Available online at www.sciencedirect.com
ScienceDirect
European Journal of Operational Research 179 (2007) 186–202
EUROPEAN JOURNAL OF OPERATIONAL RESEARCH
www.elsevier.com/locate/ejor

Decision Support

Multiperiod portfolio optimization models in stochastic markets using the mean–variance approach

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Received 26 August 2004; accepted 8 February 2005
Available online 20 July 2006

Abstract

We consider several multiperiod portfolio optimization models where the market consists of a riskless asset and several risky assets. The returns in any period are random with a mean vector and a covariance matrix that depend on the prevailing economic conditions in the market during that period. An important feature of our model is that the stochastic evolution of the market is described by a Markov chain with perfectly observable states. Various models involving the safety-first approach, coefficient of variation and quadratic utility functions are considered where the objective functions depend only on the mean and the variance of the final wealth. An auxiliary problem that generates the same efficient frontier as our formulations is solved using dynamic programming to identify optimal portfolio management policies for each problem. Illustrative cases are presented to demonstrate the solution procedure with an interpretation of the optimal policies.

Keywords: Portfolio optimization; Stochastic market; Mean–variance models; Safety-first; Coefficient of variation; Quadratic utility functions

1. Introduction

The portfolio selection problem is faced by an investor who wants to allocate his wealth among different assets within a market according to an objective based on his preferences. The investor's decisions about which portion of his wealth to invest in each asset over the investment horizon constitute his investment policy. Many factors, such as the investment horizon, characteristics of the market and the objective of the decision maker affect the optimal investment policy. In this paper, we consider a multiperiod portfolio selection problem where the returns of the assets are modulated by a Markov chain that represents the stochastic market. The main objective is to come up with optimal analytical solutions to several multiperiod formulations with different objective functions that represent the investor's preferences.

The traditional single-period mean–variance model developed by Markowitz [16] has been the basis of portfolio theory. It is the first systematic treatment of investors' conflicting objectives of high return versus low risk. The mean–variance model is a parametric optimization model for the single-period

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0377-2217/$ - see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.ejor.2005.02.079

portfolio selection problem which provides analytical solutions for both an investor trying to maximize his expected wealth without exceeding a predetermined risk level and an investor trying to minimize his risk ensuring a predetermined wealth. The analytic derivation of the mean–variance efficient portfolio frontier is given by Merton [18]. Despite its wide-ranging success, the single-period framework suffers from an important deficiency. It is difficult to apply to long-term investors having goals at particular dates in the future, for which the investment decisions should be evaluated with regard to temporal issues besides static risk-reward trade-offs.

Researchers have tried to adapt the classical mean–variance model for the multiperiod case considering the fact that investors invest continuously rather than for a single period. Mostly, it is assumed that the return of an asset in a certain period is independent of the return of the same asset in previous periods. More realistically, some sort of dependence among the returns should be considered. In this paper, this is accomplished by assuming that there are some economic, social, political and other factors affecting the asset returns. These factors form our stochastic market and it is assumed to be a Markov chain.

Early portfolio problems were mainly based on expected gain maximization. However, some objections were raised to this approach, and it is argued that some investors put more emphasis on the risk of default rather than on the investment's yield so that they want to secure at least a minimal return with a high probability. The safety-first problem, developed by Roy [23] as an alternative to the classical mean–variance concept, is of practical importance in portfolio selection, and it is one of the main problems that are analyzed in this paper. It is considered to be a simpler decision model to understand that concentrates on disastrous results where the objective is to minimize the probability that the terminal wealth of an investor is below a predetermined level. The quadratic utility function is one that has been widely used in the literature to describe investor's preferences. This problem is also analyzed where the objective is to maximize the expected utility at the end of the investment horizon. Another problem analyzed in this paper is the minimization of the coefficient of variation of the final wealth.

Multiperiod portfolio optimization models have been studied by many researchers using different approaches. This paper follows the work of Çakmak and Özekici [4] on multiperiod mean–variance portfolio optimization in Markovian markets. The correlation among returns in different periods is formulated by a stochastic market representing the underlying factors that form a Markov chain. Considering a market with one riskless and m risky assets, a multiperiod mean–variance formulation is developed. An auxiliary problem generating the same efficient frontier is used to eliminate nonseparability in the sense of dynamic programming. The analytical optimal solution is obtained for the auxiliary problem using dynamic programming. In our setting, we extend this line of research by considering various problems involving the safety-first approach, coefficient of variation and quadratic utility functions.

We want to emphasize that there is growing interest in the literature to use a stochastic market process in order to modulate various parameters of the financial model to make it more realistic. Hernández-Hernández and Marcus [10], Bielecki et al. [2], Bielecki and Pliska [3], Di Massi and Stettner [17], Stettner [25,26], and Nagai and Peng [19] provide examples on risk-sensitive portfolio optimization with observed, unobserved and partially observed states in Markovian markets. Continuous-time Markov chains with a discrete state space are used in a number of papers including, for example, Bäuerle and Rieder [1], Yin and Zhou [29], and Zhang [31] to modulate model parameters in portfolio selection and stock trading problems. Zariphopoulou [30], Fleming and Hernández-Hernández [7] use diffusion processes for modulating purposes. There are also models where only one of the parameters in modulated. Models of stochastic interest rates with some sort of a Markovian structure are also quite common as in Korn and Kraft [12] , Norberg [20], and Elliott and Mamon [5], among others.

Research on portfolio management is quite extensive; Steinbach [24] surveys single-period and multiperiod mean–variance models in financial portfolio analysis and lists 208 references. Wang and Xia [28] discuss recent developments on the portfolio selection problem and asset pricing. The case where the asset returns over the periods are statistically dependent has received only limited attention due to its apparent complexity. Hakansson and Liu [8], Hernández-Hernández and Marcus [10] and Bieleceki et al. [2] consider models where asset returns are serially correlated. Li and Ng [15] consider the mean–variance formulation in multiperiod portfolio selection and determine the optimal portfolio policy and an analytical expression of the mean–variance efficient frontier. Their model assumes independence of returns over time and an auxiliary problem is solved using dynamic programming. The solution to the auxiliary problem is then manipulated to obtain the optimal mean–variance portfolio management policy and the corresponding efficient frontier. The same approach is used by Li et al. [14] in a multiperiod safety-first formulation, and Zhu et al. [33] in a formulation involving risk control over bankruptcy. Continuous-time version of dynamic portfolio selection is provided by Zhou and Li [32].

Section 2 describes the stochastic structure of the market. Equivalent mean–variance problem formulations in generating efficient multiperiod portfolio policies are given in Section 3. The solution of the auxiliary problem found by dynamic programming is given in Section 4. Section 5 gives the solution procedure of the multiperiod portfolio problem for an arbitrary utility function like the coefficient of variation model. The quadratic utility model is analyzed in detail in Section 6. Section 7 defines the safety-first problem in the single-period setting and then provides the derivation of the analytical solution for it in the multiperiod case. The multiperiod portfolio problems are analyzed on a periodical basis in Section 8. An illustrative case demonstrating the application of the analytical solutions is given in Section 9.

2. The stochastic market

The returns of assets, except for the riskless one, are random. The exact distributions of the returns are not known, but their means, variances and covariances with each other are assumed to be known. These factors change randomly on a periodic basis and form our stochastic market. They constitute the states of a Markov chain which generates serial correlation among returns in different periods. As the state of the market changes over time, the returns also change accordingly. In short, we have a model where asset returns are modulated by the stochastic market.

We let Y_n denote the state of the market at period n so that $Y = \{Y_n; n = 0, 1, 2, \ldots\}$ is a Markov chain with a discrete state space E and transition matrix Q. Modeling a stochastic financial market by a Markov chain is a reasonable approach and this idea dates back to the paper written by Pye [21]. In the continuous time setting, Norberg [20] considers an interest rate model that is modulated by a Markov process. Recently, Elliott and Mamon [5] provide a yield curve description of a Markovian interest rate model.

Let $R(i)$ denote the random vector of asset returns in any period given that the stochastic market is in state i. The means, variances and covariances of asset returns depend only on the current state of the stochastic market. The market consists of one riskless asset with known return $r_f(i)$ and standard deviation $\sigma_f(i) = 0$ and m risky assets with random returns $R(i) = (R_1(i), R_2(i), \ldots, R_m(i))$ in state i. We let $r_k(i) = E[R_k(i)]$ denote the mean return of the kth asset in state i and $\sigma_{kj}(i) = \mathrm{Cov}(R_k(i), R_j(i))$ denote the covariance between kth and jth asset returns in state i. The excess return of the kth asset in state i is $R_k^e(i) = R_k(i) - r_f(i)$. It follows that:

$$r_k^e(i) = E[R_k^e(i)] = r_k(i) - r_f(i), \tag{1}$$

$$\sigma_{kj}(i) = \mathrm{Cov}(R_k(i) - r_f(i), R_j(i) - r_f(i)). \tag{2}$$

Our notation is such that $r_f(i)$ is a scalar and $r(i) = (r_1(i), r_2(i), \ldots, r_m(i))$ and $r^e(i) = (r_1^e(i), r_2^e(i), \ldots, r_m^e(i))$ are column vectors for all i. For any column vector z, z' denotes the row vector representing its transpose.

We define X_n as the amount of investor's wealth at period n and correspondingly X_T denotes the final wealth at the end of the investment horizon. The vector $u = (u_1, u_2, \ldots, u_m)$ gives the amounts invested in risky assets $(1, 2, \ldots, m)$ in a given period. Given any investment policy, the stochastic evolution of the investor's wealth follows the so-called wealth dynamics equation:

$$\begin{aligned} X_{n+1}(u) &= R(Y_n)'u + (X_n - 1'u)r_f(Y_n) \\ &= r_f(Y_n)X_n + R^e(Y_n)'u, \end{aligned} \tag{3}$$

where $1 = (1, 1, \ldots, 1)$ is the column vector consisting of 1's.

The assumptions regarding the model formulation can be summarized as follows: (a) there is unlimited borrowing and lending at the prevailing return of the riskless asset in any period, (b) short selling is allowed for all assets in all periods, (c) no capital additions or withdrawals are allowed throughout the investment horizon, and (d) transaction costs and fees are negligible.

3. Mean–variance model formulations

We will use the notation $E_i[Z] = E[Z|Y_0 = i]$ and $\mathrm{Var}_i(Z) = E_i[Z^2] - E_i[Z]^2$ to denote the conditional expectation and variance of any random variable Z given that the initial market state is i. In the multiperiod setting, we obtain the following two mean–variance formulations corresponding to Markowitz's [16] formulation:

$$P1(\sigma): \quad \max \quad E_i[X_T]$$
$$\text{s.t.} \quad \text{Var}_i(X_T) \leqslant \sigma,$$
$$X_{n+1}(u) = r_f(Y_n)X_n + R^e(Y_n)'u, \quad (4)$$

$$P2(\mu): \quad \min \quad \text{Var}_i(X_T)$$
$$\text{s.t.} \quad E_i[X_T] \geqslant \mu,$$
$$X_{n+1}(u) = r_f(Y_n)X_n + R^e(Y_n)'u, \quad (5)$$

given that the initial market state is i.

The multiperiod mean–variance formulations given in (4) and (5) do not have straightforward solutions, and they cannot be solved using dynamic programming due to their nonseparability. An equivalent formulation to generate efficient multiperiod portfolio policies is

$$P3(\omega): \quad \max \quad E_i[X_T] - \omega \text{Var}_i(X_T)$$
$$\text{s.t.} \quad X_{n+1}(u) = r_f(Y_n)X_n + R^e(Y_n)'u, \quad (6)$$

where $\omega > 0$. Once $P3(\omega)$ is solved parametrically for ω, it is sufficient to set $\text{Var}_i(X_T) = \sigma^2$ and $E_i[X_T] = \mu$ to identify which ω gives the optimal solution of $P1(\sigma)$ and $P2(\mu)$, respectively. The efficient frontier on $E_i[X_T]$ versus $\sqrt{\text{Var}_i(X_T)}$ graph is obtained by changing the value of ω in the objective function of problem (6).

Since $P3(\omega)$ is still not separable in the sense of dynamic programming, it is further embedded in the tractable auxiliary problem

$$P4(\lambda,\omega): \quad \max \quad E_i[-\omega X_T^2 + \lambda X_T]$$
$$\text{s.t.} \quad X_{n+1}(u) = r_f(Y_n)X_n + R^e(Y_n)'u, \quad (7)$$

where ω is a positive parameter. It turns out that $P4(\lambda,\omega)$ is separable in the sense of dynamic programming. The important relationship between these four formulations is that the optimal solution sets of former problems are included in the optimal solution sets of later formulations so that the solutions of former problems can be obtained from $P4(\lambda,\omega)$. In other words, by solving $P4(\lambda,\omega)$ we also solve $P3(\omega)$ which, in turn, leads to solving both $P1(\sigma)$ and $P2(\mu)$.

4. Solution of the auxiliary problem

The stochastic market model that we consider in this paper was introduced recently by Çakmak and Özekici [4], where the primary focus is on mean–variance formulations. They used dynamic programming to find an explicit solution of the auxiliary problem $P4(\lambda,\omega)$ where the utility function has a linear-quadratic structure. Using a recursive approach, they identify the optimal policy explicitly at each period and show that the optimal expected utility also has a linear-quadratic structure. As a matter of fact, this structure is exploited to obtain computationally tractable results. In this section, we summarize the main results and refer the reader to Çakmak and Özekici [4] for details and proofs. We will make extensive use of their results to extend this line of research by considering several utility functions of the mean and variance that have sufficient interest in portfolio optimization.

In order to solve $P4(\lambda,\omega)$, we define $v_n(i,x)$ as the optimal expected utility using the optimal policy given that the market state is i and the amount of money available for investment is x at period n. Then, the dynamic programming equation becomes

$$v_n(i,x) = \max_u \sum_{j\in E} Q(i,j)E[v_{n+1}(j, r_f(i)x + R^e(i)'u)] \quad (8)$$

for $n = 0,1,2,\ldots,T-1$ with the boundary condition $v_T(i,x) = -\omega x^2 + \lambda x$ for all i. The solution for this problem is found by solving the dynamic programming equation recursively.

We need to introduce some terminology and notation to give the optimal solution. We define the matrix

$$V(i) = E[R^e(i)R^e(i)'] \quad (9)$$

for any state i. The covariance matrix $\sigma(i)$ is shown to be positive definite for all i. This property of $\sigma(i)$ is inherited by $V(i)$ such that for any i, $V(i) = \sigma(i) + r^e(i)r^e(i)'$ is a positive definite matrix. For any state i, we set

$$f(i) = r_f(i)^2[1 - h(i)], \quad (10)$$
$$g(i) = r_f(i)[1 - h(i)], \quad (11)$$

where

$$h(i) = r^e(i)'V^{-1}(i)r^e(i). \quad (12)$$

It turns out that for any i, $f(i)$, $g(i) > 0$ and $0 < h(i) < 1$.

For any matrix M and vector f, we define the matrix M_f such that

$$M_f(i,j) = M(i,j)f(j) \quad (13)$$

for $i,j \in E$ and the vector $\overline{M}$ such that

$$\overline{M}(i) = \sum_{j\in E} M(i,j). \quad (14)$$

Using this notation M_f^n is the nth power of M_f, and $\overline{M_f^n}$ is simply the vector obtained by adding the columns of the matrix M_f^n for $n \geqslant 0$. It follows that $\overline{M}_f^0 = 1$ when $n = 0$ and $\overline{M}_f = Mf$ when $n = 1$.

If a, b and c are three vectors, then $(a/b) \cdot c$ denotes the vector where $((a/b) \cdot c)(i) = (a(i)/b(i))c(i)$. Using these notations, we finally define

$$h_n(i) = \frac{\overline{Q}_g^n(i)}{\overline{Q}_f^n(i)} h(i), \tag{15}$$

$$\bar{h}_n(i) = \left(\frac{\overline{Q}_g^n(i)}{\overline{Q}_f^n(i)}\right)^2 h(i). \tag{16}$$

We use x_0 to denote the initial wealth which is assumed to be known. The optimal solution of $P4(\lambda, \omega)$ is of the form

$$v_n(i,x) = -\omega_n(i)x^2 + \lambda_n(i)x + \alpha_n(i) \tag{17}$$

and the corresponding optimal policy maximizing the objective function is

$$u_n(i,x) = \left[\frac{1}{2}\left(\frac{\lambda}{\omega}\right)\frac{\overline{Q}_g^{T-n-1}(i)}{\overline{Q}_f^{T-n-1}(i)} - r_f(i)x\right] V^{-1}(i) r^e(i), \tag{18}$$

where

$$\omega_n(i) = \omega \overline{Q}_f^{T-n-1}(i) f(i), \tag{19}$$

$$\lambda_n(i) = \lambda \overline{Q}_g^{T-n-1}(i) g(i), \tag{20}$$

$$\alpha_n(i) = \sum_{k=n+2}^{T} Q^{k-n-2} \overline{Q}_{\bar{\alpha}_k}(i) + \bar{\alpha}_{n+1}(i) \tag{21}$$

and

$$\bar{\alpha}_n(i) = \frac{(\lambda \overline{Q}_g^{T-n}(i))^2}{4\omega \overline{Q}_f^{T-n}(i)} h(i) \tag{22}$$

for $n = 0, 1, \ldots, T-1$. In (21), the summation on the right hand side vanishes if $n = T-1$.

The optimal investment policy $u_n(i,x)$ in (18) gives the amount of money that should be invested in each asset at period n given that the market state is i and the current wealth is x. By substituting (18) into the wealth dynamics equation given in (3) and then taking expectations of X_n and X_n^2, we obtain

$$E_i[X_n] = \overline{Q}_g^{n-1}(i) g(i) x_0 + \frac{\lambda}{2\omega} \sum_{k=1}^{n} Q^{k-1} (\overline{Q}_g^{n-k} \cdot h_{T-k})(i), \tag{23}$$

$$E_i[X_n^2] = \overline{Q}_f^{n-1}(i) f(i) x_0^2 + \left(\frac{\lambda}{2\omega}\right)^2 \times \sum_{k=1}^{n} Q^{k-1} (\overline{Q}_f^{n-k} \cdot \bar{h}_{T-k})(i) \tag{24}$$

for $n = 1, \ldots, T$.

If we define

$$a_1(i) = \overline{Q}_g^{T-1}(i) g(i), \tag{25}$$

$$a_2(i) = \overline{Q}_f^{T-1}(i) f(i), \tag{26}$$

$$b(i) = \frac{1}{2} \sum_{k=1}^{T} Q^{k-1} \left(\frac{(\overline{Q}_g^{T-k})^2}{\overline{Q}_f^{T-k}} \cdot h\right)(i) \tag{27}$$

then the optimal solution satisfies the simplified expressions

$$E_i[X_T] = a_1(i) x_0 + b(i)\gamma, \tag{28}$$

$$E_i[X_T^2] = a_2(i) x_0^2 + \frac{1}{2} b(i) \gamma^2, \tag{29}$$

where $\gamma = \lambda/\omega$. Consequently, the variance of the terminal wealth is

$$\mathrm{Var}_i(X_T) = (a_2(i) - a_1(i)^2) x_0^2 - 2a_1(i) b(i) x_0 \gamma + \left(\frac{1}{2} - b(i)\right) b(i) \gamma^2. \tag{30}$$

With respect to our multiperiod portfolio optimization problem, $E_i[X_T]$ is the expected wealth at the end of the investment horizon and $\mathrm{Var}_i(X_T)$ measures the risk of the final wealth. Expectation versus standard deviation of X_T corresponds to an optimal point on the mean–variance efficient frontier.

The mean–variance efficient frontier is given by

$$\mathrm{Var}_i(X_T) = \left(a_2(i) - \frac{a_1(i)^2}{1 - 2b(i)}\right) x_0^2 + \frac{[(1 - 2b(i)) E_i[X_T] - a_1(i) x_0]^2}{2b(i)(1 - 2b(i))} \tag{31}$$

defined for $E_i[X_T] \geqslant a_1(i)x_0/(1 - 2b(i))$. Moreover, at the minimum-variance point,

$$\gamma = 2a_1(i) x_0 / (1 - 2b(i)) \tag{32}$$

and

$$E_i[X_T] = \frac{a_1(i) x_0}{1 - 2b(i)}, \tag{33}$$

$$\mathrm{Var}_i(X_T) = \left(a_2(i) - \frac{a_1(i)^2}{1 - 2b(i)}\right) x_0^2. \tag{34}$$

To simplify the notation, a_1, a_2 and b are going to be used in place of $a_1(i)$, $a_2(i)$ and $b(i)$ from now on. It must be remembered that a_1, a_2 and b are always functions of the initial market state.

5. General utility functions

All multiperiod portfolio selection problems that are considered in this paper have objectives that are functions of the mean and variance of the final wealth. The multiperiod portfolio selection problem then takes the form

$$U: \quad \max \quad U(E_i[X_T], \mathrm{Var}_i(X_T)) \\ \text{s.t.} \quad X_{n+1}(u) = r_f(Y_n)X_n + R^e(Y_n)'u. \tag{35}$$

The solution procedure for general multiperiod portfolio problems just involves the replacement of the expressions for the mean and variance, given in (28) and (30), into the objective function $U(E_i[X_T], \mathrm{Var}_i(X_T))$ of the specific problem. The only restriction regarding the use of this solution procedure is that the utility function has to be increasing with respect $E_i[X_T]$ and decreasing with respect to $\mathrm{Var}_i(X_T)$ in order to assure that the auxiliary problem gives equivalent solutions on the efficient frontier for the problem in consideration.

Investors in this paper are assumed to have an objective of maximizing their final wealth while keeping their risk as low as possible so that the utility function satisfies

$$\frac{\partial U(E_i[X_T], \mathrm{Var}_i(X_T))}{\partial E_i[X_T]} > 0 \tag{36}$$

and

$$\frac{\partial U(E_i[X_T], \mathrm{Var}_i(X_T))}{\partial \mathrm{Var}_i(X_T)} < 0. \tag{37}$$

Li and Ng [15] prove that problem U given in (35) can be embedded into problem $P3(\omega)$ given in (6) which further can be embedded into the auxiliary problem $P4(\lambda, \omega)$ given in (7) implying that a multiperiod portfolio problem of maximizing $U(E_i[X_T], \mathrm{Var}_i(X_T))$ can be embedded into $P4(\lambda, \omega)$.

The general solution procedure is rather straightforward. After replacing the optimal values of $E_i[X_T]$ and $\mathrm{Var}_i(X_T)$ given in (28) and (30) into $U(E_i[X_T], \mathrm{Var}_i(X_T))$, the objective function of the specific problem U is obtained in terms of γ. The next step is then to obtain the derivative of U with respect to γ and solve for the maximum point that will be reached at γ^*. In general

$$\frac{\mathrm{d}U}{\mathrm{d}\gamma} = \left(\frac{\partial U}{\partial E_i[X_T]} - 2E_i[X_T]\frac{\partial U}{\partial \mathrm{Var}_i(X_T)}\right)\frac{\mathrm{d}E_i[X_T]}{\mathrm{d}\gamma} + \frac{\partial U}{\partial \mathrm{Var}_i(X_T)}\frac{\mathrm{d}E_i[X_T^2]}{\mathrm{d}\gamma}, \tag{38}$$

where

$$\frac{\mathrm{d}E_i[X_T]}{\mathrm{d}\gamma} = b \quad \text{and} \quad \frac{\mathrm{d}E_i[X_T^2]}{\mathrm{d}\gamma} = b\gamma, \tag{39}$$

which are found from (28) and (29), respectively. Setting $\mathrm{d}U/\mathrm{d}\gamma$ equal to zero, the necessary optimality condition for γ is obtained

$$\left(\frac{\partial U}{\partial E_i[X_T]} - 2E_i[X_T]\frac{\partial U}{\partial \mathrm{Var}_i(X_T)}\right) + \frac{\partial U}{\partial \mathrm{Var}_i(X_T)}\gamma = 0, \tag{40}$$

which implies

$$\gamma^* = 2E_i[X_T] - \left(\frac{\partial U}{\partial E_i[X_T]} \Big/ \frac{\partial U}{\partial \mathrm{Var}_i(X_T)}\right). \tag{41}$$

Furthermore, it has to be verified that the utility function is actually maximized at this point. The optimal portfolio policy for the related problem is obtained by substituting $\gamma = \lambda/\omega$ in (18) with the optimal γ^*. Finally, the expectation and variance of the final wealth are calculated by substituting the optimal γ^* into (28) and (30), respectively.

One utility function that satisfies conditions (36) and (37) is the objective function of the coefficient of variation model. The coefficient of variation is a measure of relative dispersion and is defined formally as $\sqrt{\mathrm{Var}_i(X_T)}/E_i[X_T]$. A logical objective function for an investor dealing with the multiperiod portfolio optimization would be to minimize the coefficient of variation of the final wealth which can also be stated as

$$\mathrm{CV}: \quad \max \quad U(E_i[X_T], \mathrm{Var}_i(X_T)) = \frac{E_i[X_T]}{\sqrt{\mathrm{Var}_i(X_T)}} \\ \text{s.t.} \quad X_{n+1}(u) = r_f(Y_n)X_n + R^e(Y_n)'u. \tag{42}$$

It is clear that both (36) and (37) are satisfied.

Replacing $E_i[X_T]$ and $\mathrm{Var}_i(X_T)$ given in (28) and (30) into the objective function of problem (42), we obtain

$$U(\gamma) = \frac{a_1x_0 + b\gamma}{\sqrt{(a_2 - a_1^2)x_0^2 + \left(\frac{b}{2} - b^2\right)\gamma^2 - 2a_1bx_0\gamma}}. \tag{43}$$

The first derivative of the objective function $U(\gamma)$ with respect to γ is

$$\frac{\mathrm{d}U}{\mathrm{d}\gamma} = \frac{a_2bx_0^2 - 0.5a_1bx_0\gamma}{[(a_2 - a_1^2)x^2 + (0.5b - b^2)\gamma^2 - 2a_1bx\gamma]^{\frac{3}{2}}}. \tag{44}$$

Equating the derivative in (44) to zero gives the single optimum point to be

$$\gamma^* = \frac{2a_2x_0}{a_1}. \tag{45}$$

The derivative in (44) reveals that the optimal γ^* is a maximum point as required. This is obvious from the fact that the derivative is positive for γ values smaller than γ^* and negative for values greater than γ^*.

The coefficient of variation problem has only one single solution since the objective function does not involve any parameter which could depend on the investor's preferences. This means that the solution is the same for all investors and it gives a single point on the efficient frontier. This point corresponds to a single ω value of problem $P3(\omega)$. The optimal γ^* of $P3(\omega)$ in terms of ω is given in Çakmak and Özekici [4] as

$$\gamma^* = \frac{1 + 2\omega a_1 x_0}{\omega - 2b\omega}. \tag{46}$$

The optimal ω value can be found by equating the optimal γ^* values of both problems given in (45) and (46) to obtain

$$\omega = \frac{a_1}{2x_0(a_2 - 2a_2b - a_1^2)}. \tag{47}$$

Finally, the optimal policy to this problem can be obtained by replacing the optimal γ^* in place of $\gamma = \lambda/\omega$ in (18).

6. Quadratic utility model

The general quadratic function $X_T - AX_T^2$, where A is a positive coefficient, is the utility function that has been used in the economics and finance literature to describe investor behavior. The problem of maximizing the expectation of this utility function corresponds to the auxiliary problem $P4(\lambda, \omega)$ with $\lambda = 1$ and $\omega = A$. This means that no initial condition is required to solve this problem since the solution obtained in Section 4 is also valid in this case.

By using the definition of the variance

$$\begin{aligned} E[X_T - AX_T^2] &= E[X_T] - AE[X_T^2] \\ &= E[X_T] - A[\mathrm{Var}(X_T) + E[X_T]^2] \end{aligned} \tag{48}$$

so that the multiperiod objective can be expressed as a function of $E_i[X_T]$ and $\mathrm{Var}_i(X_T)$

$$\begin{aligned} &U(E_i[X_T], \mathrm{Var}_i(X_T)) \\ &\quad = -AE_i[X_T]^2 + E_i[X_T] - A\mathrm{Var}_i(X_T). \end{aligned} \tag{49}$$

The multiperiod portfolio problem of an investor having this utility function is

$$\begin{aligned} QU(A):\ & \max\ -AE_i[X_T]^2 + E_i[X_T] - A\mathrm{Var}_i(X_T) \\ & \text{s.t.}\quad X_{n+1}(u) = r_f(Y_n)X_n + R^e(Y_n)'u. \end{aligned} \tag{50}$$

Given the initial state i, the expectation and variance of the final wealth are already given in (28) and (30). Putting these expressions into the objective function given in (50), the objective in terms of γ turns out to be

$$U(\gamma) = -\frac{1}{2}Ab\gamma^2 + b\gamma + a_1x_0 - Aa_2x_0^2. \tag{51}$$

Taking the derivative of $U(\gamma)$ with respect to γ and equating it to zero reveals the extreme

$$\gamma^* = \frac{1}{A}. \tag{52}$$

This optimal point is also a maximum point since $\mathrm{d}^2U/\mathrm{d}\gamma^2 = -Ab < 0$.

Investors having a quadratic utility function become more risk averse as A increases since the utility function has a higher curvature which can be quantified in terms of its the second derivative.

The efficient frontier obtained in this problem corresponds to the same efficient frontier obtained from the mean–variance problem $P3(\omega)$. To get the same $(\sqrt{\mathrm{Var}_i(X_T)}, E_i[X_T])$ pair on the efficient frontiers, the optimal γ^* values of both problems are equated. Equating γ^* in (46) to γ^* in (52) reveals the relationship between the parameter of $P3(\omega)$ and the parameter of $QU(A)$ which is

$$\omega = \frac{A}{1 - 2b - 2Aa_1x_0}. \tag{53}$$

By changing the value of parameter A, we obtain the mean–variance efficient frontier.

Investors are assumed to prefer more wealth to less wealth corresponding to the nonsatiation property which means that the first derivative of the utility function with respect to $E_i[X_T]$ should be positive, i.e. the following condition must be placed on $E_i[X_T]$

$$\frac{\partial U}{\partial E_i[X_T]} = -2AE_i[X_T] + 1 > 0, \tag{54}$$

which implies

$$E_i[X_T] < \frac{1}{2A}. \tag{55}$$

An additional analysis for this problem is then to find the range of A that will assure that $E_i[X_T] < 1/2A$. Given the expected final wealth $E_i[X_T]$ in (28), this condition turns into the inequality $a_1x_0 + b\gamma <$

$1/2A$ where γ is taken to be the optimum one which is $1/A$. Finally, the range for A turns out to be

$$A < \frac{1-2b}{2a_1x_0} = A^*. \quad (56)$$

The range $A \in (0, A^*)$ will ensure that the investor always prefers more to less.

Moreover, the first derivative of the utility function $\partial U/\partial \mathrm{Var}_i(X_T) = -A$ is negative since A is taken to be positive, and this assures that the investor exhibits risk aversion.

The bounds of the range $(0, A^*)$ correspond to the minimum-variance point for A approaching A^* and to the upper end of the efficient frontier for A approaching 0 in the limit. For $A = A^*$, the optimal γ^*, which is $1/A$ as given in (52), is equal to the γ value of the minimum-variance portfolio given in (32); and for A approaching 0, the optimal γ^* approaches $+\infty$ which implies infinite $E_i[X_T]$ and $\mathrm{Var}_i(X_T)$ from (28) and (30) respectively. For intermediate values of the parameter A, the optimal portfolios move upwards on the efficient frontier as A is decreased from A^* to 0. This result is expected since a higher value of A implies higher risk aversion so that less money is invested in the risky assets which leads to lower expectation and lower variance for the final wealth. If A is further increased above A^*, which is possible since the quadratic utility problem does not require an initial condition for the proposed solution procedure, the portfolios obtained are on the minimum-variance set but not on the efficient frontier anymore.

The quadratic utility problem on its own does not have an explicit interpretation except for the fact that it can satisfy the risk-averseness and the nonsatiation property of the investor by putting some constraints on its parameter A. However, it is important to note here that there is a utility problem having an explicit interpretation that turns out to have the same objective function as the quadratic utility problem. The objective of this problem can be attached a certain meaning and it is given as

$$\min P\{|X_T - \alpha| > \epsilon\}. \quad (57)$$

This objective aims to get a final wealth X_T which is not significantly different from a specified value α, which is logically assumed to be greater than zero. That is, the investor is trying to maximize the probability that X_T is in the vicinity of α. Using Markov's inequality

$$P\{(X_T - \alpha)^2 > \epsilon^2\} \leqslant \frac{E[(X_T-\alpha)^2]}{\epsilon^2} \quad (58)$$

the objective function in (57) has the upper bound $E[X_T^2 - 2\alpha X_T + \alpha^2]/\epsilon^2$. Minimizing this upper bound is the same as minimizing $E[X_T^2] - 2\alpha E[X_T]$ since both α^2 and ϵ^2 are predetermined. Rearranging this expression yields the objective

$$\max 2\alpha\left(E[X_T] - \frac{1}{2\alpha}E[X_T^2]\right), \quad (59)$$

which is equivalent to

$$\max \left(E[X_T] - \frac{1}{2\alpha}E[X_T^2]\right). \quad (60)$$

Comparing (60) with (48) shows that this problem is the same as the quadratic utility problem with $A = 1/2\alpha$, meaning that the same solution procedure as given above can be used for this problem as well. In order for the nonsatiation property to be satisfied, the condition in (56) can be rearranged, by putting $1/2\alpha$ in place of A, to give $\alpha > (a_1x_0)/(1-2b) = k^*$, where k^* is constant for a given portfolio problem and an important notation used in the formulation of the safety-first problem.

7. Safety-first model

Elton and Gruber [6] introduce other criteria for portfolio selection as an alternative to the classical mean–variance approach. Telser [27] and Kataoka [11] develop different versions of the safety-first approach of Roy [23] by prespecifying the acceptable probability of a bad outcome. Pyle and Turnovsky [22] discuss the relationship between the three different safety-first criteria developed. Levy and Sarnat [13] try to relate the safety-first principle to the expected utility principle. Li et al. [14] and Zhu et al. [33] extend the safety-first approach to multiperiod portfolio selection problems. In a recent paper, Haque et al. [9] discuss all approaches proposed so far to solve the safety-first problem.

The objective of the safety-first model is to minimize the downside risk of an investor at the end of the investment horizon. According to the definition given by Roy [23], it is the minimization of the probability of a disaster which corresponds to receiving an undesired return. More formally, the safety-first problem deals with the minimization of the probability that the final wealth X_T is smaller than a prespecified disaster level k given by the investor (i.e., $P\{X_T \leqslant k\}$). This minimization corresponds to the maximization of $1 - P\{X_T \leqslant k\}$ which is also equivalent to maximizing $P\{X_T > k\}$. Therefore, the safety-first objective can be stated as maximizing the probability that the final wealth X_T is greater than a prespecified level k given by the investor. In other words, this problem also tries to maximize the upside potential.

Roy [23] and other researchers made use of Chebyshev's inequality to formulate the safety-first problem. The reason for using this bound is that it is very robust since it does not assume any distribution about the random variable considered. Chebyshev's inequality states that

$$P_i\{|X_T - E_i[X_T]| \geqslant E_i[X_T] - k\} \leqslant \frac{\mathrm{Var}_i(X_T)}{(E_i[X_T]-k)^2}, \tag{61}$$

which leads to

$$P_i\{X_T \leqslant k\} \leqslant \frac{\mathrm{Var}_i(X_T)}{(E_i[X_T]-k)^2}. \tag{62}$$

The objective is then to minimize the upper bound, and the safety-first portfolio selection problem for a disaster level k can be stated as

$$\begin{aligned} SF(k):\quad & \max \quad U(E_i[X_T], \mathrm{Var}_i(X_T)) = \frac{E_i[X_T]-k}{\sqrt{\mathrm{Var}_i(X_T)}} \\ & \text{s.t.} \quad X_{n+1}(u) = r_f(Y_n)X_n + R^e(Y_n)'u. \end{aligned} \tag{63}$$

This utility has to be an increasing function of $E_i[X_T]$ and a decreasing function of $\mathrm{Var}_i(X_T)$ in order to be able to apply the auxiliary problem $P4(\lambda,\omega)$. The derivative with respect to the expected final wealth $\partial U/\partial E_i[X_T] = 1/\sqrt{\mathrm{Var}_i(X_T)}$ is greater than zero, whereas the derivative with respect to the variance

$$\frac{\partial U}{\partial \mathrm{Var}_i(X_T)} = -\frac{1}{2}\frac{E_i[X_T]-k}{\sqrt{\mathrm{Var}_i(X_T)^3}} \tag{64}$$

is smaller than zero for all values of the variance but only for values of expected final wealth that are greater than the disaster level. This means that the solution procedure based on the auxiliary problem $P4(\lambda,\omega)$ is applicable only if the expected final wealth $E_i[X_T]$ is greater than the disaster level k, implying that $E_i[X_T] > k$ is the primal condition of the safety-first problem in order to be able to apply the auxiliary problem. This condition is not only a technical constraint but it is also a logical one since the value of k should reflect a disaster which logically should be set smaller than the expected wealth.

After replacing the expressions for the expectation and the variance of the final wealth given in (28) and (30) respectively into the safety-first objective function, the safety-first utility can be expressed in terms of γ as

$$U(\gamma) = \frac{a_1x_0 + b\gamma - k}{\sqrt{(a_2 - a_1^2)x_0^2 + \left(\frac{b}{2} - b^2\right)\gamma^2 - 2a_1bx_0\gamma}}. \tag{65}$$

It turns out that there is a unique extreme point that satisfies

$$\frac{\mathrm{d}U}{\mathrm{d}\gamma} = \frac{b[a_2x_0^2 - a_1x_0k - 0.5a_1x_0\gamma + 0.5k\gamma - kb\gamma]}{[(a_2 - a_1^2)x_0^2 + (0.5b - b^2)\gamma^2 - 2a_1bx_0\gamma]^{\frac{3}{2}}} = 0 \tag{66}$$

so that

$$\gamma^* = \frac{2a_2x_0^2 - 2a_1kx_0}{a_1x_0 - k + 2bk}. \tag{67}$$

The sign of the first derivative depends on disaster level k. It turns out that the utility function has a maximum point for k values smaller than

$$k^* = \frac{a_1x_0}{1-2b} \tag{68}$$

and a minimum point for k values greater than k^*.

Analyzing γ^* reveals that it is an increasing function of k; implying that the higher the value of k is, the higher the value of γ^* will be. Higher γ^*, on the other hand, leads to higher mean and variance on the efficient frontier for the final wealth. This result is not unexpected since choosing a higher disaster level requires a bigger portion of money to be invested in risky assets so as not to fall below the now-higher level and this causes both the mean and the variance of the final wealth to increase. Furthermore, as the disaster level increases, the probability of disaster also increases.

To find the allowable range of disaster level k which will assure that $E_i[X_T] > k$, the optimal γ^* given in (67) is replaced into (28) so that $E_i[X_T]$ is found in terms of the disaster level as

$$E_i[X_T] = \frac{x_0[a_1(a_1x_0 - k) + 2a_2bx_0]}{a_1x_0 - k + 2bk}. \tag{69}$$

The next step is then to find the relationship between $E_i[X_T]$ in (69) and the disaster level k. After some calculations, it turns out that the primal condition $E_i[X_T] > k$ of the safety-first problem is satisfied for disaster levels k which are smaller than the critical level k^* defined in (68). The important feature of this critical level k^* is that if the condition $k \in (-\infty, k^*)$ is satisfied, the safety-first utility function in (65) has a well-defined single optimal γ^* given in (67) that maximizes investor's utility at the end of the investment horizon. The disaster level has an upper bound so that the investor is advised to require only modest returns for his investment since the main objective should be minimizing the downside risk and not maximizing the gain.

To solve the multiperiod safety-first problem, the range of the disaster level k for which the auxiliary problem is applicable is found first. Then, γ^* is calculated for a given disaster level using (67) and the optimal policy $u_n(i,x)$ will follow directly from the expression given in (18) by replacing γ^* in place of $\gamma = \lambda/\omega$. The expected value and the variance of the final wealth X_T are found by replacing γ^* given in (67) into (28) and (30), respectively.

The efficient frontier obtained in this problem corresponds to the same efficient frontier obtained from problem $P3(\omega)$. To get the same efficient portfolio, the optimal γ^* of $P3(\omega)$ given in (46) and the optimal γ^* of the safety-first problem given in (67) are equated to obtain

$$\omega = \frac{a_1 x_0 - k + 2bk}{2x_0^2(a_2 - 2a_2 b - a_1^2)}, \tag{70}$$

which shows that the selected disaster level corresponds to a certain ω value. This result implies that solving the safety-first problem for different values of the disaster level will lead to optimal portfolios on the mean–variance efficient frontier.

The bounds of the allowable range $(-\infty, k^*)$ correspond to the minimum-variance point for k approaching $-\infty$ and to the upper end of the efficient frontier for k approaching k^* in the limit. For k approaching $-\infty$, the optimal γ^* given in (67) approaches to the γ value of the minimum-variance portfolio given in (32); and for k approaching k^*, the optimal γ^* approaches $+\infty$ which implies infinite $E_i[X_T]$ and $\mathrm{Var}_i(X_T)$ from (28) and (30), respectively. For intermediate values of the disaster level, the optimal portfolios move upwards on the efficient frontier as k is increased from $-\infty$ to k^*. This result is not unexpected since more money has to be invested in the risky assets if a higher value of k is required so as not to fall below this level. This leads to higher expectation and higher variance for the final wealth.

It is shown numerically that the critical level k^* is always greater than 1 for any multiperiod safety-first problem solved so far. This implies that the safety-first problem can be solved optimally for all disaster levels smaller than 1. This result makes a major contribution to our problem regarding the original meaning given by Roy [23]. Since k levels chosen above 1 cannot actually be thought as a disaster in its real meaning, it is much more logical to choose them to be below 1, which is really a disaster since this implies a loss for the investor at the end of the investment horizon.

From among the models considered in this paper, the safety-first problem especially is of practical importance. The following results are obtained for the safety-first problem: The investor can avoid a loss and, more importantly, secure a minimal return with a high probability by following the optimal safety-first investment policy. The realized final wealth at the end of the investment horizon is expected to be greater than the specified disaster level. The efficient frontier obtained from safety-first approach exactly matches the mean–variance efficient frontier.

As a last remark to the safety-first problem, for investors requiring a return higher than k^*, the safety-first problem is not appropriate since its main aim is to minimize the downside risk. But instead, another problem can be used for these types of investors which is already discussed at the end of Section 6 and the formulation of which is given in (57). The condition $\alpha > k^*$ implies that an investor who wants his final wealth to be in the vicinity of α, which should be higher than k^*, can use this problem and apply the corresponding policy which will better fit his objective of exceeding the critical disaster level.

8. Periodic analysis of efficient frontiers

If the length of the investment horizons of given problems are not equal, the expected value and the variance of X_T will be based on different scales and therefore a logical comparison cannot be made. Two approaches are considered in this paper that are used to transform the final results depending on T to a periodic basis. The periodic return is defined as the return with constant mean and variance that will lead to the expected final wealth and variance of the final wealth at the end of T periods by investing the initial wealth periodically using that return. More formally, we can write

$$X_T = X_0(1 + r_1)(1 + r_2)\cdots(1 + r_T), \tag{71}$$

where r_j denotes the rate of return in period j. This equality states that the final wealth X_T at the end of the investment horizon can be obtained by investing the initial wealth X_0 at a random rate of return r_j in period j.

We can make an analysis by assuming that the periodic rates of return r_j are independent and identically distributed with the same mean r and the same variance σ^2. The justification is that if the investment horizon is taken to be long enough and the stochastic market is an ergodic Markov chain, then the market will reach its steady state. This means that regardless of the initial state, the periodic returns after reaching steady state will be independent and identically distributed. Accordingly, r and σ^2 denote the periodic mean and the periodic variance that lead to the same $E[X_T]$ and $\mathrm{Var}(X_T)$ at the end of the investment horizon.

Assuming that $X_0 = 1$ without loss of generality, X_T corresponds to the compound return over T periods. Taking the expectation of (71) while $X_0 = 1$ leads to

$$E[X_T] = E[(1+r_1)(1+r_2)\dots(1+r_T)] = (1+r)^T = R^T, \tag{72}$$

which gives the relationship between the expected final wealth and the mean periodic rate of return r or the mean periodic return $R = 1 + r$. Accordingly, the mean periodic return R is $E[X_T]^{1/T}$.

Taking the variance of (71), we obtain

$$\begin{aligned}\mathrm{Var}(X_T) &= \mathrm{Var}((1+r_1)(1+r_2)\cdots(1+r_T)) \\ &= E[(1+r_1)^2\cdots(1+r_T)^2] \\ &\quad - (E[(1+r_1)\cdots(1+r_T)])^2 \\ &= (E[(1+r_j)^2])^T - ((1+r)^T)^2 \\ &= (\sigma^2 + R^2)^T - (R^2)^T \end{aligned}\tag{73}$$

since

$$E[(1+r_j)^2] = \mathrm{Var}(1+r_j) + (E[1+r_j])^2 = \sigma^2 + (1+r)^2. \tag{74}$$

After finding the mean and variance of the final wealth in terms of R and σ^2 as given in (72) and (73), respectively, a system of two equations with two unknowns is obtained, where R and σ^2 are the unknowns whereas $E[X_T]$ and $\mathrm{Var}(X_T)$ are known after solving the multiperiod portfolio problem. Using these equations, the periodic standard deviation σ and the periodic mean return r are found to be

$$\left(\sqrt{(\mathrm{Var}(X_T) + E[X_T]^2)^{1/T} - E[X_T]^{2/T}}, E[X_T]^{1/T}\right), \tag{75}$$

which then can be inserted into the mean–variance graph to get the periodic frontiers.

A question that arises is whether the transformed periodic frontiers will be efficient or not. In order for the portfolios to be efficient, they have to be solutions of the optimization problem

$$\max\{E[X_T]^{1/T} - \omega[(\mathrm{Var}(X_T) + E[X_T]^2)^{1/T} - E[X_T]^{2/T}]\}, \tag{76}$$

where ω is a positive coefficient. In order for the solutions of the multiperiod portfolio optimization problem to be efficient, the objective (76) has to be an increasing function of $E[X_T]$ and a decreasing function of $\mathrm{Var}(X_T)$ which will also lead to its maximization. The derivative with respect to $\mathrm{Var}(X_T)$

$$-\frac{\omega}{T}(\mathrm{Var}(X_T) + E[X_T]^2)^{1/T-1} \tag{77}$$

turns out to be negative whereas the sign of the derivative with respect to $E[X_T]$

$$\frac{1}{T}\Big\{E[X_T]^{1/T-1} - 2\omega E[X_T](\mathrm{Var}(X_T) + E[X_T]^2)^{1/T-1} + 2\omega E[X_T]^{2/T-1}\Big\} \tag{78}$$

is inconclusive which means that the periodic frontiers obtained from this approach are not necessarily efficient. By drawing the periodic frontiers obtained from this approach, it is verified that they are not efficient since they turn out to be convex rather than concave.

A second approach for calculating the periodic mean returns and variances uses the same equation given in (71) for defining the wealth growth. Unlike the previous one, this approach assumes that the periodic returns r_j are small so that we can take

$$X_T = X_0(1 + r_1 + r_2 + \cdots + r_T). \tag{79}$$

Making the same assumption as in the previous approach and taking $X_0 = 1$, we now obtain

$$E[X_T] = 1 + rT, \tag{80}$$

$$\mathrm{Var}(X_T) = \sigma^2 T. \tag{81}$$

This implies that the mean and variance of the periodic rate of return can be obtained as

$$r = \frac{E[X_T] - 1}{T} \tag{82}$$

and

$$\sigma^2 = \frac{\mathrm{Var}(X_T)}{T}. \tag{83}$$

The periodic standard deviation and the periodic mean return are

$$\left(\sqrt{\frac{\mathrm{Var}(X_T)}{T}}, 1+\frac{E[X_T]-1}{T}\right), \tag{84}$$

which can then be inserted into the mean–variance graph to get the periodic frontiers.

As in the previous approach, the periodic frontiers can be checked for whether they are efficient or not. In order for the portfolios which are periodically invested at the random rate to be efficient, they have to be solutions of the optimization problem

$$\max\left\{1+\frac{E[X_T]-1}{T}-\omega\frac{\mathrm{Var}(X_T)}{T}\right\}, \tag{85}$$

where ω is a positive coefficient. After rearranging, the objective function in (85) becomes

$$\frac{1}{T}\max\{E[X_T]-\omega\mathrm{Var}(X_T)\}-\frac{1}{T}+1. \tag{86}$$

This last objective function finally leads to the same form as the objective function of problem $P3(\omega)$ given in (6) since T is constant for a certain investment problem. This result shows that solutions maximizing (85) will at the same time maximize $P3(\omega)$. A consequence of this result is that an investor will get the same optimal portfolios for both problems provided that ω is the same. Moreover, optimal portfolios of the multiperiod problems will also be efficient on the transformed periodic mean–variance graph. Observing the periodic frontiers for different T values, it turns out that it is more advantageous for investors to invest their money for a planning horizon that is as long as possible, since in such a case they will get higher periodic return for the same periodic risk and alternatively incur lower periodic risk for the same periodic return. The intuition behind this observation could be that investors who are willing to tie up their money for a longer time horizon obtain their reward by getting higher periodic returns with the same periodic risk compared to the investors who prefer to invest their money for a shorter time horizon.

9. Numerical illustration

We assume that there is a stochastic market modulated by a Markov chain with only two states $E=\{1,2\}$ which consists of a single risky asset and a riskless asset. The market is in state $i=1$ initially, and we consider the problem of an investor who has a unit wealth for investment at the beginning of the investment horizon that is taken to be $T=5$ periods. The objective is to find the best allocation of investor's wealth among the two assets. The return r_f of the riskless asset, the expected value r and the standard deviation σ of the return of the risky asset for each state are given in Table 1.

The transition matrix Q of the Markov chain is

$$Q=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{3} & \frac{2}{3}\end{bmatrix}.$$

Given these input parameters, $V(i)$ for each state i is computed to be

$$V(1)=[0.0261],\quad V(2)=[0.0153].$$

One can then calculate the vectors $f(i)$, $g(i)$ and $h(i)$ with $i=1,2$ using the definitions given in (10)–(12) as

$$f(i)=\begin{bmatrix}0.9504\\1.0575\end{bmatrix},\quad g(i)=\begin{bmatrix}0.9052\\0.9976\end{bmatrix},$$
$$h(i)=\begin{bmatrix}0.1379\\0.0588\end{bmatrix}.$$

Once we have these vectors together with the transition probability matrix Q, we can use the definitions of a_1, a_2 and b given in (25)–(27) to obtain

$$a_1=\begin{bmatrix}0.7630\\0.8572\end{bmatrix},\quad a_2=\begin{bmatrix}0.9963\\1.1321\end{bmatrix},\quad b=\begin{bmatrix}0.2078\\0.1754\end{bmatrix}.$$

In order to keep track of the optimal investment strategy proposed by each multiperiod problem, a scenario is created in which it is assumed that the Markovian market follows the path $i=1,1,2,1,1$ at $n=0,1,2,3,4$, and that the expected returns given in Table 1 are realized in each period. The scenario analysis includes the computation of the optimal investment policy for each period by using (18) where corresponding optimal γ^* values are going to be used in place of λ/ω for each model. Moreover, the investor's wealth is calculated at the end of each period by using the wealth dynamics equation given in (3). The scenario analysis is performed for each problem separately, and the results are compared.

The multiperiod problem is first solved by assuming that the investor has a quadratic utility function. The range of parameter A turns out to be between 0

Table 1
Expected returns and variances for one risky asset case

State i	$r_f(i)$	$r(i)$	$\sigma(i)$
1	1.05	1.11	0.15
2	1.06	1.09	0.12

and 0.383, where $A^* = 0.383$ as given in (56). For an arbitrary A value such as $A = 0.35$, the utility function

$$U(\gamma) = -0.0364\gamma^2 + 0.2078\gamma + 0.4143$$

is obtained by using (51). The optimal γ^* given in (52) turns out to be 2.857. The expectation and the standard deviation of the final wealth are then found to be 1.357 and 0.062, respectively, using (28) and (30). According to the scenario given, the optimal investment strategy proposed by the multiperiod quadratic utility model with $A = 0.35$ and the related wealth at the end of each period are given in Table 2 under QU ($A = 0.35$).

The next problem to solve is the coefficient of variation problem with the utility function

$$U(\gamma) = \frac{0.2078\gamma + 0.7630}{\sqrt{0.0607\gamma^2 - 0.3171\gamma + 0.4141}}$$

obtained using (43). The optimal γ^* given in (45) turns out to be 2.611. The expected value of the final wealth is then found to be 1.306 with a standard deviation of 0.012 using (28) and (30). According to the scenario given, the optimal investment strategy proposed by the multiperiod coefficient of variation model and the related wealth at the end of each period are given in Table 2 under CV. This table shows that in order to minimize the relative dispersion of the final wealth, which also corresponds to the classical trade-off between minimizing risk and maximizing return without specifying any additional parameter, almost all of the current wealth has to be invested in the risk-free asset in each period.

The same input parameters are also used for the safety-first problem. The objective of this problem is to minimize the upper bound of the probability that the final wealth is below a preselected disaster level. It turns out that the problem can be solved optimally for a safety-first investor requiring a minimal return of up to $k^* = 1.306$ at the end of five periods which is computed using (68). For different applicable k values, the problem is solved and $E_i[X_T]$ and $\sqrt{\mathrm{Var}_i(X_T)}$ are put on a graph shown in Fig. 1.

Assuming a disaster level of 1.3 and using the formula (65), the utility function is found to be

$$U(\gamma) = \frac{0.2078\gamma + 0.7630 - 1.3}{\sqrt{0.0607\gamma^2 - 0.3171\gamma + 0.4141}}.$$

The k values in Fig. 1 are in the allowable range which is smaller than k^* so that the problem has an explicit solution. For $k = 1.3$, the optimal γ^* given in (67) is equal to 2.701. The expectation of X_T turns out to be 1.324 using (28) and the standard deviation of X_T is found to be 0.025 using (30). According to the scenario given, the optimal investment strategy proposed by the multiperiod safety-first model with $k = 1.3$ and the related wealth at the end of each period are given in Table 2 under SF ($k = 1.3$). It can be seen that 6–10% of the available money is periodically invested in the risky asset in order to minimize the probability that the final wealth is below the required return. The disaster level was intentionally selected high so that it was not possible for the investor to reach this wealth level trivially by only investing in the risk-free asset. Moreover, the final wealth is found to be 1.31 which is greater than which was required.

Before comparing the results of the three types of utility functions, a second disaster level for the safety-first problem is selected for further analysis of the proposed optimal policy. For $k = 1.1$, the optimal γ^* given in (67) is equal to 2.613. The expectation of X_T turns out to be 1.306 and the standard deviation of X_T is 0.012 by using (28) and (30), respectively. According to the scenario given, the optimal investment strategy proposed by the multiperiod safety-first model with $k = 1.1$ and the related wealth at the end of each period are given in Table 2 under SF ($k = 1.1$). An important observation related to the results given in this table is that some portion of the current wealth is still invested in

Table 2
Comparison of optimal policies and investor's wealth for one risky asset case

n	i	QU ($A = 0.35$)		CV		SF ($k = 1.3$)		SF ($k = 1.1$)	
		$u_n(i,x)$	X_{n+1}	$u_n(i,x)$	X_{n+1}	$u_n(i,x)$	X_{n+1}	$u_n(i,x)$	X_{n+1}
0	1	23%	1.06	0%	1.05	8%	1.05	0%	1.05
1	1	22%	1.13	2%	1.10	9%	1.11	2%	1.10
2	2	16%	1.20	0%	1.17	6%	1.18	0%	1.17
3	1	21%	1.28	2%	1.23	9%	1.25	2%	1.23
4	1	21%	1.35	3%	1.29	10%	1.31	3%	1.29

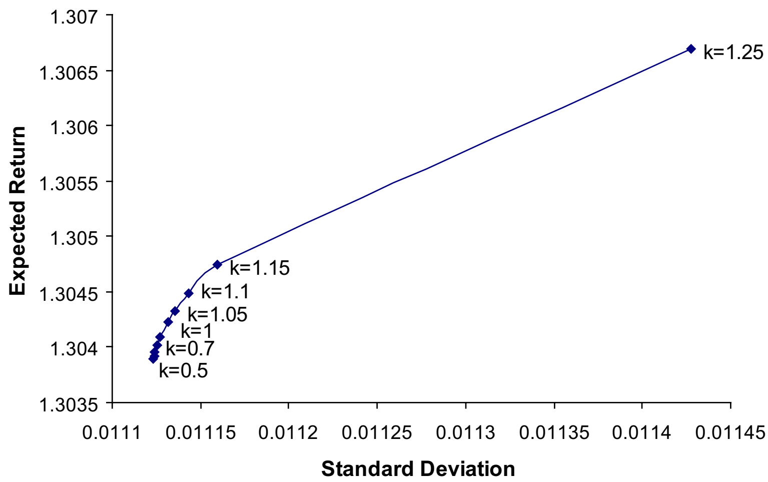

Fig. 1. Efficient frontier for safety-first problem with one risky asset.

the risky asset even when the required level 1.1 is reached at the end of the second period. If all of this money is invested in the risk-free asset for the remaining three periods, the resulting terminal wealth would be equal to $1.1(1.06)(1.05)^2 = 1.286$ under this scenario. Furthermore, this alternative investment policy would eliminate all the uncertainty coming from investing in the risky asset in the remaining periods. This implies that investing in the risk-free asset for the remaining periods is a better policy than the one given in the table since it reaches a higher return while not increasing the risk at all. The reason for obtaining a policy which does not lead to an optimal result according to our proposed solution procedure is that we are minimizing the upper bound of the undesired probability $P\{X_T \leqslant 1.1\}$ and not the probability itself which may have a different optimal policy as in this case.

Table 2 gives a comparison of the optimal policies of each problem. As mentioned in Section 6, higher A value means higher risk aversion. Therefore the parameter A of QU is chosen to be close to the upper bound 0.383, i.e. 0.35, so that the investor with this utility is relatively more risk-averse which is also assumed to be the case for investors trying to minimize the coefficient of variation of their final wealth and also for safety-first investors. From Table 2 it can be easily seen that the highest portion of available wealth is invested in the risky asset for QU which logically leads to the highest expected final wealth at the end of the investment horizon.

CV having a single objective utility function reveals an identical optimal policy as that of SF with $k = 1.1$ investing very little in the risky asset. Comparing the solutions of CV and SF ($k = 1.1$) in detail reveals that these two problems yield identical values for the expectation and variance of the final wealth as well. This result is not very unexpected if one compares the objective functions of these problems given in (42) and (63). The objectives differ from each other by the additional term $k/\sqrt{\mathrm{Var}_i(X_T)}$ which can be neglected as k becomes smaller and smaller. A further analysis done by decreasing the disaster level k until a value of zero is reached shows that the optimal solution of the safety-first problem converges to the optimal solution of the coefficient of variation problem as k goes to zero.

SF with $k = 1.3$ suggests higher investment in the risky asset. This is due to the fact that the required level of the final wealth can only be reached when the risky asset having a higher expected return than that of the risk-free asset is used, although in small quantities. The optimal policy of SF with $k = 1.1$ on the other hand shows that almost all of the current wealth is invested in the risk-free asset. This result is logical since a final wealth of 1.1 can be reached more easily compared to a final wealth of 1.3 when money is lent at the risk-free rate.

As another illustration, the multiperiod safety-first problem is solved for different T values using the same input parameters for the means, variances and the transition matrix. Assuming that the initial state is $i = 1$, corresponding efficient frontiers given in Fig. 2 are obtained.

As given in (47), (53) and (70), there exists a one-to-one relationship between the solutions of

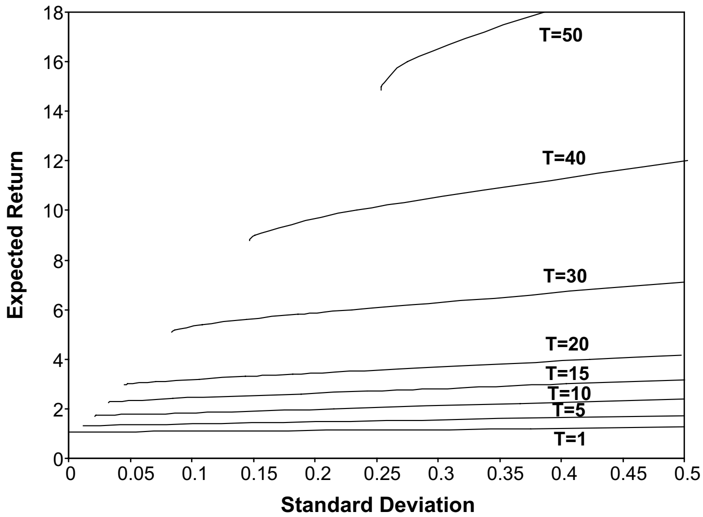

Fig. 2. Efficient frontiers for different T values.

all multiperiod problems discussed here since they are derived from the same classical mean–variance formulation. This means that same efficient frontiers in Fig. 2 are obtained for every model considered, except for the case where there is only one efficient point as the single solution of the coefficient of variation problem or where there is some condition set on the input parameters. There are some immediate conclusions that can be made about the characteristics of the efficient frontiers in Fig. 2 with respect to changes in T. As T increases, a much higher return is expected for the same standard deviation since there is more time to invest so that the initial money will accumulate to a higher level without increasing the risk. Moreover, to reach the same level of expected return, a much smaller standard deviation is needed for a longer investment horizon. This is due to the fact that since there is more time to invest, the investment in less risky assets with smaller returns will be enough to reach the required level, whereas more risky assets have to be used in order to reach the same level in a shorter time period. One particular reason for analyzing the efficient frontiers was to see whether they converge to a common frontier as T increases, which is not the case here as can be seen in Fig. 2. If there were such a common frontier, this would imply that there exists a stationary policy that is used for large values of the investment horizon T.

Since the safety-first problem is of practical importance for some investors who want to avoid a possible disaster level at the end of the planning horizon, this problem is investigated more deeply by making a sensitivity analysis. This sensitivity analysis is accomplished by changing the mean returns of both the risky and the risk-free asset one by one, and then observing the changes in the optimal policy u, which corresponds to the percentage of available money to be invested in the risky asset.

Case 1 corresponds to the input parameters in Table 1 with $k = 1.2$. Each case thereafter includes an additional change compared to the previous case. In Case 2, r_f in state 1 is increased from 1.05 to 1.12. In Case 3, r_f in state 2 is increased from 1.06 to 1.1. In Case 4, r in state 1 is decreased from 1.11 to 1.08. In Case 5, r in state 2 is decreased from 1.09 to 1.07. The aim of this analysis is to see how the optimal policy is changing as the risk-free asset is made more advantageous to investors in both states. The risk-free asset has no risk and even a better return compared to the risky asset after the input parameters are modified in different cases. The optimal policies for these cases are given in Table 3.

The policy formula (18) depends on the current state of the stochastic market. After the return of the risk-free asset for state 1 is increased for all cases, the amount invested in risky asset at the beginning of the planning horizon decreases and it even turns out to be negative, meaning that the risky asset is sold short in order to invest more in the risk-free asset. The same effect is also observed after Case 3 where the return of the risk-free asset for state 2 is increased. Since the state is 2 for $n = 2$, the risky asset is again sold short to invest more in

Table 3
Optimal policies for one risky asset case

	Case 1	Case 2	Case 3	Case 4	Case 5
u_0	0.004007	−0.00662	−0.00050	−0.00205	−0.00216
$E_1[X_1]$	1.050240	1.120066	1.120005	1.120082	1.120086
u_1	0.018909	0.009449	0.005316	0.020678	0.019903
$E_1[X_2]$	1.103887	1.25438	1.254352	1.253665	1.253701
u_2	0.002544	0.049079	−0.00131	−0.00114	−0.00520
$E_1[X_3]$	1.170197	1.331115	1.379801	1.379043	1.379227
u_3	0.024625	0.015009	0.008488	0.033098	0.03086
$E_1[X_4]$	1.230184	1.490699	1.545292	1.543204	1.543499
u_4	0.037228	0.035001	0.016148	0.060467	0.057473
$E_1[X_5]$	1.293927	1.669232	1.730566	1.725970	1.726420

the risk-free asset for all cases including and after Case 3.

It is important to note here that the expected final wealth $E_1[X_5]$ at the end of the investment horizon is much higher than the disaster level which was specified at the beginning of the planning horizon to be 1.2. This is due to the fact that while minimizing the probability of falling below 1.2, the safety-first problem at the same time aims to maximize the probability of having a final wealth greater than the prespecified level k given by the investor. Even though the risk-free asset is more advantageous to investors after the returns are modified, there is still some investment in the risky asset. The reason is that the proposed policy can reach the required level of final wealth, 1.2 in this case, even when there is an investment in the risky asset. Moreover, since the risky asset can have a return higher than its expected value in reality, this investment has the potential of realizing a higher final wealth while still satisfying the objective of not falling below the disaster level.

The multiperiod portfolio optimization model considered in this paper can be extended in several directions. A natural extension is one where the market state is not observable and there is imperfect information flow. This requires the use of so-called hidden Markov models to describe the stochastic market. Another challenge would be to analyze Bayesian models where the parameters used in our analysis are not known with certainty, but they are also random with some prior distribution which should be updated as data becomes available in time. In our paper, we analyzed multiperiod portfolio optimization problems that do not include any constraints. It may be necessary to put some constraints or include transaction costs, which could have a significant effect on the optimal solution of the problem. Another interesting issue to analyze can be to find conditions, under which long-term investment strategies having stationary optimal portfolios will exist. Dealing with other utility functions for the multiperiod portfolio optimization is another future research topic. The extension of our model to continuous time is also worth analyzing, considering the fact that many investors change their portfolios continuously rather than at discrete points in time.

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