THE HOLES IN BLACK-SCHOLES

Given the multitude of variables, it is remarkable that option formulae even occasionally get close to matching market values. So argues **Fischer Black**, co-designer of one of the classic models. Here he suggests modifications that might provide a sounder basis for profitable trading

When we calculate option values using the Black-Scholes model, and compare them with option prices, there is usually a difference. It is rare that the value of an option comes out exactly equal to the price at which it trades on the exchange.

One possible reason for the difference between value and price is that we have made a mistake in figuring the value. We may be looking at the wrong date, or using a volatility estimate that we meant to use for a different stock, or using a stock price that was reported incorrectly. Leaving aside errors like these, there are three reasons for a difference between value and price: we may have the correct value, and the option price may be out of line; we may have used the wrong outputs to the Black-Scholes formula; or the Black-Scholes formula may be wrong. Normally, all three reasons play a part in explaining a difference between value and price.

If the option price is out of line with value, it may be possible to trade profitably using the formula. Some people would say, however, that transaction costs will wipe out any possible trading profits in options no matter what formula is used.

The main input that may be wrong is the volatility. The stock price may be observed at a different time from the option price, or the interest rate we use may be outdated. These errors can be detected and corrected if they are large enough to make correction worthwhile. The volatility of the stock over the life of the option, though, must be estimated.

Different people will make different volatility estimates. An option price that is higher than the value we figure may signal that others in the market have higher volatility estimates than we do. Sometimes the market's estimate will be closer, and sometimes ours will be closer.

I want to focus in this article on the third reason for a difference between value and price: that the Black-Scholes formula may be wrong. We know some specific problems with the formula; we know how some of these problems affect the values that come out; and we know a little about how to create a better formula. Undoubtedly there will be a series of models developed over time that are better than the original Black-Scholes model.

In the original derivation of the formula, Myron Scholes and I made the following unrealistic assumptions:

- A stock's volatility is known, and never changes.
- The short-term interest rate never changes.
- Anyone can borrow or lend as much as he wants at a single interest rate, so long as he provides a portfolio as collateral with a value that exceeds any borrowing he may do.
- An investor who sells a security short will have the use of all the proceeds of the sale, and will receive any returns from investing these proceeds, even if the proceeds are used as collateral.
- There are no transaction costs for either stock or options.

THE BLACK-SCHOLES MODEL

$$C(S,X,T) = S.N(d_1) - Xe^{-rT}N(d_2)$$

$$\text{where } d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

C = call option premium

S = current asset price

X = exercise price

T = time to expiration

σ^2= instantaneous variance of the asset price (volatility)

ln = natural logarithm

N(.) = cumulative normal distribution function

r = riskless rate of interest

- An investor's trades do not affect the taxes he pays.
- Stocks pay no dividends, and investors are not allowed to exercise options early.
- There are no takeovers or other events that end the life of an option early.

Let us look now at how the values might change if we substituted more realistic assumptions.

VOLATILITY CHANGES

The volatility of a stock is not constant. The fact that it can change may have a major impact on the values of certain options, especially far-out-of-the-money options. For example, if we use a volatility estimate of 0.20 for the annual standard deviation of a six-month call option with a $40 exercise price on a $28 stock, and if we take the interest rate to be zero, we get a value of $0.00884 using the original formula. Keeping everything else the same, but doubling volatility to 0.40, we get a value of $0.465. For this out-of-the-money option, doubling the volatility estimate multiplies the option value by a factor of 53.

If we think that the volatility is 0.20 now, but that there is some chance it will change to 0.40 in the near future, we will want to use a higher value than $0.00884. We will want to give some weight to the possibility that the value will shortly be much higher than that.

One way to do this is to assign probability estimates to various volatility figures, and to use these probabilities to weight the resulting option values. Thus, if we think there is a 0.50 chance that the volatility will be 0.20, and a 0.50 chance that it will be 0.40, we will get a value in the above example of $0.237.

Taking possible changes in volatility into account will generally increase the values of all options, but it will increase out-of-the-money option values the most. It will make writing such options look less attractive.

In part, a stock's volatility changes in unexplainable ways, but it also changes in ways related to changes in the price of the stock. This relationship seems to be quite strong. A decline in the stock price implies a substantial increase in volatility, while an increase in the stock price implies a substantial decline in volatility. The effect is so strong that it is even possible that a stock with a price of $20 and a typical daily move of $0.50 will start having a typical daily move of only $0.375 if the stock price doubles to $40.

The fact that the stock price and the volatility generally change in opposite directions can be used in making estimates of volatility, and it also means that we should be using a different basic formula. John Cox and Stephen Ross have

come up with two possible alternative formulae.[1] To see the effects of using one of their formulae on the pattern of option values for at-the-money and out-of-the-money options, let us look at the values using both Black-Scholes and Cox-Ross formulae for a six-month call option on a $40 stock, taking the interest rate as zero and the volatility as 0.20 per year. For three exercise prices, the values are:

Exercise price	Black-Scholes	Cox-Ross
40.00	2.2600	2.2600
50.00	0.1550	0.0880
57.10	0.0126	0.0020

Thus the Cox-Ross formula implies lower values for out-of-the-money options than the Black-Scholes formula. But this will be offset, at least in part, by the general uncertainty about the volatility.

In addition to showing changes in volatility in general and changes in volatility related to changes in stock price, a stock may have jumps. A major news development may cause a sudden large change in the stock price, often accompanied by a temporary suspension of trading in the stock. Jumps may be thought of as momentary large increases in a stock's volatility.

Robert Merton shows that taking jumps into account will tend to increase the relative values of both out-of-the-money and in-the-money options, and will decrease the relative values of at-the-money options.[2] At least this will be true when the jumps tend to be specific to individual stocks. The effects of jumps on option values will be greatest on far-out-of-the-money options.

Merton's formula handles jumps but does not handle general or stock-price-related changes in volatility. The Cox-Ross formulae handle stock-price-related volatility changes, but do not handle jumps or general changes in volatility. But Cox and Ross also give a method that should allow several effects of this kind to be taken into account simultaneously, at least in certain cases.

Finally, the fact that a stock's volatility changes means that what seems like a close-to-riskless hedge is not. Suppose a call option moves $0.50 for a $1 move in the underlying stock, and you set up a position that is short two option contracts and long one round lot of stock. This position will be fairly well protected against stock price changes in the short run. But if the stock's volatility increases, you will lose. The option will go up even if the stock price stays where it is.

Because it may be impossible to diversify away risks like this so that investors do not care about them, options may be priced so that those who take these risks are paid by those who take the opposite side, or vice versa. These payments would not be explicit, but would be built into the prices of various kinds of options according to their exposure to risk of changes in volatility. If the direction and size of this effect could be estimated, it would imply the need for further changes in the formulae used to value options.

INTEREST RATE CHANGES

A stock's volatility changes over time, and so does the interest rate. The volatility of a stock cannot be observed; it can only be estimated. Interest rates, though, can be observed. This makes interest rate changes much easier to handle than changes in volatility.

Merton has shown that, when the interest rate is changing, one can sometimes simply substitute the interest rate on a bond with no coupons and a maturity equal to the option maturity for the short-term interest rate in the original option formula.[3] Strictly speaking, this works only when the volatility of the stock is not changing. When both the volatility and interest rate are changing, a more complicated adjustment must be made.

Perhaps we should also take into account possible changes in the interest rate when trying to set up a close-to-riskless hedged position. We might buy

[1] Cox and Ross (1976)
[2] Merton (1976)
[3] Merton (1973)

long bonds, and add them to a position that is long options and short stock, or to a position that is long out-of-the-money options and short deeper-in-the-money options, or we might sell long bonds short, to go with a position that is short options and long stock, or a position that is short out-of-the-money options and long deeper-in-the-money options.

In general, though, the effects of interest rate changes on option values do not seem nearly as great as the effects of volatility changes.

BORROWING PENALTIES

The rate at which an investor can borrow, even with securities as collateral, is higher than the rate at which he can lend. Sometimes an investor's borrowing rate is substantially higher than his lending rate. Also, margin requirements or restrictions put on by lenders may limit the amount he can borrow.

High borrowing rates and limits on borrowing amounts may cause a general increase in option values, because options provide leverage that can substitute for borrowing. If this happens, investors subject to borrowing limits may still want to buy options, but investors who can borrow freely at a rate close to the lending rate may want to get leverage by borrowing rather than buying options.

Investors who can borrow on favourable terms and investors who do not want to borrow may also find they can make consistent profits writing options against stock positions. It is not clear how large those profits will tend to be, however.

SHORT-SELLING PENALTIES

Short-selling penalties are generally even worse than borrowing penalties. An investor cannot sell a stock short on a downtick. He must go to the expense of borrowing stock if he wants to sell it short. Part of this expense involves the cash collateral he must put up with the person who lends the stock; he generally gets no interest, or interest well below market rates, on this collateral. In addition, he may have to put up margin with his broker in cash, and he may not receive interest on cash balances with his broker.

For options, the penalties tend to be much less severe. An investor who does not meet a suitability test may not be allowed to write naked options. Even the investor who meets the test may have to put up cash margin on which he receives no interest. And the broker may not encourage him to invest the money he gets from writing naked options.

But the investor does not have to borrow an option in order to sell it short, and there is no downtick rule for options. Also, professional option traders and brokerage firms find that the penalties to writing naked options generally do not affect them.

Penalties on short selling of stock may allow options to be somewhat mispriced at times. For example, deep-in-the-money call options often sell at parity – stock price minus exercise price – or below. You can make a profit from this situation if you can buy the option and sell the stock short without penalty and without transaction costs. But most investors cannot make a profit net of expenses from this kind of position.

Since buying put options is equivalent to selling stock short, penalties on short selling of stock may tend to increase the prices of put options. But there are some ways of taking advantage of this kind of mispricing, so I expect it to be minimal.

TRANSACTION COSTS

An outside investor must pay brokerage charges on his options and stock trades. An inside investor must pay floor brokerage charges, or execute the trade himself. He must pay clearing charges, and the costs of any exchange memberships he may have. These transaction costs are often substantial in relation to any potential profits from mispriced options. But they are clearly more of a barrier for outside investors than for inside investors.

Because of transaction costs, it is not possible to maintain a neutral hedge continuously, changing the ratio of your option position to your stock position as the stock price and other factors change. The fact that stock prices sometimes jump to a higher or lower level without a chance for trades to take place also makes it impossible to maintain a neutral hedge all the time. It is impossible for the same reason to maintain a neutral spread between one option on a stock and another.

But hedging and spreading are not the only forces tending to keep option prices in line. A person who wants a long position may choose between the option and the stock, based partly on whether or not the option is underpriced. And an investor might put together a diversified portfolio, with long positions in underpriced options and short positions in overpriced options, that tends to stay low in risk even though he does not adjust his option positions continuously.

TAXES

Some investors pay no taxes; some are taxed as individuals, paying taxes on dividends, interest, and capital gains; and some are taxed as corporations, also paying taxes on dividends, interest, and capital gains, but at different rates.

The very existence of taxes will affect option values. A hedged position that should give the same return as lending may have a tax that differs from the tax on interest. The fact that investor tax rates differ will affect values too. Without rules to restrict tax arbitrage, investors could use large hedged positions involving options to cut their taxes sharply or to defer them indefinitely.

The exact rules used to restrict tax arbitrage will affect option values. There may be rules to limit interest deductions, or rules to limit capital loss deductions, or rules to tax gains and losses before a position is closed out. For example, most index option positions are now taxed each year – partly at short-term capital gains rates, partly at long-term capital gains rates – whether or not the positions have been closed out.

An investor who can use capital losses to offset gains may act roughly the same way whether his tax rate is high or low. If his tax rate stays the same from year to year, he may act about the same whether he is forced to realise gains and losses or is able to choose the year when he realises his gains or losses.

But if he pays taxes on gains and cannot deduct losses, he will want to limit the volatility of his positions and have the freedom to control the timing of his gains and losses. This will affect his use of options. If many investors are in this position, it will affect option values too.

When we change the tax rules, we will change option values. I find it hard to tell, though, whether a change will increase or decrease option values. I also find it hard to tell how the changes will differ across options.

DIVIDENDS

The original Black-Scholes formula does not take account of dividends. But dividends reduce the values of call options and increase the values of put options, at least if there is no offsetting adjustment in the terms of an option. Dividends make early exercise of a call option more likely; and they make early exercise of a put option less likely.

We now have several ways to modify option formulae to take account of dividends. Some of these are discussed in the papers referred to above. None are exact, in part because option prices depend on future dividends, and future dividends are never known for sure. To find an exact solution, we need to know not only the value of possible future dividends, but also how the amount of any future dividend depends on factors that also affect the stock price.

TAKEOVERS

The original formula assumes the underlying stock will continue trading for the life of the option. Takeovers can make this assumption false.

If firm A takes over firm B through an exchange of stock, options on firm B's stock will normally become options on firm A's stock. We will use A's volatility rather than B's in valuing the option.

If firm A takes over firm B through a cash tender offer, options on B's stock will usually expire early, but the stock will be worth more than it was before. Early expiration is apt to reduce the option value, and the tender offer premium to increase the value of a call and reduce the value of a put.

The net effect of the change of a takeover will depend on the probability that the takeover occurs and the likely premium if it occurs. When the probability is high, the chance of a takeover can have a big effect on the values of different kinds of options.

Given the range of problems, it is remarkable that option formulae sometimes give values that are very close to the prices at which options trade in the market. As it stands, the Black-Scholes formula gives at least a rough approximation to the formula we would use if we knew how to take all these factors into account. Further modifications of the Black-Scholes formula will presumably move it in the direction of that hypothetical perfect formula.■