(e(q))}\mathbf{1}_{\{\tau_0^X< e(q)\}}], \\ E_x[\mathrm{e}^{\theta X_*(e(q))}\mathbf{1}_{\{\tau_0^X< e(q)\}}] &= E_x[\mathrm{e}^{\theta X(\tau_0^X)}\mathbf{1}_{\{\tau_0^X< e(q)\}}]E_0[\mathrm{e}^{\theta X_*(e(q))}], \end{aligned}$$

where the latter identity follows from the strong Markov property and the lack-of-memory property of the exponential distribution. □

By using the identity in Eqn. (4.4) we identify the forms of the integrals of $K_{\theta,q}(x)$ and $L_{\theta,q}(x)$ against a probability measure $\mu(\mathrm{d}x)$ in terms of the Wiener-Hopf factors of X.

Lemma 4.2. *Let μ be a probability measure on $\mathbb{R}_+$ with Laplace transform $\widehat{\mu}$ and assume that there is a C satisfying*

$$|\Psi^-(q,-\mathbf{i}u)(1+|u|)| < C \qquad \text{for all } u \text{ with } \Re(u)\geq 0. \tag{4.7}$$

Then we have the identities

$$E^\mu[\mathrm{e}^{-\theta X(e(q))}\mathbf{1}_{\{X_*(e(q))\geq 0\}}] = \Psi^+(q,\mathbf{i}\theta)\cdot\frac{1}{2\pi\mathbf{i}}\int_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty}\widehat{\mu}(-u)\Psi^-(q,-\mathbf{i}u)\frac{\mathrm{d}u}{u+\theta}, \tag{4.8}$$

$$E^\mu[\mathrm{e}^{-q\tau_0^X+\theta(X_{\tau_0^X}-X_0)}] = \frac{1}{2\pi\mathbf{i}}\int_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty}\widehat{\mu}(-u-\theta)\left(1-\frac{\Psi^-(q,-\mathbf{i}(u+\theta))}{\Psi^-(q,-\mathbf{i}\theta)}\right)\frac{\mathrm{d}u}{u}, \tag{4.9}$$

where a is equal to 0 and θ strictly positive. If the integral $\int_{\mathbb{R}_+} \mathrm{e}^{px}\mu(\mathrm{d}x)$ is finite for some strictly positive p, then the identities in Eqns. (4.8) *and* (4.9) *are valid for any a in the interval* $[0,p]$.

Proof. It follows from Lemma 2.3 and Lebesgue's Dominated Convergence Theorem that the map $x \mapsto K_{\theta,q}(x)$ is continuous on $\mathbb{R}_+$. The Laplace inversion theorem yields that, for any strictly positive x, $K_{\theta,q}(x)$ is equal to the integral of the rhs of the identity in Eqn. (4.4) over the Bromwich contour $\Re(u) = 0$, that is,

$$K_{\theta,q}(x) = \Psi^+(q, \mathbf{i}\theta) \cdot \frac{1}{2\pi\mathbf{i}} \int_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty} \mathrm{e}^{ux}\Psi^-(q, -\mathbf{i}u)\frac{\mathrm{d}u}{u+\theta}, \tag{4.10}$$

for $a = 0$ and $x \in \mathbb{R}_+\backslash\{0\}$. Noting that the integrand in Eqn. (4.10) is absolutely integrable (in view of the bound in Eqn. (4.7)) it follows by another application of Lebesgue's Dominated Convergence Theorem and the right-continuity of $K_{\theta,q}(x)$ at $x = 0$ that the identity in Eqn. (4.10) is also valid at $x = 0$.

In view of Eqn. (4.10) the identity in Eqn. (4.8) follows by an interchange of the order of integration. This interchange follows in turn by an application of Fubini's theorem which is justified by the estimate

$$\int_{(0,\infty)}\int_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty} \left|\mathrm{e}^{ux}\frac{\Psi^-(q,-\mathbf{i}u)}{u+\theta}\right| \mathrm{d}u\mu(\mathrm{d}x) \leq \int_{(0,\infty)} \mathrm{e}^{ax}\mu(\mathrm{d}x) \cdot \int_{\mathbb{R}} C\frac{a+\theta+|u|}{(u^2+(a+\theta)^2)(1+|u|)}\mathrm{d}u < \infty. \tag{4.11}$$

To derive the estimate, we used the bound in Eqn. (4.7), that μ is a probability measure and the observations (i) $1/(u + c) = (u^{\#} + c)/(|\Im(u)|^2 + |\Re(u) + c|^2)$ for any $c \in \mathbb{R}$ and $u \in \mathbb{C}$, with $u^{\#}$ denoting the complex conjugate of u, and (ii) $|\exp\{ux\}| = \exp\{\Re(u)x\}$ for any $x \in \mathbb{R}$ and $u \in \mathbb{C}^+$. The proof of the identity in Eqn. (4.8) is complete. The identity in Eqn. (4.9) can be proved by an analogous reasoning, of which the details are omitted. □

Deploying Lemma 4.2 we identify an equation satisfied by a quasi-invariant distribution of the process X killed upon first entrance into the negative half-axis.

Proposition 4.3. *Let λ be strictly positive and assume that μ is a λ-invariant distribution of the process $\{X_t, t < \tau_0^X\}$ with Laplace transform $\widehat{\mu}$, and that the bound in Eqn.* (4.7) *holds true.*

Then $\widehat{\mu}$ satisfies the equation

$$\widehat{\mu}(\theta) \cdot \frac{q}{q+\lambda} = \Psi^+(q, \mathbf{i}\theta) \cdot \frac{1}{2\pi\mathbf{i}} \int_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty} \widehat{\mu}(-u)\Psi^-(q, -\mathbf{i}u)\frac{\mathrm{d}u}{u+\theta} \tag{4.12}$$

with $a = 0$, for any nonnegative real θ and q.

Proof of Proposition 4.3. The statement follows directly by combining Lemma 4.2 with the definition of λ-invariant distribution in Eqn. (2.6). □

Analytical extension of the Wiener-Hopf factorisation.

Ladder processes. Related to the running supremum is the ladder process $(\overline{L}^{-1}, \overline{H})$ of X that consists of the pair of processes $(\overline{L}_t^{-1}, t \geq 0)$ and $(\overline{H}_t, t \in [0, \overline{L}_\infty)$ that is defined by the right-inverse $\overline{L}_t^{-1} = \inf\{s \geq 0 : \overline{L}_s > t\}$ of a local time $\overline{L}$ of X at its running supremum (with the convention $\inf\emptyset = \infty$) and the position $X_{\overline{L}_t^{-1}}$ of X at the epoch $\overline{L}_t^{-1}$. The Laplace exponent $\kappa^+ : \mathbb{C}^+ \to \mathbb{R}$ of the process $(\overline{L}^{-1}, \overline{H})$ is defined by

$$\exp\{-\kappa^+(u,v)t\} = E\left[\exp\{-u\overline{L}_t^{-1} - v\overline{H}_t\}I_{\{t<\overline{L}_\infty\}}\right], \quad u, v \in \mathbb{C}^+,$$

where The ladder process $(\underline{L}^{-1}, \underline{H})$ of $-X$ can be defined similarly with corresponding Laplace exponent denoted by κ^-. The Wiener-Hopf factors Ψ^+ and Ψ^- are explicitly expressed in terms of the Laplace exponents κ^+ and κ^- by

$$\Psi^+(q, \mathbf{i}\theta) = \frac{\kappa^+(q,0)}{\kappa^+(q,\theta)}, \qquad \Psi^-(q, -\mathbf{i}\theta) = \frac{\kappa^-(q,0)}{\kappa^-(q,\theta)}, \qquad q > 0, \theta \in \mathbb{C}^+, \tag{4.13}$$

where $\mathbb{C}^+ = \{u \in \mathbb{C} : \Re(u) \geq 0\}$ denotes the right-half of the complex plane. The factorisation of the characteristic exponent $q - \Psi(\theta)$ in terms of the exponents of the ladder processes is given by

$$\kappa^+(q, -\mathbf{i}\theta)\kappa^-(q, \mathbf{i}\theta) = q - \Psi(\theta), \qquad q \in \mathbb{R}_+, \ \ \Im(\theta) \in (-\overline{\theta}, -\underline{\theta}). \tag{4.14}$$

Petrov-transform. In order to derive the analytical extension of the Wiener-Hopf factors we are lead to consider the exponential family $\{P^{(\theta)}, \theta \in (\underline{\theta}, \overline{\theta})\}$ of probability measures (also called the collection of Esscher-transforms) that are absolutely continuous with respect to P with Radon-Nikodym derivative on $\mathcal{F}_t$ given by

$$\left.\frac{\mathrm{d}P^{(\theta)}}{\mathrm{d}P}\right|_{\mathcal{F}_t} = \Lambda_t^{(\theta)} = \exp(\theta(X_t - X_0) - t\psi(\theta)), \qquad \theta \in (\underline{\theta}, \overline{\theta}).$$

The Esscher transform $P^* = P^{(\theta^*)}$ with corresponding Radon-Nikodym derivative $\Lambda^* = \Lambda^{(\theta_*)}$ corresponding to the shift $\theta = \theta_*$ we will refer to as the *Petrov transform.* Under the Petrov transform the process X is still a Lévy process with characteristic exponent denoted by Ψ^* and with Laplace exponent ψ_* given by

$$\psi_*(s) = \psi(s + \theta_*) - \lambda_*, \qquad s \in (\underline{\theta} - \theta_*, \overline{\theta} - \theta_*). \tag{4.15}$$

Extension. The Laplace exponents of the ladder processes of X and $-X$ under P^* will be denoted by κ_*^+ and κ_*^-, respectively. The maps $(q, \theta) \mapsto \kappa^+(q, \mathbf{i}\theta)$ and $(q, \theta) \mapsto \kappa^-(q, \mathbf{i}\theta)$ can be analytically extended to the domain $D = \{(q, \theta) : \Re(q) > -\lambda^*, \Im(\theta) > 0\}$ using the Petrov-transform, and the definition is uniquely extended to the closure D^{cl} by using continuity of the maps. This in turn implies that the functions $\Psi^+(q, \theta)$ and $\Psi^-(q, \theta)/q$ can also be analytically extended to the domain D, and can be uniquely extended by continuity to the closure D^{cl}. For all $\theta \in \mathbb{C}^+$ and $q \in [0, \lambda^*]$ we set

$$\kappa^+(q - \lambda_*, \theta) := \kappa_*^+(q, \theta + \theta_*), \qquad \kappa^-(q - \lambda_*, \theta) := \kappa_*^-(q, \theta - \theta_*), \tag{4.16}$$

$$\Psi^+(-\lambda, \theta) := \frac{\kappa^+(-\lambda, 0)}{\kappa^+(-\lambda, -\mathbf{i}\theta)}, \qquad \Psi^-(-\lambda, \theta) := \frac{\kappa^-(-\lambda, 0)}{\kappa^-(-\lambda, \mathbf{i}\theta)}, \qquad \lambda \in (0, \lambda^*]. \tag{4.17}$$

In the following it is shown that with the given definitions in Eqn. (4.17), the Wiener-Hopf factorisation still holds true:

Lemma 4.4. *(i) For all $\theta \in \mathbb{R}_+$ and $q \in [0, \lambda^*]$, $\kappa^+(q - \lambda_*, \theta)$ and $\kappa^-(q - \lambda_*, \theta)$ are finite and non-negative.*
(ii) For any $\theta \in \mathbb{R}$ and $\lambda \in (0, \lambda^*]$ we have

$$\Psi^+(-\lambda, \theta)\Psi^-(-\lambda, \theta) = \frac{\lambda}{\lambda + \Psi(\theta)}. \tag{4.18}$$

Proof. (i) Lebesgue's Dominated Convergence Theorem, quasi left-continuity of the Lévy process X and the definitions of the Petrov transform P^* and the Laplace exponents κ^+ and κ_*^+ yield, for any q larger or equal to λ^*,

$$\exp\{-t\kappa^+(q - \lambda^*, \theta)\} = \lim_{T\to\infty} E\left[\exp\left\{-(q - \lambda^*)(\overline{L}_t^{-1} \wedge T) - \theta X_{\overline{L}_t^{-1}\wedge T}\right\}\right], \tag{4.19}$$

$$\exp\{-t\kappa_*^+(q, \theta + \theta^*)\} = \lim_{T\to\infty} E^*\left[\exp\left\{-q(\overline{L}_t^{-1} \wedge T) - (\theta + \theta^*)X_{\overline{L}_t^{-1}\wedge T}\right\}\right]. \tag{4.20}$$

The expectations on the right-hand sides of Eqns. (4.19) and (4.20) are in fact equal for any positive $T > 0$ since, in view of Doob's Optional Stopping Theorem and the fact that $\overline{L}_t^{-1} \wedge T$ is an $\mathbf{F}$-stopping time, we have

$$E\left[\Lambda_T^* \middle| \mathcal{F}_{\overline{L}_t^{-1}\wedge T}\right] = \Lambda^*_{\overline{L}_t^{-1}\wedge T},$$

whence $\kappa(q-\lambda^*, \theta) = \kappa_*^+(q, \theta+\theta^*)$ for all $\theta \in \mathbb{R}_+$ and $q \in [0, \lambda^*]$. The finiteness and nonegativity of $\kappa_*^+(q, \theta+\theta^*)$ for all $q, \theta \in \mathbb{R}_+$ imply those of $\kappa^+(q - \lambda^*, \theta)$ for all $\theta \in \mathbb{R}_+$ and $q \in [0, \lambda^*]$. The proof of the finiteness of κ^- is analogous, and is omitted.

(ii) In view of Eqns. (4.15)–(4.16), the identity in Eqn. (4.14) under the measure P^* reads as

$$\begin{aligned} q - \lambda^* - \Psi(\theta - \mathbf{i}\theta^*) &= q - \Psi_*(\theta) = \kappa_*^+(q, -\mathbf{i}\theta)\kappa_*^-(q, \mathbf{i}\theta) \\ &= \kappa^+(q - \lambda^*, -\mathbf{i}\theta + \theta^*)\kappa^-(q - \lambda^*, \mathbf{i}\theta - \theta^*) \end{aligned} \tag{4.21}$$

for all $q \in \mathbb{R}_+$ and θ with $\Im(\theta) \in [0, \theta^*]$, where Ψ_* denotes the characteristic exponent of X under the Petrov transform P^*. We obtain the identity in Eqn. (4.18) for given $\lambda \in (0, \lambda^*]$ and $\theta \in \mathbb{R}$ by taking the ratio of Eqn. (4.21) corresponding to the substitutions $(q, \theta) \to (\lambda^*-\lambda, \mathbf{i}\theta^*)$ and $(q, \theta) \to (\lambda^*-\lambda, \mathbf{i}\theta^*+\theta)$, respectively. $\square$

5. Quasi-invariant distributions for mixed-exponential Lévy processes

We next turn to a class of Lévy processes with two-sided jumps that forms a dense class in the class of all Lévy processes. A *mixed-exponential jump-diffusion* $X = \{X_t, t \in \mathbb{R}_+\}$ is a Lévy process given by

$$X_t = \mathtt{d}t + \sigma W_t + \sum_{j=1}^{N_t} U_j, \qquad t \in \mathbb{R}_+, \tag{5.1}$$

where W is a Wiener process, $\mathtt{d} \in \mathbb{R}$ denotes the drift and $\sigma > 0$ is the volatility, and N is a Poisson process with rate λ that is independent of W. The series $(U_j)_{j\in\mathbb{N}}$ consists of IID random variables that are independent of W and N and follow a *double-mixed-exponential distribution*, which is a probability distribution on $\mathbb{R}$ that has probability density function given by

$$f(x) = pf_+(x) + (1-p)f_-(x), \qquad \text{with} \qquad f_\pm(x) = \sum_{k=1}^{m_\pm} a_k^\pm \alpha_k^\pm \mathrm{e}^{-\alpha_k^\pm |x|} \mathbf{1}_{\mathbb{R}_+}(\pm x), \qquad x \in \mathbb{R}.$$

The class of double-mixed-exponential distributions is dense in the class of all probability measures on $\mathbb{R}$ (see Botta & Harris [7]). Here p is a number in the unit interval $[0,1]$ and f_+ and f_- are themselves probability density functions that are linear combinations of m^+ and m^- exponentials respectively, with real-valued weights $a_1^+, \ldots, a_{m^+}$ and $a_1^-, \ldots, a_{m^-}^-$ and strictly positive parameters $\alpha_1^+, \ldots, \alpha_{m^+}^+$ and $\alpha_1^-, \ldots, \alpha_{m^-}^-$. To ensure that f is a PDF the parameters $\{a_k^\pm, k = 1, \ldots, m^\pm\}$ need to satisfy certain restrictions; necessary and sufficient conditions for f to be a PDF are

$$p_1^\pm > 0, \qquad \sum_{k=1}^{m^\pm} p_k^\pm \alpha_k^\pm \geq 0 \qquad \text{and} \qquad \sum_{k=1}^{l} p_k^\pm \alpha_k^\pm \geq 0 \quad \forall l = 1, ..., m^\pm$$

respectively (see Bartholomew [3]). Since the Fourier transform of the PDF f of the jump-sizes is a rational function, the characteristic function $\theta \mapsto E[\mathrm{e}^{\mathbf{i}\theta X_{e(q)}}]$ of $X_{e(q)}$ is also a rational function, which is given by

$$E[\mathrm{e}^{\mathbf{i}\theta X_{e(q)}}] = \frac{1}{\left(1 - \frac{\mathbf{i}\theta}{\rho_0^-(q)}\right)} \prod_{j=1}^{m^-} \frac{\left(1 + \frac{\mathbf{i}\theta}{\alpha_j^-}\right)}{\left(1 - \frac{\mathbf{i}\theta}{\rho_j^-(q)}\right)} \frac{1}{\left(1 - \frac{\mathbf{i}\theta}{\rho_0^+(q)}\right)} \prod_{j=1}^{m^+} \frac{\left(1 + \frac{\mathbf{i}\theta}{\alpha_j^+}\right)}{\left(1 - \frac{\mathbf{i}\theta}{\rho_j^+(q)}\right)}, \qquad \theta \in \mathbb{R}, q > 0,$$

where $\rho_j^+(q), j = 1, \ldots, m^+ + 1$, and $\rho_j^-(q), j = 1, \ldots, m^- + 1$, are the roots of the Cramér-Lundberg equation

$$\Psi(-\mathbf{i}\theta) - q = \psi(\theta) - q = 0 \tag{5.2}$$

with positive and negative real parts, respectively (where multiple roots are listed as many times as their multiplicity). The analytical continuation of the characteristic exponent Ψ of X defined in Eqn. (4.1) to the set $\widetilde{\mathbb{C}} := \mathbb{C}\backslash\{-\mathbf{i}\alpha_1^+, \ldots, -\mathbf{i}\alpha_{m^+}^+, \mathbf{i}\alpha_1^-, \ldots, \mathbf{i}\alpha_{m^-}^-\}$ is again denoted by Ψ. The mixed-exponential Lévy process satisfies Assumption 2.5 precisely if the parameters satisfy the restiction

$$\psi'(0) = \mathtt{d} + p\sum_{k=1}^{m^+} \frac{a_k^+}{\alpha_k^+} - (1-p)\sum_{k=1}^{m^-} \frac{a_k^-}{\alpha_k^-} < 0. \tag{5.3}$$

Furthermore, denote by $\rho_j^+(-\lambda)$, $j = 0, \ldots, m^+$, and $\rho_j^-(-\lambda)$, $j = 0, \ldots, m^-$, the roots with real parts larger and smaller than θ^* of the equation $\psi(\theta) = -\lambda$, which is equivalently phrased as the 'shifted' Cramér-Lundberg equation $\psi_{\phi(-\lambda)}(\theta + \phi(-\lambda)) = 0$.

The positive Wiener-Hopf factor of the process X is identified as follows. The form of the positive Wiener-Hopf factor given in Eqn. (2.9) was derived in Lewis & Mordecki [22] for positive q.

Proposition 5.1. *(i) For any real θ and any real q larger or equal than $-\lambda^*$, we have*

$$\Psi^+(q,\theta) = \frac{1}{\left(1 - \frac{\mathbf{i}\theta}{\rho_0^+(q)}\right)} \prod_{j=1}^{m^+} \frac{\left(1 - \frac{\mathbf{i}\theta}{\alpha_j^+}\right)}{\left(1 - \frac{\mathbf{i}\theta}{\rho_j^+(q)}\right)}. \tag{5.4}$$

(ii) For any $\lambda \in (0, \lambda^*]$, the Laplace transform $\widehat{\mu}_\lambda$ of a λ-invariant distribution μ_λ of $\{X_t, t < \tau_0^X\}$ is given by

$$\widehat{\mu}_\lambda(\theta) = \frac{\rho_0^-(-\lambda)}{\rho_0^-(-\lambda)+\theta} \cdot \frac{\rho_0^+(-\lambda)}{\rho_0^+(-\lambda)+\theta} \prod_{j=1}^{m^+} \frac{\left(1+\frac{\theta}{\alpha_j^+}\right)}{\left(1+\frac{\theta}{\rho_j^+(-\lambda)}\right)}, \qquad \lambda \in (0, \lambda^*],\ q>0. \tag{5.5}$$

Remark. *In the case that the roots $\rho_k^+(-\lambda)$ are all distinct the probability measure μ_λ is a mixed-exponential distribution that can be obtained from the Laplace transform $\widehat{\mu}_\lambda$ by partial fraction decomposition and termwise inversion:*

$$\mu_\lambda(\mathrm{d}x) = \mathbf{1}_{\mathbb{R}_+}(x)\cdot m_\lambda(x)\mathrm{d}x, \qquad m_\lambda(x) = A_0^-\rho_0(-\lambda)\mathrm{e}^{-\rho_0^-(-\lambda)x} + \sum_{k=0}^{m_+} A_k^+\rho_k^+(-\lambda)\mathrm{e}^{-\rho_k^+(-\lambda)x}, \tag{5.6}$$

Here, the constants A_k^+, $k=0,\ldots,m_+$, and $A_0^- = A_{-1}^+$ are given by

$$A_k^+ = \left(1-\frac{\rho_k^+(-\lambda)}{\alpha_k^+}\right)\cdot \prod_{j=-1, j\neq k}^{m^+} \frac{\left(1-\frac{\rho_k^+(-\lambda)}{\alpha_j^+}\right)}{\left(1-\frac{\rho_k^+(-\lambda)}{\rho_j^+(-\lambda)}\right)}. \tag{5.7}$$

where $\rho_{-1}^+(-\lambda) := \rho_0^-(-\lambda)$ and the constants α_{-1}^+ and α_0^+ are to be taken equal to $+\infty$ (so that the factors $(1+\rho_k^+(-\lambda)/\alpha_0^+)$ and $(1+\rho_k^+(-\lambda)/\alpha_{-1}^+)$ are equal to 1).

Proof of Proposition 5.1(i). Since κ^+ is the Laplace exponent of a subordinator it follows that $\kappa^+(q,\theta)/\theta$ converges to some non-negative c as θ tends to infinity. In view of the relation $\Psi^+(q,\theta) = \kappa^+(q,0)/\kappa^+(q,\theta)$ we deduce that $\kappa^+(q,0)$ is equal to $c\cdot\limsup_{\theta\to\infty}\Psi^+(q,\theta)$. Hence, the explicit expression for Ψ^+ in Eqn. (5.4) implies that we have

$$\kappa^+(q,\theta) = \left(\rho_0^+(q)+\theta\right)\cdot \prod_{j=1}^{m^+} \frac{(\rho_j^+(q)+\theta)}{(\alpha_j^+ + \theta)}.$$

Inserting the expression for κ^+ into the definition in Eqn. (4.17) of $\Psi^+(q,\theta)$ for q in the interval $I := [-\lambda^*, 0]$ implies that the stated expression in Eqn. (5.4) remains valid for q in I. □

The proof of Proposition 5.1(ii), i.e. the quasi-invariance of the collection $\{\mu_\lambda, \lambda\in(0,\lambda^*]\}$, is based on the following auxiliary identity:

Lemma 5.2. *Consider arbitrary $\lambda\in(0,\lambda^*]$. For any $q>0$, and $\theta\geq 0$ we have the identity*

$$\frac{1}{q+\lambda}\cdot\widehat{\mu}_\lambda(\theta) = \frac{1}{q}\cdot E^{\mu_\lambda}[\mathrm{e}^{-\theta X(e(q))}\mathbf{1}_{\{e(q)<\tau_0^X\}}]. \tag{5.8}$$

The proof of this result is provided below.

Proof of Proposition 5.1(ii). The assertion holds true since it follows as a consequence of Lemma 5.2 that for any fixed strictly positive t the two measures $m_t^{(1)}$ and $m_t^{(2)}$ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ given by $m_t^{(1)}(\mathrm{d}x) = \exp(-\lambda t)\mu_\lambda(\mathrm{d}x)$ and $m_t^{(2)}(\mathrm{d}x) = P^{\mu_\lambda}(X_t\in\mathrm{d}x, t<\tau_0^X)$ coincide, in view of the following observations:
(a) The lhs and rhs of Eqn. (5.8) are equal to the double Laplace transforms of the measures on $(\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2))$ given by $m_t^{(1)}(\mathrm{d}x)\mathrm{d}t$ and $m_t^{(2)}(\mathrm{d}x)\mathrm{d}t$, respectively.
(b) The Laplace transforms $\widehat{m}_t^{(1)}$ and $\widehat{m}_t^{(2)}$ are continuous as function of t. This assertion follows by noting that, for any $x\in\mathbb{R}_+$ we have P_x-a.s. that the map $t\mapsto X_t$ is continuous at any fixed positive t since the probability of a jump ΔX_t at time t is zero and moreover the probability that τ_0^X is equal to a given positive t is zero in view of the estimate $P_x(\tau_0^X = t)\leq P_x(X_t = 0)$ and the fact that the mixed-exponential Lévy process X has absolutely continuous marginal densities. □

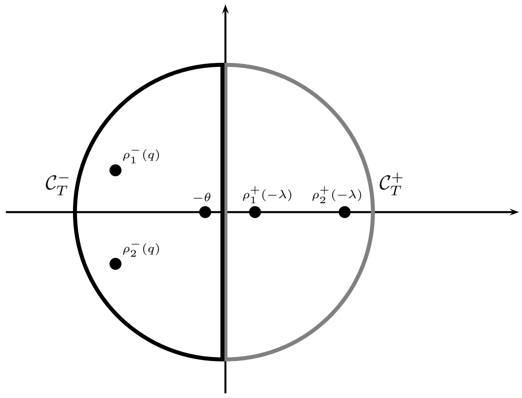

FIGURE 2. Pictured is the complex plane with an example of the two contours $\mathcal{C}_T^+$ (gray) and $\mathcal{C}_T^-$ (black) and the poles in $\mathcal{P}^+$ and $\mathcal{P}^-$. The contour $\mathcal{C}_T^+$ encloses the poles $p \in \mathcal{P}^+$ while the contour $\mathcal{C}_T^+$ encloses the poles $p \in \mathcal{P}^-$.

Residue calculus. Let $\widehat{\mu}_\lambda$ also denote the analytical continuation to the domain $\mathbb{C}^+$ of the Laplace transform $\widehat{\mu}_\lambda$ defined in Eqn. (2.10). The identity in Eqn. (5.8) is derived by an evaluation of the Bromwich integral on the rhs of the identity in Eqn. (4.8), by an application of Cauchy's residue theorem. It is immediate from Eqns. (2.10) and (5.4) that the function $f : \mathbb{C}^+ \to \mathbb{C}$ given by

$$f(u) = f_{\theta,\lambda,q}(u) = [\Psi^+(q, \mathbf{i}\theta)\widehat{\mu}_\lambda(-u)\Psi^-(q, -\mathbf{i}u)]/(u + \theta) \tag{5.9}$$

is rational with the poles given as follows:

Lemma 5.3. *The function f is rational with poles given by $\mathcal{P}^+ \cup \mathcal{P}^-$ with*

$$\mathcal{P}^+ = \{\rho_k^+(-\lambda); k = 0, \ldots, m^+ + 1\} \subset \mathbb{C}^{++}, \ \mathcal{P}^- = \{-\theta, \rho_j^-(q), j = 1, \ldots, m^- + 1\} \subset \mathbb{C}^{--},$$

where we denote $\mathbb{C}^{--} := \{u \in \mathbb{C} : \Re(u) < 0\}$ and $\mathbb{C}^{++} := \{u \in \mathbb{C} : \Re(u) > 0\}$.

Denote by $\mathcal{C}_T^+$ the contour consisting of the segment $\mathcal{I}_T = \{u \in \mathbb{C} : \Im(u) \in [-T, T], \Re(u) = 0\}$ on the imaginary axis, and the semi-circle that joins $-\mathbf{i}T$ and $\mathbf{i}T$ such that $\mathcal{C}_T^+$ is contained in the right half-plane $\mathbb{C}^+$. For T sufficiently large, the contour $\mathcal{C}_T^+$ will enclose all the poles in the set $\mathcal{P}^+$. Similarly, let $\mathcal{C}_T^-$ denote the contour consisting of the segment $\mathcal{I}_T$ and the semi-circle that joints $-\mathbf{i}T$ and $\mathbf{i}T$ such that $\mathcal{C}_T^-$ is contained in the left half-plane $\mathbb{C}^-$. We evaluate the contour integrals over the distinct contours $\mathcal{C}_T^-$ and $\mathcal{C}_T^+$, and show that both yield the same value.

Lemma 5.4. *Assume that all the elements of the sets $\mathcal{P}^+$ and $\mathcal{P}^-$ are distinct. For any $q, \theta \in \mathbb{R}_+$ and $\lambda \in (0, \lambda^*]$ the following hold true:*

$$I_o^+(T) := \oint_{\mathcal{C}_T^+} f = \frac{q}{q + \lambda}\widehat{\mu}_\lambda(\theta), \qquad I_o^-(T) := \oint_{\mathcal{C}_T^-} f = \frac{q}{q + \lambda}\widehat{\mu}_\lambda(\theta). \tag{5.10}$$

where f is given in Eqn. (5.9)*. Furthermore, it holds for any $q, \theta \in \mathbb{R}_+$ and $\lambda \in (0, \lambda^*]$*

$$\frac{1}{2\pi\mathbf{i}} \int_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty} f(u)\mathrm{d}u = \frac{q}{q + \lambda} \frac{\widehat{\mu}_\lambda(\theta)}{\Psi^+(q, \mathbf{i}\theta)}, \qquad a = 0. \tag{5.11}$$

Proof of Lemma 5.4. In view of the properties of f listed in Lemma 5.3, the integrals $I_o^+(T)$ and $I_o^-(T)$ of the function f over the contour $\mathcal{C}_T^+$ are by Cauchy's theorem equal to

$$I_o^+(T) = \frac{1}{2\pi\mathbf{i}} \sum_{p \in \mathcal{P}^+} n(\mathcal{C}_T^+, p)\mathrm{Res}_p(f) \qquad\qquad I_o^-(T) = \frac{1}{2\pi\mathbf{i}} \sum_{p \in \mathcal{P}^-} n(\mathcal{C}_T^-, p)\mathrm{Res}_p(f), \tag{5.12}$$

where $\mathrm{Res}_p(f)$ denotes the residue of the function f at the pole p and, for any $p \in \mathbb{C}$ and any curve $\Gamma : [0, 2\pi] \to \mathbb{C}$, $n(\Gamma, p)$ denotes the winding number of Γ around p. Note that we have $n(\mathcal{C}_T^+, p) = -1$ for any $p \in \mathcal{P}^+$ and $n(\mathcal{C}_T^+, p) = +1$ for any $p \in \mathcal{P}^-$ (see Figure 2).

Since by assumption the poles are all distinct, straightforward calculations show

$$\text{Res}_{p=\rho_j^+(-\lambda)}(f) = -2\pi\mathbf{i}\cdot\Psi^+(q,\mathbf{i}\theta)\cdot\left\{\Psi^-(q,-\mathbf{i}\rho_j^+(\lambda))\prod_{k=0,k\neq j}^{m^++1}\frac{1-\frac{\rho_j^+(-\lambda)}{\alpha_k^+}}{1-\frac{\rho_j^+(-\lambda)}{\rho_k^+(-\lambda)}}\right\}\cdot\frac{\rho_j^+(-\lambda)}{\rho_j^+(-\lambda)+\theta}, \tag{5.13}$$

$$\text{Res}_{p=-\theta}(f) = 2\pi\mathbf{i}\cdot\widehat{\mu}_\lambda(\theta)\Psi^+(q,\mathbf{i}\theta)\Psi^-(q,\mathbf{i}\theta) = 2\pi\mathbf{i}\cdot\widehat{\mu}_\lambda(\theta)q(q-\psi(\theta))^{-1}, \tag{5.14}$$

$$\text{Res}_{p=\rho_\ell^-(q)}(f) = -2\pi\mathbf{i}\cdot\Psi^+(q,\mathbf{i}\theta)\cdot\left\{\widehat{\mu}_\lambda(-\rho_\ell^-(q))\prod_{k=0,k\neq\ell}^{m^-+1}\frac{1+\frac{\rho_\ell^-(q)}{\alpha_k^-}}{1-\frac{\rho_\ell^-(q)}{\rho_k^-(q)}}\right\}\cdot\frac{\rho_\ell^-(q)}{\rho_\ell^-(q)+\theta}, \tag{5.15}$$

for $j=0,\ldots,m^++1$ and $\ell=1,\ldots,m^-+1$. We claim that the following hold true:

$$\Psi^+(q,\mathbf{i}\theta)^{-1}\frac{1}{2\pi\mathbf{i}}\sum_{p\in\mathcal{P}^+}(-1)\cdot\text{Res}_p(f) = \Psi^+(q,\mathbf{i}\theta)^{-1}\frac{1}{2\pi\mathbf{i}}\sum_{p\in\mathcal{P}^-}\text{Res}_p(f) \tag{5.16}$$

$$= \frac{q}{q+\lambda}\frac{\widehat{\mu}_\lambda(\theta)}{\Psi^+(q,\mathbf{i}\theta)}. \tag{5.17}$$

The identities in Eqn. (5.17) can be seen to hold true by combining Eqn. (5.13) with the following two observations:

(a) the facts $\psi(\rho_j^+(-\lambda))=-\lambda$ and $\psi(\rho_j^-(q))=q$ and the Wiener-Hopf factorisation imply the identities

$$\Psi^-(q,-\mathbf{i}\rho_j^+(-\lambda)) = \frac{q}{q+\lambda}\Psi^+(q,-\mathbf{i}\rho_j^+(-\lambda))^{-1},\quad \Psi^+(-\lambda,\mathbf{i}\rho_\ell^-(q)) = \frac{\lambda}{\lambda+q}\Psi^-(-\lambda,\mathbf{i}\rho_\ell^-(q))^{-1},$$

for any $\lambda\in(0,\lambda^*]$;

(b) the right-hand side of Eqn. (5.17) is a rational function in θ with the coefficients of its partial fraction decomposition into the functions $1/[1+\theta/\rho_j^+(q)]$, $j=0,\ldots,m^++1$, given by the respective terms in Eqn. (5.13) between the curly brackets (using part (a), the forms of $\widehat{\mu}_\lambda$ and Ψ^+ in Eqns. (5.4) and (5.5));

(c) the right-hand side of Eqn. (5.17) can be decomposed as

$$\frac{q}{q+\lambda}\frac{\widehat{\mu}_\lambda(\theta)}{\Psi^+(q,\mathbf{i}\theta)} = \widehat{\mu}_\lambda(\theta)\frac{q}{q-\psi(\theta)}\left[1-\frac{\lambda}{\lambda+q}\cdot\frac{\lambda+\psi(\theta)}{\lambda}\right],$$

where the map

$$\theta\mapsto\Psi^-(q,\mathbf{i}\theta)\widehat{\mu}_\lambda(\theta)\frac{\lambda}{\lambda+q}\cdot\frac{\lambda+\psi(\theta)}{\lambda}$$

is a rational function with the coefficients of its partial fraction into the functions $1/[1+\theta/\rho_\ell^-(q)]$ for any $\ell=1,\ldots,m^-+1$, given by the expressions in the curly brackets in Eqn. (5.15) (using part (a), the forms of $\widehat{\mu}_\lambda$ and Ψ^+ in Eqns. (5.4) and (5.5) and Wiener-Hopf factorisation in Eqn. (4.18)).

Combining Eqns. (5.12) and (5.17) shows

$$I_o^+(T) = I_o^-(T) = \frac{q}{q+\lambda}\widehat{\mu}_\lambda(\theta).$$

To complete the proof we next show that $I_o^+(T)$ and $I_o^-(T)$ both tends to the integral on the right-hand side of Eqn. (4.8) as $T\to\infty$: Since the function f is continuous and satisfies the growth-condition $|f(u)| = O(|u|^{-2})(|u|\to\infty)$, while the length of the semi-circles is proportional to T, it follows that the integrals $I_c^+(T)$ and $I_c^-(T)$ over the semi-circles only (that is, over $\mathcal{C}_T^+\backslash\mathcal{I}_T$ and $\mathcal{C}_T^-\backslash\mathcal{I}_T$) tend to zero as T tends to infinity. Thus, we conclude that $I_o^+(T)$ and $I_o^-(T)$ both converge to the right-hand side of Eqn. (4.8) as T tends to infinity, and the proof is complete. □

Proof of Lemma 5.2. Fix q,λ and $\theta\in\mathbb{R}_+$ arbitrary. If the elements of the sets $\mathcal{P}^+$ and $\mathcal{P}^-$ are distinct, the identity follows by combining the identities in Eqns. (5.11) and (4.8) (for $\mu=\mu_\lambda$). By approximating (in the sense of weak convergence) the process X by a sequence $(X^{(n)})_n$ of mixed-exponential Lévy processes for which the corresponding roots are distinct, and using the definition of weak-convergence and the fact that both sides in Eqn. (5.8) are equal to the integral of μ against a bounded continuous function, it follows that the identity in Eqn. (5.8) remains true in the case of multiple roots. □

6. Proof of Theorem 2.6

The proof, given at the end of this section, combines the results of three lemmas. The first two of these, Lemmas 6.1 and 6.2, concern approximation of general Lévy processes by mixed-exponential jump diffusions, while the third, Lemma 6.3, establishes uniqueness of λ-invariant distributions.

Lemma 6.1. *Let X be a Lévy process that satisfies Assumption 2.5 and has Petrov coefficient λ_*.*

(i) *There exists a sequence $(X^{(n)})_n$ of mixed-exponential Lévy processes satisfying Assumption 2.5 such that $X^{(n)}$ converges weakly to X in the Skorokhod J_1 topology as n tends to infinity.*

(ii) *For any positive t the convergence in distribution of $(X^{(n)})^*(t)$ and $X^{(n)}_*(t)$ to $X^*(t)$ and $X_*(t)$ holds, where $(X^{(n)})_n$ denotes the sequence from (i).*

(iii) *The sequence $(X^{(n)})_n$ in part (i) can be chosen such that the corresponding Petrov-coefficients and Laplace exponents of the ladder processes under the Petrov transform satisfy $\lambda^{(n)}_* \uparrow \lambda_*$ and $\kappa^{(n)+}_*(q,\theta) \to \kappa^+_*(q,\theta)$ for any non-negative q and θ, and we have $\overline{\theta}^{(n)} \geq \overline{\theta}$.*

Proof. (i) It is well known (see e.g. Jacod & Shiryaev [19, Cor. VII.3.6]) that to prove weak convergence of a sequence of Lévy processes in the Skorokhod topology J_1 on the Skorokhod space $D(\mathbb{R})$ it suffices to show connvergence in distribution of the values at one fixed time. Moreover, the marginal distribution at this fixed time can be approximated by a sequence of compound Poisson distributions. The jump-distribution in turn can be approximated arbitrarily closely by a double-mixed exponential distribution since these distributions form a sense class in the sense of weak-covergence in the set of all probability distributions on the real line, as noted before. Uniform integrability of the distributions of $X^{(n)}_1$ and the fact that $E[X_1]$ is strictly negative imply that for all n sufficiently large the expectation of $X^{(n)}_1$ is strictly negative, which implies that $X^{(n)}$ satisfies Assumption 2.5 for all n sufficiently large .

(ii) Since the real-valued maps on the Skorokhod space $D(\mathbb{R})$ given by $\omega \mapsto \omega_*(t)$ and $\omega \mapsto \omega^*(t)$ are continuous in the Skorokhod topology at any $t \in \{s : \Delta\omega(s) = 0\}$ (see [19, Ch. VI.2]) and we have $P(\Delta X(t) = 0)$ (since X is quasi-left-continuous), it follows that, as n tends to infinity, $(X^n)_*(t)$ and $(X^n)^*(t)$ converge in distribution to $X_*(t)$ and $X^*(t)$, for any non-negative t.

(iii) Consider first a direct construction of the sequence of processes $(X^{(n)})_n$ weakly converging to the process X. Let $(\overline{H}^{(n)})_n$ and $(\underline{H}^{(n)})_n$ be sequences of mixed-exponential Lévy-subordinators that weakly converge to the up-crossing and down-crossing ladder processes $\overline{H}$ and $\underline{H}$ of X under the Petrov transform P^*. Then the Laplace exponents of $(\overline{H}^{(n)})_n$ and $(\underline{H}^{(n)})_n$ converge pointwise to those of $\overline{H}$ and $\underline{H}$ under P^*, *i.e.*, $\kappa^{+(n)}(0,\theta) \to \kappa^+_*(0,\theta)$ and $\kappa^{-(n)}(0,\theta) \to \kappa^-_*(0,\theta)$. Let $\widetilde{X}^{(n)}$ be the Lévy process with characteristic exponent given by $-\Psi^{(n)}_*(\theta) := \kappa^{(n)+}(0,-\mathbf{i}\theta)\kappa^{(n)-}(0,\mathbf{i}\theta)$. In view of the Wiener-Hopf factorisation in Eqn. (4.14), it follows that $\Psi^{(n)}_*(\theta)$ converges to $\Psi_*(\theta)$ for any real θ, which implies that $\widetilde{X}^{(n)}$ converges weakly to the process X under the measure P^*. Thus, the mixed-exponential Lévy processes $X^{(n)}$ that have the same law as the processes $\widetilde{X}^{(n)}$ under the probability measure $P^{(-\theta^*)}$ weakly converges to the process X under the measure P as n tends to infinity. By construction the ladder processes of $X^{(n)}$ under the P^* have Laplace exponents $\kappa^{+(n)}_*(0,\theta) = \kappa^{+(n)}(0,\theta)$ and $\kappa^{-(n)}_*(0,\theta) = \kappa^{-(n)}(0,\theta)$, so that the convergence follows by construction.

Next we show that also the other two conditions can be satisfied by suitable choice of $X^{(n)}$. Let $(X^{(n)})_n$ be chosen in such a way that (a) the Laplace exponents $\psi_n(-\theta^*)$ at $-\theta^*$ are strictly positive and (b) $\psi_n(\overline{\theta} - \theta^*)$ is finite (where ψ_n denotes the Laplace exponent of $X^{(n)}$). Then the Petrov-coefficient $\lambda^{(n)}_*$ and the supremum of the domain $\overline{\theta}^{(n)}$ corresponding to the process $\widetilde{X}^{(n)}$ are larger or equal than the values λ^* and $\overline{\theta}$ corresponding to X, in view of the transformation in Eqn. (4.15) of the Laplace exponent under the Petrov transform. This completes the proof. □

Let $(X^{(n)})_n$ be a sequence of Lévy processes as in Lemma 6.1(iii). For any λ in the interval $(0, \lambda^*]$, let $(\mu^{(n)}_\lambda)_n$ be the collection of measures on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ that are characterised by their Laplace transforms

$$\widehat{\mu}_n(\theta) = \frac{\bar{\phi}_n(\lambda)}{\bar{\phi}_n(\lambda) + \theta}\Psi^+_n(-\lambda, \mathbf{i}\theta), \qquad \theta \in \mathbb{R}_+, \tag{6.1}$$

where $\bar{\phi}_n$ and Ψ_n^+ are understood to be taken with respect to the process $X^{(n)}$.

Lemma 6.2. *For any λ in the interval $(0, \lambda^*]$, the sequence $\mu_\lambda^{(n)}$ admits a limit in distribution μ_λ, and we have*

$$E^{\mu_\lambda^{(n)}}\left[\exp\{-\theta X_{e(q)}\}\mathbf{1}_{\{e(q)<\tau_0^{X^{(n)}}\}}\right] \stackrel{n\to\infty}{\longrightarrow} E^{\mu_\lambda}\left[\exp\{-\theta X_{e(q)}\}\mathbf{1}_{\{e(q)<\tau_0^{X}\}}\right]. \tag{6.2}$$

In particular, the measure μ_λ satisfies Eqn. (5.8).

Proof. For any $\theta \in [0, \gamma]$ we have the convergence of $\psi_n(\theta)$ to $\psi(\theta)$, where ψ_n denotes the Laplace exponent of $X^{(n)}$, since, for any fixed positive t, $X_t^{(n)}$ converges in distribution to X_t. As the functions ψ_n and ψ are strictly convex, it follows that we also have the convergence of the sequence $(\bar{\phi}_n(\lambda))_n$ to $\bar{\phi}(\lambda)$, for any λ in the interval $(0, \lambda^*]$. Furthermore, since $\kappa_*^{(n)+}(0, \theta)$ converges to $\kappa_*^+(0, \theta)$ [by definition of the sequence $(X^{(n)})_n$], the definition in Eqn. (4.17) of the Wiener-Hopf factors $\Psi^{(n)+}(-\lambda, \theta)$ and $\Psi^+(-\lambda, \theta)$ for λ in the interval $(0, \lambda^*]$ imply that $\Psi^{(n)+}(-\lambda, \theta)$ converges to $\Psi^+(-\lambda, \theta)$, as n tends to infinity.

Hence, it follows from Eqns. (2.10) and (6.1) that $\widehat{\mu}^{(n)}(\theta)$ converges to $\widehat{\mu}(\theta)$ for every $\theta > 0$, so that Lévy's Continuity Theorem implies that the sequence of measures $(\mu^{(n)})_n$ converges to μ in distribution as n tends to infinity.

Note next that we have the relation

$$\begin{aligned} E^{\mu^{(n)}}\left[\mathrm{e}^{-\theta X_{e(q)}^{(n)}}\mathbf{1}_{\{e(q)<\tau_0^{X^{(n)}}\}}\right] &= \int_{[0,\infty)} P(x + X_{e(q)}^{(n)} > e'(\theta), -X_*^{(n)}(e(q)) > x)\mu^{(n)}(\mathrm{d}x) \\ &= P(M^{(n)} + X_{e(q)}^{(n)} > e'(\theta), -X_*^{(n)}(e(q)) + M^{(n)} > 0), \end{aligned} \tag{6.3}$$

where $e'(\theta)$ denotes an independent $\mathrm{Exp}(\theta)$ random time, $M^{(n)} \sim \mu^{(n)}$ is independent of $X_*^{(n)}(e(q))$. Moreover, the same identity holds with $\mu^{(n)}$ and $X^{(n)}$ replaced by μ and X, respectively. Noting that (a) M^n converges in distribution to M, (b) the sequence $(X^{(n)}(e(q)), -X_*^{(n)}(e(q))_n$ converges weakly to $(X(e(q)), -X_*(e(q))$, and (c) $M + X_*(e(q))$ is absolutely continuous (which is the case since M is equal to the convolution of some distribution with an exponential random distribution), we conclude that the probability in the rhs of Eqn. (6.3) converges to $P(X(e(q)) + M > e'(\theta), -X_*(e(q)) + M > 0)$ as n tends to infinity, which is equivalent to the statement in Eqn. (6.2). Finally, by combining Lemma 5.2, Eqn. (6.2) with the fact that $\mu_\lambda^{(n)}$ converges in distribution to μ_λ, it follows that μ_λ satisfies Eqn. (5.8). □

With the above results in hand, we now move to the question of uniqueness of the quasi-invariant distributions.

Lemma 6.3. *For any λ in the interval $(0, \lambda^*]$, there exists a unique probability measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ that satisfies the relation*

$$\mu(A) = \frac{q+\lambda}{q} P^\mu[X_{e(q)} \in A, \tau_0^X < e(q)] \qquad A \in \mathcal{B}(\mathbb{R}_+), q > 0. \tag{6.4}$$

The proof rests on a contraction argument.

Proof of Lemma 6.3. First consider the case $\lambda \in (0, \lambda^*)$. By changing the measure using Petrov's transform Λ^{θ^*} the expression on the rhs of Eqn. (6.4) can be expressed as

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} (q+\lambda)\mathrm{e}^{-qt}\mathrm{e}^{-\lambda^* t} E_x^{\theta^*}[\mathrm{e}^{-\theta^*(X_t - x)}\mathbf{1}_{\{\tau_0^X < t\}}]\mathrm{d}t\mu(\mathrm{d}x)$$

Denote by m_* the measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ given by $m_*(\mathrm{d}x) = \mathrm{e}^{\theta^* x}\mu(\mathrm{d}x)$, and by $\mathcal{M}$ the collection of measures on the measure space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ that satisfy the integrability condition

$$\int_{\mathbb{R}_+} \mathrm{e}^{-\theta^* x} m(\mathrm{d}x) \leq 1.$$

Then the equality in Eqn. (6.4) can be rephrased as $m_* = \mathcal{H}m_*$, where $\mathcal{H}$ is the operator $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ given by

$$(\mathcal{H}\pi)(A) = \frac{q+\lambda}{q^*}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} q_*\mathrm{e}^{-q_* t} P_x^{\theta_*}[X_t \in A, \tau_0^X < t]\mathrm{d}t\,\pi(\mathrm{d}x), \qquad A \in \mathcal{B}(\mathbb{R}_+), \pi \in \mathcal{M}, \tag{6.5}$$

where $q^* = q + \lambda_*$. The operator $\mathcal{H}$ is a contraction on the Banach space $\mathcal{M}$ endowed with the norm given by $\|\pi - \pi'\| := \sup_\Upsilon |\pi(f) - \pi'(f)|$, where the supremum is taken over the collection of functions given by

$\Upsilon := \{f \in L^0 : |f(x)| \leq \mathrm{e}^{-\theta^* x}\ \forall x \in \mathbb{R}_+\}$ which is a subset of set L^0 of Borel-measurable real-valued functions with domain $\mathbb{R}_+$. Indeed, it is a direct consequence of the definition of $\mathcal{H}$ that we have the estimate

$$\|\mathcal{H}\pi - \mathcal{H}\pi'\| \leq \frac{q+\lambda}{q^*}\|\pi - \pi'\| < \|\pi - \pi'\|, \qquad \pi, \pi' \in \mathcal{M},$$

where in the second inequality we used that $q + \lambda$ is strictly smaller than q^*. Thus, Banach's Contraction Theorem in conjunction with Lemma 6.2 implies that μ_λ is the unique measure π in $\mathcal{M}$ that satisfies the relation $\pi = \mathcal{H}\pi$. Hence, μ_λ is the unique probability measure satisfying Eqn. (6.4), for any λ in the interval $(0, \lambda^*)$.

We next consider the boundary case $\lambda = \lambda^*$. The proof in this case follows by a modification of above argument. In particular, the Implicit Function Theorem implies that, for any strictly positive and sufficiently small ϵ and any λ satisfying $\lambda - \lambda^* \in (0, \epsilon]$, there exists an υ in a neighbourhood of θ^* in the complex plane such that $\Psi(-\mathbf{i}\upsilon) = -\lambda$. Fix such an ϵ and a corresponding $\upsilon = \upsilon_\epsilon$. By repeating above argument, replacing the Petrov-transform $\Lambda^{(\theta^*)}$ by the complex-valued change of measure $\Lambda^{(\upsilon_\epsilon)}$, we find that the corresponding map $\mathcal{H}$ is still a contraction but now on the space $\mathcal{M}_\epsilon$ given by

$$\mathcal{M}_\epsilon = \left\{\pi = \pi_1 + \mathbf{i}\pi_2 : \left|\int_{\mathbb{R}_+} \mathrm{e}^{-\upsilon_\epsilon x}\pi(\mathrm{d}x)\right| \leq 1\right\}$$

and with respect to the norm $\|\pi - \pi'\|_\epsilon := \sup_{\Upsilon_\epsilon} |\pi(f) - \pi'(f)|$ where the supremum is taken over the set of functions $\Upsilon_\epsilon := \{f \in L^0(\mathbb{C}) : |f(x)| \leq |\mathrm{e}^{-\upsilon_\epsilon x}|\ \forall x \in \mathbb{R}_+\}$, where $L^0(\mathbb{C})$ denotes the set of Borel-measurable complex-valued functions with domain $\mathbb{R}_+$. Thus, also in the case $\lambda = \lambda^*$, an application of Banach's Contraction Theorem in combination with Lemma 6.2 shows that μ_λ is the unique probability measure satisfying Eqn. (6.4). □

Proof of Theorem 2.6. For any λ in the interval $(0, \lambda^*]$, it follows by combining Lemmas 6.2 and 6.3 that the probability measure μ_λ is the unique λ-invariant distribution for the process $\{X_t, t < \tau_0^X\}$. □

7. Application to credit-risk modeling

The structural approach that was initially proposed by Black & Cox [5] is to model the time of default of a firm as the first time that the value of the equity of the firm falls below the value of its debt, which is equal in the setting of [5] to the first-hitting time of a geometric Brownian motion to some level. Subsequent studies such as [1, 17] present stylized 'default barrier models' for the time of default as the epoch of first-passage of a stochastic process over a default-barrier.

A Credit Default Swap (CDS) is a commonly traded financial contract that provides insurance against the event that a specific company defaults on its financial obligations. An important problem for a financial institution is to ensure that the model-value of a traded credit derivative such as the CDS that it holds in its portfolio is consistent with market quotes. In a default-barrier model for the value of the CDS one is led to the inverse problem of identifying the boundary that will equate model and market values.

Apart from featuring in the valuation of credit derivatives such as the CDS, the credit risk of a company may also affect the value of other assets in the portfolio, especially in the cases where the company in question acts as counterparty in a trade. The quantification of this type of risk, named *counterparty risk*, requires the joint modeling of asset values and the risk of default of the company in question (see Cesari *et al.* [10] for background on counterparty risk). Various aspects of the modeling of counterparty risk in default barrier models have been investigated for instance in [6, 9, 15, 23]; in these papers the model and market quotes can be matched by calibration of the model parameters. Next we present an explicit example of the valuation of a call option under counter-party risk in a default-barrier model that is by construction *consistent* with a given risk-neutral probability of default, using the solution to the RIFPT problem given in Corollary 2.7.

7.1. **Valuation of a call option under counterparty risk.** This problem involves three entities, a company A, whose stock price is denoted S_t, a bank B and the bank's counterparty C. The problem under consideration is the fair valuation of the counterparty risk to B of a trade in which C has sold to the bank a European call option on the stock of company A. This source of risk refers to the potential loss that the bank, as the owner of the call option, incurs when its counterparty C goes into default before the maturity T of the call option, and fails to deliver the pay-off of the call option. If τ denotes the epoch of default of C then the fair value π of the potential loss of the holder of the option (discounted to time 0 at the risk-free rate r) and the so-called *expected positive exposure* P_t are given by

$$\pi = E[V_\tau \mathbf{1}_{\{\tau\leq T\}}], \tag{7.1}$$

$$P_t = E[V_\tau|\tau = t], \qquad t \in [0,T], \tag{7.2}$$

where V_τ denotes the value at time τ of a T-maturity call-option with strike K on the value of stock, discounted to time 0:

$$V_\tau = \mathrm{e}^{-r\tau} E[\mathrm{e}^{-r(T-\tau)}(S_T - K)^+|\mathcal{F}_\tau]. \tag{7.3}$$

The conditional expectation in Eqn. (7.2) is understood as the regular version of the conditional expectation $E[V_\tau|\tau]$ (under Assumption 7.1(iii) below this conditional expectation can in fact be defined in the usual way for continuous random variables). We will phrase the model in terms of two independent Lévy processes X and Z satisfying Assumption 2.5. Throughout this section we work under the following additional assumptions:

Assumption 7.1. (i) We have $\underline{\theta}_X < -1$, $\overline{\theta}_X > 1+\alpha$, $\overline{\theta}_Z > 1+\alpha$ for some $\alpha > 0$.

(ii) The CDF H has a continuous density h, and satisfies $\overline{H}(T) > 0$ and $\lambda^*_X > -\log\overline{H}(T)/H(T)$, where λ^*_X denotes the Petrov coefficient of X.

(iii) For any λ in $(0,\lambda^*_X]$, $A \in \mathcal{B}(\mathbb{R})$ and $x > 0$, the function $t \mapsto P(X_{\tau^X_{-x}} \in A, \tau^X_{-x} \leq t)$ is continuously differentiable.

Let the credit-worthiness of the counterparty C be described by the credit-index process Y, in the sense that default of C occurs at the first moment that the process Y falls below the level 0, that is given in terms of X by

$$Y_t = Y_0 + X_{I(t)}, \qquad I(t) = I_{\mu^X_{\lambda^0}}(t) = T\cdot\frac{\log\overline{H}(t)}{\log\overline{H}(T)}, \qquad t \in [0,T], \tag{7.4}$$

$$Y_0 \sim \mu^X_{\lambda^0}, \qquad \lambda^0 = -T^{-1}\cdot\log\overline{H}(T), \tag{7.5}$$

where, as before, Y_0 is independent of X and $\mu^X_{\lambda^0}$ denotes the λ^0-invariant distribution corresponding of the process $\{X_t, t < \tau^X_0\}$. Here we have chosen λ^0 so as to normalise the ratio $I(T)/T$ to unity. Note that the CDF of the first-passage time τ^Y_0 of the process Y defined in Eqn. (7.4) is given by H (in view of Corollary 2.7 and Assumption 7.1(ii)).

In the case that the price process S is independent of credit index process Y we note that the expectation in Eqn. (7.1) is equal to the integral of the expectation $E[V_t]$ against the measure $H(\mathrm{d}t)$. Next we consider the complementary case that S and Y are dependent. More specifically, we assume that S is given by

$$\begin{cases} S_t = S_0\exp\{(r-d)t + L_t - \kappa_t(-\mathbf{i})\}, & t\in[0,T], \qquad S_0 > 0,\\ L_t = \rho X_{I(t)} + Z_t, & \rho\in[-1,1],\\ \kappa_t(u) = \Psi_Z(u)t + \Psi_X(u\rho)I(t), & \Im(u)\in[-1-\alpha,0], \end{cases} \tag{7.6}$$

where Ψ_Z and Ψ_X denote the characteristic exponents of the Lévy processes X and Z and r and d denote the risk-free rate and the dividend yield, respectively. The degree of dependence between the stock price process S and the credit index process Y is controlled by the parameter ρ. Note that κ_t has been specified such that the discounted stock-price process $\mathrm{e}^{-rt}[\mathrm{e}^{dt}S_t]$ with reinvested dividends is a martingale. In the following result an explicit expression is derived for π and $P(t)$ in terms of the inverse Fourier-transform $\mathcal{F}^{-1}_\xi$ and the inverse Laplace-transform $\mathcal{L}^{-1}_q$ with respect to ξ and q, respectively.

Proposition 7.2. *The values π and P_t, $t \in [0,T]$, are given by*

$$\pi = \int_0^{I(T)} N(t)\mathrm{d}t, \qquad P_t = \frac{N(t)}{\lambda^0 \overline{H}(t)} \tag{7.7}$$

$$N(t) = \mathcal{F}_\xi^{-1}\left(D_{t,T}(\xi) \cdot C_{I(t)}(\xi)\right)(k), \tag{7.8}$$

with $k = \log K/s'$, $s' = sc'$, $c' = \exp(-rT + (r-d)(T-t) - \kappa_T(-\mathbf{i}) + \kappa_t(-\mathbf{i}))$ and

$$D_{t,T}(\xi) = \frac{\exp\{\Psi_Z(\xi - (1+\alpha)\mathbf{i})(T-t) + \Psi_X(\xi - (1+\alpha)\mathbf{i})(I(T) - I(t))\}}{(1+\alpha+\mathbf{i}\xi)(\alpha+\mathbf{i}\xi)}, \tag{7.9}$$

and with $C_{I(t)}(\xi)$ given by the rhs of Eqn. (7.11) below, with the substitutions $t \to I(t)$ and $u \to 1 + \alpha + \mathbf{i}\xi$. Here α is the constant appearing in Assumption 7.1.

The proof relies on the following lemma:

Lemma 7.3. *For any u with $\Re(u) \in (\underline{\theta}_X, \overline{\theta}_X)$ and $t \in [0,T]$ we have, with $\tau = \tau_0^Y$,*

$$E\left[\mathrm{e}^{uX_{I(\tau)}}\middle|\tau = t\right] = \frac{1}{\lambda^0 \overline{H}(t)}\mathcal{L}_q^{-1}\left(\widehat{f}_{t,u}(q)\right)(I(t)), \qquad \widehat{f}_{t,u}(q) = \int_{\mathbb{R}_+} \mu_{\lambda^0}^X(\mathrm{d}x) E\left[\mathrm{e}^{uX_{\tau_{-x}^X} - q\tau_{-x}^X}\right], \tag{7.10}$$

where $\widehat{f}_{t,u}(q)$ is given in terms of the Wiener-Hopf factor Ψ^- of X in Eqn. (4.9) above. In particular, for u satisfying in addition $\Re(u) \in (\underline{\theta}_Z, \overline{\theta}_Z)$ we have

$$E_s[S_\tau^u|\tau = t] = \frac{s^u}{\lambda^0 \overline{H}(t)} \cdot \exp\{(r-d)tu - \kappa_t(-\mathbf{i})u + \Psi_Z(-\mathbf{i}u)t\} \cdot \widehat{f}_t(u). \tag{7.11}$$

Proof of Lemma 7.3: Denote by $\{p_{t,x}(\mathrm{d}y), t \in \mathbb{R}_+\}$ the collection of measures on $(\mathbb{R}_-, \mathcal{B}(\mathbb{R}_-))$ satisfying the equality $p_{t,x}(\mathrm{d}y)\mathrm{d}t = P(X_{\tau_{-x}^X} \in \mathrm{d}y, \tau_{-x}^X \in \mathrm{d}t)$ (the existence of such a collection is guaranteed by Assumption 7.1(iii)). Since the CDF of τ_0^Y is given by H, it follows by Bayes' lemma that the conditional expectation in the lhs of Eqn. (7.10) can be expressed as

$$E[\mathrm{e}^{uX_{I(\tau)}}|\tau = t] = \frac{1}{h(t)}\int_{\mathbb{R}_+} \mu_{\lambda^0}^X(\mathrm{d}x)\int_{\mathbb{R}} \mathrm{e}^{ux} p_{I(t),x}(\mathrm{d}y) I'(t). \tag{7.12}$$

Since we have $I'(t) = h(t)/[\lambda^0 \overline{H}(t)]$, it follows that the rhs of Eqn. (7.12) and Eqn. (7.10) are equal. The identity in Eqn. (7.11) is a direct consequence of the form of S in given in Eqn. (7.6) and the independence of Z and τ. □

Proof of Proposition 7.2. Note first that the form of π is obtained by integrating P_t against $h(t)$ over the interval $[0,T]$ and performing the change of variables $u = I(t)$, noting that $I'(t) = h(t)/[\lambda \overline{H}(t)]$.

The independence of the increments of $\log S$ implies

$$P_t = E[G(\tau, S_\tau)|\tau = t], \qquad G(t,s) = s' \cdot E[(\mathrm{e}^{L_T - L_t} - \mathrm{e}^k)^+],$$
$$s' = sc', \qquad c' = \exp(-rT + (r-d)(T-t) - \kappa_T(-\mathbf{i}) + \kappa_t(-\mathbf{i})), \qquad k = \log(K/s').$$

By a standard Fourier-transform argument it can be shows that $G(t,s)$ admits an explicit integral representation representation in terms of the characteristic exponents of X and Z. More specifically, since the dampened function $k \mapsto \exp(\alpha k) \cdot G(t,s)$ and its Fourier transform are integrable, the Fourier Inversion Theorem implies

$$G(t,s) = [\mathcal{F}_\xi^{-1}(G_{t,s}^\wedge)](k), \qquad G_{t,s}^\wedge(\xi) = s \cdot D_{t,T}(\xi), \qquad \xi \in \mathbb{R}, \tag{7.13}$$

where $D_{t,T}(\xi)$ is given in Eqn. (7.9). Using the integral representation of the Fourier-inverse and by an interchange of the expectation and integration (justified by Fubini's theorem) we find that P_t, $t \in [0,T]$, is equal to

$$P_t = c' \int_{\mathbb{R}} E_s[S_\tau^\theta|\tau = t] \cdot \left(\frac{K}{c'}\right)^{-\alpha - \mathbf{i}\xi} D_{t,T}(\xi)\mathrm{d}\xi. \tag{7.14}$$

The expression for P_t in Eqn. (7.8) follows by inserting the expression in Eqn. (7.11) in Lemma 7.3(ii). □

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