The Pricing of Options for Jump Processes

by
John C. Cox*
and
Stephen A. Ross*

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RODNEY L. WHITE CENTER
FOR FINANCIAL RESEARCH

University of Pennsylvania

The Wharton School

Philadelphia, Pa. 19174

The problem of valuing options, financial claims whose returns are contingent on the performance of underlying primary assets, has long occupied a place in the financial literature. While options themselves are rather specialized instruments, the potential applicability of option pricing theory to all securities fully justifies this interest and indeed makes it one of the most important areas in finance (see Ross [1974]).

The first material of serious academic interest on options was the remarkable 1900 dissertation of Bachelier. Subsequent work on the subject (see, for example, the papers by Sprenkle, Ayres, Boness and Kruizenga collected in Cootner) was done without awareness of Bachelier's treatment, until the 1965 paper by Samuelson. Samuelson, following Bachelier's approach, postulated that the behavior of the stock could be represented by a particular stochastic process and then linked the value of the option to that of the stock by requiring that the option be priced to have a constant expected return at each instant of time. While very stimulating in many aspects, unfortunately, there was not an economic justification for this approach, and the resulting theory was unsatisfactory in several respects. Some of the difficulties were remedied by the subsequent work of Samuelson and Merton.

In an important paper Black and Scholes put forth a sound economic rationale that linked the values of the option and the stock and yielded a pricing relation depending only on observable variables. They also noted the scope of possible applications, many of which have been developed and expanded by Merton. The applicability of the Black and Scholes analysis depends in large measure on the acceptability of their central assumption on the stochastic process that generates stock returns. They assume that changes in the price of the stock, S, are governed by a diffusion process of the form

$$\frac{dS}{S} = \mu dt + \sigma dz, \tag{1}$$

where z is a Wiener process. This can be taken to mean that the percentage change in the stock price over a small interval will be given by a deterministic drift component, μdt, plus a random increment, normally distributed with mean zero and variance $\sigma^2 dt$. A fundamental restriction imposed by the use of a diffusion process to describe the stock price is that within a small interval of time, $[t, t+dt)$, S_t will move in a random fashion, but with high probability, approaching 1 as $dt \to 0$, S_{t+dt} will be in an arbitrarily small neighborhood of S_t. In such a process only local changes in the stock price are permitted. The Black and Scholes approach can be extended to other diffusion processes, as Cox has recently done, but this basic restriction remains, and there are a number of assets for which this may not be a very adequate description of uncertainty.

New information tends to arrive at a market in discrete lumps rather than in a smooth flow, and assets in such markets are likely to have discontinuous jumps in value, thus violating the basic assumption of a diffusion process. Such behavior is characteristic of many economic situations and is particularly relevant when political events or stochastically applied government or institutional constraints are of primary importance. For example, foreign exchange rates with any sort of fixing typically change significantly only with devaluations and revaluations. Some of the properties of equilibrium in markets of this type have been studied by Ross [1975].

Price movements of this sort can be captured by assuming that the asset follows a jump process rather than a diffusion process. Unlike a diffusion process, a jump process is characterized by the property that with high probability, approaching 1 as $dt \to 0$, its movement within the interval [t, t+dt) will be certain, but with a low and continuing probability it will jump to a new value. In its simplest form a jump process can be written as

$$
\begin{aligned}
\frac{dS}{S} &= \mu dt + (k-1)d\tilde{\Pi} \\
&= \mu dt + \begin{cases} k-1 & \lambda dt \\ 0 & 1-\lambda dt \end{cases}
\end{aligned}
\qquad (2)
$$

where $\tilde{\Pi}$ is a Poisson process and $d\tilde{\Pi}$ takes the value 0 with probability $1 - \lambda dt$ and 1 with probability λdt. The parameter λ is called the intensity of the process since it measures the rate of probability flow for the jump and $k-1$ is called the jump amplitude. From (2) if no jump occurs then S_t moves at the exponential rate μ, but if a jump occurs S_t changes by $(k-1)S_t$ to $S_t + (k-1)S_t = kS_t$.[1]

In this paper we will study the problem of option valuation when the stock follows a jump process. Although both our process and that of Black and Scholes can be thought of as simplified abstractions of more complex processes, they are respectively representative descriptions of two fundamentally different forms of uncertainty in continuous time stochastic processes. Furthermore, the two approaches lead to option pricing models which are similar in some respects but quite different in others.

Throughout the paper we will make the same assumptions about market structure as did Black and Scholes. Specifically, we will assume:

(1) the instantaneous interest rate, r, is known and constant through time, and individuals can borrow or lend as much as they want at this rate,[2]

(2) the competitive assumption that the scale of individual participants relative to the total market is sufficiently small that each individual acts as if he can buy or sell as much of the stock as he pleases without affecting the price,

(3) the stock may be sold short with the seller receiving the proceeds,

(4) there are no transactions costs or taxes, and

(5) the stock pays no dividends.

With the fifth assumption, the valuation formula will be the same for both an American and a European call. A call is an option whose value when exercised is the larger of S-E or 0, where E is a fixed exercise price. A European call can be exercised only at the expiration date, while an American call can be exercised at any time the owner chooses to do so. If it is never to the owner's advantage to exercise before the expiration date, as will be the case when there are no dividends, then the two types must have the same value. A put is the opposite of a call in that it offers a positive return when exercised of E-S if S falls short of E and 0 if S exceeds E. Merton has shown that premature exercising may be optimal with an American put even if there are no payouts on the stock, so the opportunity to do so must command a premium over the European put. In this paper we will confine our attention to the European case.

The Valuation Formula

Consider an option written on a stock whose price movement follows equation (2). If we assume that the price of the option, P, is a continuous function,

$$P = P(S, t), \tag{3}$$

of S and t, then the option value will jump when the stock jumps. In fact P(S, t) will follow a jump process of the form

$$\frac{dP}{P} = \begin{cases} \xrightarrow{\lambda dt} \dfrac{P(kS, t) - P(S, t)}{P(S, t)} \\ \xrightarrow{1-\lambda dt} \left\{\mu \dfrac{S}{P} \dfrac{\partial P}{\partial S} + \dfrac{1}{P} \dfrac{\partial P}{\partial t}\right\} dt, \end{cases} \tag{4}$$

in other words, with probability $(1 - \lambda dt)$ dP will just be a differential movement caused by the exponential growth (or decay) of S at the rate μ and the passage of time, but with probability λdt, P will jump to

P(kS, t) as a consequence of the jump in S. This suggests that it might be possible to form a portfolio to hedge against the jump.

Let α_p and α_s denote the percentages of wealth put in the option and the stock respectively. If α_p and α_s are set so as to keep

$$\alpha_p \left\{\frac{P(kS,\ t) - P(S,\ t)}{P(S,\ t)}\right\} + \alpha_s \left\{k - 1\right\} > 0, \tag{5}$$

then the value of the portfolio immediately after a jump will be at least as great as it was before the jump. If there is no jump, then the return on the portfolio will be given by

$$\alpha_p \left\{\mu\frac{S}{P}\frac{\partial P}{\partial S} + \frac{1}{P}\frac{\partial P}{\partial t}\right\} dt + \alpha_s \mu dt. \tag{6}$$

Since by (5) a jump can only raise the return, to prevent arbitrage possibilities the return if there is no jump, (6), must be less than the return on the riskless asset, $(\alpha_p + \alpha_s)rdt$. Since the post jump capital gain, (5), can be made as close to zero as desired, a hedged position with

$$\alpha_p \left\{\frac{P(kS,\ t) - P(S,\ t)}{P(S,\ t)}\right\} + \alpha_s \left\{k - 1\right\} = 0, \tag{7}$$

must earn exactly the riskless rate or

$$\alpha_p \left\{\mu\frac{S}{P}\frac{\partial P}{\partial S} + \frac{1}{P}\frac{\partial P}{\partial t} - r\right\} dt + \alpha_s \left\{\mu - r\right\} dt = 0\ . \tag{8}$$

From (7)

$$\frac{\alpha_s}{\alpha_p} = \frac{P(kS,\ t) - P(S,\ t)}{(1 - k)\ P(S,\ t)}\ , \tag{9}$$

i.e., we hold s and p in proportion to the relative post jump capital gains on P and S respectively. Such a hedged position can be maintained after a jump by a discontinuous adjustment to a new position and between jumps the hedge will change smoothly with the trend movement, μ, of S and the passage of time. Substituting (9) into (8) we obtain the basic differential equation for the option,

$$\mu S \frac{\partial P}{\partial S} + \left(\frac{\mu - r}{1 - k}\right) P(kS) + \left(\frac{rk - \mu}{1 - k}\right) P = -\frac{\partial P}{\partial t} \qquad (10)$$

To solve (10) we, of course, need to specify the option in greater detail. Let us suppose, to begin with, that P is a European call option with exercise price, E, and maturity T. Now, we can add the boundary condition

$$P(S, T) = \max \{S_T - E, 0\} \qquad (11)$$

to (10) and try for a solution to this system. It is interesting to note that (10) has appeared in economics before. Wold and Whittle derived a formally identical equation as a description of the income distribution.

Attempting to solve this system by direct analytic methods is a surprisingly difficult task. Unlike the second order partial differential equations that emerge in option pricing problems from the diffusion process assumptions, (10) is a mixed differential difference equation and the mathematics of these equations has not yet been fully developed. One formal approach would be to transform (10) by the Laplace and then the Mellin integral transforms and then attempt an inversion of the resulting algebraic equation. Unfortunately, though, the roots of the resulting algebraic equation are not fully known.[3]

The economics of the problem, however, offers a simpler approach to solving (10) and (11). Equation (10) was derived without any explicit reference to the preference or demand structures in the market. Implicit in the derivation is only the assumption that more is better, i.e., that individual utility functions increase monotonically. If (10) and (11) have a solution, this same solution must hold for all possible individual preference structures (that permit equilibrium solutions). This suggests that if we can find a solution for any particular market structure, then this must also be the solution to (10) and (11). Consider, then, a market with a single risk neutral (over terminal wealth) investor with a T period horizon. For such a market to be in equilibrium, the expected T period return on the riskless bond, the stock, and the option must all be identical.

The return on the riskless bond is simply given by e^{rT}. To evaluate the return on the stock we must examine the probability distribution of, j, the number of jumps that occur in the interval $[0, T)$. It is well known (see Feller) that j is a Poisson random variable with

$$\text{Prob } (j = i) = \frac{e^{-\lambda T}(\lambda T)^i}{i!} \quad . \tag{12}$$

Since

$$\frac{S_T}{S_i} = k^j e^{\mu T} \quad , \tag{13}$$

it follows from (12) that

$$E\left\{\frac{S_T}{S_0}\right\} = E\left\{k^j e^{\mu T}\right\}$$

$$= e^{\mu T} \sum_{i=0}^{\infty} e^{-\lambda T} \frac{(k\lambda T)^i}{i!} \qquad (14)$$

In equilibrium,

$$e^{rT} = E\left\{\frac{S_T}{S_0}\right\},$$

or, from (14)

$$\lambda = \frac{r - \mu}{k - 1}\ .^4 \qquad (15)$$

The expected return on the option is given by

$$\frac{1}{P(S,0)} \quad E\left\{\max\left\{S_T - E,\ 0\right\}\right\}$$

$$= \frac{1}{P(S,\ 0)} \begin{cases} e^{\mu T} S \sum\limits_{n}^{\infty} e^{-\lambda T} \dfrac{(k\lambda T)^i}{i!} - E \sum\limits_{n}^{\infty} e^{-\lambda T} \dfrac{(\lambda T)^i}{i!} & \text{if } k > 1 \\ \\ e^{\mu T} S \sum\limits_{0}^{n-1} e^{-\lambda T} \dfrac{(k\lambda T)^i}{i!} - E \sum\limits_{0}^{n-1} e^{-\lambda T} \dfrac{(\lambda T)^i}{i!} & \text{if } k < 1, \end{cases} \qquad (16)$$

where n is the minimum number of jumps required for $S_T > E$ if $k > 1$ and n-1 is the maximum number of jumps for $S_T > E$ if $K < 1$; from (13)

$$n = \left[\frac{\log {}^{E}/_{S} - \mu T}{\log k}\right] + 1, \qquad (17)$$

where $[X]$ denotes the largest integer smaller than X ($n = 0$ if $X \leq 0$).

In a risk neutral market, the return on the option (16) must be the same as that on the risk free asset, e^{rT}. From (16) and (15), then, we have

$$P(S, t) = \begin{cases} S \sum_{n}^{\infty} e^{-y} \frac{y^i}{i!} - Ee^{-r(T-t)} \sum_{n}^{\infty} e^{-z} \frac{z^i}{i!} & \text{if } k > 1 \\ S \sum_{0}^{n-1} e^{-y} \frac{y^i}{i!} - Ee^{-r(T-t)} \sum_{0}^{n-1} e^{-z} \frac{z^i}{i!} & \text{if } k < 1, \end{cases} \tag{18}$$

where

$$y = \frac{(r - \mu)\, k(T - t)}{k - 1} \quad ,$$

and

$$z = \frac{y}{k} \, ,$$

and

$$n = \left[\frac{\log {}^{E}/_{S} - \mu(T - t)}{\log k} \right] + 1.$$ [5]

The summations in (18) are cumulative and complementary Poisson distribution functions and are well tabulated, often as chi-square distribution functions with $2n$ degrees of freedom or incomplete λ functions. For example, if we let $F(X;\nu)$ and $G(X;\nu)$ be respectively cumulative and complementary chi-square distributions with ν degrees of freedom, then (18) can be rewritten as

$$P(S, t) = \begin{cases} SF(2y;2n) - Ee^{-r(T - t)}F(2z;2n) & \text{if } k > 1 \\ SG(2y;2n) - Ee^{-r(T-t)}G(2z;2n) & \text{if } k < 1. \end{cases} \tag{19}$$

To verify the heuristic argument that led to (18), we have to show that it does indeed solve (10) and (11). By definition, (18) is simply the discounted expected terminal value of the option so it must satisfy the boundary condition. To verify that (18) satisfies (10) we have to perform the substitution. If S is in an interval on which n does not change, P(S, t) as defined by (18) is easily shown to satisfy (10). Suppose, though, that S is such that

$$n = \frac{\log {}^{E}/_{S} - \mu(T-t)}{\log k}. \tag{20}$$

Now, the right and left derivatives of P(S, t) will be different since an increase or decrease in S will change the number of terms in the defining series. (It is easily verified, though, that P(S, t) is continuous.) How, then, can (18) possibly solve (10)?

The answer is that it cannot, but that it is still the correct valuation! The reason is that (10) is not quite correct. When S satisfies (20) it becomes important to remember that the derivative, $\frac{\partial P}{\partial S}$, in (10) enters from the differential movement in S when there is no jump. If $\mu > 0$, then S will be increasing and we should use $(\frac{\partial P}{\partial S})^{+}$, the right derivative, and if $\mu < 0$, then S will be falling and the relevant differential effect on P is captured by $(\frac{\partial P}{\partial S})^{-}$.

Notice that (18) defines the option price solely in terms of the observable variables, t, r, S, E, μ, and k. What is striking about the pricing equation (18), and can be seen from (10) as well, is that the intensity of the process λ which determines the expected number of jumps plays no role in the valuation formula. The comparative statics analysis with respect to the parameters that enter (18) yields intuitive results.

The option value is an increasing function of the stock value and as $S \to \infty$, $P(S, t) \to S - Ee^{-r(T-t)}$. Furthermore, $P(S, t)$ must also be a convex function of S and it is easy to see that $P \geq S - Ee^{-r(T-t)}$ Similarly, P increases with increases in r and μ, and falls as E is increased. The effect of k is somewhat more subtle, but it can be shown that P is an increasing concave function of k. To see this we have to use the requirement that $(k - 1)$ and $(r - \mu)$ must have the same sign to prevent arbitrage. Finally, as t increases and the expiration date comes closer, P declines and as $T \to \infty$, $P \to S$.[6]

Comparison with the Black and Scholes Diffusion Results

It is useful to begin with a brief description of the Black and Scholes analysis so as to compare our results. Black and Scholes used an arbitrage argument similar to ours to obtain their valuation formula for the price, P, of an option written on a stock whose price movement followed the diffusion process, (1). Assuming that such a valuation formula exists and is of the form $P = P(S, t)$ then the random return on the option in the interval $[t, t + dt)$ will be perfectly correlated with the return on the underlying stock, S. This follows directly from the diffusion assumption. Since S_t will change only be a small amount on the interval, the stochastic calculus can be used to give the return on the option as

$$\frac{dP}{P} = \frac{1}{P} \{ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + \mu S \frac{\partial P}{\partial S} + \frac{\partial P}{\partial t} \} dt + \sigma \frac{S}{P} \frac{\partial P}{\partial S} dz . \tag{21}$$

Following Black and Scholes we can now form a portfolio of the stock and the option that is perfectly hedged against the risk, dz, and since the resulting portfolio is riskless, it must earn the riskless rate of return, rdt. This enables us to write the fundamental Black and Scholes differential equation for the option,

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = -\frac{\partial P}{\partial t} . \tag{22}$$

The particular option studied by Black and Scholes was the European call which satisfies the terminal value condition (11). Equations (22) and (11) describe a differential equation system with its boundary value condition and by transforming the equation into the heat equation Black and Scholes found the solution:

$$P(S,\ t) = S\Phi(d_1) - Ee^{-r(T-t)}\Phi(d_2), \tag{23}$$

where

$$d_1 = \frac{\log {}^S/_E + (r + \tfrac{1}{2}\sigma^2)\ (T - t)}{\sigma\sqrt{T - t}}$$

and (24)

$$d_2 = d_1 - \sigma\sqrt{T - t},$$

and Φ $(\cdot)$ is the cumulative unit normal distribution.

Perhaps the most surprising feature of the Black and Scholes valuation formula, (23), is the fact that it does not depend upon, μ, the deterministic drift of the stock. This is in marked contrast to the valuation formula, (18), for the jump process which, as we have seen, does not depend upon the intensity of the process, λ, but is dependent on the deterministic drift term, μ.

However, the form of (23) is not unlike that of (18) and this suggests that some specific parallels can be drawn in spite of the difference in probabilistic structure. The ability to form a perfect hedge leads to a relationship between the option price and the stock price which is free of individual preferences; and, as a consequence, the Black and Scholes valuation formula, (23), is derivable by the same argument we used for the jump process. In a risk neutral market, if S follows the diffusion process (11) we must have $\mu = r$, and (23) is simply the discounted terminal expected value in this risk neutral market,

$$e^{-r(T-t)}E \{ \max (S_T - E, 0) \}.$$

It is important to realize that this does not imply that any part of the results is peculiar to a risk neutral market. In examining a relationship which holds for all risk structures, we are certainly free to work with any particular one which may be convenient.

In a risk neutral market the expected rate of return on the option and the stock must be equal and equal to the risk free rate; but this will not, in general, be true in any other market, and the determination of the expected rate of return on the stock would require a general equilibrium analysis including all other assets. However, as Black and Scholes pointed out, their analysis does imply a relationship between the instantaneous expected return on the option, γ, and the (exogenously determined) instantaneous expected return on the stock, § (= μ for the diffusion process), as given by

$$\gamma - r = \left[\frac{S}{P}\frac{\partial P}{\partial S}\right] (\S - r). \tag{25}$$

Returning to (2) and (10), we see that a similar result obtains for our model,

$$\gamma - r = \left[\frac{P(kS) - P(S)}{(k-1)P(S)}\right] (\S - r). \tag{26}$$

Once again it is possible to write the excess expected return on the option as a multiple of the excess return on the stock. The multiplicity factor will be a function of S and t and will have qualitative behavior similar to that of the Black and Scholes factor when similar comparisons are being made. The nature of our process allows us to look at some other situations which have no analogue in the Black and Scholes model, and some interesting relationships arise. For example, if we consider jumps which will take the firm to the verge of bankruptcy if they occur, we find that if the option has any value, its expected rate of return approaches that of the stock.

An important parallel between the models can be seen by considering certain limiting forms of the jump process. Figure 1 shows how the graphs of the Black and Scholes pricing relation (23), and the jump pricing relation, (18), can be quite similar for some choices of parameter values. It can be shown that under an appropriate limiting process (18) converges to (23). The straight line segments of (18) become smaller and the convergence is uniform.

The reason for this similarity is, perhaps, somewhat surprising. The jump process may appear to be a rather special sort of process compared to the diffusion process in that the jump is always in the same direction and when a jump is not occurring, S is moving at a certain exponential rate. (Of course, μ could be a function of S, but the instantaneous movement of S would be sure.) In fact, though, the diffusion process is simply a limiting case of the jump process, and can be approximated arbitrarily closely by a jump process.

To see this suppose we let the rate of probability flow, λ, for the jump process approach infinity changing k and μ_j in such a way that the following relations are satisfied:

$$\lambda(\log k)^2 = \sigma^2$$

and (27)

$$\mu_j + \lambda \log k = \mu_d - \tfrac{1}{2}\sigma^2,$$

where μ_j and μ_d are the drifts for the jump process and the diffusion process respectively. Equations (27) just insure that the instantaneous mean and variance of the two processes which govern the movement of log S are the same. Using (13) and recalling that j denotes the Poisson distributed number of jumps in the interval $[0, T)$, we have by the central limit theorem that

$$\frac{\log \frac{S_T}{S_0} - (\mu T + \lambda \log kT)}{\sqrt{\lambda T}\ \log k}$$

$$= \frac{j - \lambda T}{\sqrt{\lambda T}}$$

$$\rightarrow N(0, 1),$$

as $\lambda \rightarrow \infty$, satisfying (27). It follows that $\log\left(S_T/S_0\right)$ approaches the lognormal random variable of the diffusion process, (1), with parameters μ_d and σ^2.

It is also easy to show that as $\lambda \to \infty$, according to (27), the differential valuation equation for the jump process, (10), approaches the Black and Scholes differential valuation equation, (22). As a consequence, the Black and Scholes pricing formula, (23), will also be a limiting form of the jump pricing formula, (18). In a very real sense, then, the diffusion process is a special case of the jump process.[7] Furthermore, as we have seen, outside of the limiting case the jump process behaves in a qualitatively different fashion than the diffusion process.

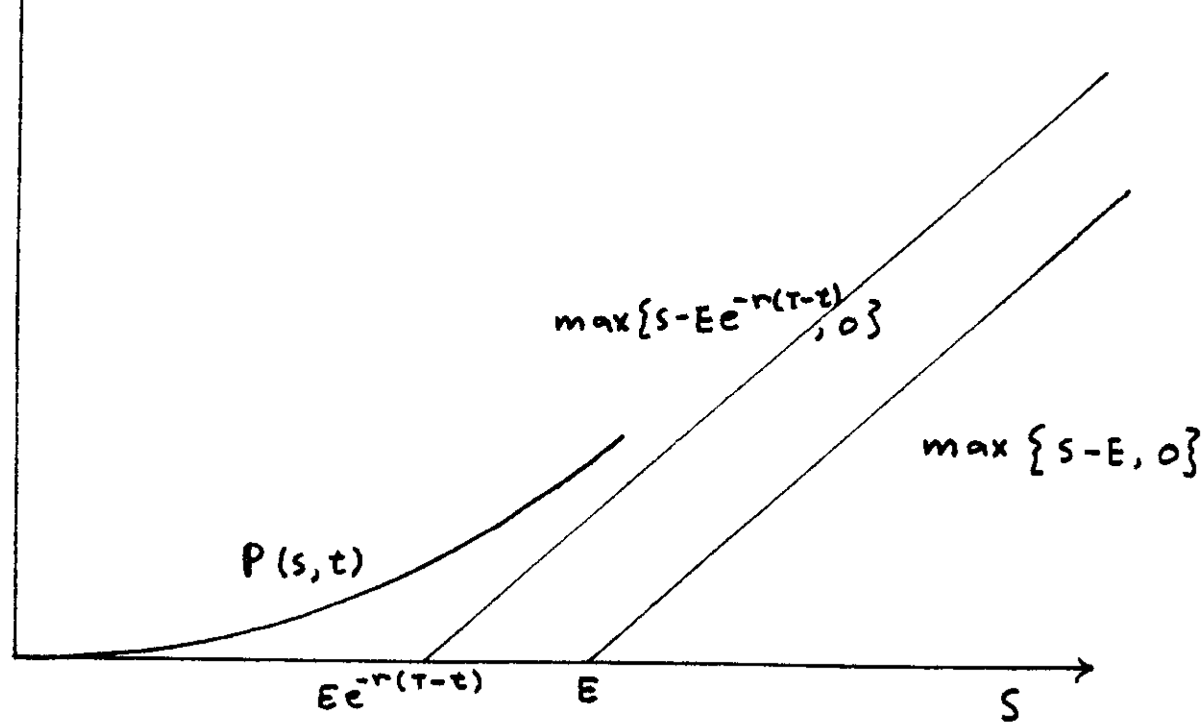
Extensions and Conclusions

The model developed above for the jump process is suitable for a broad range of applications. If our interest is in valuing the securities of a corporation, then we could assume that the total value of the assets of the corporation follows a jump process and then consider individual securities as combinations of options on this total value. The value of each security would then have to satisfy an equation like (10), with different securities distinguished by their terminal conditions. We could, for example, find the valuation formula for the European put, denoted as $O(S, t)$, by repeating our analysis with the appropriate terminal condition for the put, max $(E - S, 0)$. However, this is not necessary, since Stoll, under assumptions which implicitly limited his analysis to the European case, has pointed out that to prevent arbitrage there must be a direct relationship between put and call prices. In our terms this relationship would give $O(S, t) = P(S, t) + Ee^{-rt} - S$, and it is readily verifiable that this expression satisfies the equation and the terminal condition. Similarly, the options can be evaluated by applying the relevant terminal conditions or by considering them as an appropriate combination of calls.

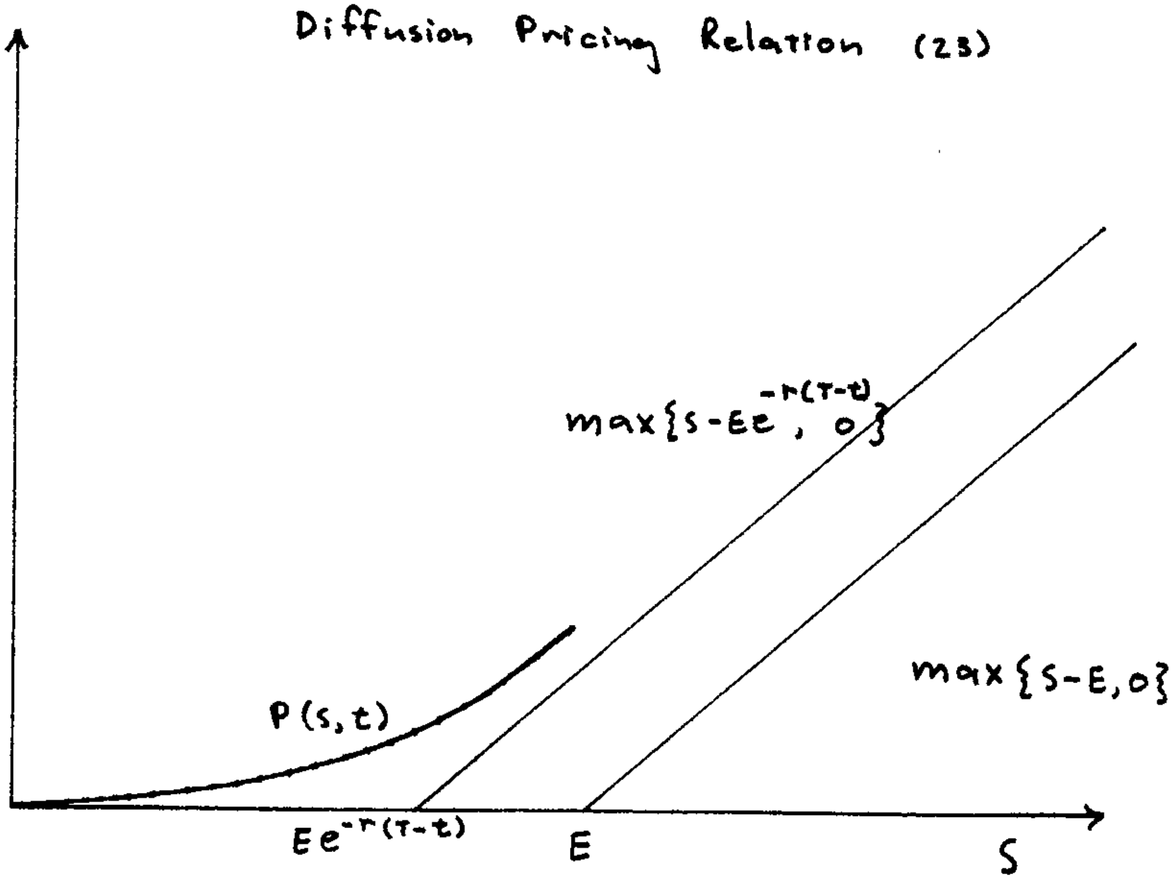
In addition, our model can be expanded in a straightforward way to include some more general kinds of behavior. We could let μ, λ, and k be functions of S, and λ could be allowed to depend on an additional random variable. However, a direct application of the hedging mechanism does require that k not be random except through its possible dependence on S. Cases for which this is not appropriate, as well as cases involving both diffusion and jump components, apparently will require explicit consideration of individual preferences or the characteristics of other assets, and work in this direction is currently in progress.

We fully realize, then, that the process assumed in this paper is a relatively simple one, and that a more complex process would undoubtedly provide a more accurate description of the actual behavior of asset prices. In the case of foreign investment cited in the introduction, for example, while the jump process captures the risks associated with a foreign devaluation, a smoother process should be grafted onto the simple jump to pick up the frictional portfolio risk. Nevertheless, we feel that the model developed here has the ability to capture effects which will in many instances make it preferable to existing alternatives, and that it will serve as a stepping stone for further development.

Figure 1



Diffusion Pricing Relation (23)



Jump Pricing Relation (18)

Footnotes

*The authors are respectively Lecturer of Finance at the University of Pennsylvania and Professor of Economics at the University of Pennsylvania. The authors are grateful for the research support of the Rodney L. White Center for Financial Research at the University of Pennsylvania and the National Science Foundation Grant No. 20292.

[1]It is possible to make k, itself, a random variable and we will consider this possibility below, along with the possibility of combining the jump process with the diffusion process.

[2]This assumption can be substantially weakened but at a large pedagogic cost.

[3]The reader interested in the mathematics of such an approach is referred to Widder, and Bellman and Cooke. It is verified in Bellman and Cooke that the solution is unique.

[4]We have used the fact that

$$\sum_{0}^{\infty} e^{-x} \frac{x^i}{i!} = 1.$$

Equation (15) can also be derived by setting the instantaneous expected stock return, $\lambda(k-1) + \mu$, equal to r.

[5]To obtain (18) from (16) simply substitute (T-t) for T.

[6]These results have to be qualified somewhat. In ranges of the parameter values where $P = S - Ee^{-r(T-t)}$, or $P = 0$, P will be insensitive to the omitted parameters. In addition, if $\gamma = 0$ uncertainty is eliminated and we must have $\mu = r$ to prevent arbitrage. Equation (18) now reduces to simply $\max(S_0 - Ee^{-r(T-t)}, 0)$.

[7]Whether this implies that the jump process is actually more general than the diffusion process is not established, though. By considering complex diffusion processes which allow for a random dependence of σ on S and t, for example, it might be possible to arbitrarily approximate a jump process.

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