Efficient Divide-and-Conquer Parsing of Practical Context-Free Languages

Long Version

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Abstract

We present a divide-and-conquer algorithm for parsing context-free languages efficiently. Our algorithm is an instance of Valiant's (1975), who reduced the problem of parsing to matrix multiplications. We show that, while the conquer step of Valiant's is $O(n^3)$ in the worst case, it improves to $O(\log^3 n)$, under certain conditions satisfied by many useful inputs. These conditions occur for example in program texts written by humans. The improvement happens because the multiplications involve an overwhelming majority of empty matrices. This result is relevant to modern computing: divide-and-conquer algorithms can be parallelized relatively easily.

Categories and Subject Descriptors F.4.2 [*Grammars and Other Rewriting Systems*]: Parsing

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1. Introduction

Recent years have seen the rise of parallel computer architectures for the masses. Multicore CPUs and GPUs are legion. One would expect functional programs to be a perfect match for these architectures. Indeed, thanks to the absence of side-effects, functional programs are conceptually easy to parallelise. However, functional programmers have traditionally relied heavily on lists as the data-structure of choice. This tradition hinders the adaptation of functional programs to the age of parallelism. Indeed, the very linear structure of lists imposes a sequential treatment of them. In an eloquent 2009 ICFP invited talk, Guy Steele harangued the functional programming crowds to stop using lists and use sequences, represented as balanced trees. If a computation over them follows the divide-and-conquer skeleton, and uses an associative operator to cheaply combine intermediate results at each node, their fractal structure allows to take advantage of many processors in parallel; in fact as many as there are leaves in the tree.

An additional benefit of the structure its ability to support incremental computation. That is, if one remembers the intermediate results of the computation for each node, then after changing a single leaf in the tree, it suffices to recompute the results for the nodes which are on the path from the root to the given leaf. If the tree is balanced, this means that one only has to run the association operator only a few times to update the result after a single incremental change.

Some problems are naturally solved by divide-and-conquer algorithms. This is the case for example of vector operations, which treat each element independently of the others. However, many problems require creativity to discover divide-and-conquer solutions. This is the case of the problem of parsing context free languages.

Valiant (1975) discovered a divide-and-conquer algorithm for context-free recognition. However, on the face of it, the cost of the conquer step is cubic. This means that the conquer step dominates the cost of the algorithm: what we gain by running sub-problems in parallel is dwarfed by the cost of what we must run sequentially. Therefore the divide-and-conquer structure does not yield a significant performance benefit. In this paper, we show that on most inputs, one can carefully implement Valiant's algorithm to get a polylogarithmic conquer step, yielding good overall performance.

We make the following contributions:

- We give a new correctness proof of the Valiant algorithm. We re-construct the Valiant algorithm by calculating it from its specification, in a style reminiscent of Bird and de Moor (1997) (Sec. 3.4).
- We characterize a new subclass of context-free languages, corresponding to strings parsed in a hierarchical manner (Def. 9, Sec. 4.1).
- We show that, for languages of the new subclass, the time complexity of the conquer step is $O(\log^3 n)$ (Sec. 4.2).
- We propose a refinement of context-free grammars (and the corresponding modifications of Valiant's parser) which allows to parse iterative structures hierarchically instead of linearly (Sec. 5).
- We conjecture that any given context-free programming language, understood as the set of strings actually written in it (not the language prescribed by its grammar) belong to the subclass that we propose if iteration is represented hierarchically as we prescribe.
- We have implemented the parsing technique described above, and have integrated it in the BNFC parser generator (Forsberg and Ranta 2012).

2. The Divide-And-Conquer Skeleton

We aim to construct a divide and conquer algorithm which processes sequences of input symbols (taken in a finite alphabet Σ) — strings. In the following section, we specify this aim precisely by using the theory of sequences as initial algebras (Bird 1986).

Definition 1. *A sequence-algebra is a triplet of:*

- *A carrier type a*
- *A constant nil of type a*
- *A ternary operation bin of type $a \to \Sigma \to a \to a$.*

which satisfies the associative law:

$$bin\ a\ x\ (bin\ b\ y\ c) = bin\ (bin\ a\ x\ b)\ y\ c \tag{1}$$

The type of sequences of Σ, written Seq, can be defined as the initial sequence-algebra. Concretely, one way to implement Seq is as a tree structure, but all different balancing are considered equivalent. In actual implementations, sequences will be represented by more complex data structures; perhaps featuring dynamic re-balancing such as finger trees (Hinze and Paterson 2006). The associative law (1) guarantees that re-balancing will not be observable by user code. We will write Nil and Bin (with capitals) for the operations of the initial sequence-algebra:

$$Nil : Seq$$
$$Bin : Seq \to \Sigma \to Seq \to Seq$$

Assume a function $f : Seq \to A$. The construction of a divide-and-conquer algorithm computing f can be specified as finding a sequence-algebra $\mathcal{A} = (A, nil_A, bin_A)$ such as f is an homomorphism between Seq and $\mathcal{A}$.

That is, we need a carrier type A, a constant nil_A and a function bin_A such that (1) is satisfied and

$$nil_A = f\ Nil$$
$$bin_A\ (f l)\ x\ (f r) = f\ (Bin\ l\ x\ r)$$

Given such an algebra $\mathcal{A}$ and a sequence t, one can compute ft as the catamorphism of $\mathcal{A}$ applied to t.

Assuming an implementation of Seq as trees, one can obtain a parallel algorithm by spawning a new thread of execution at each node. In an actual implementation, the shape of the tree structure will be dictated by the architecture of the computer running the code. The implementation is free to choose the structure: any choice yields the same result, as guaranteed by (1).

An incremental algorithm can be obtained by caching the intermediate results in each node. An update at a leaf of the tree needs to run the bin function d times, where d is the depth of the tree.

Furthermore, and crucially, in order for parallelisation and incrementalization to yield benefits in terms of performance, the cost of running bin, assuming $f\ l$ and $f\ r$ are already computed, must be less than that of running f on the subtrees l and r.

In all the cases considered in the remainder, the associative law will be a consequence of the other homomorphism laws. This is because we are interested only in values which are generated by a sequence-homomorphism.

Lemma 1. *Given $f : Seq \to A$, and $bin : A \to \Sigma \to A \to A$ such that*

$$bin\ (f\ l)\ x\ (f\ r) = f\ (Bin\ l\ x\ r)$$

then $(A', f\ Nil, bin)$ is a sequence-algebra, where A' is the image of Seq under f.

Proof. The missing associative law is obtained as follows:

$$\begin{aligned}
& bin\ a\ x\ (bin\ b\ y\ c) \\
= \ & \{\text{-by } A' \text{ being inverse image of } f\ \text{-}\} \\
& bin\ (f\ s)\ x\ (bin\ (f\ t)\ y\ (f\ u)) \\
= \ & \{\text{-by assumption on } bin\ \text{-}\} \\
& f\ (Bin\ s\ x\ (Bin\ t\ y\ u) \\
= \ & \{\text{-by } Seq \text{ being a sequence-algebra -}\} \\
& f\ (Bin\ (Bin\ s\ x\ t)\ y\ u) \\
= \ & \{\text{-by assumption on } bin\ \text{-}\} \\
& bin\ (bin\ (f\ s)\ x\ (f\ t))\ y\ (f\ u) \\
= \ & \{\text{-by definition of } a, b, c\ \text{-}\} \\
& bin\ (bin\ a\ x\ b)\ y\ c
\end{aligned}$$

□

3. Context Free Parsing

In this section we briefly review the basics of context free (CF) parsing and introduce our notation.

3.1 Conventions and notations

We assume a CF grammar $\mathcal{G}$, given by a quadruple (Σ, N, P, S), where Σ is a finite set of terminals, N is a finite set of non-terminals of which S is the starting symbol, and P a finite set of productions.

We furthermore assume an input $w \in \Sigma^*$ — a sequence of terminal symbols of length $|w|$. The input symbol at position i is denoted $w[i]$. A sub-string of w starting at position i (included) and ending at position j (excluded), is denoted $w[i..j]$. Metasyntactic variables standing for arbitrary strings of terminals will have the form $w_1, w_2, \ldots$. The letters $A, B, C, \ldots$, stand for arbitrary non-terminals, while $\alpha, \beta, \ldots$ stand for arbitrary strings (elements of $(\Sigma \cup N)^*$) and t stands for a terminal symbol. Each production rule associates a non-terminal with a string it can generate.

Definition 2. $\alpha A \beta \longrightarrow \alpha\gamma\beta$ *iff.* $(A ::= \gamma) \in P$

Definition 3. $\xrightarrow{*}$ *stands for the reflexive transitive closure of* $\longrightarrow$.

The input string w belongs to the language $\mathcal{L}$ generated by $\mathcal{G}$ iff. $S \xrightarrow{*} w$.

Any CF grammar $\mathcal{G}$ defining a language $\mathcal{L}$ can be converted to a grammar $\mathcal{G}'$ in Chomsky Normal Form (CNF) defining the same language $\mathcal{L}$, with $|\mathcal{G}'| \leq |\mathcal{G}|^2$ (Chomsky 1959). Hence we will assume without loss of generality a grammar in CNF. In CNF, the rules are restricted to the following forms

$$S ::= \epsilon \qquad \text{(nullary)}$$
$$A ::= t \qquad \text{(unary)}$$
$$A_0 ::= A_1 A_2 \qquad \text{(binary)}$$

but it is easy to handle the empty string specially, so we conventionally exclude it from the input language and thus exclude the nullary rule. In sum, we assume that P contains only unary and binary rules. The reader avid of details is directed to Lange and Leiß (2009) for a pedagogical account of reduction to CNF.

Given a grammar specified as above, the problem of parsing is reduced to finding a binary tree such that each leaf corresponds to a symbol of the input and a suitable unary rule; and each branch corresponds to a suitable binary rule. Essentially, parsing is equivalent to consider all possible bracketings of the input, and verify that they form a valid parse.

3.2 Charts as matrices, parsing as closure

In this section we show how to specify parsing as an equation on matrices. We start by abstracting away from the grammar, via a ring-like structure. We define the operations $0, +, \cdot$ and σ as follows.

Definition 4 $(0, +, \cdot$ on $\mathcal{P}(N))$**.**

$$\begin{aligned} 0 &= \varnothing \\ x + y &= x \cup y \\ x \cdot y &= \{A \mid A_0 \in x, A_1 \in y, A ::= A_0 A_1 \in P\} \\ \sigma_i &= \{A \mid A ::= w[i] \in P\} \end{aligned}$$

The $(.)$ operation fully characterizes the binary production rules of the grammar, while σ captures the unary ones. We have the following properties: $(0,+)$ forms a commutative monoid (the usual group of sets with union); 0 is absorbing for $(\cdot)$; and $(\cdot)$ distributes over $(+)$. However, and crucially, $(\cdot)$ is *not* associative.

$$\begin{aligned} x + 0 &= x \\ 0 + x &= x \\ (x + y) + z &= x + (y + z) \\ x \cdot (y + z) &= x \cdot y + x \cdot z \\ x \cdot 0 &= x \\ 0 \cdot x &= x \end{aligned}$$

We will then use matrices of sets of non-terminals to record which non-terminals can generate a given substring. The intention is that $A \in C_{ij}$ iff. $A \xrightarrow{*} w[i..j]$. See Fig. 1 for an illustration. In parsing terminology, a structure containing intermediate parse results is called a chart. We call the set of charts $\mathcal{C}$.

Definition 5. $\mathcal{C} = \mathcal{P}(N)^{\mathbb{N}\times\mathbb{N}}$

We lift the operations $0, +, \cdot$ from sets of non-terminals to matrices of sets of nonterminals, in the usual manner.

Definition 6 $(0, +, \cdot$ on $\mathcal{C})$**.**

$$\begin{aligned} 0_{ij} &= 0 \\ (A + B)_{ij} &= A_{ij} + B_{ij} \\ (A \cdot B)_{ij} &= \sum_k A_{ik} \cdot B_{kj} \end{aligned}$$

The properties are lifted accordingly. The operation σ is used to compute an upper diagonal matrix corresponding to the input w, as follows.

Definition 7 (Initial matrix)**.** *The initial matrix, written $I(w)$, is a square matrix of dimension $|w| + 1$ such that*

$$\begin{aligned} I(w)_{i,i+1} &= \sigma_i \\ I(w)_{i,j} &= 0 \qquad\qquad \text{if } j \neq i + 1 \end{aligned}$$

Let $W^{(1)} = I(w)$. Note that $W^{(1)}_{i,i+1} = \sigma_i$ contains all the non-terminals which can generate the substring $w[i..i+1]$. Let $W^{(2)} = W^{(1)}W^{(1)} + I(w)$. It is easy to see that $W^{(2)}_{i,i+2} = \sigma_i \cdot \sigma_{i+1}$, hence it contains all the non-terminals which can generate the substring $w[i..i+2]$. Consider now $W^{(3)} = W^{(2)} \cdot W^{(2)} + I(w)$. We have

$$\begin{aligned} W^{(3)}_{i,i+3} &= W^{(2)}_{i,i+2} \cdot W^{(2)}_{i+2,i+3} + W^{(2)}_{i,i+1} \cdot W^{(2)}_{i+1,i+3} \\ &= (\sigma_i \cdot \sigma_{i+1}) \cdot \sigma_{i+2} + \sigma_i \cdot (\sigma_{i+1} \cdot \sigma_{i+2}) \end{aligned}$$

and

$$\begin{aligned} W^{(3)}_{i,i+4} &= W^{(2)}_{i,i+2} \cdot W^{(2)}_{i+2,i+4} \\ &= (\sigma_i \cdot \sigma_{i+1}) \cdot (\sigma_{i+2} \cdot \sigma_{i+3}) \end{aligned}$$

Hence $W^{(3)}$ contains all possible parsing of 3 symbols, and all *balanced* parsings of 4 symbols. By iterating n times, one obtains all the parsings of n symbols. (However, as a hint to our method for efficient parsing, it suffices to repeat the process $\log n + 1$ times to obtain all balanced parsings of n symbols).

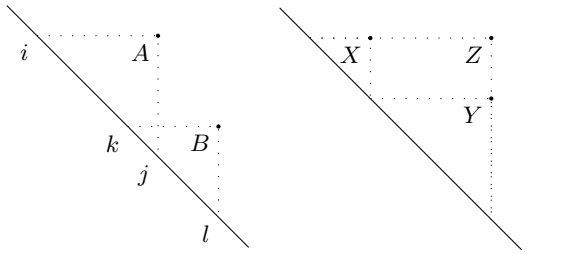


Figure 1. Example charts. In each chart a point at position (x, y) corresponds to a substring starting at x and ending at y. The first parameter x grows downwards and the second y one rightwards. The input string w is represented by the diagonal line. Dots in the upper-right part represent nonterminals. The first chart witnesses $A \xrightarrow{*} w[i..j]$ and $B \xrightarrow{*} w[k..l]$. An instance of the rule $Z ::= XY$ is exemplified on the second chart.

Definition 8 (Transitive closure)**.** *If it exists, the transitive closure of a matrix W, written W^+, is the matrix C such that*

$$C = C \cdot C + W$$

A consequence of the above is $C \supseteq C \cdot C + I(w)$. It is clear by now that, consequently, every possible bracketing of the products $I(w) \cdot \cdots \cdot I(w)$ is contained in C, and thus all possible parsings of $w[i..j]$ are found in C_{ij}. Conversely, because $C \subseteq C \cdot C + W$, if C_{ij} contains a non-terminal then it must generate $w[i..j]$. Algorithms which parse by computing a chart are known as chart parsers.

The above procedure specifies a recognizer: by constructing $I(w)^+$ one finds if w is parsable, but not the corresponding parse. Even though we focus on the recognition problem in this paper, it is straightforward to specify parsers by using matrices of parse trees instead of non-terminals, and adapting the operations accordingly.

In order to construct an efficient parallel parser, we must construct a sequence-homomorphism from input strings to charts. Thanks to Lem. 1, it suffices find an operator bin which combines two charts $I(w_1)^+$, $I(w_2)^+$ and a terminal t into a chart $I(w_1 t w_2)^+$.

3.3 Cocke–Younger–Kasami

A straightforward manner to turn the above specification into an algorithm is as follows.

Let us first remark that the product of two upper triangular matrices is upper triangular. Hence the closure of an upper triangular matrix must also be upper triangular. Hence, in every chart ever considered, every element at the diagonal and below it equals zero. Expanding index-wise the equation $C = C \cdot C + I(w)$ yields:

$$C_{ij} = I(w)_{ij} + \sum_{k=0}^{n} C_{ik} \cdot C_{kj}$$

Because C is upper triangular, $C_{ik} = 0$ if $k \leq i$ and $C_{kj} = 0$ if $k > j$. Hence the sum can be limited to the interval $[i + 1..j]$

$$C_{ij} = I(w)_{ij} + \sum_{k=i+1}^{j} C_{ik} \cdot C_{kj}$$

Observing that the summand equals 0 on the upper diagonal and $I(w)_{ij} = 0$ otherwise, we distinguish on that condition and obtain the two equations:

$$C_{i,i+1} = \sigma_i \tag{2}$$

$$C_{ij} = \sum_{k=i+1}^{j} C_{ik} \cdot C_{kj} \qquad \text{if } j > i + 1 \tag{3}$$

These equations give a method to compute C_{ij} by induction on $j - i$. The equations can be re-interpreted in term of parses and non-terminals as follows. Either

- we parse a single token w_i, and the nonterminals generating it are given directly from unary rules, or
- we parse a longer string. In this case we split it at any intermediate position k, and combine the intermediate results (found in C_{ik} and C_{kj}) in every possible way according to binary rules.

By applying the above rules naively, computation time is exponential in the length of the input; however by memoizing each intermediate result (for example by using lazy dynamic programming (Allison 1992)) the complexity is merely cubic. The resulting dynamic programming algorithm is known as CYK, owing to its independent discoverers: Cocke (1969), Kasami (1965) and Younger (1967).

In the CYK algorithm, any element of the chart is computed only on the basis of elements strictly closer to the diagonal. Hence it can be used to combine charts. The combination of two charts and a terminal $C = bin(A, w[i], B)$, is defined as follows. Elements of C in the upper left corner are copied from A; elements of the bottom right corner are copied from B; and elements from the top right corner are computed using σ_i and the CYK formula (Eq. (3)).

Are we done? Unfortunately, no. The above operator has to compute a matrix of size $n \times m$, and computing each element takes time linear in $n+m$. The complexity of the association is therefore cubic; and dwarfs the time spent on computing the sub-charts A and B. However, for parallelisation to be effective, we need the running time of the combination operator to be less than the running time of the recursive calls.

3.4 Valiant

A more subtle way to turn the transitive closure specification into an algorithm is the following. Our task is to find a function $^+$ which maps a matrix W to its transitive closure $C = W^+$. As above, we do so by refinement of the definition of transitive closure, but we adopt a divide and conquer approach rather than a direct one.

If W is a 1 by 1 matrix, $C = W = 0$. Otherwise, let us divide W and C in blocks as follows (for efficiency the blocks should be roughly of the same size; but the reasoning here holds for any sizes):

$$W = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \qquad C = \begin{bmatrix} A' & X' \\ 0 & B' \end{bmatrix}$$

Then the condition that C is the transitive closure of W becomes

$$\begin{bmatrix} A' & X' \\ 0 & B' \end{bmatrix} = \begin{bmatrix} A' & X' \\ 0 & B' \end{bmatrix} \cdot \begin{bmatrix} A' & X' \\ 0 & B' \end{bmatrix} + \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$$

Applying matrix multiplication and sum block-wise:

$$\begin{aligned} A' &= A'A' + A \\ X' &= A'X' + X'B' + X \\ B' &= B'B' + B \end{aligned}$$

Because A and B are smaller than W (and still upper triangular), we know how to compute A' and B' recursively ($A' = A^+$, $B' = B^+$). There remains to find an algorithm to compute the top-right corner X' of the matrix. That is (renaming variables for convenience) the problem is reduced to finding a recursive function V which maps A, B and X to $Y = V(A, X, B)$, such that $Y = AY + YB + X$. In terms of parsing, the function V combines the chart A of the first part of the input with the chart B of the second part of the input, via a *partial* chart X concerned only with strings starting in A and ending in B, and produces a full chart Y. Let us divide each matrix in blocks again:

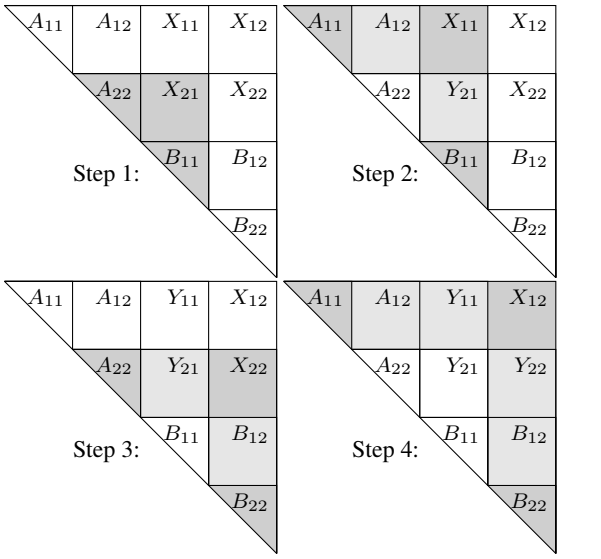


Figure 2. The recursive step of function V. The charts A and B are already complete. To complete the matrix X, that is, compute $Y = V(A, X, B)$, one splits the matrices and performs 4 recursive calls. Each recursive call is depicted graphically. In each figure, to complete the dark-gray square, multiply the light-gray rectangles and add them to the dark-gray square, then do a recursive call on triangular matrix composed of the completed dark-gray square and the triangles.

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \quad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$$

(Again we assume that splitting can be done; the base cases can be obtained by dropping the first rows and/or the second columns in the above splits.) The condition on Y then becomes

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \cdot \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} + \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

By applying matrix multiplication and sum block-wise:

$$\begin{aligned} Y_{11} &= A_{11}Y_{11} + A_{12}Y_{21} + Y_{11}B_{11} + 0 + X_{11} \\ Y_{12} &= A_{11}Y_{12} + A_{12}Y_{22} + Y_{11}B_{12} + Y_{12}B_{22} + X_{12} \\ Y_{21} &= 0 + A_{22}Y_{21} + Y_{21}B_{11} + 0 + X_{21} \\ Y_{22} &= 0 + A_{22}Y_{22} + Y_{21}B_{12} + Y_{22}B_{22} + X_{22} \end{aligned}$$

By commutativity of $(+)$ and 0 being its unit:

$$\begin{aligned} Y_{11} &= A_{11}Y_{11} + X_{11} + A_{12}Y_{21} + Y_{11}B_{11} \\ Y_{12} &= A_{11}Y_{12} + X_{12} + A_{12}Y_{22} + Y_{11}B_{12} + Y_{12}B_{22} \\ Y_{21} &= A_{22}Y_{21} + X_{21} + 0 + Y_{21}B_{11} \\ Y_{22} &= A_{22}Y_{22} + X_{22} + Y_{21}B_{12} + Y_{22}B_{22} \end{aligned}$$

Because each of the sub-matrices is smaller and because of the absence of circular dependencies, Y can be computed recursively:

$$\begin{aligned} Y_{21} &= V(A_{22}, X_{21} &, B_{11}) \\ Y_{11} &= V(A_{11}, X_{11} + A_{12}Y_{21} &, B_{11}) \\ Y_{22} &= V(A_{22}, X_{22} + Y_{21}B_{12} &, B_{22}) \\ Y_{12} &= V(A_{11}, X_{12} + A_{12}Y_{22} + Y_{11}B_{12}, B_{22}) \end{aligned}$$

We have ignored the base cases so far because they are straightforward, except for the following point. When computing $V(A, X, B)$ on matrices of dimension 1×1, it is guaranteed that A and B are equal to 0. Indeed, in that case X is just above the diagonal. Therefore A and B are on it and must then be 0. The result matrix is therefore equal to X.

In sum, with the above definitions, we have the following expression for V in the recursive case

$$V\left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}\right) = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}.$$

In the base cases, some or all of the top and/or right sub-matrices are empty and the corresponding recursive calls are omitted. In terms of parsing, initially the partial chart X contains at the bottom-left position a single non-zero element corresponding to the symbol at the interface of A and B. Recursive calls progressively fill this chart, quadrant by quadrant. The above algorithm was first described by Valiant (1975). A graphical summary is shown in Fig. 2, and Haskell implementation is show if Fig. 3.

From Valiant's function V, one can construct the bin operator (completing the sequence homomorphism) as follows:

$$bin(A, t, B) = \begin{bmatrix} A & V(A, X, B) \\ 0 & B \end{bmatrix} \text{ where } X = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & \iddots & \vdots \\ 0 & 0 & & \vdots \\ \sigma_i & 0 & \cdots & 0 \end{bmatrix}$$

An advantage of Valiant's algorithm over CYK is that it treats whole subcharts at once, via matrix-level multiplication and addition, while CYK explicitly refers to each element of C individually. In particular, when using a sparse-matrix representation, the multiplication of an empty chart with any other chart is instantaneous. The ability to handle this case efficiently is key: we observe that in many cases, charts are sparse, and composition of charts is efficient.

When using a representation supporting sparse matrices, the implementation of Valiant's algorithm is an elegant functional program, as can be seen in Fig. 3.

4. Sparse Matrix assumption and Complexity Analysis

4.1 Model of the input

In practice, matrices representing charts are expected to be sparse for large inputs, that is, a given substring is unlikely to be generated by a given non-terminal. Indeed, in most cases, the substring starts in the middle of a construction and ends in the middle of some other, usually unrelated other construction. This effect is illustrated in Fig. 4. In the remainder of the paper, we assume that inputs conform to this assumption. Before explaining where it is coming from, we give its formal definition.

Definition 9 (Assumption)**.** *There exists a constant α such that, for any input, the distribution of non-zero elements in the chart C corresponding to it is bounded as follows. For any square subchart A of C above the diagonal,*

$$\#A \leq \left\lceil \alpha \sum_{(i,j) \in \mathrm{dom}(A)} \frac{1}{(j-i)^2} \right\rceil$$

where $\#A$ is the number of non-zero elements in matrix A.

We stress that the assumption involves not a grammar *per se*, but the language itself (i.e. the set of possible input strings we consider), when seen as strings generated by a given grammar in CNF. So, for any given α, every non-trivial grammar will admit

```
import Prelude (Eq (..))
class RingLike a where
  zero :: a
  (+) :: a → a → a
  (·) :: a → a → a
data M a = Q (M a) (M a) (M a) (M a) | Z | One a
q Z Z Z Z = Z
q a b c d = Q a b c d
one x = if x ≡ zero then Z else One x
instance (Eq a, RingLike a) ⇒ RingLike (M a) where
  zero = Z
  Z + x = x
  x + Z = x
  One x + One y = one (x + y)
  Q a11 a12 a21 a22 + Q b11 b12 b21 b22
    = q (a11 + b11) (a12 + b12)
        (a21 + b21) (a22 + b22)
  Z · x = Z
  x · Z = Z
  One x · One y = one (x · y)
  Q a11 a12 a21 a22 · Q b11 b12 b21 b22
    = q (a11 · b11 + a12 · b21) (a11 · b12 + a12 · b22)
        (a21 · b11 + a22 · b21) (a21 · b21 + a22 · b22)
v :: (Eq a, RingLike a) ⇒ M a → M a → M a → M a
v a          Z          b = Z
v Z          (One x)    Z = One x
v (Q a11 a12 Z a22) (Q x11 x12 x21 x22) (Q b11 b12 Z b22)
  = q y11 y12 y21 y22
    where y21 = v a22  x21                    b11
          y11 = v a11 (x11 + a12 · y21       ) b11
          y22 = v a22 (x22 +          y21 · b12) b22
          y12 = v a11 (x12 + a12 · y22 + y11 · b12) b22
```

Figure 3. Data structure for charts as sparse matrices (M), and implementation of the function V. The tricky parts compared to the mathematical development of Sec. 3.4 is the handling of empty matrices. Care must be taken to create empty matrices (Z) whenever they contain only zero elements. This is done by using the smart constructors q and one in matrix multiplication. The input matrices a and b are empty iff. the matrix x has dimension one. For concision, this implementation supports only matrices of size 2^n for some n. It can be extended to matrices of arbitrary dimension in a straightforward manner by adding constructors for row and column matrices, to be used as leaves. An implementation supporting arbitrary matrix dimensions, as well as the optimization explained in Sec. 7.2 can be found in the BNFC repository:
`https://github.com/BNFC/bnfc/blob/master/source/runtime/Data/Matrix/Quad.hs`

strings that break the assumption, but usually the set of strings we consider behaves well in practice.

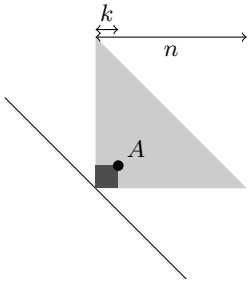
The above formula merits justification. Before using it to evaluate the complexity of the parsing algorithm, we will build a more precise intuition for it by examining its consequences and its possible causes.

By symbolically evaluating the sum, we can deduce what the above assumption implies for each of the following shapes for A:

- if A is a square of size n touching the diagonal, then $\#A \leq \lceil \alpha \log n \rceil$
- if A is a square of size n at distance kn to the diagonal with $k \geq 1$, then $\#A \leq \lceil \alpha(2\log(k+1) - \log(k+2) - \log(k)) \rceil$. Hence, remarkably, a square chart of dimension n which is at least n elements away from the diagonal contains at most a number of elements which is independent from n.

A stochastic way to formulate the idea behind our assumption is the following. When considering k random substrings of size n in a corpus of strings representative of the language, one finds on average that $\frac{\alpha k}{n^2}$ of them correspond to a single nonterminal. That is, by doubling the size of the substring considered, it will be four times less likely to be parsable. This condition agrees with experience.

In order for the assumption to be satisfied, it is also sufficient to assume that the parse trees corresponding to the inputs are balanced. That is, if every node in the chart corresponding to a substring of size n is combinable with exactly one other symbol of size n (precluding ambiguity), then the assumption is verified. In fact the assumption can be relaxed as follows. Consider the triangle-shaped subchart T^n which touches the diagonal and a non-terminal A at distance k from it. We assume all symbols in the triangle but closer to the diagonal combine to form A. If the symbol A can be combined with exactly one other symbol of size βk with $0 < \beta \leq 1$, it will yield exactly one symbol at distance $(1+\beta)k$. Inductively we compute that is of the order of $\frac{\log(n)}{\log(1+\beta)}$ nodes in the triangle, which is compatible with our condition, with $\alpha = 1/\log_2(1+\beta)$.



Experiments The assumption we make is not strictly speaking verifiable experimentally, since for any chart there exists an α such that the assumption is verified. However, one can gain confidence in the assumption by plotting the probability of a string to be parsable against its size. One should observe that this probability decreases with the square of the size. In practical terms, given a chart corresponding to a large input, if one observes a drastic cut-off in the density of non-zero elements when departing from a certain distance from the diagonal, then the input is compatible with our assumption. In Fig. 4, we show a chart corresponding to a fragment of C code, obtained using our algorithm. This chart, along with all other inputs for which we have run this experiment, exhibits the expected features. The assumption is also confirmed, albeit indirectly by observing that the cost analysis which depends on it holds in practice.

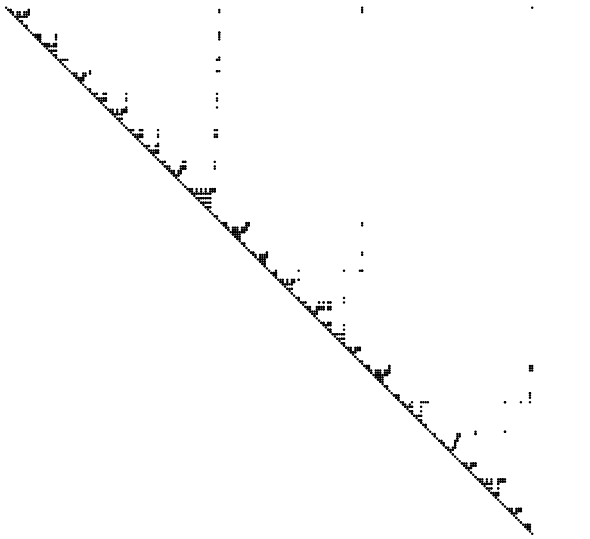


Figure 4. The chart corresponding to a fragment of a C program. The input program can be found in appendix. Two remarkable features merit commentary. First, the staircase shapes, which are explained in Sec. 5.4. Second, some small sub-matrices near the diagonal appear to be dense. These regions correspond to argument lists in the C program, and this iteration structure is implemented by linear recursion rather than our special encoding of Sec. 5.

Non-suitable inputs Any input which uses nesting in linear proportion to the size of its input will violate our assumption. For example, the lisp program composed of n successive applications of cons does not satisfy our assumption.

$$(\mathsf{cons}\ x\ (\mathsf{cons}\ x\ (\ldots(\mathsf{cons}\ x\ \mathsf{nil})\ldots)))$$

It appears however that few programs are written in this style, except perhaps for machine-generated ones. Linear constructions are often present, but they are then supported by special syntax. Indeed the above lisp program is invariably written as:

$$(\mathsf{list}\ x\ x \ldots x)$$

Hence we provide special treatment for such special iteration syntaxes. We show in Sec. 5 how to deal with them, while respecting our assumption.

4.2 Cost estimation

For simplicity we consider only inputs of sizes which are powers of 2. This implies that we only need to consider square matrices in our analysis. We will estimate the cost as the number of elementary multiplications (multiplications on sets of non-terminals) to be performed.

We first remark that because charts are always divided in the middle, a subchart X considered by the algorithm is always square, and at a distance kn to the diagonal, where k is some natural number and n is the dimension of X. When $k = 0$ we say that X touches the diagonal and when $k > 0$ we say that X is far from the diagonal. This distinction is crucial, because matrices touching the diagonal have $O(\log n)$ elements in them, whereas matrix far away have a constant number of elements in them.

4.2.1 Cost of matrix multiplications

We start by estimating the cost of the multiplication of two matrices A and B of size n, $M_n(A, B)$. Assuming that $p = \#A$ and $q = \#B$, then matrix multiplication needs to perform $O(pq)$ elementary multiplications. Indeed, each non-zero element in A

needs to be multiplied at most with every non-zero element in B. We now assume that A and B are also a subcharts, and distinguish the following cases.

1. Both A and B touch the diagonal. We have $p = O(\log n)$ and $q = O(\log n)$ and thus $M_n(A, B) = O(\log^2 n)$.
2. Either A or B is at least n away from the diagonal. In this case there either p or q is bounded by a constant and thus $M_n(A, B) = O(\log n)$.
3. Both A and B are at least n away from the diagonal. In this case there both p and q are bounded by a constant and thus $M_n(A, B) = O(1)$.

Before proceeding we recall a standard result:

Theorem 1 (Master Theorem, Cormen et al.)**.** *Assume a function T_n constrained by the recurrence*

$$T_n = aT_{\frac{n}{b}} + f(n)$$

(Such an equation will typically come from a divide-and-conquer algorithm, where a is the number of sub-problems at each recursive step, n/b is the size of each sub-problem, and $f(n)$ is the running time of dividing up the problem space into a parts, and combining the sub-results together.)

If we let $e = \log_b a$ and $f(n) = O(n^c \log^d n)$, then

$$\begin{aligned} T_n &= O(n^e) && \text{if } c < e \\ T_n &= O(n^c \log^{d+1} n) && \text{if } c = e \\ T_n &= O(n^c) && \text{if } c > e \end{aligned}$$

4.2.2 Cost of the conquer step

We proceed to estimate the running cost V_n of the valiant function V on a matrix of size n. We do so assuming that we know the resulting chart $Y = V(A, X, B)$. That is, V_n maps Y to the cost of running $V(A, X, B)$. We have the following recurrence:

$$\begin{aligned} V_n(0) &= 0 \\ V_n \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} &= V_{\frac{n}{2}}(Y_{21}) + V_{\frac{n}{2}}(Y_{11}) + V_{\frac{n}{2}}(Y_{22}) + V_{\frac{n}{2}}(Y_{12}) \\ &\quad + M_{\frac{n}{2}}(A_{12}, Y_{21}) + M_{\frac{n}{2}}(Y_{21}, B_{12}) \\ &\quad + M_{\frac{n}{2}}(A_{12}, Y_{22}) + M_{\frac{n}{2}}(Y_{11}, B_{12}) \end{aligned}$$

Because A and B are upper-triangular matrices, the subcharts A_{12} and B_{12} touch the diagonal. We distinguish two cases: either Y touches the diagonal or it is at least at a distance n from it.

Y is far away In the latter case we pose $F_n = V_n$ (F for far), and all sub-matrices of Y are far from the diagonal. Hence all matrix multiplications involve at most one matrix touching the diagonal, and the total combined cost of multiplications is $O(\log n)$. The recurrence specializes then to:

$$\begin{aligned} F_n(0) &= 0 \\ F_1(Y) &= 1 \\ F_n \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} &= F_{\frac{n}{2}}(Y_{11}) + F_{\frac{n}{2}}(Y_{12}) + F_{\frac{n}{2}}(Y_{21}) + F_{\frac{n}{2}}(Y_{22}) \\ &\quad + O(\log n) \end{aligned}$$

Because Y has a constant number of non-zero elements, most recursive calls will return immediately, and on average only one recursive call need to be counted. Hence we use the Master Theorem with $a = 1, b = 2$ and $f(n) = O(\log n)$. We are therefore in the case $c = e$, and obtain $F_n = O(\log^2 n)$.

Y touches the diagonal In the former case, let $V_n = C_n$ (C for close), and Y_{21} touches the diagonal. Hence two of the matrix multiplications involve pairs of matrices at the diagonal, and the combined cost of multiplications is $O(\log^2 n)$. Only the recursive call involving Y_{12} touches the diagonal, and the three others involve matrices far from it. Therefore the recurrence specializes to:

$$\begin{aligned} C_1 &= 1 \\ C_n &= C_{\frac{n}{2}} + 3F_{\frac{n}{2}} + O(\log^2 n) \\ &= C_{\frac{n}{2}} + O(\log^2 n) \end{aligned}$$

We use the Master Theorem with $a = 1$, $b = 2$ and $f(n) = O(\log^2 n)$. We are therefore in the case $c = e$, and obtain $C_n = O(\log^3 n)$.

4.2.3 Total cost

We can proceed to compute the total cost of our algorithm on an input string of size $n = |w|$, again using the Master Theorem. We always divide the input into two parts, so $b = 2$. We assume that the input is provided as a balanced tree representing the matrix $I(w)$, and so the cost of the divide step is zero. Therefore $f(n)$ is the cost of the conquer step only. This step involves a matrix close to the diagonal, so $f(n) = C_n = O(\log^3 n)$, and in turn $c = 0$ and $d = 3$.

- If we assume a sequential execution of sub-problems then $a = 2$. In turn, $e = 1$ and $T(n) = O(n)$.
- If we assume perfect parallelisation of sub-problems, or an incremental situation, where one of the sub-solution can be reused, then $a = 1$. In turn, $e = 0$ and $T(n) = O(\log^4 n)$.

We remark that Valiant's evaluation for $V(n)$ is $O(n^\gamma)$, for some γ between 2 and 3 (the exact value depends on the matrix multiplication algorithm used). In his case $c = \gamma$ and $d = 0$, yielding $T(n) = O(n^\gamma)$ in all cases: according to Valiant's original analysis, making an incremental or parallel version of his algorithm would lead no benefit, while our analysis reveals that a big payoff is at hand.

4.2.4 Experiments

We have conducted two sets of experiments on the running time of the algorithm. All timings were obtained using the CRITERION library (O'Sullivan 2013), on an Intel Core 2 at 2.13GHz. All programs were compiled with GHC 7.6.1. In the first set, we have measured the performance on a practical language on practical inputs, to confirm that the function is fast enough to use as an incremental parser in an interactive setting. To do so, we have run our BNFC implementation on a C grammar to produce the σ and $(\cdot)$ functions, and tested the running time of the V function on a large C program, extracted from the Linux kernel scheduler (`https://github.com/torvalds/linux/blob/master/kernel/sched/core.c` — preprocessor directives as well as `typedef`s found in it were expanded by hand.) The input was divided into a left part and a right part of equal sizes, and a middle symbol. The complete charts for the left and the right part were computed, then we measured the time of the V function on the charts and the singleton chart containing the middle symbol. After collecting 100 samples, CRITERION reported a mean runtime of 320.1469 μs, with a standard deviation of 23.06691 μs. This is well within acceptable limits for interactive use: most people cannot perceive a delay less than a millisecond.

In the second set of experiments, we tested the V function on generated inputs of various sizes, to confirm our calculation of the worst case running time. The grammar is that corresponding to the encoding of t^* (the nonterminal t repeated an arbitrary number of times) using the technique described in Sec. 5 (which ensures that our assumption is verified with α close to 1). The inputs were a repetition of that terminal symbol. The results are shown

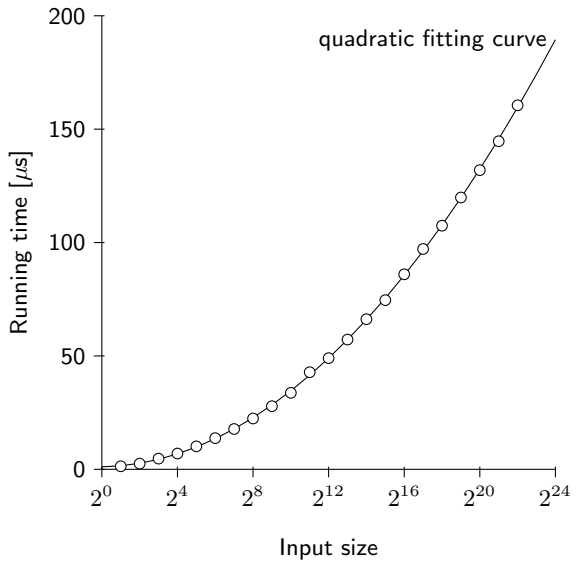


Figure 5. Running time of the V function in function of the size of the input, using semi-logarithmic scale. The grammar is that corresponding to the encoding of t^* using the technique described in Sec. 5. The next data point (input size 2^{23}) could not be obtained due to running out of memory. The curve is the graph of a quadratic function which fits the measurements.

in Fig. 5. We observe that the measurements, when drawn on a semi-logarithmic scale, fit a quadratic curve.

The observed running therefore appears better than the theoretical cost estimation, which predicts a cubic curve. The best explanation we have for this discrepancy is that we made a pessimistic assumption in our analysis: we have estimated the cost of the matrix product to be at worst $O(\log^2 n)$ if each argument matrix has $O(\log n)$ elements. However, this occurs only if the elements are lined up so that every pair needs to be considered. This case may occur in practice, but we conjecture that it happens so infrequently that it does not contribute significantly to the total running time. This experiment thus suggests that a more precise analysis of the algorithm can be done, which we leave for future work.

5. Iteration in Context-Free Grammars

5.1 The problem with iteration

While we have worked hard to ensure the efficient handling of the non-associative aspect of CF parsing, we have neglected so far that most CF languages feature regular iteration; that is, associative concatenation rules. Without special treatment, such associative rules cause severe inefficiencies in the algorithm as presented so far.

Iteration is technically known as Kleene star, and is written here as a postfix *. In context-free grammars, it can be (and usually is) encoded as either as left or right recursion. For example a rule $A ::= Y^*$ is typically encoded as follows.

$$A ::= \epsilon$$
$$A ::= AY$$

The problem with this encoding is two-fold. First, inputs consisting mostly of a sequence of Y necessarily violate our assumption on inputs: the depth of the parse tree grows linearly with the size of the input.

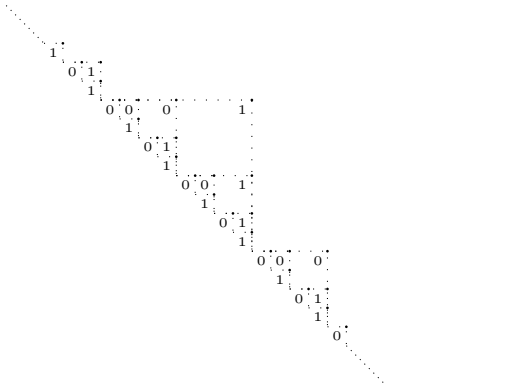


Figure 6. Matching a list using the oracle-sensitive algorithm. We assume that only one non-terminal Y is involved and thus show only the bit-tags. Considering only the non-terminals which cannot be combined using the rule $Y ::= Y^0Y^1$, the charts features a sequence of Y^1 (of increasing size), followed by a sequence of Y^0 (of decreasing size).

Second, the generated AST will necessarily be linear. Consequently, as we have seen in the introduction, this linear shape would preclude efficient parallel or incremental processing of the AST by computations consuming it.

One could possibly imagine working around the first problem with creative algorithmic devices. However it is clear that the second problem is intrinsic to the encoding of iteration as linear recursion. Hence we take the stance that special support for iteration is necessary in any parallel or incremental parser.

5.2 Towards an efficient encoding

Instead of a linear, unary encoding of iterations, one can attempt a binary tree encoding. One might propose the following encoding:

$$A ::= AA$$
$$A ::= Y$$

However this encoding accepts all possible associations of sequences of Ys, in particular also linear ones. One might attempt to mend the rules by using a more clever encoding, say:

$$A_{k+1} ::= A_k A_k$$

Ignoring that it codes only lists of size 2^n for some n, our second condition on inputs is still be violated. Indeed, in a sequence of Y, any subsequence of length 2^n for some n will be recognized. This means that there will be a lot of overlap between possible parse trees.

In the remainder of the section we describe a way to keep the rule $A ::= AA$, but tweak the parsing algorithm so that for any sequence of Ys only a single association is considered.

5.3 Oracle-sensitive parsing

Overview Each nonterminal will come with a bit indicating whether it should be used either as a left or right-child in the parse tree. The bit will be chosen by an oracle upon production of the nonterminal, so that the tree will be balanced. We write Y^b for the non-terminal Y annotated with bit b. The main rule constructing trees is then written:

$$Y ::= Y^0Y^1$$

This restricts which trees are explored. After parsing with this rule, we obtain a sequence of Y^1 (unmatched right children) of growing size followed by a sequence of Y_0 (unmatched left children), as

depicted in Fig. 6. These nodes will then be collected using special rules. Assuming that C_0 and D_0 delimit the list of non-terminals Y^*, the collecting rules would be written:

$$\begin{array}{ll} C ::= C_0 & D ::= D_0 \\ \quad ::= C\,Y^1 & \quad ::= Y^0 D \end{array}$$

And the final list can be produced by the rule $L ::= CD$.

The delimiters C_0 and D_0 are necessary so that only one collection of Y^1 and only one collection of Y^0 are needed; thereby ensuring a good performance. Without delimiters, every combination of sequences of Y^1 and Y^0 would need to be considered. An intermediate situation is where only one delimiter is present, say the opening one. In that case, only one list of Y^1 is considered, but many sequences of Y^0 would be considered.

Oracle-Sensitive Grammar Formalism In general, we extend productions so that non-terminals on a right-hand-side are tagged with a bit. Formally, we extend the syntax of the productions as follows, where $b_1, b_2, \ldots$ range over bits:

- $A ::= B^{b_1} C^{b_2}$
- $A ::= t$, for $t \in \Sigma$

We allow, as a shorthand, to write non-annotated non-terminal in a production right-hand-side. The production then stands for a pair of rules with either annotation. That is $A ::= \alpha_0 B \alpha_1$ is a shorthand for the pair of rules $A ::= \alpha_0 B^0 \alpha_1$ and $A ::= \alpha_0 B^1 \alpha_1$.

Algorithm An implementation will take a grammar written using a special construction for iteration and translate it to the above formalism appropriately. The parsing algorithm *per se* remains the same as previously, except for the operator combining non-terminals, which is changed as follows.

Definition 10.

$$x \cdot y = \{A^b \mid B^{b_1} \in x, C^{b_2} \in y, A ::= B^{b_1} C^{b_2} \in P\}$$

where the output bit b comes from the oracle.

The transitive closure function modified to use the above version of the $(\cdot)$ operator is called T in the remainder.

Formalization and proof We proceed to prove that the above implementation indeed recognizes the intended language. Firstly, we must define the meaning of our extended grammar formalism and show that it corresponds to our needs.

The main issue is that the algorithm behaves non-deterministically, in the sense that the grammar-writer does not have access to the bits generated by the oracle. The rest of the section is structured as follows:

1. we define a generation relation restricted to a given source bits ρ;
2. we show that the algorithm decides the relation for a specific (but intangible) ρ;
3. we narrow the acceptable grammars to those which are oblivious to ρ (describe languages independent of ρ);
4. we provide a toolkit which enables to identify and construct oblivious grammars;
5. and finally we show that our encoding of iteration preserves obliviousness.

Oracle We define a new generation relation $\overset{\rho}{\longmapsto}$, indexed by a stream of bits ρ. This stream of bits wholly models the oracle.

The meaning of production rules annotated with bits can then be given. We first define a 1-step generation relation indexed by a single bit.

Definition 11 (bit-indexed generation)**.**

- *if* $(A ::= B^{b_1} C^{b_2}) \in P$, *then* $w_0 A^b \alpha \overset{b}{\longmapsto} w_0 B^{b_1} C^{b_2} \alpha$
- *if* $(A ::= x) \in P$, *then* $w_0 A^b \alpha \overset{b,\rho}{\longmapsto} w_0 x \alpha$

Crucially, the rules require the relation to act on the first non-terminal in a string. This forces the bit-stream ρ to be used in a deterministic way. Otherwise, the relation could use each bit of ρ in a arbitrary place, essentially bypassing the instructions of the oracle, transmitted via the bitstream ρ.

Definition 12 (stream-indexed generation)**.** *The relation* $\alpha \overset{\rho}{\longmapsto} w$ *is inductively defined as follows.*

- $w \overset{\rho}{\longmapsto} w$
- *If* $\alpha \overset{b}{\longmapsto} \gamma$ *and* $\gamma \overset{\rho}{\longmapsto} w$ *then* $\alpha \overset{b,\rho}{\longmapsto} w$

Algorithm The algorithm decides the $\overset{\rho}{\longmapsto}$ relation, but only for one particular bit-stream ρ (which the grammar-writer has no control over).

Theorem 2.

1. *if* $A^b \in T(I(w))_{ij}$ *then* $\exists \rho. A^b \overset{\rho}{\longmapsto} w_{ij}$
2. *if* $A^b \notin T(I(w))_{ij}$ *then* $\exists \rho. A^b \overset{\rho}{\not\longmapsto} w_{ij}$

Proof. By induction on the decomposition structure of the matrix (done by T). □

Obliviousness Ultimately, we do not want the language defined using our formalism to depend on the actual stream ρ of bits generated by the oracle, since this is out of the control of the grammar writer. That is, if a string is recognized using some ρ, it should be recognized with every ρ.

We first remark that the set of strings generated by any given tagged non-terminal will always depend on ρ. Hence instead we have to consider the strings generated by sets of non-terminals (and in general sets of strings).

We thus define the following relations, using Γ, Δ and Ξ to range over sets of strings.

Definition 13.

- $\Gamma \overset{\exists}{\longmapsto} w$ *iff.* $\exists \rho. \exists \alpha \in \Gamma.\ \alpha \overset{\rho}{\longmapsto} w$
- $\Gamma \overset{\forall}{\longmapsto} w$ *iff.* $\forall \rho. \exists \alpha \in \Gamma.\ \alpha \overset{\rho}{\longmapsto} w$

Definition 14. *A set of strings* Γ *is called* oracle-oblivious *if the set of strings of terminals generated by it is insensitive to non-determinism; that is, for any* w_0, *if* $\Gamma \overset{\exists}{\longmapsto} w_0$ *then* $\Gamma \overset{\forall}{\longmapsto} w_0$.

Definition 15. *We note* $\tilde{A}$ *the set* $\{A^0, A^1\}$.

Definition 16 (well-formed grammar)**.** *A oracle-sensitive grammar is well-formed if* $\tilde{S}$ *is oracle-oblivious.*

We can then show that obliviousness fulfills its purpose: the sensitivity to ρ introduced in the algorithm is indeed hidden by obliviousness.

Theorem 3. *If* $\tilde{A}$ *is oracle-oblivious then*

$$\tilde{A} \overset{\forall}{\longmapsto} w_{ij} \quad \textit{iff} \quad A^b \in T(I(w))_{ij}, \textit{for some bit } b$$

Proof. The left-to-right direction is the contrapositive of Th. 2.2. The right-to-left direction is obtained by composing Th. 2.1 with the definition of obliviousness for $\tilde{A}$. □

A kit for well-formed grammars Given a grammar definition using bit-annotations arbitrarily, it is hard to decide whether it is well-formed. Hence we define the following relation, which enables us to reason about obliviousness compositionally.

Definition 17. $\Gamma \overset{*}{\Longrightarrow} \Delta$ *iff for every* w_0,

- *if* $\Gamma \overset{\exists}{\longmapsto} w_0$ *then* $\Delta \overset{\exists}{\longmapsto} w_0$.
- *if* $\Delta \overset{\forall}{\longmapsto} w_0$ *then* $\Gamma \overset{\forall}{\longmapsto} w_0$.

The above relation is constructed to transport obliviousness:

Lemma 2. *If* $\Gamma \overset{*}{\Longrightarrow} \Delta$ *and* Δ *is oracle oblivious, then so is* Γ.

Proof. Direct consequence of the definition. □

Lemma 3.

1. $\overset{*}{\Longrightarrow}$ *is reflexive and transitive*
2. *If* $\Gamma \overset{*}{\Longrightarrow} \Delta$ *then* $\Gamma\Xi \overset{*}{\Longrightarrow} \Delta\Xi$ *and* $\Xi\Gamma \overset{*}{\Longrightarrow} \Xi\Delta$
3. *Assume a non-terminal* A *and* Γ *its set of productions. Then* $\tilde{A} \overset{*}{\Longrightarrow} \Gamma$.

Proof. 1. and 3. are a direct consequences of the definitions. The proof of 2. is tedious but straightforward, and similar in style to the proof of Lem. 4 and thus omitted. □

The above lemma means that, if productions are written without bit annotations (they generate all possible annotations), then they preserve obliviousness. Hence, a grammar written without annotations is necessarily well formed. Because our encoding of iteration also preserves obliviousness, this in turn means that, if one uses annotations only to encode iteration in the pattern we prescribe, the grammar will be well-formed.

Encoding iteration As a reminder, we encode $L ::= C_0 Y_0 * D_0$, as

$$\begin{aligned} Y &::= Y_0 \\ &::= Y^0 Y^1 \\ C &::= C_0 \\ &::= C\, Y^1 \\ D &::= D_0 \\ &::= Y^0 D \\ L &::= CD \end{aligned}$$

Theorem 4. $\tilde{L} \overset{*}{\Longrightarrow} \tilde{C}_0 \tilde{Y}_0^* \tilde{D}_0$

Proof. We construct the relation in the following stages.

1. $\tilde{L}$
2. $\tilde{C}_0 \{Y^1\}^* \{Y^0\}^* \tilde{D}_0$
3. $\tilde{C}_0 \tilde{Y}^* \tilde{D}_0$
4. $\tilde{C}_0 \tilde{Y}_0^* \tilde{D}_0$

Most of the steps are consequences of Lem. 3. Only the step between stages 2. and 3. requires special treatment: it depends on the relation

$$\{Y^1\}^* \{Y^0\}^* \overset{*}{\Longrightarrow} \tilde{Y}^*$$

Proving it requires to preservation lemmas for every w_0:

- if $\{Y^1\}^* \{Y^0\}^* \overset{\exists}{\longmapsto} w_0$ then $\tilde{Y}^* \overset{\exists}{\longmapsto} w_0$.
- if $\tilde{Y}^* \overset{\forall}{\longmapsto} w_0$ then $\{Y^1\}^* \{Y^0\}^* \overset{\forall}{\longmapsto} w_0$.

The first one is an easy consequence of the ability to chose any possible ρ in the $\overset{\exists}{\longmapsto}$ relation. The second one is the angular stone of our method, and is proved in the following lemma. □

Lemma 4. *Let* $w \in \Sigma^*$ *and* $\alpha \in \tilde{Y}^*$. *If* $\alpha \overset{\forall}{\longmapsto} w$ *then there exists* $\beta \in \{Y^1\}^*$ *and* $\gamma \in \{Y^0\}^*$ *such that* $\beta\gamma \overset{\forall}{\longmapsto} w$.

Proof. By induction on the length of α. If α is in the required form, we have the result. If not, then the subsequence $Y^0 Y^1$ can be found at least once in α:

$$\alpha = \alpha_0 Y^0 Y^1 \alpha_1$$

We can decompose w into two parts w_0 and w_1 such that

$$\alpha_0 \overset{\forall}{\longmapsto} w_0$$
$$Y^0 Y^1 \alpha_1 \overset{\forall}{\longmapsto} w_1$$

But, for any b, we have $Y^b \alpha_1 \overset{b}{\longmapsto} Y^0 Y^1 \alpha_1$. Therefore, $Y^b \alpha_1 \overset{\forall}{\longmapsto} w_1$ and in turn $\alpha_0 Y^b \alpha_1 \overset{\forall}{\longmapsto} w$.

We can then use the induction hypothesis on $\alpha_0 Y^b \alpha_1$ to obtain β and γ satisfying the conditions of the theorem. □

5.4 Performance

The above encoding yields good performance in practice, even with a straightforward implementation of the oracle providing the stream of bits ρ. Indeed, Fig. 6 shows the chart generated from a sample C program. It exhibits the drastic cut-off in non-zero node density formalized in Def. 9, except for a few linear shapes, as one can observe. These are caused by our implementation of the oracle, which is naive. In our implementation, the bit which is generated is a parameter of the function V, and it is flipped (deterministically) for some recursive calls. This means that, inside a given subchart, all instances of associative rules either right-associate or left-associate, yielding a linear arrangement of results in the chart. Yet, this strategy for bit generation is the best we have found with respect to observed performance. The reason might be that more even distributions of results in the chart worsens the locality of non-zero data, yielding smaller zero subcharts.

6. Related work

6.1 Our own previous work

Claessen (2004) wrote a paper titled "parallel parsing processes", but which has only tenuous connections with the present work. The paper of 2004 presents a parsing technique based on usual sequential parsers, but where disjunction is represented by processes running concurrently. An advantage of that technique is that the parser processes the input string in chunks that can be discarded as soon as the parser has analyzed them.

Bernardy (2009) has shown how to combine the above idea with the online parsers of Hughes and Swierstra (2003). This makes the resulting parsing algorithm suitable for incremental parsing in an editing environment such as Yi (Bernardy 2008). However the method is brittle, because grammars need to be expressed in a special-purpose formalism, and error-correction must be "bake-in" the grammar. In contrast, the method presented here works with context free grammars as usually written; only iterative structures need to be changed to use the special construction of Sec. 5. One does not have to worry about error recovery since all substrings are parsed.

6.2 Special support for iteration

The assumption we make on inputs, which is tied to the balancing of the parse trees is partially inspired by work by Wagner and Graham (1998). They show that linear parse trees cannot be handled efficiently, since updating a structure requires time proportional to its depth. Wagner and Graham then deduce that efficient incremental parsing requires a special purpose support for iteration, as we have done in Sec. 5.

6.3 General CF Parsing

Perhaps the most well known method for parsing general CF languages is that of Tomita (1986). This method has in common with ours that it achieves linear performance on well-behaved inputs, while degrading gracefully to the best possible performance (cubic) in the worst case.

The main difference between the methods is that Tomita's algorithm processes the input sequentially, while we can process it any bottom-up order. This means that the condition for well-behaved inputs is different for either methods. In Tomita's case, the condition is that, at any point during the parsing, the amount of ambiguity is small (bound by a constant), implying that the next action of the parser is most of the time determined by the next symbol in the input. In our case, it is captured by Def. 9, which essentially means that the input should be hierarchical. Tomita's condition does not imply ours: linearly arranged inputs can be deterministic. Checking the other implication is left for future work. It is not easy to conclude: our condition imposes non-local conditions which may or may not restrict non-determinism in a linear processing of the input.

The chief advantage of our method is its divide-and-conquer structure, which means that is can be used in a standard parallel or incremental framework. Tomita inherits essential use of the sequential processing of the input from LR parsing, making his technique not amenable to parallelization.

6.4 Parallel Parsing

There is a wealth of previous work devoted to efficient recognition and parsing of context-free languages on abstract parallel machines, so much that a comprehensive survey of the field does not fit in this paper. The situation can however be summarized as follows: to the best of our knowledge, before this work, algorithms proposed for parallel parsing either need an unrealistic number of processors, or they target a language class which is too restrictive to be of practical interest.

Too many processors Sikkel and Nijholt (1997) describe a parallel algorithm (in section 6.3) which can recognize a string of length n in $O(\log n)$ time, but it requires $O(n^6)$ processors in the worst case.

A line of work involving Rytter gives a dozen of complexity results for various sub-classes of CF and various abstract machines. The most closely related results are perhaps the following.

Chytil et al. (1991) present a simple parallel algorithm recognizing unambiguous context-free languages on a CREW PRAM in time $\log^2 n$ with only n^3 processors. The similarity with our work is that the authors restrict the languages they accept to a well-behaved subset of CF to obtain sensible running time. In our opinion the present work captures better the actual sets of inputs found in the actual practice of CF parsing.

Too restrictive grammars Rytter and Giancarlo (1987) analyze an algorithm which can parse a bracket grammar in $O(\log n)$ time and $O(n/\log n)$ processors. This is fast and does not use too many processors, but is restricted to languages where the grouping of non-terminals is completely explicit in the input: each production rule starts with an opening bracket and ends with a closing bracket.

Parallel implementations of Existing Chart Parsing Algorithm Dozens of papers have been written about parallel implementations of usual chart parsing algorithms, again too many to survey extensively. Most work in that area focuses on practical aspects such as balancing the load between execution units or optimizing the flow of communication between them. Overall the issues addressed appear largely disjoint from our concerns.

6.5 Automatic Parallelisation

Gibbons (1996) (following the work of Bird (1986)) states that if a function can be expressed both as a leftwards and rightwards function (foldl and foldr), then it can also be expressed as a sequence homomorphism. Morita et al. (2007) use this theorem to derive such sequence homomorphism algorithmically. They present a tool which can produce a sequence homomorphism when given functions expressed both as foldl and foldr.

It would be interesting to check if the method could derive an efficient parallel parsing algorithm. As far as we understand, the method might (possibly with extensions) be able to discover the Valiant algorithm from a leftwards and a rightwards CYK algorithm. However, discovering the interest of a sparse matrix representation out of reach: it requires a creative step which is hard to capture in an automatic tool.

Mainstream parsing algorithms (such as LL(k) or LALR(k)) also seem hard to parallelise using an automatic method. First, it is not clear how one can reverse such a parser, since the definition of the algorithm is tightly coupled with direction of parsing (as their name indicates). Second, Morita et al. (2007) do not give an upper bound on the efficiency of the generated combination operator (bin), but only measure the performance of the generated code on a number of examples. As we understand there may be situations where the method produces an associative operator of linear (or worse) complexity, thereby defeating the effectiveness of parallelisation.

6.6 Simultaneous incremental and parallel computation

Burckhardt et al. (2011) propose a model of computation which captures both incremental and parallel execution. Their model is based on concurrently running tasks which commit their results atomically upon completion. Our work is instead based on the well-known sequence homomorphism as model of parallel and incremental computation.

7. Discussion

7.1 Destructive updates

We were tempted to solve the problem of iteration by using destructive updates. That is, to have associative rules such as $Y ::= YY$ consume their arguments. That is, when a Y non-terminal is added to the chart using the above rule, the two Y non-terminals that compose it would be removed. We have attempted this solution, but faced a couple of issues, which will not surprise an audience of functional programmers.

- On the theoretical side, reasoning about parsing with destructive updates of the chart has proven too complicated to fit in this paper. The generation relation describing which strings are recognized by a parser is hard to define, let alone reason about. A major difficulty is to combine destructive updates with a notion of non-determinism similar to that described in Sec. 5. Indeed, the user has no control on which particular consuming rule will fire first, since this depends on the particular of the implementation of Valiant's algorithm (the order in which matrix multiplications are run, etc.) and the exact positioning of the substrings.
- On the practical side, the presence of updates makes for a more complicated implementation. It would also mean to abandon (so far unexploited) parallel opportunities in the matrix multiplication and the V function.

7.2 Optimization

In many grammars, a fair proportion of non-terminals occur only either on the left, or on the right of binary productions. Assume for

example that A only ever occurs on the left. It is wasteful in this case to consider A for right-combinations, as does the algorithm we have presented so far.

This optimization is available to many CF parsing algorithms, but it is especially useful to us, because it acts in synergy with the detection of empty matrices. Indeed, by having separate matrices of left-combinable and right-combinable non-terminals, each matrix will be sparser. This means some combinations can be discarded *in blocks*, that is, at the level of matrices instead at the level of individual non-terminals.

An additional benefit of this optimization is that it pays for the cost of tagging non-terminals with an extra bit, as we suggest in Sec. 5. Indeed, 0-tagged non-terminals occur only on the left of binary productions, and 1-tagged non-terminals occur only on the right in our encoding of iteration. Therefore this optimization eliminates all the cost of tagging: instead of tagging a non-terminal with a bit, it suffice to insert it only in the relevant matrix.

7.3 Implementation

An implementation of the parsing method presented here, including special support for iteration as presented in Sec. 5 and the optimization presented above, is implemented as a new back-end for the BNFC tool, (Forsberg and Ranta 2012) available in version 2.6. The tool takes a grammar in BNF with annotations for efficient repetition. When running the tool with the option `-cnf`, it produces a Haskell implementation of CNF tables and an instance of the Valiant's algorithm using it. As other BNFC back-ends, our implementation produces full parsers, not mere recognizers.

7.4 Unexploited parallelism

The parallelisation that we suggest can take advantage of at most a number of processors proportional to the length of the input. When parsing using Valiant's algorithm, there is more parallelism to take advantage of (for example two of the recursive calls in the V function are independent from each other). However, running in parallel all recursive calls to V would require asymptotically more processors than the length of the input. We do believe that this is *not* a reasonable assumption to make when parsing a whole input. However, in the case of incremental parsing, where only a tiny fraction of the input will be re-parsed, one might want to take advantage of such extra parallelism opportunities.

7.5 Unexploited incrementality

We have suggested that the incremental version of the parser should run the V function $O(\log n)$ times when changing one symbol in the input. In fact, it might be possible to use a better implementation of the chart data structure, which would support an incremental update with a single run of the V function. Indeed, when changing a single symbol of the input, only the part of the chart which depends on that symbol (the square whose bottom-left corner is the symbol in question) needs to be recomputed. This improved re-use of results is left for future work.

7.6 Chomsky normal-form

Even though we assume that we transform the grammar to CNF for ease of presentation, this is not actually the best form to use in an implementation. In fact, it is better to convert the grammar to 2NF (where productions have at most 2 symbols) and derive the operations $(\cdot)$ and σ using a slightly modified algorithm, using the method described by Lange and Leiß (2009), as we have done in our implementation.

The conversion from Backus-Naur Form (BNF) to CNF (or 2NF) involves a division of long productions into binary ones. This is usually done by chaining the binary rules linearly. If the productions of the input grammar are long, this impacts negatively the performance of our algorithm, which performs best on balanced inputs. Fortunately it is not difficult to divide long productions into a balanced tree of binary rules.

The CNF grammar is suitable not only for recognition of languages, but also for parsing: the parse trees obtained by the converted grammar are essentially a binarization of the trees obtained by the grammar in BNF. The aspect which cannot be preserved by the conversion is the presence of cycles of unit rules. However, the elimination of such cycles can only be seen as a benefit: they introduce an unbounded amount of ambiguity in the grammar, and are a symptom of a mistake in the grammar specification.

7.7 A new class of languages

The assumption we make on the input (depending on a constant α), defines implicitly a new class of languages. The class lies between regular and context-free languages. We call the class α-balanced context-free languages, or BCF(α). The use of the parameter α contrasts with that of the parameter k in classes such as LL(k) or LR(k). While LL(k) or LR(k) restricts the form that a CF grammar can take, BCF(α) does not. Instead, it restricts the form that the strings of the language can take as a whole.

We have found that for a given grammar, programs are written with a shallow nesting structure, instead of a deep one (with the exception of regular iteration) and hence we have anecdotal evidence that any given programming languages is a member of BCF(α). Together with observation that the parsing problem for BCF(α) is simpler than that of general-context free languages, this makes BCF(α) worthy of study.

In fact, because the assumption we make is not one which is enforced by usual CF grammars, but we still observe it to hold in practice, it must mean that the assumption is self-imposed by the writers of these inputs, namely programmers. This is not too surprising, as our assumption can be violated only by programs which exhibit an amount of nesting comparable to the total length of the input. As folklore goes, programmers are adverse to deeply-nested constructions. Indeed, understanding a program with n levels of nesting requires to remember n levels of context. The link between the ability for a computer to efficiently parse an input in parallel and incrementally and for a human to do so is intriguing, and we hope that the present paper sheds an interesting light on it.

7.8 Generalization

The body of the paper does not depend on the particulars of CF recognition: we abstract over it via an arbitrary association operator. This means that other applications can be devised. A natural extension is to support CF *parsing*, as we have done in our implementation. More exotic extensions are also possible. A first example would be to support symbol tables, which are necessary for proper parsing of C. In this extension, non-terminals would be associated with two symbol sets, one that they assume comes from the environment and one which they provide to the environment. The combination operator would reconcile these two sets. A second example is probabilistic parsing. Here, a probability would be associated with each non-terminal and production rule, and the association operator would simply multiply the probabilities.

In fact, our method can be seen as a general way to turn a non-associative operator into an associative one by computing all possible associations. The efficiency is recovered by the ability to filter out most of the results; either because the original operator discards them, or because there is (possibly hidden) associativity which can be taken advantage of.

7.9 The old as new

It strikes us that a parsing algorithm published in 1975 finds an application in the area of parallelisation for computer architectures of

the 2010 decade. Further, Valiant gives no indication that the algorithm described should find any practical parsing application. As it seems, he aims only to tie the complexity of context-free recognition to that of matrix multiplication (via the transitive closure operation).

Indeed, in the case of parsing (in contrast to mere recognition), subtraction of matrices is not defined. Hence one cannot use the efficient Strassen algorithm (Strassen 1969) for multiplication, and in turn the complexity of general context-free parsing using Valiant's method is cubic, and fails to beat the simpler CYK algorithm.

Our contribution is to recognize that Valiant's algorithm performs well for parsing practical inputs, given a special handling of iteration and a sparse matrix representation (even when using the naive matrix multiplication algorithm). If we also account for the ease of making parallel and incremental implementations of the algorithm thanks to its divide and conquer structure, we must classify Valiant's algorithm as a practical method of parsing.

In fact, Valiant's algorithm offers such a combination of simplicity and performance that we believe it deserves a prominent place in textbooks, on par with LALR algorithms.

8. Conclusions

At the start of this work, we set out to find an associative operator with sub-linear complexity that could be used to implement a divide-and-conquer algorithm for parsing. The goal was to obtain a parallelizable parsing algorithm that would double as an incremental parsing algorithm. We managed to find such an operator, but the desired complexity only holds under certain assumptions that luckily do seem to hold in practice. The conditions hold when the recursive nesting depth of a program text only grows, say logarithmically in terms of the total length of the program. An unanticipated result of our work is thus the definition of a new class of languages. We were also forced to come up with a special way of dealing with iteration (frequently occurring in grammars) so it would not break this practical assumption.

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Appendix: Bounds on Non-Empty Matrices

We have

$$\#A \leq \left\lceil \alpha \sum_{(i,j)\in \mathrm{dom}(A)} \frac{1}{(j-i)^2} \right\rceil$$

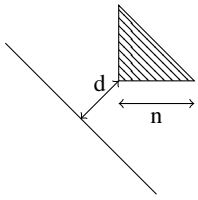
Assume $S(n,d)$ is a square matrix of size n at distance d to the diagonal. In this section we compute asymptotic bounds for $\#S(n,kn)$, for any k.

The strategy is to symbolically evaluate $P\,(A)$, from which it is easy to infer bounds for $\#A$, where

$$P\,(A) = \sum_{(i,j)\in \mathrm{dom}(A)} \frac{1}{(j-i)^2}$$

Consider a lower triangle $T(n,d)$, of size n and at distance d to the diagonal, for which the above sum is easy to evaluate symbolically. From a result on triangles one can recover a result on squares:

$$P\,(S(n,d)) = P\,(T(2n,d)) - 2P\,(T(n,n+d)) \qquad (4)$$



We have:

$$\begin{aligned} P\,(T(n,d)) &= \sum_{(i,j)\in T(n,d)} \frac{1}{(j-i)^2} \\ &= \sum_{k=1}^{n}\sum_{l=1}^{k} \frac{1}{(d+k)^2} \\ &= \sum_{k=1}^{n} \frac{k}{(d+k)^2} \\ &= \psi^0(d+n+1) - \psi^0(d+1) + \\ &\quad d(\psi^1(d+n+1) - \psi^1(d+1)) \end{aligned}$$

Where ψ is the polygamma function. We then use the approximation $\psi^k(n) \sim \frac{d^k}{dn}\log n$ together with (4) and get

$$\begin{aligned} P\,(S(n,kn)) \sim\ & 2(kn+n)\left(\frac{1}{kn+n+1} - \frac{1}{kn+2n+1}\right) \\ & - kn\left(\frac{1}{kn+1} - \frac{1}{kn+2n+1}\right) \\ & - \log(kn+1) + 2\log(kn+n+1) - \log(kn+2n+1) \end{aligned}$$

- if $k > 0$, we have

  $$\lim_{n\to\infty} P\,(S(n,kn)) = 2\log(k+1) - \log(k+2) - \log(k)$$

  and the limit converges from below. So we the above expression is an asymptotic bound for $P\,(S(n,kn))$.

- if $k = 0$, we have

  $$\begin{aligned} S(n,kn) &= S(n,0) \\ &\sim 2n\left(\frac{1}{1+n} - \frac{1}{1+2n}\right) + 2\log(1+n) - \log(1+2n) \\ &\sim \log n \end{aligned}$$

Appendix: C Program Fragment

```
__BEGIN_PROGRAM

void start_bandwidth_timer(struct hrtimer period_timer , int period)
{
  unsigned long delta;
  int soft, hard, now;

  for (;;) {
    if (hrtimer_active(period_timer))
      break;

    now = hrtimer_cb_get_time(period_timer);
    hrtimer_forward(period_timer, now, period);

    soft = hrtimer_get_softexpires(period_timer);
    hard = hrtimer_get_expires(period_timer);
    delta = into_ns(ktime_sub(hard, soft));
    hrtimer_start_range_ns(period_timer, soft, delta,
                                        HRTIMER_MODE_ABS_PINNED, 0);
  }
}


static void update_rq_clock_task(struct rq *rq, long delta);

void update_rq_clock(struct rq *rq)
{
  long delta;

  if (rq->skip_clock_update > 0)
    return;

  delta = sched_clock_cpu(cpu_of(rq)) - rq->clock;
  rq->clock += delta;
  update_rq_clock_task(rq, delta);
}

static int sched_feat_show(struct seq_file * m, void v)
{
  int i;

  for (i = 0; i < SCHED_FEAT_NR; i++) {
    if (!(sysctl_sched_features & (1 << i)))
      seq_puts(m, "NO_");
    seq_printf(m, "%s ", sched_feat_names[i]);
  }
  seq_puts(m, "\n");

  return 0;
}

__END_PROGRAM
```

Fragment of a C program corresponding to the chart in Fig. 4. It is excerpt by hand from the linux kernel scheduler (beginning of the file `https://github.com/torvalds/linux/blob/master/kernel/sched/core.c`)