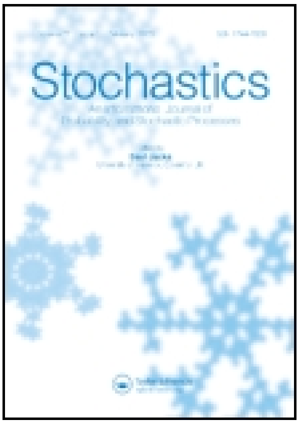

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EQUIVALENT MARTINGALE MEASURES AND NO-ARBITRAGE IN STOCHASTIC SECURITIES MARKET MODELS

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We characterize those vector-valued stochastic processes (with a finite index set and defined on an arbitrary stochasic base) which can become a martingale under an equivalent change of measure.

This question is important in a widely studied problem which arises in the theory of finite period securities markets with one riskless bond and a finite number of risky stocks. In this setting, our characterization gives a criterion for recognizing when a securities market model allows for no arbitrage opportunities ("free lunches"). Intuitively, this can be interpreted as saying "if one cannot win betting on a process, then it must be a martingale under an equivalent measure," and provides a converse to the classical notion that "one cannot win betting on a martingale."

1. INTRODUCTION

Classical martingale systems theorems (Halmos [9] and Doob [7]) formalize the intuitive idea behind martingales, namely that "one can't win betting on a martingale". More precisely, using Burkholder's martingale transforms (Burkholder [3]), we have the following. Let $X=(X_t:t=0,1,\ldots,T)$ be an R^d-valued martingale $(1\leqq d<\infty)$ on some stochastic base $(\Omega,\mathbf{F},P)$ and let $V=(V_t:t=1,2,\ldots,T)$ denote an R^d-valued **F**-predictable stochastic process ("betting strategy"). Then the martingale transform $V\circ X=((V\circ X)_t:t=0,1,\ldots,T)$ of V with respect to X is defined by

$$(V\circ X)_t=V_1\cdot X_o+\sum_{s=1}^{t}V_s\cdot(X_s-X_{s-1})$$

where $(V\circ X)_t$ represents the "accumulated gain up to time t" when following the strategy V ($V_s\cdot X_s$ is the Euclidean scalar product of the vectors V_s and X_s). If $V\circ X$ is integrable then classical martingale systems results show that $V\circ X$ is again a martingale with respect to P and $\mathbf{F}$. Thus there are no "smart" betting strategies which can change the character of the fair game X in favor of the gambler. Put differently, if X and V are as above then the following condition holds:

$$\text{for } t=1,2,\ldots,T \text{ and for } \mathbf{F}\text{-predictable } V,$$
$$V_t\cdot(X_t-X_{t-1})\geqq 0\ P-a.s. \Rightarrow V_t\cdot(X_t-X_{t-1})=0\ P-a.s. \tag{1.1}$$

Observe that (1.1) still holds if P is replaced by an equivalent probability measure Q on $(\Omega,\mathbf{F})$ (i.e. P and Q have the same null sets), but that the martingale property of $V\circ X$ (under P) will in general be destroyed when P is replaced by Q.

In this paper, we show that condition (1.1) is not only sufficient but also necessary for a process X to be a martingale under an equivalent probability measure Q on $(\Omega,\mathbf{F})$; such a Q is called an *equivalent martingale measure* for $(\mathbf{F},X)$. The results in this paper can thus be viewed as a converse to the classical martingale systems theorem and can be interpreted as saying: "if one can't win betting on a process then it must be a martingale under an equivalent change of measure". Our approach is based on a pathwise analysis of condition (1.1) and extends two recent developments in this area: (i) a similar but more elementary sample path investigation of (1.1) when there are only finitely many states of nature (see Taqqu and Willinger [13]), and (ii) an analysis of the same change of measure problem in the case of a single-period random process X (see Willinger and Taqqu [15]). Willinger and Taqqu [15] have also solved the problem of existence of a *unique* equivalent martingale measure. Whereas the uniqueness problem can be solved using only elementary probability tools, our extension relies on some abstract measurable selection theorems. Our results also include the special case $d=1$ studied by Back and Pliska [2]. Their proof, however, does not generalize to higher dimensions.

Characterizing stochastic processes which can be transformed into martingales by means of an equivalent change of measure is of particular interest in the analysis of stochastic models of securities markets (see, for example, Harrison and Kreps [10], Harrison and Pliska [11], Duffie and Huang [8], Taqqu and Willinger [13], Back and Pliska [2] and Willinger and Taqqu [15]). Indeed, Harrison and Kreps [10] and Harrison and Pliska [11] first demonstrated a fundamental relationship between the question of existence of equivalent martingale measures for the securities price process (modeling the prices of one riskless bond and $1\leq d<\infty$ risky stocks over time) and the economic notion of "no arbitrage". Their results were generalized by Duffie and Huang [8]. In view of our earlier comments, this relationship is natural since an arbitrage opportunity represents a riskless plan for making profits without initial investments (a "free lunch"). All these references use the notion of agent's preferences, rely on a global functional-analytic argument, and require some additional assumptions on the price process, such as integrability.

In this securities market setting, our results (in particular, Theorem 2.6) extend those of Harrison and Pliska [11] and Taqqu and Willinger [13] who consider finite-period, frictionless securities market models when there are only finitely many states of nature. Here we allow an arbitrary probability space, make no integrability assumption on the price process, and show how one can recognize that a given securities market model satisfies the economically meaningful assumption of "no arbitrage". In the one-dimensional case (i.e., one risky stock and one riskless bond), this problem was studied by Back and Pliska [2]. Although their method of proof does not generalize to higher dimensions, these authors conjectured that the same result holds for finite-period, vector-valued securities price processes. This conjecture is proved in our Theorem 2.6. Also note that Back and Pliska work with a more restrictive class of "feasible" trading strategies than we do (we impose no "positive wealth constraints", see Section 3.1); in the present finite-period setting, it is easy to see that this restriction is not essential.

A pathwise analysis along the lines suggested in this paper of the no-arbitrage assumption for continuous-time price processes, where trading in stock and bonds can take place continuously in time, remains an open problem. See, however, Harrison and Pliska [11] and Back and Pliska [2] for examples of what can go wrong when trading continuously.

The paper is structured as follows. In Section 2 we prove that condition (1.1) is necessary and sufficient for the existence of an equivalent martingale measure Q for X and briefly mention the known results concerning uniqueness of such a Q. In Section 3 we apply our results in the context of stochastic modelling of finite-period, frictionless securities markets and show how the existence of an equivalent martingale measure for the securities price process is related to the economically meaningful "no arbitrage" assumption.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF AN EQUIVALENT MARTINGALE MEASURE

The purpose of this section is to give necessary and sufficient conditions under which a discrete-time, $\mathbf{R}^d$-valued process with finite time horizon has an equivalent martingale measure. We first show that condition (1.1) is sufficient for the existence of an equivalent martingale measure for a "one-step" process $X=(X_0,X_1)$. This special case contains most of the technical difficulties and the problem of a finite-period process $X=(X_t:t=0,1,\ldots,T)$ follows as a corollary of the results for the "one-step" process. Our proofs are essentially self-contained, using only standard results from convex analysis, and some general results concerning measurable selection. In particular, the proofs do not rely on results of the finite-probability space setting (see Taqqu and Willinger [13]), nor on the one-dimensional result of Back and Pliska [2]. Both are special cases of our proof, but a reader only interested in these settings should consult these references. (Note that Back and Pliska [2] assume integrability of the process, an apparently unnatural condition since it is not preserved under a change of equivalent measure; see, however, Remark 3.4).

We would like to point out that most of the technical difficulties in our proof come from the fact that with $d>1$, an explicit construction as in Back and Pliska [2] is no longer feasible. However, if in addition the process were assumed to be bounded, a discrete approximation argument as in Willinger and Taqqu ([15], Theorem 2.3.1), together with appropriate use of measurable selection would be possible.

2.1 The One-step Case

Let $(\Omega, \mathscr{F}, P)$ be a (complete) probability space. If $\tilde{P}$ is a probability measure on $(\Omega, \mathscr{F})$ then P and $\tilde{P}$ are *equivalent* (on $\mathscr{F}$) provided for all $F \in \mathscr{F}, P(F)=0$ if and only if $\tilde{P}(F)=0$. Note that if $\mathscr{G}$ is a sub-σ-algebra of $\mathscr{F}$, then P and $\tilde{P}$ may be equivalent on $\mathscr{G}$ but not on $\mathscr{F}$. When P and $\tilde{P}$ are equivalent, $d\tilde{P}/dP$ denotes the Radon–Nikodym derivative of $\tilde{P}$ with respect to P. In this case, $d\tilde{P}/dP>0, P$—*a.s.* If P and $\tilde{P}$ are equivalent on $\mathscr{F}$ and Y is $\tilde{P}$-integrable, recall that

$$\tilde{E}(Y|\mathscr{G}) = E\left(Y\frac{d\tilde{P}}{dP}\bigg|\mathscr{G}\right)\bigg/E\left(\frac{d\tilde{P}}{dP}\bigg|\mathscr{G}\right), \tag{2.1}$$

where $\tilde{E}(\cdot)$ denotes expectation with respect to $\tilde{P}$.

The set $\mathbf{R}^d$ with its Euclidean norm $\|\cdot\|$ will be equipped with its usual topology and its Borel σ-algebra $\mathscr{B}(\mathbf{R}^d)$. We will add to $\mathbf{R}^d$ an element ∞, and set $\bar{\mathbf{R}}^d = \mathbf{R}^d \cup \{\infty\}$. If the open neighborhoods of ∞ are the complements of compact sets then $\bar{\mathbf{R}}^d$ is compact and metrisable. We equip $\bar{\mathbf{R}}^d$ with this topology.

Given two elements $x, y \in \mathbf{R}^d$, $x \cdot y$ will denote their Euclidean scalar product. Each $\alpha \in \mathbf{R}^d$ defines a hyperplane $H^\alpha = \{x \in \mathbf{R}^d : \alpha \cdot x = 0\}$. We also define $H^\alpha_\geqq = \{x \in \mathbf{R}^d : \alpha \cdot x \geqq 0\}$, $H^\alpha_> = \{x \in \mathbf{R}^d : \alpha \cdot x > 0\}$. $H^\alpha_<$ and $H^\alpha_\leqq$ are defined analogously.

The following theorem gives a slightly sharper result than Theorem 2.3.1. of Willinger and Taqqu [15]; indeed, it is not difficult to see that the two statements would be equivalent if the function g below were only required to be measurable instead of continuous.

THEOREM 2.1 *Let ν be an arbitrary probability measure on $\mathbf{R}^d$. Then the following two conditions are equivalent.*

a) For all $\alpha \in \mathbf{R}^d$, $\nu(H^\alpha_\geqq)=1$ implies $\nu(H^\alpha)=1$.

b) There is a continuous function $g: \bar{\mathbf{R}}^d \to [0,1]$, such that $g(x)>0, \forall x \in \mathbf{R}^d$, and

$$\int_{\mathbf{R}^d} \|x\| g(x) \nu(dx) \leqq 1 \text{ and } \int_{\mathbf{R}^d} x g(x) \nu(dx) = 0.$$

Proof We do not need the implication (b)⇒(a), so its easy proof is omitted (it is similar to the first few lines of the proof of Theorem 2.4). In order to show that (a) implies (b), observe that it is sufficient to prove (b) with $\int_{\mathbf{R}^d} \|x\| g(x) \nu(dx) \leqq 1$ replaced by $\int_{\mathbf{R}^d} \|x\| g(x) \nu(dx) < \infty$. (Indeed, if the integral happens to be greater than 1, simply divide g by the value of the integral and this does not affect the other properties of g). To this end we consider two cases.

i) ν has bounded support.

Consider the two convex sets $C_1=\{0\}$ and $C_2=\{\int_{\mathbf{R}^d} xg(x)\nu(dx) | g:\mathbf{R}^d\to(0,1], g$ continuous$\}$ and suppose condition (b) does not hold (note: the condition that ν has bounded support is necessary to ensure that the integrals in the definition of C_2 are well-defined). Then $C_1\cap C_2=\emptyset$, and so there exists a hyperplane H^α in $\mathbf{R}^d$ which *properly* separates C_1 and C_2 (recall that H^α properly separates C_1 and C_2 provided $C_2\subseteq H^\alpha_{\geqq}$, but $C_2\not\subseteq H^\alpha$; see Rockafellar ([12], Section 11)). Thus we can assume

$$\alpha\cdot\left(\int_{\mathbf{R}^d} xg(x)\nu(dx)\right)\geqq 0, \text{ for all positive and continuous } g.$$

Let $(g_n)_{n\geqq 0}$ denote a sequence of bounded and continuous functions $g_n:\mathbf{R}^d\to(0,1]$ such that

$$\lim_{n\to\infty} g_n(y)=1_{\{\alpha\cdot y<0\}}, \text{ for all } y\in\mathbf{R}^d.$$

By the dominated convergence theorem (which can be applied since ν has bounded support),

$$0\leqq\lim_{n\to\infty}\left(\int_{\mathbf{R}^d}\alpha\cdot xg_n(x)\nu(dx)\right)=\int_{\mathbf{R}^d}\alpha\cdot x1_{\{\alpha\cdot x<0\}}\nu(dx),$$

that is, $\alpha\cdot x\geqq 0$, for ν-almost all $x\in\mathbf{R}^d$, and so $\nu(H^\alpha_{\geqq})=1$. However, if $\nu(H^\alpha)$ were equal to 1, then we would have $\alpha\cdot x=0$ $\nu-a.s.$, so C_2 would be contained in H^α. This would contradict the fact that H^α properly separates C_1 and C_2. Thus, $\nu(H^\alpha)<1$ and (a) fails.

ii) ν is an arbitrary probability measure on $\mathbf{R}^d$.

Define a one-to-one transformation $\psi:\mathbf{R}^d\to B(0,1)$ (the open unit ball centered at the origin) by $\psi(x)=x/(1+\|x\|)$, and let $\tilde{\nu}$ be the image of ν under ψ. Then $\tilde{\nu}$ defines a probability measure on $\mathbf{R}^d$ with bounded support. Also note that condition (a) holds for ν if and only if it holds for $\tilde{\nu}$. Thus, if (a) holds for ν, (i) guarantees the existence of a continuous function $\tilde{g}:\mathbf{R}^d\to(0,1]$ with $\int_{\mathbf{R}^d} x\tilde{g}(x)\tilde{\nu}(dx)=0$. It is then easy to verify that the function $g:\bar{\mathbf{R}}^d\to[0,1]$ defined by

$$g(x)=\begin{cases}\tilde{g}(\psi(x))/(1+\|x\|), & \text{if } x\neq\infty\\ 0 & \text{if } x=\infty\end{cases}$$

satisfies condition (b) of the theorem. □

Let $\mathscr{G}$ be a sub σ-algebra of $\mathscr{F}$. Recall (see, for example Ash [1]) that the

$\mathbf{R}^d$-valued random variable Y has a regular conditional probability distribution given $\mathscr{G}$, that is, there exists a function $\mu:\Omega\times\mathscr{B}(\mathbf{R}^d)\to\mathbf{R}$ such that

a) $\omega\to\mu(\omega,B)$ is $\mathscr{G}$-measurable, $\forall B\in\mathscr{B}(\mathbf{R}^d)$;

b) for P-almost all $\omega\in\Omega, B\to\mu(\omega,B)$ is a probability measure on $(\mathbf{R}^d,\mathscr{B}(\mathbf{R}^d))$;

c) $\mu(\cdot,B)=P(Y\in B|\mathscr{G})$ *a.s.*, $\forall B\in\mathscr{B}(\mathbf{R}^d)$.

In order to prove Lemma 2.3 below, we need the following technical results. They are stated here for later reference and can be proved using standard measurability arguments such as the monotone class theorem (see Dellacherie and Meyer ([5], Theorem I.19)) and properties of the conditional expectation operator for random variables with or without finite expectations (see Ash ([1], Chap. 6.4 and 6.5)).

LEMMA 2.2

a) Let $(S,\mathscr{S})$ be a measurable space. Suppose $F:\Omega\times\mathbf{R}^d\times S\to\mathbf{R}$ is $\mathscr{G}\times\mathscr{B}(\mathbf{R}^d)\times\mathscr{S}$-measurable and non-negative. Then the map $F^*:\Omega\times S\to\mathbf{R}_+\cup\{+\infty\}$ defined by

$$F^*(\omega,s)=\int_{\mathbf{R}^d}F(\omega,x,s)\mu(\omega,dx)$$

is $\mathscr{G}\times\mathscr{S}$-measurable.

b) Suppose $h:\Omega\to S$ is $\mathscr{G}$-measurable. Set $U(\omega)=F(\omega,Y(\omega),h(\omega))$ and $V(\omega)=F^*(\omega,h(\omega))$. Then V is P-integrable if and only if U is, and in this case $V=E(U|\mathscr{G})$ a.s.

c) For $K\subset\Omega\times\mathbf{R}^d$, set $K^\omega=\{x\in\mathbf{R}^d:(\omega,x)\in K\}$. Now suppose $K\in\mathscr{G}\times\mathscr{B}(\mathbf{R}^d)$. Then the map $\omega\to\mu(\omega,K^\omega)$ is $\mathscr{G}$-measurable and

$$P\{\omega\in\Omega: Y(\omega)\in K^\omega\}=\int_\Omega\mu(\omega,K^\omega)\,dP(\omega).$$

LEMMA 2.3 *Let Y be an arbitrary $\mathbf{R}^d$-valued random variable. Then the following conditions are equivalent:*

For all $\mathscr{G}$-measurable $\mathbf{R}^d$-valued random variables Z,

$$Z\cdot Y\geqq 0\;P-a.s.\Rightarrow Z\cdot Y=0\;P-a.s. \tag{2.2}$$

For almost all $\omega\in\Omega$, for all $\alpha\in\mathbf{R}^d$, $\mu(\omega,H^\alpha_\geqq)=1\Rightarrow\mu(\omega,H^\alpha)=1$. (2.3)

Proof of Lemma 2.3 (2.3)$\Rightarrow$(2.2). Let Z be an $\mathbf{R}^d$-valued $\mathscr{G}$-measurable random variable such that $Z\cdot Y\geqq 0$ $P-a.s.$ Note that

$$\{Z\cdot Y\geqq 0\}=\{\omega\in\Omega:\ Y(\omega)\in H^{Z(\omega)}_\geqq\},$$

$$\{Z\cdot Y=0\}=\{\omega\in\Omega:\ Y(\omega)\in H^{Z(\omega)}\}.$$

Using Lemma 2.2(c) we see that

$$1 = P\{Z \cdot Y \geqq 0\} = \int_{\Omega} \mu(\omega, H_{\geqq}^{Z(\omega)})\, dP(\omega),$$

and so $\mu(\omega, H_{\geqq}^{Z(\omega)}) = 1$ for $\omega \in \Omega \backslash N$, where N is a P-null set. By (2.3), for almost all $\omega \in \Omega \backslash N$, $\mu(\omega, H^{Z(\omega)}) = 1$. Thus by Lemma 2.2(c)

$$P\{Z \cdot Y = 0\} = \int_{\Omega} \mu(\omega, H^{Z(\omega)})\, dP(\omega) = 1.$$

This proves (2.2).

(2.2)⇒(2.3). Set

$$U = \{(\omega, \alpha) \in \Omega \times \mathbf{R}^d : \mu(\omega, H_{\geqq}^{\alpha}) = 1 \text{ and } \mu(\omega, H^{\alpha}) < 1\}.$$

Then $U \in \mathscr{G} \times \mathscr{B}(\mathbf{R}^d)$.

Let $pr: \Omega \times \mathbf{R}^d \to \Omega$ be the canonical projection: $pr(\omega, x) = \omega$. To prove (2.3), we must show that $pr(U)$ has P-probability zero (since $(\Omega, \mathscr{F}, P)$ is complete, $pr(U)$ is $\mathscr{F}$-measurable: cf. Dellacherie and Meyer ([5], Theorem III. 44–45)).

Suppose $P(pr(U)) > 0$. We shall show that this leads to a contradiction. Using measurable selection (cf. for example, the theorem mentioned above) we see that there is a $\mathscr{G}$-measurable $\bar{\mathbf{R}}^d$-valued random variable $\tilde{Z}$ such that

$$P\{\omega \in \Omega : (\omega, \tilde{Z}(\omega)) \in U\} = P(pr(U)) > 0.$$

Set $Z(\omega) = \tilde{Z}(\omega)$ if $(\omega, \tilde{Z}(\omega)) \in U, Z(\omega) = 0$ otherwise. Now we shall show that $P\{Z \cdot Y \geqq 0\} = 1$ but $P\{Z \cdot Y = 0\} < 1$, contradicting (2.2). Indeed, we have

$$\begin{aligned} P\{Z \cdot Y \geqq 0\} &= \int_{\Omega} \mu(\omega, H_{\geqq}^{Z(\omega)})\, dP(\omega) \\ &= \int_{pr(U)} 1\, dP(\omega) + \int_{(pr(U))^c} 1\, dP(\omega) \\ &= 1, \end{aligned}$$

but

$$\begin{aligned} P\{Z \cdot Y = 0\} &= \int_{\Omega} \mu(\omega, H^{Z(\omega)})\, dP(\omega) \\ &= \int_{pr(U)} \mu(\omega, H^{Z(\omega)})\, dP(\omega) + \int_{(pr(U))^c} 1\, dP(\omega) \\ &< P(pr(U)) + P((pr(U))^c) \\ &= 1. \quad \square \end{aligned}$$

THEOREM 2.4 *Let $\mathscr{G}\subset\mathscr{H}$ be two (complete) sub-σ-algebras of $\mathscr{F}$*, and let *Y be an arbitrary $\mathscr{H}$-measurable random variable. Then* (2.2) *is equivalent to the following: there exists an $\mathscr{H}$-measurable real random variable D such that $D>0$ a.s., $E(\|Y\|D)<+\infty$ and $E(YD|\mathscr{G})=0$. In addition, when* (2.2) *holds, it is always possible to choose D such that $D\leqq 1$ a.s.*

In order to prove this theorem we use the following technical lemma; its easy proof is left to the reader.

LEMMA 2.5 *Let $C[\bar{\mathbf{R}}^d]$ be the space of continuous real functions on $\bar{\mathbf{R}}^d$, with the norm $\|g\|_\infty=\sup_{x\in\mathbf{R}^d}|g(x)|$, and let $\mathscr{B}(C[\bar{\mathbf{R}}^d])$ be its Borel σ-algebra. Then the following properties hold.*

a) $\{g\in C[\bar{\mathbf{R}}^d]:0<g(x)\leqq 1,\forall x\in\mathbf{R}^d\}\in\mathscr{B}(C[\bar{\mathbf{R}}^d])$.

b) The map $(x,g)\to g(x)$ is $\mathscr{B}(\mathbf{R}^d)\times\mathscr{B}(C[\bar{\mathbf{R}}^d])$-measurable.

c) Suppose $F:\Omega\times\mathbf{R}^d\times C[\bar{\mathbf{R}}^d]\to\mathbf{R}^d$ is $G\times\mathscr{B}(\mathbf{R}^d)\times\mathscr{B}(C[\bar{\mathbf{R}}^d])$-measurable. Set

$$\tilde{F}^*(\omega,g)=\int_{\mathbf{R}^d}\|F(\omega,x,g)\|\,\mu(\omega,dx),$$

and

$$F^*(\omega,g)=\begin{cases}\int_{\mathbf{R}^d} F(\omega,x,g)\mu(\omega,dx) & \text{if } \tilde{F}^*(\omega,g)<+\infty,\\ \infty & \text{otherwise.}\end{cases}$$

Then F^* is $\mathscr{G}\times\mathscr{B}(C[\bar{\mathbf{R}}^d])$-measurable.

d) Suppose $h:\Omega\to C[\bar{\mathbf{R}}^d]$ is $\mathscr{G}$-measurable. Set $U(\omega)=F(\omega,Y(\omega),h(\omega))$, $V(\omega)=F^*(\omega,h(\omega))$, $\tilde{U}(\omega)=\|F(\omega,Y(\omega),h(\omega))\|$, $\tilde{V}(\omega)=\tilde{F}^*(\omega,h(\omega))$. If $\tilde{V}$ is P-integrable, then U and V are P-integrable and $V=E(U|\mathscr{G})$ $P-a.s.$

Proof of Theorem 2.4 Suppose there exists a random variable D with the properties stated in the theorem: we show that this implies (2.2). Indeed, let Z be a $\mathscr{G}$-measurable, $\mathbf{R}^d$-valued random variable such that $Z\cdot Y\geqq 0\,P-a.s.$ We must show that $P\{Z\cdot Y>0\}=0$. Now if $P\{Z\cdot Y>0\}>0$, we would have $P\{Z\cdot(DY)>0\}>0$, and thus

$$0<E(Z\cdot(DY))=E(Z\cdot E(DY|\mathscr{G}))=0,$$

a contradiction.

Now suppose (2.2) holds, or equivalently, by Lemma 2.3, that (2.3) holds. We shall prove the existence of a random variable D with the desired properties.

Set $F(\omega,x,g)=xg(x)$, $\tilde{F}(\omega,x,g)=\|x\|\,|g(x)|$. Using Lemma 2.5, we see that the set

$$H=\{(\omega,g)\in\Omega\times C[\bar{\mathbf{R}}^d]:0<g(x)\leqq 1,\forall x\in\mathbf{R}^d,\text{ and}$$

$$\int_{\mathbf{R}^d}\|x\|\,g(x)\mu(\omega,dx)\leqq 1,\ \int_{\mathbf{R}^d}xg(x)\mu(\omega,dx)=0\}$$

is in $\mathscr{G} \times \mathscr{B}(C[\bar{\mathbf{R}}^d])$. Since (2.3) holds, we use Theorem 2.1 to see that for P-almost all $\omega \in \Omega$, there is a $g_\omega \in C[\bar{\mathbf{R}}^d]$ such that $(\omega, g_\omega) \in H$. This means that the projection of H on Ω has P-probability one.

Now since $\bar{\mathbf{R}}^d$ is compact and metrisable, $C[\bar{\mathbf{R}}^d]$ is a separable and complete space, and thus we can apply a measurable selection theorem (see, for example, Dellacherie and Meyer ([5], Theorem III, 44–45)) to get a $\mathscr{G}$-measurable map $G:\Omega \to C[\bar{\mathbf{R}}^d]$ such that $(\omega, G(\omega)) \in H$ for P-almost all $\omega \in \Omega$. We write $G(\omega, x)$ instead of $G(\omega)(x)$. The map $(\omega, x) \to G(\omega, x)$ is $\mathscr{G} \times \mathscr{B}(\mathbf{R}^d)$-measurable, since it is the composition of the two measurable maps $(\omega, x) \to (x, G(\omega))$ and $(x, g) \to g(x)$ (see Lemma 2.5(b)).

Set $D(\omega) = G(\omega, Y(\omega))$: D is $\mathscr{H}$-measurable (again since it is the composition of two $\mathscr{H}$-measurable maps), and $0 < D \leqq 1$ *a.s.* In order to finish the proof, we use Lemma 2.5(d) with $U(\omega) = F(\omega, Y(\omega), G(\omega)) = Y(\omega)D(\omega)$ and $V(\omega) = F^*(\omega, G(\omega))$. Indeed, since

$$\tilde{F}^*(\omega, G(\omega)) = \int_{\mathbf{R}^d} \|F(\omega, x, G(\omega))\| \, \mu(\omega, dx)$$

$$= \int_{\mathbf{R}^d} \|x\| \, G(\omega, x) \mu(\omega, dx) \leqq 1 \ P - a.s.,$$

and

$$F^*(\omega, G(\omega)) = \int_{\mathbf{R}^d} F(\omega, x, G(\omega)) \, \mu(\omega, dx)$$

$$= \int_{\mathbf{R}^d} x G(\omega, x) \mu(\omega, dx) = 0 \ P - a.s.,$$

Lemma 2.5(d) applies and yields $E(YD \mid \mathscr{G}) = 0 \ P - a.s.$ □

2.2 Discrete time, finite horizon

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, $\mathbf{F} = (\mathscr{F}_k : k = 0, \ldots, n)$ a *filtration*, that is, each $\mathscr{F}_k$ is a complete sub-σ-algebra of $\mathscr{F}$ and $\mathscr{F}_k \subset \mathscr{F}_{k+1}$, $k = 0, \ldots, n-1$. Let $X = (X_k : k = 0, \ldots, n)$ be an $\mathbf{R}^d$-valued stochastic process which is *adapted* to $(\mathscr{F}_k : k = 0, \ldots, n)$, that is X_k is $\mathscr{F}_k$-measurable, $k = 0, \ldots, n$.

Recall that $\tilde{P}$ is called an *equivalent martingale measure for* X if $\|X_k\|$ is $\tilde{P}$-integrable $(k = 0, 1, \ldots, n)$, $\tilde{E}(X_{k+1} \mid \mathscr{F}_k) = X_k$ *a.s.* $(k = 0, 1, \ldots, n-1)$, and $\tilde{P}$ and P are equivalent. A consequence of Theorem 2.4 is a necessary and sufficient condition for the existence of an equivalent martingale measure for X.

THEOREM 2.6 *The following two conditions are equivalent.*

For $k = 1, \ldots, n$, *for all* $\mathscr{F}_{k-1}$*-measurable variables* Z,

$$Z \cdot (X_k - X_{k-1}) \geqq 0 \; P-a.s. \Rightarrow Z \cdot (X_k - X_{k-1}) = 0 \; P-a.s. \tag{2.4}$$

There exists an equivalent martingale measure $\tilde{P}$ for X. (2.5)

Proof The implication (2.5)⇒(2.4) is an immediate consequence of Theorem 2.4. Before proving the converse implication, recall that standard properties of conditional expectation for real random variables, such as $E(D_1 D_2 | \mathscr{G}) = D_1 E(D_2 | \mathscr{G})$ when D_1 is $\mathscr{G}$-measurable, or $E(E(D|\mathscr{G})|\mathscr{H}) = E(D|\mathscr{H})$ when $\mathscr{G} \subset \mathscr{H}$, are valid when $E(D_1 D_2), E(D_2)$ and $E(D)$ exist, but are not necessarily finite (see Ash ([1], Theorems 6.5.12 and 6.5.10)); in particular, they always hold when D_1, D_2 and D are non-negative.

Now suppose (2.4) holds, and set $\mathscr{F}_{n+1} = \mathscr{F}_n$, $X_{n+1} = X_n$, $D_{n+1} = 1$, $Y_{n+1} = 0$. Fix $k \leqq n$ and suppose by backwards induction that we have defined $D_{k+1}, \ldots, D_{n+1}$ and $Y_{k+1}, \ldots, Y_{n+1}$ in such a way that for $k+1 \leqq l \leqq n+1$

$$D_l \text{ is } \mathscr{F}_l\text{-measurable and } 0 < D_l \leqq 1 \; P-a.s., \tag{2.6}$$

and for $k+1 \leqq l \leqq n$

$$Y_l = (X_l - X_{l-1}) \, E(D_{l+1} \ldots D_{n+1} | \mathscr{F}_l) \; P-a.s., \tag{2.7}$$

and

$$E(\|Y_l\| \, D_l) < +\infty \text{ and } E(Y_l D_l | F_{l-1}) = 0 \; P-a.s. \tag{2.8}$$

Using Theorem 2.4, we then construct an $\mathscr{F}_k$-measurable random variable D_k with $0 < D_k \leqq 1$ $P-a.s.$, in such a way that if Y_k is defined by (2.7) with $l = k$, then (2.8) holds with $l = k$. Indeed, since $0 < D_l \leqq 1$ $P-a.s.$, we have $0 < E(D_{k+1} \ldots D_{n+1} | \mathscr{F}_k) \leqq 1$ $P-a.s.$, and so by (2.4), Y_k satisfies (2.2). Since Y_k is $\mathscr{F}_k$-measurable, Theorem 2.4 with $\mathscr{G} = \mathscr{F}_{k-1}$ and $\mathscr{H} = \mathscr{F}_k$ implies the existence of an $\mathscr{F}_k$-measurable random variable D_k such that (2.8) holds with $l = k$.

By backwards induction, we thus have defined random variables $D_1, \ldots, D_{n+1}$ and $Y_1, \ldots, Y_{n+1}$ such that (2.7) and (2.8) hold for $1 \leqq l \leqq n+1$. Finally, we set $D_0 = 1/(1 + \|X_0\|)$, and $D = D_0 \ldots D_{n+1}$. Observe that $0 < D \leqq 1$ $P-a.s.$ Let $\tilde{P}$ be the probability measure which is equivalent to P and whose Radon–Nikodym derivative $d\tilde{P}/dP$ equals $D/E(D)$. We shall show that $\tilde{P}$ is a martingale measure for X.

Indeed,

$$\begin{aligned}\tilde{E}(\|X_0\|) &= E(\|X_0\| D/E(D)) = E(\|X_0\| \, D_0 \ldots D_{n+1})/E(D) \\ &\leqq E(\|X_0\| D_0)/E(D) \\ &\leqq 1/E(D)\end{aligned}$$

and

$$\begin{aligned}\tilde{E}(\|X_l - X_{l-1}\|) &= E(\|X_l - X_{l-1}\| D/E(D)) \\ &= E(D_0 \dots D_l \|X_l - X_{l-1}\| D_{l+1} \dots D_{n+1})/E(D) \\ &= E(D_0 \dots D_l \|Y_l\|)/E(D) \\ &\leqq E(D_l \|Y_l\|)/E(D) \\ &< +\infty\end{aligned}$$

by (2.8), $1 \leqq l \leqq n$. Thus $\|X_l\|$ is $\tilde{P}$-integrable, $0 \leqq l \leqq n$.

Finally, to check $\tilde{E}(X_l | \mathscr{F}_{l-1}) = X_{l-1}$, it is sufficient by (2.1) to show that $E((X_l - X_{l-1}) D/E(D) | \mathscr{F}_{l-1}) = 0$. Now

$$\begin{aligned}E((X_l - X_{l-1}) D/E(D) | \mathscr{F}_{l-1}) &= D_0 \dots D_{l-1} E((X_l - X_{l-1}) D_l \dots D_{n+1} | \mathscr{F}_{l-1})/E(D) \\ &= D_0 \dots D_{l-1} E((X_l - X_{l-1}) D_l E(D_{l+1} \dots D_{n+1} | \mathscr{F}_l) | \mathscr{F}_{l-1})/E(D) \\ &= D_0 \dots D_{l-1} E(Y_l D_l | \mathscr{F}_{l-1})/E(D) \\ &= 0 \ a.s.\end{aligned}$$

by (2.8), $1 \leqq l \leqq n$. □

Remark 2.7 1) In the case of a discrete-time stochastic process $X = (X_k : k \in \mathbf{N})$ with infinite time horizon, condition (2.4) of Theorem 2.6 does not necessarily guarantee the existence of an equivalent martingale measure $\tilde{P}$ for X. This is illustrated by the following simple example which is well-known in a variety of contexts. Suppose $X_k = Y_1 + \dots + Y_k$ where for some $0 < p < 1$, $p \neq 1/2$, $(Y_k : k \in \mathbf{N})$ is a sequence of i.i.d. random variables with $P\{Y_k = 1\} = p$, $P\{Y_k = -1\} = 1 - p$. Set $\mathscr{F}_k = \sigma(Y_1, \dots, Y_k), k \in \mathbf{N}$. Now suppose there exists an equivalent martingale measure $\tilde{P}$ for X. Then by definition, $\tilde{E}(Y_k | \mathscr{F}_{k-1}) = 0 \ a.s.$ Since Y_k only takes the two values ± 1, this implies in particular that $\tilde{P}(Y_k = +1 | \mathscr{F}_{k-1}) = 1/2 \ a.s.$, that is, under $\tilde{P}$, the Y_k's are independent with mean zero. Now observe that by the strong law of large numbers, X_k/k converges to $2p - 1, P - a.s.$, whereas under $\tilde{P}, X_k/k$ converges to $0, \tilde{P} - a.s.$ contradicting the assumption that P and $\tilde{P}$ are equivalent. Thus, though X satisfies condition (2.4), there exists no equivalent martingale measure for X.

2) In contrast to the existence problem, Willinger and Taqqu [15] show that the uniqueness problem of an equivalent martingale measure $\tilde{P}$ for X can be dealt with using elementary probability techniques. In particular, they proved that $\tilde{P}$ is unique if and only if condition (2.9) below holds.

For $k = 0, 1, \dots, n-1$, there exists a finite minimal partition $\mathscr{P}_k$ of Ω with $\mathscr{F}_k = \sigma(\mathscr{P}_k)$ (up to P-null sets) and such that for all $A \in \mathscr{P}_k$,

$$\dim(\text{span}(\{X_{k+1}(\omega) - X_k(\omega) : \omega \in A\})) = \text{cardinality } (A' \in \mathscr{P}_{k+1} : A' \subseteq A) - 1 \quad (2.9)$$

where without loss of generality, we assume $P\{A\}>0$ for all $A\in\mathscr{P}_k$.

Whereas Theorem 2.6 imposes no restrictions on the underlying filtration **F** and is exclusively concerned with the proper "geometry" of X, condition (2.9) explicitly depicts the fundamental role of the fine structure of **F**. In fact, (2.9) not only implies that if $\tilde{P}$ is unique then **F** is necessarily minimal (i.e., $\mathbf{F}=\mathbf{F}^X=(\mathscr{F}_k^X:k=0,1,\ldots,n)$ with $\mathscr{F}_k^X=\sigma(X_0,X_1,\ldots,X_k)$, $0\leqq k\leqq n)$ but it also imposes stringent constraints of the form (2.9) on the relationship between X and **F**. It is this lack of a tight control on X and **F** that requires the use of measurable selections in the general case (see Theorem 2.4).

3. THE ANALYSIS OF FINITE-PERIOD STOCHASTIC SECURITIES MARKETS

In this section we illustrate the main results of Section 2 in the context of a stochastic model for the buying and selling of securities in discrete and finite time. The model was introduced by Harrison and Pliska [11] and further discussed in the setting of finite probability spaces by Taqqu and Willinger [13]. Here we show that condition (2.4) of Theorem 2.6 arises naturally from and is equivalent to the economically meaningful assumption of "no arbitrage". Moreover, uniqueness of an equivalent martingale measure for the underlying securities price process is related to the so-called "completeness"-property of the market which enables one to uniquely price any financial instrument in the market.

3.1 The Stochastic Model

For a fixed time horizon $T<\infty$ (terminal date of all economic activities), consider an $\mathbf{R}^{d+1}$-valued $(1\leqq d<\infty)$ stochastic process $S=(S_t:t=0,1,\ldots,T)$ defined on some complete probability space $(\Omega,\mathscr{F},P)$. Each component-process $S^k=(S_t^k:t=0,1,\ldots,T), 0\leqq k\leqq d,$ is assumed to be strictly positive so that $S_t^k(\omega)$ can be intepreted as the price of security k at time t if $\omega\in\Omega$ represents the true state of nature. The $0th$ security is called the *bond* and without loss of generality (see Harrison and Kreps [10]), we set $S_t^0=1$ for all t; that is, we assume that the stock prices have been discounted by the price of the riskless bond. S is also assumed to be adapted to a given filtration $\mathbf{F}=\{\mathscr{F}_t:t=0,1,\ldots,T\}$ and for convenience, we take $\mathscr{F}_T=\mathscr{F}$. **F** describes how information is revealed to the investors when securities are traded over time; starting with an initial knowledge $\mathscr{F}_0$ about the true state of nature, investors learn without forgetting $(\mathscr{F}_t\subseteq\mathscr{F}_{t+1}, t=0,1,\ldots,T-1)$ until they have complete information by time T $(\mathscr{F}_T=\mathscr{F})$.

The buying and selling of securities over time must be done according to certain trading strategies. A *trading strategy* is an **F**-predictable, $\mathbf{R}^{d+1}$-valued stochastic process $\phi=(\phi_t:t=1,2,\ldots,T)$ with components $\phi^0,\phi^1,\ldots,\phi^d$. $\phi_t^k(\omega)$ represents the number of shares of stock k held by an investor between times $t-1$ and t, namely during the time period $[t-1, t)$ if $\omega \in \Omega$ occurs. Thus, the vector ϕ_t denotes the investor's *portfolio at time t* and the components of ϕ_t may assume positive as well as negative values. When investors readjust their portfolio ϕ_t at time t, that is, buy and sell securities so as to form a new portfolio ϕ_{t+1}, they must do so without any knowledge of the future since ϕ is required to be **F**-predictable (i.e., ϕ_{t+1} is $\mathscr{F}_t$-measurable). The *value-process* $V(\phi) = (V_t(\phi): t = 0, 1, \ldots, T)$ associated with a trading strategy ϕ is defined by

$$V_t(\phi) = \begin{cases} \phi_1 \cdot S_0 = \sum_{k=0}^{d} \phi_1^k S_0^k & \text{if } t = 0 \\ \\ \phi_t \cdot S_t = \sum_{k=0}^{d} \phi_t^k S_t^k & \text{otherwise,} \end{cases} \quad P-a.s.$$

Thus, $V_t(\phi)$ represents the value of the portfolio ϕ_t at time t and before any changes are made at that time. A trading strategy ϕ is called *self-financing* if all changes in the value of ϕ_t are due to net gains realized on investments; that is, if

$$\phi_t \cdot S_t = \phi_{t+1} \cdot S_t \ P-a.s., t = 1, \ldots, T-1.$$

We denote by Φ the set of all self-financing trading strategies.

Finally, we state the following assumptions commonly found in the economics literature: (i) there are no transaction costs, (ii) all securities are perfectly divisable, (iii) the securities do not pay dividends in $[0, T]$, and (iv) short sales of all securities are allowed without any restrictions. Subsequently, the stochastic model corresponding to the stochastic base $(\Omega, \mathbf{F}, P)$, the price process S, and the set Φ of allowable trading strategies, and satisfying conditions (i)–(iv) will be denoted by $(\mathbf{T}, \mathbf{F}, S)$ and called a *(finite-period, frictionless) securites market*, where $\mathbf{T} = \{0, 1, \ldots, T\}$ denotes the set of all trading dates.

Remark 3.1 We do not impose any kind of wealth constraint as, for example, in Harrison and Pliska [11] or Back and Pliska [2], but allow for unbounded short sales. In discrete time, restrictions on short sales have little effect on subsequent results and are not needed from a mathematical point of view (see also the comment in Back and Pliska ([2], p. 3)).

3.2 The "No-Arbitrage-Assumption" and the Martingale-Property

An abitrage opportunity ("free lunch") represents a riskless plan for making profit without any investment. Prohibiting arbitrage opportunities is, therefore, economically reasonable and is necessary for any kind of economic equilibrium to exist. More formally, we have

DEFINITION 3.2 An *arbitrage opportunity* is a self-financing trading strategy $\phi \in \Phi$

such that $V_0(\phi)=0$ and $V_T(\phi)\geqq 0$ with probability one, and $V_T(\phi)>0$ with positive probability. The market model $(\mathbf{T},\mathbf{F},S)$ is said to contain *no arbitrage opportunities* if for all $\phi\in\Phi$ with $V_0(\phi)=0$ and $V_T(\phi)\geqq 0\ P-a.s.$, we have $V_T(\phi)=0\ P-a.s.$

Although an arbitrage opportunity as described above is defined "globally" (that is, it involves the trading dates 0 and T only), "no arbitrage" also holds "locally", namely at any trading date $t=1,2,\ldots,T$, as we will see below. In addition to illustrating this pathwise nature of the "no-arbitrage"-property, Theorem 3.3 below relates "no arbitrage" to condition (2.4) of Theorem 2.6 and hence to the martingale property of the price process S under a new equivalent probability measure $\tilde{P}$. Let $\mathbf{P}$ denote the set of all equivalent martingale measures for S and let $\tilde{S}=(\tilde{S}_t: t=0,1,\ldots,T)$ be the $\mathbf{R}^d$-valued, $\mathbf{F}$-adapted process obtained from S by deleting the 0*th* component-process $S_t^0\equiv 1$ (i.e., $S=(1,\tilde{S})$).

THEOREM 3.3 *The following three conditions are equivalent.*

The market model $(\mathbf{T},\mathbf{F},S)$ *contains no arbitrage opportunities.* (3.1)

For all $t\in\{1,2,\ldots,T\}$ *and all* $\mathbf{R}^d$*-valued* $\mathscr{F}_{t-1}$*-measurable random variables* α,

$$\alpha\cdot(\tilde{S}_t-\tilde{S}_{t-1})\geqq 0\ P-a.s. \Rightarrow \alpha\cdot(\tilde{S}_t-\tilde{S}_{t-1})=0\ P-a.s. \tag{3.2}$$

$\mathbf{P}\neq\emptyset$; *that is, there exists an equivalent martingale measure* $\tilde{P}$ *for* S. (3.3)

Proof 1) The proof of (3.1)⇒(3.2) is similar to that of Taqqu and Willinger ([13], Lemma 3.2) except that the probability space is no longer finite. Assume that there exists $t\in\{0,1,\ldots,T-1\}$ and $\alpha=(\alpha^1,\alpha^2,\ldots,\alpha^d)\in\mathscr{F}_t$ such that $\alpha\cdot(\tilde{S}_{t+1}-\tilde{S}_t)\geqq 0\ P-a.s.$ and $\alpha\cdot(\tilde{S}_{t+1}-\tilde{S}_t)>0$ with positive probability. Set $W=\{\omega\in\Omega: P\{\alpha\cdot(\tilde{S}_{t+1}-\tilde{S}_t)>0|\mathscr{F}_t\}(\omega)>0\}$ and note that, by assumption, $P(W)>0$. We will construct a trading strategy $\phi\in\Phi$ with $V_0(\phi)=0$ and $V_T(\phi)\geqq 0\ P-a.s.$, and such that $V_T(\phi)>0$ with positive probability; that is, ϕ defines an arbitrage opportunity, contradicting the assumption that the market model $(\mathbf{T},\mathbf{F},S)$ is "arbitrage-free".

In order to construct ϕ with the desired properties, define ϕ_s at every point in time and for each $\omega\in\Omega$ as follows:

for $s\leqq t$: $\quad \phi_s(\omega)=0$ for all $\omega\in\Omega$,

$$\text{for } s=t+1\text{: on } W\text{, set } \phi_{t+1}^k(\omega)=\begin{cases}\alpha^k(\omega) & \text{if } k\in\{1,2,\ldots,d\},\\ -\sum_{k=1}^d \alpha^k(\omega)S_t^k(\omega) & \text{if } k=0,\end{cases}$$

and on W^c, set $\phi_{t+1}(\omega)=0$,

$$\text{for } t+1<s<T: \phi_s^k(\omega)=\begin{cases} V_{t+1}(\phi)(\omega) & \text{if } k=0 \text{ and } \omega\in W \\ 0 & \text{otherwise.}\end{cases}$$

Clearly, ϕ is **F**-predictable. To see that ϕ is self-financing, we check the relation $\phi_s\cdot S_s=\phi_{s+1}\cdot S_s$ which clearly holds for $s<t$ and $s>t$. For $s=t$, we have $\phi_t(\omega)=0$ for all $\omega\in\Omega$; for $\omega\in W$,

$$\phi_{t+1}(\omega)\cdot S_t(\omega)=-\left(\sum_{k=1}^{d}\alpha^k S_t^k\right)(\omega)+\sum_{k=1}^{d}\alpha^k(\omega)S_t^k(\omega)=0.$$

Since $\phi_{t+1}\cdot S_t=0$ on W^c, $\phi_{t+1}\cdot S_t=\phi_t\cdot S_t$ holds for all $\omega\in\Omega$.

Next observe that $V_0(\phi)=0\ P-a.s.$ and $V_T(\phi)\geqq 0\ P-a.s.$ In fact, for all $s>t+1$,

$$V_s(\phi)(\omega)=\begin{cases} V_{t+1}(\phi)(\omega)=\alpha(\omega)\cdot(\tilde{S}_{t+1}(\omega)-\tilde{S}_t(\omega))\geq 0 & \text{if } \omega\in W, \\ 0 & \text{otherwise,}\end{cases}$$

and hence,

$$P\{V_T(\phi)\geqq 0\}=P\{\alpha\cdot(\tilde{S}_{t+1}-\tilde{S}_t)\geqq 0\}=1.$$

Moreover, by the definition of W,

$$\begin{aligned} P\{V_T(\phi)>0\}&=P\{\{V_T(\phi)>0\}\cap W\} \\ &=P(1_W P\{\alpha\cdot(\tilde{S}_{t+1}-\tilde{S}_t)>0|\mathscr{F}_t\})>0, \end{aligned}$$

which shows that ϕ is an arbitrage opportunity.

2) The equivalence (3.2)⇔(3.3) holds by Theorem 2.6.

3) Finally, in order to prove (3.3)⇒(3.1), let $\tilde{P}\in\mathbf{P}$ and let $\phi\in\Phi$ be such that $V_0(\phi)=0$ and $V_T(\phi)\geqq 0\ P-a.s.$ Then $V_T(\phi)=0\ P-a.s.$ because repeated applications of (i) the properties of conditional expectations mentioned at the beginning of the proof of Theorem 2.6 (recall that S is positive), (ii) the martingale property of S under $\tilde{P}$, and (iii) the self-financing property of $\phi\in\Phi$, enable us to write

$$\tilde{E}(V_T(\phi))=\tilde{E}(V_0(\phi))=0. \quad \square$$

Remarks 3.4 1) To our knowledge, Theorem 3.3 is the first result which provides a criterion (condition (3.2)) for determining whether or not a market model with several securities contains arbitrage opportunities. Indeed, previous results mention the equivalence of (3.1) and (3.3) under various restrictive assumptions on the price process, such as integrability: see Harrison and Kreps

([10], Theorem 2) and Duffie and Huang ([8], Theorem 5.1 and Proposition 6.4). In these last references, the authors also consider the continuous-time case. For one-dimensional price processes, a criterion analogous to (3.2) was obtained by Back and Pliska [2] who assume P-integrability of the price process, an assumption crucial to their proof. On the one hand, such a requirement seems somewhat unnatural since (i) it is, in general, not preserved under an equivalent change of measure, and (ii) for the formulation of an arbitrage opportunity (see Definition 3.2), integrability of S under P is irrelevant. On the other hand one can always assume the existence of an equivalent probability measure P' on $(\Omega, \mathscr{F})$ with bounded Radon–Nikodym derivative dP'/dP such that S is integrable under P' (see Dellacherie and Meyer ([6], Theorem VII.57)). Thus, assuming integrability of S under P becomes a modeling issue and is not necessary from a mathematical point of view.

2) Theorem 3.3 not only provides a probabilistic characterization (condition (3.3)) but also a geometric characterization (condition (3.2)) of "no arbitrage". Indeed, condition (3.2) states implicitly that along almost all sample paths of $\tilde{S}$ (or S), the support of the conditional distribution of the increment $\tilde{S}_{t+1} - \tilde{S}_t$ given $\mathscr{F}_t$ cannot be concentrated on only one "side" of any $\mathscr{F}_t$-measurable hyperplane in $\mathbf{R}^d$ (or $\mathbf{R}^{d+1}$).

3) The question of uniqueness of an equivalent martingale measure $\tilde{P}$ for S (see Remark 2.7.2) also has an economic interpretation. Namely, let X denote a non-negative, $\mathscr{F}$-measurable random variable (*contingent claim*) and interpret X as representing a contract that pays $X(\omega)$ dollars if, at time T, $\omega \in \Omega$ denotes the true state of nature. We would like to know what prices at time zero are "reasonable" for X if the market model $(\mathbf{T}, \mathbf{F}, S)$ contains no arbitrage opportunities. Clearly, if X is *attainable*; that is, there exists $\phi \in \Phi$ such that $X = V_T(\phi)\, P-a.s.$, then "no arbitrage" implies that the (time zero) price $\pi(X)$ is given by $\pi(X) = V_0(\phi)$. But which claims are attainable? The market model $(\mathbf{T}, \mathbf{F}, S)$ is called *complete* if all contingent claims are attainable. Taqqu and Willinger [13] showed the equivalence between the economically desirable completeness-property of the market model $(\mathbf{T}, \mathbf{F}, S)$, the uniqueness of $\tilde{P}$, and condition (2.9).

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