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Decision Support

# Portfolio selection in stochastic markets with HARA utility functions

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ABSTRACT

In this paper, we consider the optimal portfolio selection problem where the investor maximizes the expected utility of the terminal wealth. The utility function belongs to the HARA family which includes exponential, logarithmic, and power utility functions. The main feature of the model is that returns of the risky assets and the utility function all depend on an external process that represents the stochastic market. The states of the market describe the prevailing economic, financial, social, political and other conditions that affect the deterministic and probabilistic parameters of the model. We suppose that the random changes in the market states are depicted by a Markov chain. Dynamic programming is used to obtain an explicit characterization of the optimal policy. In particular, it is shown that optimal portfolios satisfy the separation property and the composition of the risky portfolio does not depend on the wealth of the investor. We also provide an explicit construction of the optimal wealth process and use it to determine various quantities of interest. The return-risk frontiers of the terminal wealth are shown to have linear forms. Special cases are discussed together with numerical illustrations.



## 1. Introduction

One of the most important problems faced by investors involve the allocation of their wealth among different investment opportunities in a market consisting of risky assets. Determination of optimal portfolios is a rather complex problem depending on the objective of the investor. In this setting, the objective of the investor is to maximize the expected value of a utility function of the terminal wealth. The risk preferences of the investor is given and measured by the utility function. The most widely used measures of risk aversion were introduced by Pratt (1964) and Arrow (1965). Mossin (1968) examined some utility functions that lead to myopic policies, and Merton (1969) considered special utility functions with logarithmic and power structures. Hakansson (1971), in discrete-time setting, investigates the optimization of logarithmic and power utility in a random market. More recently, Dokuchaev (2007) considers a model where the expected utility of the terminal wealth with power and logarithmic utility functions are maximized in a discrete-time market with serial correlations. Also, Breuer and Gürtler (2006) investigate the performance of funds using different utility functions.

In most of the multiperiod problems, the rates of return of the assets during consecutive periods are assumed to be uncorrelated. In a realistic setting, this is not correct and the dependence among the rates of return in consecutive periods should also be considered. This dependence or correlation is often achieved through a stochastic market process that affects all deterministic and probabilistic parameters of the model. A tractable and realistic approach is provided by using a Markov chain that represents the economic, financial, social, political and other factors which affect the returns of the assets. The use of a modulating stochastic process as a source of variation in the model parameters and of dependence among the model components has proved to be quite useful in operations research and management science applications. The concept was introduced by Çınlar and Özekici (1987) in a reliability setting where the failure rate and hazard functions of a device depend on the prevailing environmental conditions. There is now considerable amount of literature on modulation in a variety of applications. An example in queueing is provided by Prabhu and Zhu (1989) where customer arrival and service rates are modulated by a Markov process. Song and Zipkin (1993) consider an inventory model with a demand process that fluctuates with respect to stochastically changing economic conditions. A general discussion on the idea can be found in Özekici (1996). The interested reader is referred to Asmussen (2000) and Rolski et al. (1999) for further applications in queueing, insurance and finance. Çakmak and Özekici (2006) applied the idea to multiperiod mean–variance portfolio optimization problem. Considering a market with one riskless and some risky assets, a multiperiod mean–variance formulation is developed. An auxiliary problem generating the efficient frontier is used to eliminate nonseparability in the sense of dynamic programming, and an analytical optimal solution is obtained. Following their work, Çelikyurt

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and Özekici (2007) analyzed the multiperiod mean–variance model by considering the safety-first approach, coefficient of variation of the terminal wealth and quadratic utility functions. Using dynamic programming, efficient frontiers and optimal portfolio management policies are obtained.

Our primary aim in this paper is to extend the discussion in Çanakoğlu and Özekici (2009) who consider the portfolio selection problem in a stochastic market with exponential utility functions. We will extend their utility based approach to multiperiod portfolio optimization by considering investors with logarithmic and power utility where we suppose that the asset returns all depend on a stochastic market depicted by a Markov chain. This extension completes the analysis for the hyperbolic absolute risk aversion (HARA) class. The stochastic structure of the market is described in Section 2 and the dynamic programming formulation is provided in Section 3. The case involving logarithmic utility functions is investigated in Section 4 where explicit characterizations are obtained for the optimal policy, and the optimal wealth process is also analyzed. This analysis is repeated for power utility functions in Section 5 and an illustration is given in Section 6. The detailed proofs of the main results and some of the technical derivations are excluded from the main manuscript, and they can be found in the appendix.

## 2. The stochastic market

The returns of risky assets in a financial market are random and there are various underlying economic, financial, social, political and other factors that affect their distributions in one way or another. As the state of a market changes over time, the returns will change accordingly. It is fair to say that in today's financial markets most of the risks, or variances of asset returns, are due to the changes in local or global factors. Investment decisions are affected by these factors as well as the correlation among asset returns. Modeling a stochastic financial market by a Markov chain is a reasonable approach and this idea dates back to Pye (1966). In the continuous-time setting, Norberg (1995) considers an interest rate model that is modulated by a Markov process. Recently there is growing interest in the literature to use a stochastic market process in order to modulate various parameters of the financial model to make it more realistic. Bielecki et al. (1999), Bielecki and Pliska (1999), Massi and Stettner (1999), Stettner (1999), and Nagai and Peng (2002) provide examples on risk-sensitive portfolio optimization with observed, unobserved and partially observed states in Markovian markets. Continuous-time Markov chains with a discrete state space are used in a number of papers including, for example, Bäuerle and Rieder (2004), and Zhang (2001) to modulate model parameters in portfolio selection and stock trading problems. There are also models where only one of the parameters in modulated. Models of stochastic interest rates with some sort of a Markovian structure are also quiet common as in Korn and Kraft (2001) and Elliott and Mamon (2003), among others.

Suppose that the state of the market in period $n$ is denoted by $Y_n$ and $Y = \{Y_n; n = 0, 1, 2, \ldots\}$ is a Markov chain with a discrete state space $E$ and transition matrix $Q$. Let $R(i)$ denote the random vector of asset returns in any period given that the stochastic market is in state $i$. The means, variances and covariances of asset returns depend only on the current state of the stochastic market. The market consists of one riskless asset with known return $r_f(i)$ and standard deviation $\sigma_f(i) = 0$, and $m$ risky assets with random returns $R^n(i) = (R_1^n(i), R_2^n(i), \ldots, R_m^n(i))$ in period $n$ if the state of the market is $i$. We assume that the random returns in consecutive periods are conditionally independent given the market states. In other words, $R^n(i)$ is independent of $R^k(j)$ for all periods $k \neq n$ and states $i, j$. Moreover, $R^n(i)$ and $R^k(i)$ are independent and identically distributed random vectors whenever $k \neq n$. This implies that the distributions of the asset returns depend only on the state of the market independent of time. For this reason, we will let $R(i) = R^n(i)$ denote the random return vector in any period $n$ to simplify our notation.

We let $r_k(i) = E[R_k(i)]$ denote the mean return of the $k$th asset in state $i$ and $\sigma_{kj}(i) = \mathrm{Cov}(R_k(i), R_j(i))$ denote the covariance between $k$th and $j$th asset returns in state $i$. The excess return of the $k$th asset in state $i$ is $R_k^e(i) = R_k(i) - r_f(i)$. It follows that

$$r_k^e(i) = E\left[R_k^e(i)\right] = r_k(i) - r_f(i), \tag{1}$$
$$\sigma_{kj}(i) = \mathrm{Cov}(R_k^e(i), R_j^e(i)). \tag{2}$$

Our notation is such that $r_f(i)$ is a scalar and $r(i) = (r_1(i), r_2(i), \ldots, r_m(i))$, $r^e(i) = (r_1^e(i), r_2^e(i), \ldots, r_m^e(i))$ are column vectors for all $i$. For any column vector $z$, $z'$ denotes the row vector representing its transpose.

We define the matrix

$$V(i) = E\left[R^e(i)R^e(i)'\right] = \sigma(i) + r^e(i)r^e(i)'$$

for any state $i$. Note that the covariance matrix $\sigma(i)$ is positive definite for all $i$ so one can easily see that $V(i)$ is also positive definite.

We let $X_n$ denote the amount of investor's wealth at period $n$ and the vector $u = (u_1, u_2, \ldots, u_m)$ denotes the amounts invested in risky assets $(1, 2, \ldots, m)$. Given any investment policy, the stochastic evolution of the investor's wealth follows the so-called wealth dynamics equation

$$X_{n+1}(u) = R(Y_n)'u + \left(X_n - 1'u\right)r_f(Y_n) = r_f(Y_n)X_n + R^e(Y_n)'u, \tag{3}$$

where $1 = (1, 1, \ldots, 1)$ is the column vector consisting of 1's.

We will use the notation $E_i[Z] = E[Z \mid Y_0 = i]$ and $\mathrm{Var}_i(Z) = E_i[Z^2] - E_i[Z]^2$ to denote the conditional expectation and variance of any random variable $Z$ given that the initial market state is $i$. Throughout this paper, unless otherwise stated, a vector $z$ is a column vector so that its transpose, denoted by $z'$, is always a row vector. The assumptions regarding the model formulation can be summarized as follows: (a) there is unlimited borrowing and lending at the prevailing return of the riskless asset in any period, (b) short selling is allowed for all assets in all periods, (c) no capital additions or withdrawals are allowed throughout the investment horizon, and (d) transaction costs and fees are negligible.

## 3. Dynamic programming with utility functions

A utility function $U$ is a non-decreasing real valued function defined on the real numbers. Pratt (1964) and Arrow (1965) suggest the ratio

$$r(x) = -\frac{U''(x)}{U'(x)} \tag{4}$$

as a measure of absolute risk aversion. Using (4), hyperbolic absolute risk aversion (HARA) is described by the absolute risk aversion function

$$r(x) = \frac{1}{a + bx}, \tag{5}$$

where HARA utility functions with an identical parameter $b$ belong to the same class. This also implies that the risk tolerance function of the investor, defined as the inverse of the risk aversion function, has the linear form $a + bx$ for HARA utility functions. It should be noted that $b = 0$ refers to constant absolute risk aversion where the utility function has an exponential form and, for $b = 1$, the utility function is logarithmic. For other values of $b$ the utility function is in the form of a power function. The exponential case with $b = 0$ is analyzed in detail by Çanakoğlu and Özekici (2009) and our primary objective is to extend their discussion to the general HARA case with $b \neq 0$. We assume that the utility of the investor also depends on the market state so that the utility function is $U(i,x)$ if the state of the market is $i$ and the wealth is $x$ at the terminal time.

Dynamic programming is the method used in the derivation of the optimal solution of the multiperiod portfolio selection problem that can be stated as

$$\max_u E_i[U(Y_T, X_T)],$$

where the investor maximizes his expected utility of the terminal wealth $X_T$ at some terminal time $T$. Let $g_n(i,x,u)$ denote the expected utility using the investment policy $u$ in period $n$ and the optimal policies from period $n+1$ to period $T$ given that the market is in state $i$ and the amount of money available for investment is $x$ at period $n$. Then,

$$v_n(i,x) = \max_u g_n(i,x,u)$$

is the optimal expected utility using the optimal policy given that the market is in state $i$ and the amount of money available for investment is $x$ at period $n$. Using the dynamic programming principle

$$g_n(i,x,u) = E[v_{n+1}(Y_{n+1}, X_{n+1}(u))|Y_n = i, X_n = x]$$

and we can write the dynamic programming equation (DPE) as

$$v_n(i,x) = \max_u E[v_{n+1}(Y_{n+1}, X_{n+1}(u))|Y_n = i, X_n = x],$$

which can be rewritten as

$$v_n(i,x) = \max_u \sum_{j \in E} Q(i,j) E\left[v_{n+1}\left(j, r_f(i)x + R^e(i)'u\right)\right] \tag{6}$$

for $n = 0, 1, \ldots, T-1$ with the boundary condition $v_T(i,x) = U(i,x)$ for all $i$ and $x$. The solution for this problem is found by solving the DPE recursively for different cases involving logarithmic and power utility functions.

## 4. Logarithmic utility function

In this section, we assume that the utility of the investor in state $i$ is given by the logarithmic function

$$U(i,x) = \begin{cases} K(i) + C(i)\log(x+\beta), & x+\beta > 0, \\ -\infty, & x+\beta \leqslant 0, \end{cases} \tag{7}$$

with $C(i) > 0$ where we can easily see that Pratt–Arrow's measure of absolute risk aversion is simply equal to $r(x) = 1/(\beta + x) > 0$ for all $i$ so that $b = 1$ and $a = \beta$ in (5). Note that $\beta$ is the same for all market states so that risk classification of the investor does not depend on the stochastic market. Similarly, we assume that the return for the riskless asset is the same for all market states so that $r_f(i) = r_f$ for all $i$.

We will first consider an optimization problem of the form

$$\max_u E\left[\log\left(R^{e'}u + c\right)\right], \tag{8}$$

where $c > 0$ is any constant and $R^e$ is any random vector. Now, let

$$A(c) = \left\{u : P\left\{R^{e'}u + c > 0\right\} = 1\right\}$$

be the set of all investment policies that gives finite expected utility so that $|E[\log(R^{e'}u + c)]| < +\infty$ for $u \in A(c)$. It can be seen that $u = (u_1, u_2, \ldots, u_n) = (0, 0, \ldots, 0) \in A(c)$ satisfies this condition trivially for all $c > 0$. So, $A(c)$ is not empty. Also, let $u$, $w \in A(c)$, then $R^{e'}u + c > 0$, and $R^{e'}w + c > 0$ implies that

$$\lambda R^{e'}u + (1-\lambda)R^{e'}w + c > 0$$

so that $\lambda u + (1-\lambda)w \in A(c)$ for all $0 \leqslant \lambda \leqslant 1$. Therefore, the solution set $A(c)$ is nonempty and convex. The gradient vector of the objection function $g(u) = E[\log(R^{e'}u + c)]$ is given by

$$\nabla_k g(u) = \frac{\partial g(u)}{\partial u_k} = E\left[\frac{R_k^e}{R_k^{e'}u + c}\right]$$

while the Hessian matrix is

$$\nabla^2_{k,l} g(u) = \frac{\partial^2 g(u)}{\partial u_k \partial u_l} = -E\left[\frac{R^e_k R^e_l}{(R^{e\prime}_k u + c)^2}\right]$$

for all $k, l$.

The first order optimality condition to find the optimal solution of (8) is obtained by setting the gradient vector equal to zero so that

$$E\left[\frac{R^e_k}{R^{e\prime} u + c}\right] = 0 \quad (9)$$

for all $k$.

Let $z = (z_1, \ldots, z_m)$ be any non-zero column vector where $z_i$'s are real numbers. Then, one can see that

$$z'\nabla^2 g(u) z = -E\left[\left(z_1 R^e_1 + z_2 R^e_2 + \cdots + z_m R^e_m\right)^2 / \left(R^{e\prime} u + c\right)^2\right] \leqslant 0.$$

Thus, the Hessian matrix $\nabla^2 g(u)$ is negative semi-definite and if there is a solution $u \in A(c)$ satisfying the first order condition (9), it must be optimal. Throughout this paper, we assume that the excess returns are such that there is a solution of the first order condition (9) in $A(c)$ for all $\{R^e(i)\}$ and $c > 0$. The appendix includes some insight on this assumption.

We will now show that the log utility function is meaningful for an investor with $x_n + \beta / r_f^{T-n} > 0$ at period $n$ where $x_n$ is the wealth in period $n$. Suppose $x_n + \beta / r_f^{T-n} \leqslant 0$; then using the strategy of only buying risk-free bonds in each period, the investor will have a terminal wealth of $r_f^{T-n} x_n$ with utility equal to $-\infty$ since $r_f^{T-n} x_n + \beta = r_f^{T-n}(x_n + \beta / r_f^{T-n}) \leqslant 0$. For any other strategy, the final wealth should satisfy

$$P\{X_T \leqslant r_f^{T-n} x_n\} > 0$$

according to no arbitrage condition. Otherwise, if $P\{X_T \leqslant r_f^{T-n} x_n\} = 0$ (or $P\{X_T > r_f^{T-n} x_n\} = 1$), then an arbitrage opportunity exists by selling bonds. We can therefore write

$$P\{X_T + \beta \leqslant r_f^{T-n} x_n + \beta\} > 0$$

and

$$P\{X_T + \beta \leqslant 0\} > 0$$

which means that the investor has $-\infty$ terminal utility for any investment strategy and any policy is therefore optimal.

At the beginning, if $x_0 + \beta / r_f^T \leqslant 0$, then any policy leads to $-\infty$ utility. We therefore suppose that $x_0 + \beta / r_f^T > 0$. Then, the policy of investing only on the risk-free asset for $n$ periods leads to $x_n = x_0 r_f^n$ and

$$x_n + \frac{\beta}{r_f^{T-n}} = x_0 r_f^n + \frac{\beta}{r_f^{T-n}} = r_f^n \left(x_0 + \frac{\beta}{r_f^T}\right) > 0.$$

Since the investor selects a policy optimally to maximize the expected terminal utility, we can assume without loss of generality that $x_n + \beta / r_f^{T-n} > 0$ (or $r_f^{T-n} x_n + \beta > 0$) for any $n = 0, 1, \ldots, T$.

**Theorem 1.** *Let the utility function of the investor be the logarithmic function (7) and suppose that the riskless asset return does not depend on the market state. Then, the optimal solution of the dynamic programming Eq. (6) is*

$$v_n(i, x) = K_n(i) + C_n(i) \log(x + \beta_n)$$

*and the optimal portfolio is*

$$u^*_n(i, x) = \alpha(i)(r_f x + \beta_{n+1}) \quad (10)$$

*where*

$$\beta_n = \frac{\beta}{r_f^{T-n}}, \quad K_n = Q^{T-n} K + \left(\sum_{m=0}^{T-n-1} Q^m \widehat{Q}_\alpha Q^{T-n-1-m}\right) C, \quad C_n = Q^{T-n} C \quad (11)$$

*and*

$$\widehat{Q}_\alpha(i, j) = E\left[\log\left(r_f\left(1 + R^e(i)'\alpha(i)\right)\right)\right] Q(i, j)$$

*for $n = 0, 1, \ldots, T-1$; where $\alpha(i)$ satisfies*

$$E\left[\frac{R^e_k(i)}{1 + R^e(i)'\alpha(i)}\right] = 0 \quad (12)$$

*for all assets $k = 1, 2, \ldots, m$ and all i.*

**Proof.** See Appendix B. □

In Theorem 1, we have found a closed-form solution (10) for the optimal portfolio. We can further characterize the optimal policy if the return distributions are discrete. Suppose that there is a single risky asset and the return distribution is such that excess return of the asset is equal to $u(i)$ with probability $p(i)$ and $d(i)$ with probability $(1 - p(i))$ with $d(i) < 0 < u(i)$. This assumption is required for the no arbitrage requirement to hold. From (12), the optimality condition is

$$p\left(\frac{u(i)}{1 + u(i)\alpha(i)}\right) + (1 - p(i))\left(\frac{d(i)}{1 + d(i)\alpha(i)}\right) = 0,$$

which can be solved to find

$$\alpha(i) = -\frac{d(i) + p(i)(u(i) - d(i))}{d(i)u(i)} = -\frac{r^e(i)}{d(i)u(i)}.$$

Then, we can see that $\alpha(i) = 0$ if and only if $r^e(i) = 0$. Also, $\alpha(i) > 0$ if $r^e(i) > 0$ and $\alpha(i) < 0$ if $r^e(i) < 0$.

Suppose that there is a single risky asset and the excess return of the asset is equal to $a_k(i)$ with probability $p_k(i)$ for $k = 1, 2, \ldots, n_i$ in state $i$ where $\sum_{k=1}^{n_i} p_k(i) = 1$. Then, from (12), the optimality condition is

$$\sum_{k=1}^{n_i} p_k(i)\left(\frac{a_k(i)}{1 + a_k(i)\alpha(i)}\right) = 0. \tag{13}$$

It is clear that (13) is a polynomial equation with power $n_i$, and solving this will give the optimal $\alpha(i)$ values.

Note that the structure of the optimal solution in (10) is such that the optimal distribution of wealth invested in the risky assets depend only on the state of the market independent of time. If the market is in state $i$ in period $n$, then the total amount of money invested on the risky assets is

$$1'u_n^*(i,x) = 1'\alpha(i)(r_f x + \beta_{n+1}) = \left(r_f x + \frac{\beta}{r_f^{T-(n+1)}}\right)\sum_{k=1}^{m}\alpha_k(i)$$

and the proportion on wealth allocated for asset $k$ in the risky portfolio is

$$w_k(i) = \frac{\alpha_k(i)}{\sum_{k=1}^{m}\alpha_k(i)}, \tag{14}$$

which is totally independent of both time $n$ and wealth $x$. The optimal policy specified by (10) is not static in time since it depends on $n$, and it is not memoryless in wealth since it depends on $x$. However, (14) clearly indicates that the composition of the risky part of the optimal portfolio only depends on the market state. The risky portfolio composition is both static and memoryless. It satisfies the separation property in the sense that it represents the single fund of risky assets that logarithmic investors choose. The amount of total wealth allocated for risky assets depend on the level of wealth, but the composition of the risky assets depend only on the market state. This composition, however, is random due to the randomly changing market conditions in time. Our results are of course consistent with similar work in the literature on logarithmic utility functions, but the stochastic market approach makes our model more realistic without causing substantial difficulty in the analysis. Another important observation is that the structure of the optimal portfolio is not affected by the transition matrix $Q$ of the stochastic market. It only depends on the joint distribution of the risky asset returns as prescribed by (12) in a given market state, irrespective of future expectations on the stochastic market.

### *4.1. Evolution of wealth and the logarithmic frontier*

The evolution of the wealth process $X$ using the optimal policy can be analyzed by the wealth dynamics equation

$$X_{n+1} = r_f X_n + R^e(Y_n)'u_n^*(Y_n, X_n) = r_f X_n + R^e(Y_n)'\alpha(Y_n)\left(r_f X_n + \beta_{n_{+1}}\right) = r_f X_n(1 + A(Y_n)) + r_f^{n+1-T}A(Y_n)\beta, \tag{15}$$

where we define $A(i)$ as the random variable

$$A(i) = R^e(i)'\alpha(i) = \sum_{k=1}^{m}\alpha_k(i)R_k^e(i) \tag{16}$$

for any state $i$.

Note that the wealth process satisfies

$$X_{n+1} + \beta_{n+1} = (1 + A(Y_n))r_f(X_n + \beta_n),$$

since $\beta_{n+1} = r_f\beta_n$. Recall that we initially assumed that $X_0 + \beta_0 = x_0 + \beta/r_f^T > 0$ since the objective function value is $-\infty$ for any policy otherwise. Now, suppose that $X_n + \beta_n > 0$ for some $n$. According to our assumption on excess returns we know that the optimal investment policy $u^* \in A(c)$ satisfies the condition

$$P\left\{R^{e'}(Y_n)u^* + c > 0\right\} = 1$$

with $c = r_f X_n + \beta_{n+1} = r_f(X_n + \beta_n) > 0$. Since $u^* = \alpha(Y_n)(r_f X_n + \beta_{n+1})$, we get

$$P\left\{R^{e'}(Y_n)\alpha(Y_n)\left(r_f X_n + \beta_{n+1}\right) + r_f X_n + \beta_{n+1} > 0\right\} = 1$$

and

$$P\left\{(1 + A(Y_n))r_f(X_n + \beta_n) > 0\right\} = P\{X_{n+1} + \beta_{n+1} > 0 = 1.$$

This argument clearly shows that if $X_0 + \beta_0 > 0$ as we initially assume, then $X_n + \beta_n > 0$ for all $n$ using the optimal policy. We are therefore justified in supposing implicitly that this condition is always satisfied before the statement of Theorem 1.

It is clear that $A(i)$ is a linear combination of the excess returns of the risky assets with mean

$$a(i) = E[A(i)] = r^e(i)'\alpha(i) \tag{17}$$

and second moment

$$s(i) = E\left[A(i)^2\right] = E\left[\alpha(i)'R^e(i)R^e(i)'\alpha(i)\right] = \alpha(i)'V(i)\alpha(i), \tag{18}$$

which gives the variance

$$\mathrm{Var}(A(i)) = \alpha(i)'V(i)\alpha(i) - \alpha(i)'r^e(i)r^e(i)'\alpha(i) = \alpha(i)'\sigma(i)\alpha(i).$$

Our construction of the optimal wealth process in Appendix C shows that the optimal wealth process satisfies

$$X_n = r_f^n X_0 \prod_{k=0}^{n-1}(1 + A(Y_k)) + r_f^{n-T}\beta\mathbb{C}_n(A(Y_0, Y_1, \ldots, Y_{n-1})), \tag{19}$$

where $\mathbb{C}_n$ is given by

$$\mathbb{C}_n(x_1, x_2, \ldots, x_n) = \prod_{k=1}^{n}(1 + x_k) - 1.$$

We can clearly see from C.6 and C.8 in Appendix C that both the mean and the standard deviation of $X_T$ depend linearly on $\beta$. This shows that the logarithmic frontier is the straight line

$$E_i[X_T] = r_f^T x_0 + \left(\frac{m_l(i,T)}{v_l(i,T)}\right)\mathrm{SD}_i(X_T), \tag{20}$$

where $SD_i(X_T) = \sqrt{\mathrm{Var}_i(X_T)}$, and $m_l(i,T)$ and $v_l(i,T)$ are given explicitly by C.10 and C.12 respectively. In other words, the expected value and standard deviation of the terminal wealth fall on this straight line when they are calculated and plotted for different values of $\beta$. Also, it cuts the zero-risk level at $E_i[X_T] = r_f^T x_0$ as expected. The reason for this is that for zero-risk level the investor puts all of his money on the riskless asset. The return of the riskless asset until the terminal time $T$ is $r_f^T$, and the wealth at the terminal time will be $r_f^T x_0$ for sure. The risk premium, or Sharpe ratio, for the logarithmic investor is given by $m_l(i,T)/v_l(i,T)$.

The case with exponential utility is considered in Çanakoğlu and Özekici (2009) where $U(i,x) = K(i) - C(i)\exp(-x/\beta)$ and the optimal solution has the simpler structure

$$u_n^*(i,x) = \alpha(i)\beta_{n+1}, \tag{21}$$

where $\beta_n = \beta/r_f^{T-n}$ and $\alpha(i)$ satisfies

$$E\left[R^e(i)\exp(-R^e(i)'\alpha(i))\right] = 0. \tag{22}$$

The optimal portfolio is separable in the sense that the amounts of money invested in the risky assets by exponential investors are independent of their wealth levels. For computational purposes we only need to find $\alpha(i)$ for any market state $i$ to determine the single fund of risky assets. The total investment also depends only on the period $n$ in a simple way as prescribed by (21). They also show that the terminal wealth is on the exponential frontier represented by the straight line

$$E_i[X_T] = r_f^T x_0 + \left(\frac{m_e(i,T)}{v_e(i,T)}\right)\mathrm{SD}_i(X_T), \tag{23}$$

where

$$m_e(i,T) = E_i\left[\sum_{k=0}^{T-1} A(Y_k)\right] = \sum_{k=0}^{T-1}\sum_{j\in E} Q^k(i,j)\bar{a}(j) = \sum_{k=0}^{T-1} Q^k\bar{a}(i) \tag{24}$$

and

$$v_e^2(i,T) = \sum_{k=0}^{T-1}\left(Q^k s(i) - \left(Q^k\bar{a}(i)\right)^2\right) + 2\sum_{k=0}^{T-1}\sum_{m=k+1}^{T-1}\left(\sum_{j\in E}\sum_{l\in E} Q^k(i,j)Q^{m-k}(j,l)\bar{a}(j)\bar{a}(l) - Q^k\bar{a}(i)Q^m\bar{a}(i)\right). \tag{25}$$

Therefore, in all cases involving logarithmic, and exponential utility functions the relationship between the expected value and standard deviation of the terminal wealth is represented by a linear frontier. These are given by (20), and (23) respectively for these two cases.

### *4.2. Simple logarithmic utility function*

We now consider the special case of a simple logarithmic utility function with $\beta = 0$ so that

$$U(i,x) = C(i)\log(x) + K(i), \tag{26}$$

with $C(i) > 0$ where we can easily see that $r(x) = 1/x$. However, we remove the restriction that $r_f(i) = r_f$ and the riskless return depends on the market state.

**Theorem 2.** *Let the utility function of the investor be the simple logarithmic function (26). Then, the optimal solution of the dynamic programming Eq. (6) is*

$$v_n(i,x) = K_n(i) + C_n(i)\log(x)$$

*and the optimal portfolio is*

$$u_n^*(i,x) = \alpha(i)r_f(i)x, \tag{27}$$

*where*

$$C_n = Q^{T-n}C, \quad K_n = Q^{T-n}K + \left(\sum_{m=0}^{T-n-1} Q^m \widehat{Q}_\alpha Q^{T-n-1-m}\right)C$$

*and*

$$\widehat{Q}_\alpha(i,j) = E\left[\log(r_f(i)\left(1 + R^e(i)'\alpha(i)\right))\right]Q(i,j)$$

*for all $n = 0, 1, \ldots, T-1$; and $\alpha(i)$ satisfies*

$$E\left[\frac{R_k^e(i)}{1 + R^e(i)'\alpha(i)}\right] = 0 \tag{28}$$

*for all assets $k = 1, 2, \ldots, m$ independent of period $n$ and all $i$.*

**Proof.** This is similar to the proof of Theorem 1 and it will not be repeated here. □

In this special case with $\beta = 0$, it is clear that the optimal policy in (27) is myopic since there is no dependence on $n$. At any time $n$, the total amount of money invested in the risky assets depends only on the market state $i$ and wealth $x$. Since the total risky investment is $1'u_n^*(i,x) = 1'\alpha(i)r_f(i)x$, it follows that $r_f(i)\sum_{k=1}^m \alpha_k(i)$ is the proportion of total wealth that is invested in the risky assets if the market is in state $i$. Moreover, as in the general logarithmic case, the composition of the risky portfolio (14) also depends only on the market state $i$ independent of the available wealth $x$.

The evolution of the wealth process $X$ using the optimal policy can be analyzed by the wealth dynamics equation

$$X_{n+1} = r_f(Y_n)X_n + R^e(Y_n)'u^*(Y_n, X_n) = X_n r_f(Y_n)(1 + A(Y_n)) = X_n B(Y_n),$$

where $B(i) = r_f(i)(1 + A(i))$. Clearly, the solution is

$$X_n = X_0 \prod_{k=0}^{n-1} B(Y_k) \tag{29}$$

for $n \geqslant 1$, and this simple structure can be exploited to analyze the terminal wealth $X_T$. In particular, given $X_0 = x_0$

$$E_i[X_T] = x_0(1 + E_i[\mathbb{C}_T(b(Y_0) - 1, \ldots, b(Y_{T-1}) - 1)]) = x_0 Q_g^{T-1} g(i), \tag{30}$$

where $b(i) = r_f(i)(1 + a(i))$ and $g(i) = b(i) - 1$. The second moment is

$$E_i[X_T^2] = x_0^2(1 + E_i[\mathbb{C}_T(b_2(Y_0) - 1, \ldots, b_2(Y_{T-1}) - 1)]) = x_0^2 Q_f^{T-1} f(i), \tag{31}$$

where $b_2(i) = r_f(i)^2 E[(1 + A(i))^2] = r_f(i)^2(1 + 2a(i) + s(i)), f(i) = b_2(i) - 1$ and $Var_i(X_T)$ is the difference of (31) and the square of (30). The log-return at the terminal time $T$ is

$$\ln(X_T/X_0) = \sum_{k=0}^{T-1} \ln(B(Y_k)),$$

so that the mean is

$$E_i[\ln(X_T/X_0)] = \sum_{k=0}^{T-1} Q^k(i,j)E[\ln(B(j))] = \sum_{k=0}^{T-1} Q^k(i,j)\left(\ln\left(r_f(j)\right) + E[\ln(1 + A(j))]\right),$$

which can be determined using the distributions of $\{A(i) = R^e(i)'\alpha(i)\}$. The simple structure of (29) can be exploited to determine various quantities of interest associated with the terminal wealth.

## 5. Power utility function

Suppose that the utility function is the power function

$$U(i,x) = K(i) + C(i)\frac{(x-\beta)^\gamma}{\gamma} \tag{32}$$

and Pratt–Arrow ratio can be calculated as $r(x) = (1-\gamma)/(x-\beta)$ for all $i$ so that $b = 1/(1-\gamma)$ and $a = \beta/(\gamma - 1)$ in (5). In this paper, we assume that the utility function (32) is well-defined for all possible values of $x$. For example, if $(x - \beta) < 0$ is possible, then we exclude $\gamma = 1/2$ in our analysis. If we need to include these values of $\gamma$, we can define the utility function to be $-\infty$ whenever (32) is not well-defined and make appropriate assumptions on excess returns $\{R^e(i)\}$ as in Section 4. For $U(i,x)$ to be a legitimate utility function some additional restrictions may be imposed, but we do not dwell with such technical issues here. Note that $\gamma$ and $\beta$ is the same for all market states so that risk classification of the investor does not depend on the stochastic market. Similarly, we assume that the return for the riskless asset is the same for all market states so that $r_f(i) = r_f$ for all $i$.

We will first consider an optimization problem of the form

$$\max_u c_0 E\left[\frac{\left(R^{e'}u - c\right)^\gamma}{\gamma}\right], \tag{33}$$

where $R^e$ is any random vector. The gradient vector of the objection function $g(u) = E[(R^{e'}u - c)^\gamma/\gamma]$ is given by

$$\nabla_k g(u) = \frac{\partial g(u)}{\partial u_k} = c_0 E\left[R_k^e\left(R^{e'}u - c\right)^{\gamma-1}\right],$$

while the Hessian matrix is

$$\nabla_{k,l}^2 g(u) = \frac{\partial^2 g(u)}{\partial u_k \partial u_l} = (\gamma - 1)c_0 E\left[R_k^e R_l^e\left(R^{e'}u - c\right)^{\gamma-2}\right]$$

for all $k, l$. The first order optimality condition to find the optimal solution of (8) is obtained by setting the gradient vector equal to zero so that

$$E\left[R_k^e\left(R^{e'}u - c\right)^{\gamma-1}\right] = 0 \tag{34}$$

for all $k = 1, 2, \ldots, m$. Let $z = (z_1, \ldots, z_m)$ be any non-zero column vector where $z_i$'s are real numbers. Then, one can see that

$$z'\nabla^2 g(u)z = (\gamma - 1)c_0 E\left[\left(z_1 R_1^e + z_2 R_2^e + \cdots + z_m R_m^e\right)^2\left(R^{e\prime}u - c\right)^{\gamma-2}\right]. \tag{35}$$

Throughout this paper, we assume that the excess returns $\{R^e(i)\}$ and the parameters of the utility function are such that there is always an optimal solution of (33) that satisfies the first order conditions (34). Note that this requirement does not necessarily impose concavity restriction on the objective function. We only require that the optimal solution is at an interior point which satisfies the necessary conditions of optimality (34). Our purpose is to identify the structure of the optimal policy and we will not dwell will these technical details on optimization. This is of course an important issue and we do not intend to undermine its significance. We now consider some possible cases to illustrate how one can approach this technical problem. If $\gamma - 2$ is even, then the Hessian matrix $\nabla^2 g$ in (35) is negative semi-definite provided that $(\gamma - 1)c_0 \leqslant 0$ and the optimal solution satisfies (34) since we have an unconstrained concave maximization problem. If $\gamma - 2$ is not even and $(\gamma - 1)c_0 \leqslant 0$, then the objective function is concave over the set

$$A(c) = \left\{u : P\left\{\left(R^{e'}u - c\right)^{\gamma-2} \geqslant 0\right\} = 1\right\} \tag{36}$$

and we need additional restrictions on the excess returns $\{R^e(i)\}$; like the existence of a solution of the first order condition (34) in $A(c)$ for all $c$. In case $(\gamma - 1)c_0 \geqslant 0$, it suffices to reverse the inequality in (36).

**Theorem 3.** *Let the utility function of the investor be the power utility function (32) and suppose that the riskless asset return does not depend on the market state. Then, the optimal solution of the dynamic programming Eq. (6) is*

$$v_n(i, x) = K_n(i) + C_n(i)\frac{(x - \beta_n)^\gamma}{\gamma}$$

*and the optimal portfolio is*

$$u_n^*(i, x) = \alpha(i)\left(r_f x - \beta_{n+1}\right), \tag{37}$$

*where*

$$\beta_n = \frac{\beta}{r_f^{T-n}}, \quad C_n = \widehat{Q}_\alpha^{T-n} C, \quad K_n = Q^{T-n} K \tag{38}$$

*and*

$$\widehat{Q}_\alpha(i, j) = E\left[\left(r_f\left(1 + R^e(i)'\alpha(i)\right)\right)^\gamma\right] Q(i, j)$$

*for all $n = 0, 1, \ldots, T - 1$; and $\alpha(i)$ satisfies*

$$E\left[R_k^e(i)\left(1 + R^e(i)'\alpha(i)\right)^{\gamma-1}\right] = 0 \tag{39}$$

*for all assets $k = 1, 2, \ldots, m$ and all $i$.*

**Proof.** This is similar to the proof of Theorem 1 and it will not be repeated here. □

Note that the wealth dynamics equation for the power utility function is not the same as the wealth dynamics equation (15) for the logarithmic case since the structure of the optimal policy in (10) and (37) are different where for the latter $\beta$ has a minus sign. However, using a similar analysis as in Section 4.1 we can easily determine

$$E_i[X_T] = r_f^T x_0 + \left(\beta - r_f^T x_0\right) m_\gamma(i, T) \tag{40}$$

and

$$\mathrm{Var}_i(X_T) = \left(\beta - r_f^T x_0\right)^2 v_\gamma^2(i, T), \tag{41}$$

where

$$m_\gamma(i, T) = -E_i\left[\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)\right] \tag{42}$$

and

$$v_\gamma^2(i,T) = \text{Var}_i\left(\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)\right). \tag{43}$$

Likewise, similar interpretations can be made on the structure of the optimal policy. In particular, the optimal policy is not myopic, but the risky composition of the portfolio is both myopic and memoryless. Moreover, this composition only depends on the state of the market. Although we obtain similar characterizations and interpretations, note that the optimal policies for logarithmic and power cases are not identical since (12) and (39) have different solutions. In particular, the solution of (39) clearly depends on the risk aversion coefficient $\gamma$.

For the power utility case, (40) and (41) imply that we can also write

$$E_i[X_T] = r_f^T x_0 + \left(\frac{m_\gamma(i,T)}{v_\gamma(i,T)}\right)\text{SD}_i(X_T) \tag{44}$$

to represent the power frontier with slope or risk premium $m_\gamma(i,T)/v_\gamma(i,T)$.

If $\gamma = 1$, then the utility function (32) becomes linear and the investor tries to maximize the expected terminal wealth. The optimal solution then is uninteresting and trivial since the investor will invest an infinite amount of money on the asset (including the riskless asset) with the highest expected return in any market state.

More interestingly, when $\gamma = 2$, the utility function (32) has a quadratic form. In this case, the assumption is satisfied and there is a unique solution satisfying the first order condition (39), which simplifies to

$$E[R^e(i)] + E\left[R^e(i)R^e(i)'\alpha(i)\right] = 0$$

and the optimal solution can be found explicitly as

$$\alpha(i) = -V(i)^{-1}r^e(i), \tag{45}$$

where $V(i) = E[R^e(i)R^e(i)'] = \sigma(i) + r^e(i)r^e(i)'$ is the matrix of second moments, and $r^e(i) = E[R^e(i)]$ is the expected value of the return vector in state $i$. Furthermore, it follows form (16) that $A(i) = -R^e(i)'V(i)^{-1}r^e(i)$ which gives

$$a(i) = -E[R^e(i)'V(i)^{-1}r^e(i)] = -r^e(i)'V(i)^{-1}r^e(i) \tag{46}$$

and $m_2(i,T)$ can be computed using (46) with $g(i) = 1 - r^e(i)'V(i)^{-1}r^e(i)$. Note from (18) that

$$s(i) = \alpha(i)'V(i)\alpha(i) = r^e(i)'V(i)^{-1}V(i)V(i)^{-1}r^e(i) = r^e(i)'V(i)^{-1}r^e(i) = -a(i)$$

and the mean becomes

$$E_i\left[\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)^2\right] = E_i\left[\mathbb{C}_T(a(\overline{Y}_{T-1}))\right] - 2E_i\left[\mathbb{C}_T\left(a\left(\overline{Y}_{T-1}\right)\right)\right] = -E_i\left[\mathbb{C}_T(a(\overline{Y}_{T-1}))\right] = m_2(i,T),$$

so that the variance term is

$$v_2^2(i,T) = m_2(i,T) - m_2(i,T)^2 = m_2(i,T)(1 - m_2(i,T)).$$

Therefore, for the quadratic model with $\gamma = 2$, we obtain the mean–variance efficient frontier using (44) given by the straight line

$$E_i[X_T] = r_f^T x_0 + \left(\sqrt{\frac{m_2(i,T)}{1 - m_2(i,T)}}\right)\text{SD}_i(X_T), \tag{47}$$

where the slope, or the risk premium, is $m_2(i,T)/v_2(i,T)$. Çakmak and Özekici (2006) discussed the mean–variance problem where the objective is to maximize the linear-quadratic objective function $E_i[-X_T^2 + \beta X_T]$ parameterized by $\beta$. When the riskless interest rate is fixed, our result (47) coincides with the efficient frontier obtained by Çanakoğlu and Özekici (2009).

As a special case, suppose now that the utility function is the CRRA (constant relative risk aversion) function with $\beta = 0$ so that

$$U(i,x) = K(i) + C(i)\left(\frac{x^\gamma}{\gamma}\right). \tag{48}$$

We can easily see that $r(x) = (1-\gamma)/x$. We remove the restriction that $r_f(i) = r_f$ and the riskless return depends on the market state.

**Theorem 4.** *Let the utility function of the investor be the CRRA function (48). Then, the optimal solution of the dynamic programming Eq. (6) is*

$$v_n(i,x) = K_n(i) + C_n(i)\left(\frac{x^\gamma}{\gamma}\right)$$

*and the optimal portfolio is*

$$u_n^*(i,x) = \alpha(i)r_f(i)x, \tag{49}$$

*where*

$$K_n = Q^{T-n}K, \quad C_n = \widehat{Q}_\alpha^{T-n}C$$

*and*

$$\widehat{Q}_\alpha(i,j) = E\left[(r_f(i)\left(1 + R^e(i)'\alpha(i)\right))^\gamma\right]Q(i,j)$$

*for all* $n = 0, 1, \ldots, T-1$; *and* $\alpha(i)$ *satisfies*

$$E\left[R_k^e(i)\left(1 + R^e(i)'\alpha(i)\right)^{\gamma-1}\right] = 0 \tag{50}$$

*for all assets* $k = 1, 2, \ldots, m$ *independent of period* $n$ *and all* $i$.

**Proof.** This is similar to the proof of Theorem 1. □

Note that the structure of the optimal policy (49) is identical to (27). Therefore, the results and interpretations presented for the simple logarithmic case also hold. The optimal policies are of course different since the solutions of (28) and (50) are not identical. A numerical illustration of our results and comparison of the frontiers are included in the appendix.

## 6. Illustration

In this section, we address the computational issues and demonstrate how our results can be put to work by considering a numerical illustration for the logarithmic, power ($\gamma = 0.5, 2$ (quadratic), and 4) and exponential utility cases. Consider a market with three risky assets and one riskless asset where the returns of the risky assets follow an arbitrary multivariate distribution. The illustration is based on data obtained during January 1991 to December 2006 from weekly return information of three assets (IBM, Dell and Microsoft) traded in New York Stock Exchange; and the daily effective federal funds rate. The states of the market are classified by considering whether the SP500 index went up or down during the previous 2 weeks. Therefore, there are 4 states labeled as $1 \equiv (\text{down}, \text{down})$, $2 \equiv (\text{down}, \text{up})$, $3 \equiv (\text{up}, \text{down})$, and $4 \equiv (\text{up}, \text{up})$. The weekly interest rates for all states were approximately equal to 0.08% and our assumption is satisfied. Using historical data which consists of 829 weekly closings we calculated the number of transitions from one state to another and estimated the transition probability matrix $Q$ of the Markov chain as

$$Q = \begin{bmatrix} 0.410 & 0 & 0.590 & 0 \\ 0.388 & 0 & 0.612 & 0 \\ 0 & 0.445 & 0 & 0.555 \\ 0 & 0.494 & 0 & 0.506 \end{bmatrix}.$$

The return of the riskless asset and the expected return of each risky asset for each state are also estimated from the appropriate weekly closings of the assets and they are as follows:

| $i$ | $r_f$ | $\mu_1(i)$ | $\mu_2(i)$ | $\mu_3(i)$ |
|---|---|---|---|---|
| 1 | 1.0008 | 1.0105 | 1.0096 | 0.9995 |
| 2 | 1.0008 | 1.0071 | 1.0097 | 1.0061 |
| 3 | 1.0008 | 1.0039 | 1.0114 | 1.0052 |
| 4 | 1.0008 | 1.0011 | 1.0033 | 0.9990 |

and the covariance matrices for each state are

$$\sigma(1) = \begin{bmatrix} 2.425 & 1.809 & 0.607 \\ 1.809 & 5.990 & 0.684 \\ 0.607 & 0.684 & 1.893 \end{bmatrix},$$
$$\sigma(2) = \begin{bmatrix} 2.046 & 1.310 & 0.542 \\ 1.310 & 4.855 & 0.906 \\ 0.542 & 0.906 & 1.657 \end{bmatrix},$$
$$\sigma(3) = \begin{bmatrix} 2.109 & 1.417 & 1.074 \\ 1.417 & 4.663 & 1.169 \\ 1.074 & 1.169 & 1.982 \end{bmatrix},$$
$$\sigma(4) = \begin{bmatrix} 1.607 & 1.229 & 0.430 \\ 1.229 & 4.556 & 0.486 \\ 0.430 & 0.486 & 1.446 \end{bmatrix}.$$

Note that these values are obtained by multiplying the actual numbers by 1000 for simplification.

We consider the problem of investors with initial wealth $x_0 = 1$ who want to maximize the expected utility of terminal wealth. We consider cases with logarithmic, power ($\gamma = 0.5$, 2 and 4) and exponential utility functions where the time horizon is $T = 4$ periods.

It is difficult to calculate optimal $\alpha$ values numerically for an arbitrary distribution using 12, 22 and 39. Our approach is to use Taylor series expansion of the utility function around the expected value $\overline{W} = E[W]$ of the terminal wealth $W = X_T$. The reader is referred to Jondeau and Rockinger (2006) for a detailed discussion on the benefits, advantages and disadvantages of using Taylor series expansion in optimal portfolio allocation. In particular, they give a convincing argument for using the first 4 moments in the approximation. When we checked our data we recognized that the return distributions have non-zero skewness and excess kurtosis, so we decided to use the first four moments. The details of the computational process and results are given in Appendix D. The numerical values are computed and the Sharpe ratios, or the risk premiums, are

$$m_l(i,T)/v_l(i,T) = [0.336 \quad 0.316 \quad 0.287 \quad 0.258],$$
$$m_{0.5}(i,T)/v_{0.5}(i,T) = [0.305 \quad 0.289 \quad 0.267 \quad 0.244],$$
$$m_2(i,T)/v_2(i,T) = [0.371 \quad 0.344 \quad 0.309 \quad 0.273],$$
$$m_4(i,T)/v_4(i,T) = [0.364 \quad 0.339 \quad 0.305 \quad 0.270],$$
$$m_e(i,T)/v_e(i,T) = [0.352 \quad 0.324 \quad 0.294 \quad 0.254].$$

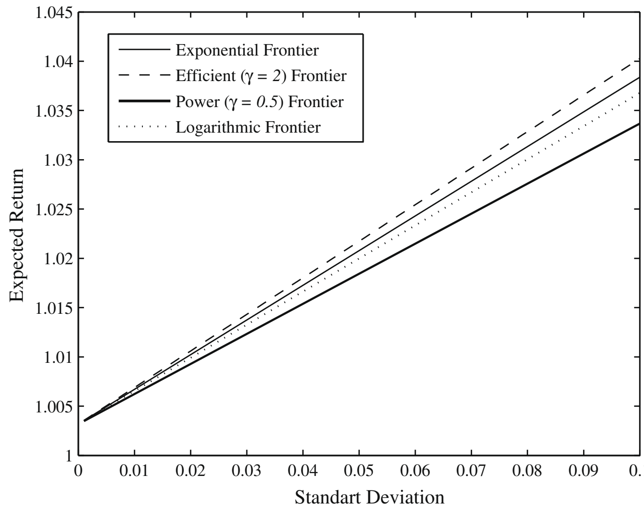


**Fig. 1.** Frontiers for $i = 1, T = 4$.

Investors with different utility functions will have differing risk preferences measured by $\beta$ in their utility functions. The return and risk of the terminal wealth for these investors will be on the respective frontier in Fig. 1. The slopes measure the risk premiums and, as expected, the risk premiums for the mean–variance frontier are highest.

Çanakoğlu and Özekici (2009) considered the exponential utility case and obtained the exponential frontier by solving the optimality condition (22) directly under the assumption that the asset returns follow a multivariate normal distribution. They obtained the frontier to be

$$m_e(i,T)/v_e(i,T) = [\,0.353 \quad 0.324 \quad 0.295 \quad 0.255\,],$$

which is very close to our approximate values. However, note that Taylor series approximation does not require knowledge on the asset return distributions.

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## Appendix A. Insight on the assumption on asset returns

We shall not dwell with the implications of our assumption on asset returns; but to get some insight, we consider the case when there is only a single risky asset. Let

$$m_l = \sup\{y; P\{R^e \leqslant y\} = 0\}$$

and

$$m_h = \inf\{y; P\{R^e_k \leqslant y\} = 1\}$$

so that $P\{R^e \in [m_l, m_h]\} = 1$. This also implies that the condition $R^e u + c > 0$ is satisfied if and only if $u \in (-c/m_h, -c/m_l)$. It should be noted that $m_l \leqslant 0 \leqslant m_h$ must be satisfied; otherwise, there exists arbitrage opportunity in the market either by shortselling the riskless asset (if $m_l > 0$) or by shortselling the risky asset (if $m_h < 0$). We know that $\nabla^2 g$ is always negative and $g$ is concave on $A(c)$. Another observation is that the optimal solution found from the first order condition is $u = 0$ if and only if $r^e = 0$. So, if $r^e = 0$ we can always solve the optimization problem trivially. In the following analysis we will consider the cases when $r^e \neq 0$. There are four possible cases depending on the support of the distribution of $R^e$ as analyzed below.

**Case 1** $(m_l = -\infty, m_h = +\infty)$: In this case $u \neq 0$ implies that $P\{R^{e'} u + c < 0\} > 0$ and $E[\log(R^{e'} u + c)] = -\infty$, thus $A(c) = \{0\}$ and the only solution with finite objective function value is $u = 0$. Therefore, $u = 0$ is also the optimal solution which does not necessarily satisfy the first order condition (9) except for the case $r^e = 0$ as mentioned earlier.

**Case 2** $(m_l = -\infty, m_h < +\infty)$: In this case $u > 0$ or $u \leqslant -c/m_h$ implies that $P\{R^{e'} u + c < 0\} > 0$ and $E[\log(R^{e'} u + c)] = -\infty$. So, $A(c) = (-c/m_h, 0]$ and for the solution of the first order condition (9) to be in the interior of $A(c)$, we need to have $\nabla g(0) < 0$ and $\nabla g(-c/m_h) > 0$. This requires $r^e < 0$ and $E[R^e/(m_h - R^e)] > 0$. However, if $r^e \geqslant 0$, then the optimal solution is at the boundary $u = 0$. Similarly, if $E[R^e/(m_h - R^e)] \leqslant 0$, then the optimal solution is at the other boundary $u = -c/m_h$.

**Case 3** $(m_l > -\infty, m_h = +\infty)$: In this case $u < 0$ or $u \geqslant -c/m_l$ implies that $P\{R^{e'} u + c < 0\} > 0$ and $E[\log(R^{e'} u + c)] = -\infty$. So, $A(c) = [0, -c/m_l)$ and for the solution of the first order condition (9) to be in the interior of $A(c)$, we need to have $\nabla g(0) > 0$ and $\nabla g(-c/m_l) < 0$. This requires $r^e > 0$ and $E[R^e/(m_l - R^e)] < 0$. However, if $r^e < 0$, then the optimal solution is at the boundary $u = 0$. Similarly, if $E[R^e/(m_h - R^e)] \geqslant 0$, then the optimal solution is at the other boundary $u = -c/m_l$.

**Case 4** $(m_l > -\infty, m_h < +\infty)$: In this case $u \leqslant -c/m_h$ or $u \geqslant -c/m_l$ implies that $P\{R^{e'} u + c < 0\} > 0$ and $E[\log(R^{e'} u + c)] = -\infty$. So, $A(c) = (-c/m_h, -c/m_l)$ for the solution of the first order condition to be interior of $A(c)$, we need to have $\nabla g(-c/m_h) > 0$ and

$\nabla g(-c/m_l) < 0$. This requires $E[R^e/(m_h - R^e)] > 0$ and $E[R^e/(m_l - R^e)] < 0$. But, if $E[R^e/(m_h - R^e)] \leqslant 0$, then the optimal solution is at the boundary $u = -c/m_h$. Similarly, if $E[R^e/(m_h - R^e)] \geqslant 0$, then the optimal solution is at the other boundary $u = -c/m_l$.

## Appendix B. Proof of Theorem 1

We use induction starting with the boundary condition $v_T(i,x) = C(i)\log(x+\beta) + K(i)$ and obtain

$$g_{T-1}(i,x,u) = \sum_{j\in E} Q(i,j)E[U(j, r_f x + R^e(i)'u)] = QK(i) + QC(i)E[\log(r_f x + R^e(i)'u + \beta)]$$

for all available investment strategies. Let $u^*$ be the optimal amount of money that should be invested in the risky asset so that

$$v_{T-1}(i,x) = \max_u g_{T-1}(i,x,u) = g_{T-1}(i,x,u^*).$$

One can see that the objection function $g_{T-1}(i,x,u)$ is in the form of the objection function in (8) where $c = r_f x + \beta > 0$. So, the objective function is concave since $QC(i) = \sum_{j\in E} Q(i,j)C(j) > 0$ and, with our assumption on $\{R^e(i)\}$, the optimal policy can be found using the first order condition

$$E\left[\frac{R^e_k(i)}{r_f x + R^e(i)'u^*_{T-1}(i,x) + \beta}\right] = 0$$

for all $k = 1, 2, \ldots, m$.

Defining the vector function $\alpha(i,x) = (\alpha_1(i,x), \alpha_2(i,x), \ldots, \alpha_m(i,x))$ such that $\alpha(i,x) = u^*(i,x)/(r_f x + \beta)$ we obtain $u^*_{T-1}(i,x) = \alpha(i,x)(r_f x + \beta)$ so the optimality condition can be rewritten as

$$E\left[\frac{R^e(i)}{r_f x + R^e(i)'\alpha(i,x)\big(r_f x + \beta\big) + \beta}\right] = E\left[\frac{R^e(i)}{\big(r_f x + \beta\big)\big(1 + R^e(i)'\alpha(i,x)\big)}\right] = 0$$

and, since $r_f x + \beta > 0$, we have

$$E\left[\frac{R^e_k(i)}{1 + R^e(i)'\alpha(i,x)}\right] = 0. \tag{B.1}$$

Since (B.1) holds for every $x$ we can say that $\alpha$ does not depend on $x$ and $\alpha_k(i,x) = \alpha_k(i)$ for all $k = 1, 2, \ldots, m$. We can write the optimal policy as $u^*_{T-1}(i,x) = \alpha(i)(r_f x + \beta)$ where $\alpha(i)$ satisfies

$$E\left[\frac{R^e_k(i)}{1 + R^e(i)'\alpha(i)}\right] = 0$$

for all $k = 1, 2, \ldots, m$. When the value function at time $T-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{T-1}(i,x) &= \sum_{j\in E} Q(i,j)E[K(j) + C(j)\log(r_f x + R^e(i)'\alpha(i)\big(r_f x + \beta\big) + \beta)] = QK(i) + QC(i)\big[E\big[\log\big(r_f(1 + R^e(i)'\alpha(i))\big)\big] + \log(x + \beta/r_f)\big] \\ &= QK(i) + \widehat{Q}_\alpha C(i) + QC(i)\log(x + \beta/r_f) = K_{T-1}(i) + C_{T-1}(i)\log(x + \beta_{T-1}) \end{aligned}$$

and the value function is still logarithmic like the utility function. This follows by noting that $K_{T-1} = QK + \widehat{Q}_\alpha C$ and $C_{T-1} = QC$ in (11). This completes the proof for $n = T-1$.

Suppose now that the induction hypothesis holds for periods $T, T-1, T-2, \ldots, n$. Then, for period $n-1$,

$$g_{n-1}(i,x,u) = \sum_{j\in E} Q(i,j)E[v_n(j, r_f x + R^e(i)'u + \beta_n)] = QK_n(i) + QC_n(i)E[\log(r_f x + R^e(i)'u + \beta_n)]. \tag{B.2}$$

Let $u^*$ be the optimal policy such that

$$v_{n-1}(i,x) = \max_u g_{n-1}(i,x,u) = g_{n-1}(i,x,u^*).$$

It is clear, once again, that the objective function $g_{n-1}(i,x,u)$ is in the form of the objection function in (8) with $c = r_f x_n + \beta_n > 0$ and it is concave since $QC_n = Q^{T-n+1}C > 0$. The optimal solution can be found by using the first order condition

$$E\left[\frac{R^e_k(i)}{r_f x + R^e(i)'u^*_{n-1}(i,x) + \beta_n}\right] = 0$$

for $k = 1, 2, \ldots, m$.

Letting $\alpha(i,x) = u^*_{n-1}(i,x)/(r_f x + \beta_n)$ we obtain $u^*_{n-1}(i,x) = \alpha(i,x)\ (r_f x + \beta_n)$ and

$$E\left[\frac{R^e_k(i)}{1 + R^e(i)'\alpha(i,x)}\right] = 0$$

where $\alpha(i,x)$ does not depend on the period $n$ and on $x$ as in Eq. (B.1). Therefore, we can write $\alpha(i,x) = \alpha(i)$ and the optimal policy is $u^*_{n-1}(i,x) = \alpha(i)(r_f x + \beta_n)$ where $\alpha(i)$ satisfies

$$E\left[\frac{R^e_k(i)}{1 + R^e(i)'\alpha(i)}\right] = 0$$

for all $k = 1, 2, \ldots, m$. If we insert the optimal policy in the value function using (B.2), we can see that

$$\begin{aligned} v_{n-1}(i,x) &= QK_n(i) + QC_n(i)E[\log(r_f x + R^e(i)'\alpha(i)(r_f x + \beta_n) + \beta_n)] = QK_n(i) + QC_n(i)\left(E\left[\log\left(r_f(1 + R^e(i)'\alpha(i))\right)\right] + \log(x + \beta_n/r_f)\right) \\ &= QK_n(i) + \widehat{Q}_\alpha C_n(i) + QC_n(i)\log(x + \beta_n/r_f) = K_{n-1}(i) + C_{n-1}(i)\log(x + \beta_{n-1}) \end{aligned}$$

and the value function is still logarithmic. Note that the recursions $K_{n-1} = QK_n + \widehat{Q}_\alpha C_n$ and $C_{n-1} = QC_n$ with boundary values $K_T = K$ and $C_T = C$ give the explicit solutions in (11). This completes the proof.

## Appendix C. Optimal wealth process

As a computational formula that we will use frequently in the following analysis, we define

$$E_i[h_n(g(Y_0), \ldots, g(Y_n))] = \sum_{i_1, \ldots, i_n \in E} Q(i, i_1) \cdots Q(i_{n-1}, i_n) h_n(g(i), \ldots, g(i_n)), \tag{C.1}$$

which provides an explicit expression to compute expectations for any deterministic functions $h_n$ and $g$ of the random vector $\overline{Y}_n = (Y_0, Y_1, \ldots, Y_n)$ of Markovian states. For notational simplification in our analysis, we will let

$$g(\overline{Y}_n) = (g(Y_0), g(Y_1), \ldots, g(Y_n))$$

for any function $g$ defined on $E$. We will use the representation (C.1) whenever necessary to economize on the notation and note that this provides an exact computational formula. In particular, if $h_n(x_0, x_1, \ldots, x_n) = \prod_{k=0}^n x_k$, then letting $f_n(i) = E_i[h_n(g(\overline{Y}_n))]$ we obtain

$$f_n(i) = E_i\left[\prod_{k=0}^n g(\overline{Y}_k)\right] = g(i)\sum_{j \in E} Q(i,j) f_{n-1}(j) = \sum_{j \in E} Q_g(i,j) f_{n-1}(j) = Q_g f_{n-1}(j), \tag{C.2}$$

where we define the matrix $Q_g$ such that $Q_g(i,j) = g(i)Q(i,j)$ for all $i,j$. Using the boundary condition $f_0(i) = g(i)$ and the recursion (C.2), the explicit solution is

$$f_n(i) = E_i\left[\prod_{k=0}^n g(Y_k)\right] = Q_g^n g(i) \tag{C.3}$$

and $f_n = Q_g^n g$ is simply the product of the matrix $Q_g^n$ by the vector $g$. For computational analysis we use (C.3) whenever appropriate.

Define

$$\mathbb{C}_n(x_1, x_2, \ldots, x_n) = \prod_{k=1}^n (1 + x_k) - 1$$

as the sum of all combinations of the products of $n$ variables for $n \geqslant 1$, and set $\mathbb{C}_0 = 0$. Note that using (C.3) we can compute

$$E_i\left[\mathbb{C}_n\left(h\left(\overline{Y}_{n-1}\right)\right)\right] = E_i[\mathbb{C}_n(h(Y_0), h(Y_1), \ldots, h(Y_{n-1}))] = Q_g^{n-1} g(i) - 1$$

explicitly for $n \geqslant 1$ and any function $h$ by setting $g(i) = 1 + h(i)$.

Now, we will show that the wealth process is

$$X_n = r_f^n X_0 \prod_{k=0}^{n-1} (1 + A(Y_k)) + r_f^{n-T} \beta \mathbb{C}_n\left(A\left(\overline{Y}_{n-1}\right)\right) \tag{C.4}$$

using induction where the product on the right hand side is set to 1 when $n = 0$. The induction hypothesis holds trivially for $n = 0$. Suppose (C.4) holds for some $n \geqslant 0$. If we write $X_{n+1}$ using the wealth dynamics equation (15)

$$\begin{aligned} X_{n+1} &= r_f X_n(1 + A(Y_n)) + r_f^{n+1-T} A(Y_n)\beta = r_f^{n+1} X_0 \prod_{k=0}^n (1 + A(Y_k)) + r_f^{n+1-T}\beta\left[(1 + A(Y_n))\mathbb{C}_n\left(A\left(\overline{Y}_{n-1}\right)\right) + A(Y_n)\right] \\ &= r_f^{n+1} X_0 \prod_{k=0}^n (1 + A(Y_k)) + r_f^{n+1-T}\beta\mathbb{C}_{n+1}\left(A\left(\overline{Y}_n\right)\right) \end{aligned}$$

and we see that the induction hypothesis also holds for $n+1$. So, we conclude that the wealth process can be written as in (C.4) and, for $n = T$, we can find the terminal wealth as

$$X_T = r_f^T X_0 \prod_{k=0}^{T-1} (1 + A(Y_k)) + \beta\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right) = r_f^T X_0 + \left(r_f^T X_0 + \beta\right)\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right). \tag{C.5}$$

It is clear from (C.4) and (C.5) that the random variables $\{A(i)\}$ will play a key role in any probabilistic analysis involving the wealth process $X$. Given $X_0 = x_0$, the expected value of the terminal wealth satisfies

$$E_i[X_T] = r_f^T x_0 + \left(r_f^T x_0 + \beta\right) m_l(i, T), \tag{C.6}$$

where

$$m_l(i, T) = E_i\left[\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)\right] \tag{C.7}$$

and the variance of the terminal wealth satisfies

$$\mathrm{Var}_i(X_T) = \left(r_f^T x_0 + \beta\right)^2 v_l^2(i,T), \tag{C.8}$$

where

$$v_l^2(i,T) = \mathrm{Var}_i(\mathbb{C}_T(A(\overline{Y}_{T-1}))). \tag{C.9}$$

Using the fact that $Y$ is a Markov chain with transition matrix $Q$ and the distributions of the random variables $\{A(i)\}$, one can easily obtain computational formulas. In particular,

$$m_l(i,T) = E_i\left[\prod_{k=0}^{T-1}(1+A(Y_k)) - 1\right] = E_i\left[E_i\left[\prod_{k=0}^{T-1}(1+A(Y_k)) - 1 \middle| Y_1,\ldots,Y_{T-1}\right]\right]$$

and, since the returns in different periods are independent given the market states, we obtain

$$m_l(i,T) = E_i\left[\prod_{k=0}^{T-1}(1+a(Y_k)) - 1\right] = E_i\left[\mathbb{C}_T\left(a\left(\overline{Y}_{T-1}\right)\right)\right] = Q_g^{T-1}g(i) - 1 \tag{C.10}$$

with $g(i) = 1 + a(i)$.

To determine $v_l^2(i,T)$, we first calculate the second moment

$$\begin{aligned}
E_i\left[\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)^2\right] &= E_i\left[E_i\left[\left(\prod_{k=0}^{T-1}(1+A(Y_k)) - 1\right)^2 \middle| Y_1,\ldots,Y_{T-1}\right]\right] \\
&= E_i\left[E_i\left[\prod_{k=0}^{T-1}(1+A(Y_k))^2 - 2\prod_{k=0}^{T-1}(1+A(Y_k)) + 1 \middle| Y_1,\ldots,Y_{T-1}\right]\right] \\
&= E_i\left[\prod_{k=0}^{T-1}(1+2a(Y_k)+s(Y_k)) - 1 - 2\left(\prod_{k=0}^{T-1}(1+a(Y_k)) - 1\right)\right] \\
&= E_i\left[\mathbb{C}_T\left(2a\left(\overline{Y}_{T-1}\right) + s\left(\overline{Y}_{T-1}\right)\right)\right] - 2E_i\left[\mathbb{C}_T\left(a\left(\overline{Y}_{T-1}\right)\right)\right],
\end{aligned} \tag{C.11}$$

since the returns in different periods are independent given the market states. Therefore, we can write

$$E_i\left[\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)^2\right] = E_i\left[\mathbb{C}_T\left(2a\left(\overline{Y}_{T-1}\right) + s\left(\overline{Y}_{T-1}\right)\right)\right] - 2E_i\left[\mathbb{C}_T\left(a\left(\overline{Y}_{T-1}\right)\right)\right]$$

and the variance can be found as

$$v_l^2(i,T) = E_i\left[\mathbb{C}_T\left(2a\left(\overline{Y}_{T-1}\right) + s\left(\overline{Y}_{T-1}\right)\right)\right] - 2E_i\left[\mathbb{C}_T\left(a\left(\overline{Y}_{T-1}\right)\right)\right] - E_i\left[\mathbb{C}_T\left(a\left(\overline{Y}_{T-1}\right)\right)\right]^2 = Q_{g_1}^{T-1}g_1(i) - \left(Q_g^{T-1}g(i)\right)^2, \tag{C.12}$$

where $g_1(i) = 1 + 2a(i) + s(i)$ and $g(i) = 1 + a(i)$. The mean (C.6) and variance (C.8) of the terminal wealth can thus be computed explicitly using (C.10) and (C.12) where $\{(a(i), s(i))\}$ are determined from (17) and (18). The distribution of the final wealth other than just the mean and variance is also important. Using (C.5), this distribution can be characterized through its Fourier transform

$$E_i[\exp(j\lambda X_T)] = \exp\left(j\lambda r_f^T x_0\right)E_i\left[\exp\left(j\lambda\left(r_f^T x_0 + \beta\right)\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)\right)\right] = \exp\left(j\lambda r_f^T x_0\right)E_i\left[\mathscr{F}_T\left(Y_0, Y_1,\ldots,Y_{T-1};\lambda\left(r_f^T x_0 + \beta\right)\right)\right],$$

where

$$\mathscr{F}_T(i, i_1,\ldots,i_{T-1};\gamma) = E[\exp(j\gamma\mathbb{C}_T(A(i), A(i_1),\ldots,A(i_{T-1})))]$$

is the Fourier transform of $\mathbb{C}_T(A(i), A(i_1),\ldots,A(i_{T-1}))$ for independent random variables $A(i), A(i_1),\ldots,A(i_{T-1})$. When $T = 2$, for example, this transform becomes

$$E_i[\exp(j\lambda X_2)] = \exp\left(j\lambda r_f^2 x_0\right)\sum_{k\in E} Q(i,k)\mathscr{F}_2\left(i,k;\lambda\left(r_f^2 x_0 + \beta\right)\right),$$

where $\mathscr{F}_2(i,k;\gamma) = E[\exp(j\gamma(A(i) + A(k) + A(i)A(k)))]$ for independent random variables $A(i)$ and $A(k)$. The mean, variance and Fourier transform of the final wealth can be computed once the means, variances and Fourier transforms of the product of any combination of independent random variables in $\{A(i)\}$ are known.

## Appendix D. Computational results on the illustration

In this section we will describe the tools used in Section 6. Taylor series expansion is

$$U(W) = \sum_{j=0}^{+\infty} U^{(j)}\left(\overline{W}\right)\frac{\left(W - \overline{W}\right)^j}{j!},$$

where $U^{(j)}(\overline{W})$ is the $j$th derivative of the utility function at $\overline{W}$. Taking expectations we can write

$$E[U(W)] = U\left(\overline{W}\right) + \frac{1}{2!}U^{(2)}\left(\overline{W}\right)\mu_p^2 + \frac{1}{3!}U^{(3)}\left(\overline{W}\right)\mu_p^3 + \frac{1}{4!}U^{(4)}\left(\overline{W}\right)\mu_p^4 + E\left[R_4\left(W,\overline{W}\right)\right], \tag{D.1}$$

where $R_4(W, \overline{W})$ is the remainder for the first 4 moments and $\mu_p^n$ is the $n$th moment of the portfolio defined as

$$\mu_p^n = E\left[\left(W - \overline{W}\right)^n\right].$$

Using the definitions in Jondeau and Rockinger (2006) for any market state, the second moment can be expressed as

$$\mu_p^2 = \alpha' M_2 \alpha,$$

where $M_2 = \sigma$ is the covariance matrix. Similarly,

$$\mu_p^3 = \alpha' M_3 (\alpha \otimes \alpha),$$

where $\otimes$ is the Kronecker product, and $M_3$ is the $3 \times 9$ co-skewness matrix defined as

$$M_3 = \left[\begin{array}{ccc|ccc|ccc} s_{111} & s_{112} & s_{113} & s_{211} & s_{212} & s_{213} & s_{311} & s_{312} & s_{313} \\ s_{121} & s_{122} & s_{123} & s_{221} & s_{222} & s_{223} & s_{321} & s_{322} & s_{323} \\ s_{131} & s_{132} & s_{133} & s_{231} & s_{232} & s_{233} & s_{331} & s_{332} & s_{333} \end{array}\right]$$

with

$$s_{ijk} = E\left[(R_i - \mu_i)\left(R_j - \mu_j\right)(R_k - \mu_k)\right]$$

for $i, j, k = 1, 2, 3$. Finally,

$$\mu_p^4 = \alpha' M_4 (\alpha \otimes \alpha \otimes \alpha),$$

where $M_4$ is the $3 \times 27$ co-kurtosis matrix with elements

$$k_{ijkl} = E\left[(R_i - \mu_i)\left(R_j - \mu_j\right)(R_k - \mu_k)(R_l - \mu_l)\right]$$

for $i, j, k, l = 1, 2, 3$.

For the logarithmic utility function $U(x) = \log(x)$, we can write (D.1) as

$$E[U(W)] \cong \log\left(\overline{W}\right) - \frac{1}{2\overline{W}^2}\mu_p^2 + \frac{1}{3\overline{W}^3}\mu_p^3 - \frac{1}{4\overline{W}^4}\mu_p^4. \tag{D.2}$$

According to Loistl (1976), Taylor series for power and logarithmic functions converge for $0 < W < 2\overline{W}$ and we suppose that this is indeed the case here. For the logarithmic utility case in Theorem 1, the optimal policy $\alpha$ has the same solution for the maximization problem $\max E[\log(1 + R^{e'}\alpha(i))]$. Therefore, it suffices to take $W = 1 + R^{e'}\alpha$ in the Taylor series expansion (D.1). If we check our data, both covariances and expected excess returns are in the order of 0.01. So, for $W = (1 + R^{e'}\alpha(i))$, we can suppose that $0 < W < 2\overline{W}$ and the series converges most of the time. We can therefore use the Taylor series expansion

$$E\left[U\left(1 + R^{e'}\alpha\right)\right] \cong \log\left(1 + r^{e'}\alpha\right) - \frac{1}{2(1 + r^{e'}\alpha)^2}\mu_p^2 + \frac{1}{3(1 + r^{e'}\alpha)^3}\mu_p^3 - \frac{1}{4(1 + r^{e'}\alpha)^4}\mu_p^4. \tag{D.3}$$

If we take the gradient of (D.3) with respect to $\alpha$, and set it equal to zero, we find the first order condition

$$\frac{r^e}{(1 + r^{e'}\alpha)} + \frac{r^e}{(1 + r^{e'}\alpha)^3}\mu_p^2 - \frac{r^e}{(1 + r^{e'}\alpha)^4}\mu_p^3 + \frac{r^e}{(1 + r^{e'}\alpha)^5}\mu_p^4 - \frac{1}{(1 + r^{e'}\alpha)^2}M_2\alpha + \frac{1}{(1 + r^{e'}\alpha)^3}M_3(\alpha \otimes \alpha) - \frac{1}{(1 + r^{e'}\alpha)^4}M_4(\alpha \otimes \alpha \otimes \alpha) = 0.$$

We determined the optimal $\alpha$ values numerically using MATLAB for each market state and the optimal solution is

$$\alpha_l = \begin{bmatrix} 4.258 & 1.968 & -0.590 & 0.033 \\ 0.528 & 0.931 & 2.069 & 0.771 \\ -2.196 & 2.053 & 1.406 & -1.469 \end{bmatrix},$$

where the rows correspond to three assets and the columns correspond to four market states. Furthermore, the proportions of the risky assets in the risky part of the portfolio are

$$w_l = \begin{bmatrix} 1.644 & 0.397 & -0.205 & -0.050 \\ 0.203 & 0.188 & 0.717 & -1.161 \\ -0.847 & 0.415 & 0.488 & 2.211 \end{bmatrix}$$

obtained by normalizing $\alpha$ values. The exact amounts to be invested can easily be determined using (37) by simply multiplying the $\alpha$ values by the discounted value of $\beta$.

When we make a similar analysis through Taylor series approximation (D.1) for the power utility function $U(x) = x^{0.5}$, we obtain

$$E\left[U\left(1 + R^{e'}\alpha\right)\right] \cong \left(1 + r^{e'}\alpha\right)^{0.5} - \frac{1}{8}\left(1 + r^{e'}\alpha\right)^{-1.5}\mu_p^2 + \frac{3}{48}\left(1 + r^{e'}\alpha\right)^{-2.5}\mu_p^3 - \frac{15}{384}\left(1 + r^{e'}\alpha\right)^{-3.5}\mu_p^4, \tag{D.4}$$

which give the optimality condition

$$\frac{1}{2}r^e\left(1 + r^{e'}\alpha\right)^{-0.5} + \frac{3}{16}r^e\left(1 + r^{e'}\alpha\right)^{-2.5}\mu_p^2 - \frac{1}{4}\left(1 + r^{e'}\alpha\right)^{-1.5}M_2\alpha - \frac{15}{96}r^e\left(1 + r^{e'}\alpha\right)^{-3.5}\mu_p^3 + \frac{9}{48}\left(1 + r^{e'}\alpha\right)^{-2.5}M_3(\alpha \otimes \alpha)$$
$$+ \frac{105}{768}\left(1 + r^{e'}\alpha\right)^{-4.5}\mu_p^4 - \frac{15}{96}\left(1 + r^{e'}\alpha\right)^{-3.5}M_4(\alpha \otimes \alpha \otimes \alpha) = 0$$

by setting the gradient of (D.4) equal to zero. The optimal $\alpha$ values are computed numerically using MATLAB so that

$$\alpha_{0.5} = \begin{bmatrix} 7.886 & 3.655 & -1.092 & 0.116 \\ 1.115 & 1.785 & 3.874 & 1.486 \\ -4.031 & 3.789 & 2.809 & -2.859 \end{bmatrix}.$$

For the power utility function $U(x) = x^2$ case, Taylor series expansion (D.1) is exact with

$$E\left[U\left(1 + R^{e'}\alpha\right)\right] = \left(1 + r^{e'}\alpha\right)^2 + \mu_p^2,$$

which now gives the optimality condition

$$2r^e\left(1 + r^{e'}\alpha\right) + 2M_2\alpha = 0$$

or

$$\alpha = -V^{-1}r^e,$$

which is equal to (45) since $V = M_2 + r^e r^{e\prime}$. The optimal solution is

$$\alpha_2 = \begin{bmatrix} -4.032 & -1.893 & 0.594 & 0.022 \\ -0.431 & -0.893 & -2.075 & -0.803 \\ 2.090 & -1.990 & -1.244 & 1.481 \end{bmatrix}.$$

For the power utility function $U(x) = x^4$, we can write

$$E\left[U\left(1 + R^{e'}\alpha\right)\right] = \left(1 + r^{e'}\alpha\right)^4 + 6\left(1 + r^{e'}\alpha\right)^2\mu_p^2 + 4\left(1 + r^{e'}\alpha\right)\mu_p^3 + \mu_p^4 \tag{D.5}$$

and, by taking the gradient, the first order conditions are

$$4r^e\left(1 + r^{e'}\alpha\right)^3 + 12r^e\left(1 + r^{e'}\alpha\right)\mu_p^2 + 12\left(1 + r^{e'}\alpha\right)^2 M_2\alpha + 12r^e\mu_p^3 + 12\left(1 + r^{e'}\alpha\right)M_3(\alpha\otimes\alpha) + 4M_4(\alpha\otimes\alpha\otimes\alpha) = 0.$$

We determined the optimal $\alpha$ values numerically using MATLAB. The optimal solution is

$$\alpha_4 = \begin{bmatrix} -0.727 & -0.334 & 0.103 & 0.002 \\ -0.076 & -0.156 & -0.360 & -0.133 \\ 0.377 & -0.352 & -0.218 & 0.249 \end{bmatrix}.$$

For the exponential utility function $U(x) = \exp(-x)$, the Taylor series approximation becomes

$$E\left[U\left(1 + R^{e'}\alpha\right)\right] \cong \exp\left(-\left(1 + r^{e'}\alpha\right)\right)\left(1 + \frac{1}{2}\mu_p^2 + -\frac{1}{6}\mu_p^3 + \frac{1}{24}\mu_p^4\right) \tag{D.6}$$

and the optimality condition is

$$\exp\left(-\left(1 + r^{e'}\alpha\right)\right)\left[-r^e\left(1 + \frac{1}{2}\mu_p^2 - \frac{1}{6}\mu_p^3 + \frac{1}{24}\mu_p^4\right) + M_2\alpha - \frac{1}{2}M_3(\alpha\otimes\alpha) + \frac{1}{6}M_4(\alpha\otimes\alpha\otimes\alpha)\right] = 0$$

or

$$-r^e\left(1 + \frac{1}{2}\mu_p^2 - \frac{1}{6}\mu_p^3 + \frac{1}{24}\mu_p^4\right) + M_2\alpha - \frac{1}{2}M_3(\alpha\otimes\alpha) + \frac{1}{6}M_4(\alpha\otimes\alpha\otimes\alpha) = 0.$$

Using MATLAB, the optimal solution is

$$\alpha_e = \begin{bmatrix} 4.409 & 2.031 & -0.615 & 0.005 \\ 0.474 & 0.940 & 2.144 & 0.791 \\ -2.292 & 2.133 & 1.352 & -1.489 \end{bmatrix}.$$

We determined the logarithmic frontier using the explicit formulas (C.10), and (C.12), power frontier (44) with $\gamma = 0.5$, 2, and 4, and the exponential frontier using the explicit formulas (24), (25), (42), and (43). Note that the mean–variance efficient frontier is also the power frontier with the quadratic utility function with $\gamma = 2$. Numerical values are computed to be

$$\begin{aligned}
m_l(i,T) &= [0.140 \quad 0.121 \quad 0.097 \quad 0.076], \quad & v_l(i,T) &= [0.418 \quad 0.383 \quad 0.338 \quad 0.294],\\
m_{0.5}(i,T) &= [0.276 \quad 0.236 \quad 0.19 \quad 0.147], \quad & v_{0.5}(i,T) &= [0.906 \quad 0.817 \quad 0.71 \quad 0.603],\\
m_2(i,T) &= [0.120 \quad 0.106 \quad 0.087 \quad 0.069], \quad & v_2(i,T) &= [0.326 \quad 0.308 \quad 0.282 \quad 0.254],\\
m_4(i,T) &= [0.022 \quad 0.019 \quad 0.016 \quad 0.012], \quad & v_4(i,T) &= [0.061 \quad 0.057 \quad 0.051 \quad 0.046],\\
m_e(i,T) &= [0.137 \quad 0.118 \quad 0.096 \quad 0.076], \quad & v_e(i,T) &= [0.387 \quad 0.365 \quad 0.324 \quad 0.297].
\end{aligned}$$

and the slopes, or the risk premiums, are

$$\begin{aligned}
m_l(i,T)/v_l(i,T) &= [0.336 \quad 0.316 \quad 0.287 \quad 0.258],\\
m_{0.5}(i,T)/v_{0.5}(i,T) &= [0.305 \quad 0.289 \quad 0.267 \quad 0.244],\\
m_2(i,T)/v_2(i,T) &= [0.371 \quad 0.344 \quad 0.309 \quad 0.273],\\
m_4(i,T)/v_4(i,T) &= [0.364 \quad 0.339 \quad 0.305 \quad 0.270],\\
m_e(i,T)/v_e(i,T) &= [0.352 \quad 0.324 \quad 0.294 \quad 0.254].
\end{aligned}$$

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