THE THEORY OF SPECULATION

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Translated by D. May from
Annales scientifiques de l'École Normale Supérieure, Sér. 3, 17 (1900), p. 21-86.
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INTRODUCTION

The influences which determine the movements of the Stock Exchange are innumerable. Events past, present or even anticipated, often showing no apparent connection with its fluctuations, yet have repercussions on its course.

Beside fluctuations from, as it were, natural causes, artificial causes are also involved. The Stock Exchange acts upon itself and its current movement is a function not only of earlier fluctuations, but also of the present market position.

The determination of these fluctuations is subject to an infinite number of factors: it is therefore impossible to expect a mathematically exact forecast. Contradictory opinions in regard to these fluctuations are so divided that at the same instant buyers believe the market is rising and sellers that it is falling.

Undoubtedly, the Theory of Probability will never be applicable to the movements of quoted prices and the dynamics of the Stock Exchange will never be an exact science.

However, it is possible to study mathematically the static state of the market at a given instant, that is to say, to establish the probability law for the price fluctuations that the market admits at this instant. Indeed, while the market does not foresee fluctuations, it considers which of them are more or less probable, and this probability can be evaluated mathematically.

Up to the present day, no investigation into a formula for such an expression appears to have been published: that will be the object of this work.

I have thought it necessary to recall initially some theoretical notions relating to Stock Exchange operations and to adjoin certain new insights indispensable to our subsequent investigations.

THE OPERATIONS OF THE STOCK EXCHANGE.

Stock Exchange Operations. — There are two kinds of forward-dated operations[1]:

- Forward contracts[2],
- Options[3].

These operations can be combined in infinite variety, especially since several types of options are dealt with frequently.

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[1]Trans: French *les opérations à terme*.
[2]Trans: French *les opérations fermes*.
[3]Trans: French *les opérations à prime*. To be precise: European-style call options.

Forward Contracts. — Operations for forward contracts are completely analogous to those for cash, but are adjusted only for price differences at a date fixed in advance and called the *liquidation day*[4]. This falls on the last day of each month.

The price established on the liquidation day, and which is reported for all the operations for the month, is the *adjustment price*[5].

The buyer of a forward contract limits neither his profit nor his loss. He gains the difference between the purchase price and the sale price, if the sale is made above the purchase price; he loses the difference if the sale is made below it.

There is a loss for the seller of a forward contract who repurchases higher than he originally sold; there is a profit in the contrary case.

Contangos. [6] — A cash buyer redeems his coupons and may retain his securities indefinitely. Because a forward-dated operation expires at liquidation, the buyer of a forward contract, in order to maintain his position until the following liquidation day, must pay to the seller a compensation called a *contango*[7].

The contango varies at each liquidation; on Rentes[8] it is on average 0.18fr per 3fr coupon, but may be higher or zero; it may even be negative, in which case it is then called *backwardation*[9]. In this case, the seller compensates the buyer.

On the day of coupon detachment, the buyer of a forward contract receives from the seller the amount of the coupon payment. At the same time, the price falls by an equal amount. Buyer and seller then find themselves immediately after coupon detachment in the same relative position as before this operation.

It can be seen that though the buyer has the advantage of receiving the coupons, in contrast, he must, in general, pay the contangos. The seller, on the other hand, receives the contangos, but he pays the coupons.

Deferrable Rentes. [10] — On Rentes, the coupon of 0.75fr per quarter represents 0.25fr per month, while the contango is almost always less than 0.20fr.

The difference is thus in favour of the buyer; thence comes the concept of purchasing Rentes to be carried over indefinitely.

This operation is called a *deferrable Rente*. The probability of profiting from it will be examined later on.

Equivalent Prices. [11] — To better give an account of the mechanism of coupons and contangos, let us make an abstraction of all other causes of fluctuations in prices.

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[4]Trans: French *le jour de la liquidation*.

[5]Trans: French *le cours de compensation*.

[6]Trans: French *les reports*.

[7]For the complete definition of contangos, I refer to specialist works.

[8]Trans: French *la rente*. French Third Republic government bond issued in the form of a perpetual annuity, typically with a coupon of 3% of face value, payable quarterly.

[9]Trans: French *le déport*.

[10]Trans: French *les rentes reportables*.

[11]Trans: French *les cours équivalents*.

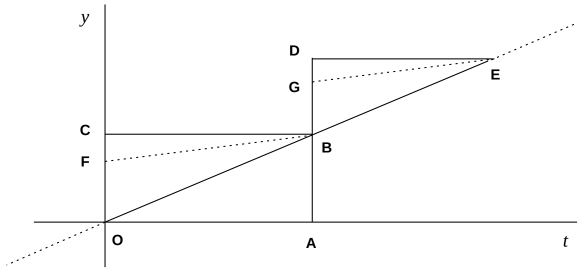
Since every quarter a coupon for 0.75fr is detached from a Rente, representing a payment of interest for the buyer, the cash price of the Rente must logically rise each month by 0.25fr. For the current quoted price there is a corresponding price which, in thirty days, would be higher by 0.25fr, in fifteen days 0.125fr, etc.

All these prices can be considered *equivalent*.

The consideration of equivalent prices is much more complicated in the case of forward contracts. It is of course obvious that if the contango is nil, the forward-dated operation must behave as the operation for cash and that the price must logically rise by 0.25fr per month.

Now consider the case where the contango would be 0.25fr. Taking the x-axis as representing time (*Figure 1*), the length OA represents one month between two liquidation dates, one of which corresponds to point O and the other to point A.

Figure 1.



Let the ordinates represent prices.

If AB is equivalent to 0.25fr, the logical path of the cash price of a Rente is represented by the straight line OBE[12].

Now consider the case where the contango would be 0.25fr. Just before liquidation, the cash and forward prices will be the same, at point O. Then, the buyer of the forward contract will pay 0.25fr in advance for contango. The forward price will jump abruptly from O to C and will follow the horizontal line CB throughout the month. At B, it will merge anew with the cash price to increase abruptly by 0.25fr to D, etc.

In the case where the contango is a given quantity corresponding to the length OF, the price must follow the line FB, then GE, and so on.

Therefore, in this case, the forward contract on a Rente, from one liquidation to the next, logically must rise by a quantity represented by FC which could be called the *complement of contango*[13].

All the prices from F to B along the line FB are *equivalent* for the different epochs to which they correspond.

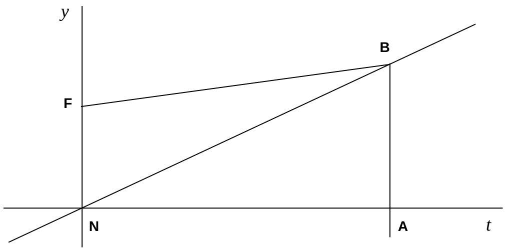
In fact, the spread between the forward and cash prices does not extend itself in an absolutely regular manner, and FB is not a straight line, but the construction that was just made at the start of the month may be repeated at an arbitrary time represented by the point N.

[12]It is assumed that there is no detachment of coupons in the interval under consideration, which anyway would not alter the demonstration.

[13]Trans: French *complêment du report*.

Let NA be the time that will elapse between the epoch N under consideration and the liquidation date represented by the point A (*Figure 2*).

FIGURE 2.



During the interval of time NA, the cash price of a Rente must logically rise by AB, in proportional to NA. Let NF be the spread between the cash price and the forward price. All prices corresponding to the line FB are *equivalent*.

True Prices. [14] — Let the equivalent price corresponding to an epoch be called the *true price* corresponding to that epoch.

Knowledge of the true price is of very great importance. I shall now proceed to examine how it is determined.

Let b designate the quantity by which a Rente must logically increase within the interval of a single day. The coefficient b generally varies little, its value each day can be precisely determined.

Suppose that n days separate us from the liquidation date, and let C be the spread between the forward and cash prices.

In n days, the cash price must rise by $25n/30$ centimes. The forward price, being higher by the quantity C, must rise during these n days only by the quantity $25n/30 - C$, that is to say, during one day by

$$\frac{1}{n}\left(\frac{25n}{30} - C\right) = \frac{1}{6n}(5n - 6C).$$

Therefore

$$b = \frac{1}{6n}(5n - 6C).$$

The average of the last five years gives $b = 0.264$ centimes.

The true price corresponding to m days will be equal to the currently quoted price, increased by the quantity mb.

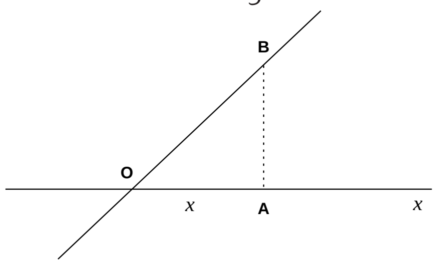
Geometrical Representation of Forward-Dated Operations. — An operation can be represented geometrically in a very simple fashion, the x-axis representing different prices and the y-axis the corresponding profits (*Figure 3*).

Suppose that I had made a forward purchase at a price represented by O, that I take as the origin. At price $x = \text{OA}$, the operation gives profit x; and as the corresponding ordinate must be equal to the profit, AB = OA. A forward purchase is thus represented by the line OB inclined at 45° to the line of prices.

A forward sale would be represented in an inverse fashion.

[14] Trans: French *les cours vrais*.

FIGURE 3.



Options. — In the purchase or sale of a forward contract, buyers and sellers expose themselves to theoretically unlimited losses. In the market for options, the buyer pays more for the asset than in the case of the market for forward contracts, but his loss on a fall in the price is limited in advance to a specified sum which is the amount for the option.

The seller of an option has the advantage of selling for a higher price, but he can profit by only the amount of the option premium.

There are also *options for a fall*[15] which limit the loss of the seller. In this case, the operation is transacted at a price below that of a forward contract.

These options for a fall are negotiated only in speculation on commodities; in speculation on securities, an option for a fall is obtained by the sale of a forward contract and the simultaneous purchase of an option. In order to limit the ideas, I shall be concerned only with *options for a rise*[16].

Suppose, for example, that the 3% Rente is quoted at 104fr at the beginning of the month. If we buy a forward contract on 3,000, we expose ourselves to a loss which may be considerable if there were a heavy price fall.

To avoid this risk, we can purchase an option at 50c[17] on paying, no longer 104fr, but 104.15fr, for example. Our purchase price is higher, it is true, but our loss remains limited, whatever be the fall in the price, to 50c per 3fr, that is to say, to 500fr.

The operation is the same as if we had purchased a forward contract at 104.15fr. This forward contract cannot fall by more than 50c, that is to say, descend below 103.65fr.

The price of 103.65fr, in the present case, is the *foot of the option*[18].

It can be seen that the price of the foot of the option is equal to the price at which it is negotiated, diminished by the amount of the option premium.

Declaration of Options. [19] — The day before liquidation, that is to say, the penultimate day of the month, is the occasion of the *declaration of options*. Let us resume the preceding example and suppose that at the time of the declaration

[15]Trans: French *les primes à la baisse*. That is, options anticipating a price fall.

[16]Trans: French *les primes à la hausse*. That is, options anticipating a price rise.

[17]We say an *option at* for *a premium of* and we use the notation 104.15/50 to denote an operation transacted at a price of 104.15fr for a premium of 50c.

[18]Trans: French *le pied de la prime*.

[19]Trans: French *la réponse des primes*.

the price of Rentes were below 103.65fr, then we would *abandon*[20] our option, which would be to the benefit of our seller.

If, on the contrary, the price at the declaration were above 103.65fr, then our operation would be transformed into a forward contract. In this case it is said that the option is *exercised*[21].

In summary, an option is exercised or abandoned according as the price at the declaration is below or above the foot of the option.

It can be seen that operations with options do not run until liquidation. If an option is exercised at the declaration of options, it becomes a forward contract and is settled the next day.

In all that will follow, it will be assumed that the adjustment price coincides with the price at the declaration of options. This hypothesis can be justified, for nothing prevents liquidation of operations at the declaration of options.

Spread of Options. [22] — The *spread* between the price of a forward contract and that of an option depends on a great number of factors and varies unceasingly.

At the same time, the spread is correspondingly greater as the premium is smaller; for example, an option /50c is obviously better value than the option /25c.

The spread of an option decreases more or less regularly from the beginning of the month until the day before the declaration of the option, when this spread becomes very small.

But, according to the circumstances, it can change very irregularly and become greater a few days before the declaration of the option than at the start of the month.

Options for the Following Liquidation Date. — Options are negotiated not only for the current liquidation date, but also for the following liquidation date. The spread for these is necessarily greater than that of options for the current liquidation date, but it is less than might be supposed from the difference between the price of an option and that of a forward contract. It is necessary to deduct from the apparent spread the magnitude of the contango for the current liquidation date.

For example, the average spread for an option /25c at 45 days from declaration is on average 72c. But as the average contango is 17c the spread is actually only 55c.

The detachment of a coupon lowers the price of the option by a value equal to the amount of the coupon. For example, if I buy, on the 2nd of September, an option /25c for 104.50fr for the current liquidation date, the price of my option will become 103.75fr on 16 September after detachment of the coupon.

The price of the foot of the option will be 103.50fr.

Options for the Next Day. — We deal, especially off the Exchange[23], in options /5c and sometimes /10c for the next day.

The declaration of these small options is held each day at 2pm.

[20]Trans: French *abandonner la prime*.

[21]Trans: French *levér la prime*.

[22]Trans: French *l'écart des primes*.

[23]Trans: French *en coulisse*.

Options in General. — In a market for options of a given expiry date, there are two factors to be considered: the amount of the premium and its spread from the price of the forward contract.

It is quite evident that the greater the size of the premium, the smaller the spread.

To simplify the negotiation of options, they can be reduced to three types based on the three simplest assumptions for the amount of the premium and for its spread:

(1) The amount of the premium is constant and the spread is variable. It is this kind of option that is negotiated on securities. For example on the 3% Rente option premia /50c, /25c and /10c are negotiated.
(2) The spread of the option is constant and the amount of the premium is variable. This is what happens to options for a fall on securities (that is to say, the sale of a forward contract against purchase of an option).
(3) The spread of the premium is variable and so is its size. However, these two quantities are always equal. This is how options on commodities are negotiated. It is evident that by employing the latter system we can negotiate at any given moment only a single option for the same expiry date.

Remark on Options. — We will examine what law governs the spread of options. Nevertheless, we can, at the present time, make this rather intriguing comment:

The value of an option must be greater according as its spread be narrower. This obvious fact does not suffice to demonstrate that the use of options is rational.

Indeed, I realised several years ago, it was possible to imagine operations where one of the contractors would profit at every price.

Without reproducing the calculations, elementary but rather laborious, I shall be content to present an example.

The following operation:

Purchase of one unit	/1fr,
Sale of four units	/50c,
Purchase of three units	/25c,

would produce a profit at all prices provided that the spread from /25c to /50c be at most a third of the spread from /50c to /1fr.

It will be seen that spreads like these never occur together in practice.

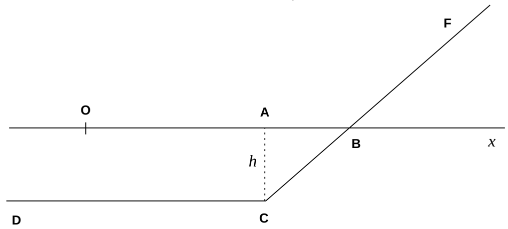
Geometric Representation of Operations with Options. — We propose to represent geometrically the purchase of an option (*Figure 4*).

Take, for example, for the origin the price of a forward contract at the instant where the option at h was negotiated. Let E_1 be the relative price of this option or its spread.

Above the foot of the option, that is to say, at price $(E_1 - h)$ represented by the point A, the operation is similar to a forward contract negotiated at price E_1; it is represented by the line CBF. Below price $E_1 - h$, the loss is constant and, consequently, the operation is represented by the broken line DCF.

The sale of an option would be represented in an inverse manner.

Figure 4.



True Spread. [24] — So far we have spoken only of quoted spreads, the only ones with which we are ordinarily concerned. However, they are not the ones that will be introduced in our theory, but rather *true spreads*, that is to say, the spreads between prices of options and true prices corresponding to declaration of options. The price in question being above the quoted price (unless the contango is above 25c, which is rare), it follows that the true spread of an option is lower than its quoted spread.

The true spread of an option negotiated n days before the declaration of options will be equal to its quoted spread diminished by the quantity nb.

The true spread of a option for the following liquidation date will be equal to its quoted spread diminished by the quantity $[25 + (n - 30)b]$.

Call-of-More Operations. [25] — In certain markets there are operations which are in some way intermediate between forward contracts and options, these are *call-of-more* operations.

Suppose that 30fr be the price of a commodity. Instead of buying a unit at a price of 30fr for a given expiry date, a *call-of-twice-more*[26] can be bought at the same expiry date for 32fr, for example. This means that for any difference below a price of 32fr, only one unit is lost, while for any difference above, two units are gained.

A *call-of-thrice-more*[27] could be bought for 33fr, for example. That is to say that, for any difference below a price of 33fr one unit is lost, while for any difference over this price three units are earned. Calls-of-more of multiple orders can be imagined, the geometrical representation of these operations presents no difficulty.

Calls-of-more for a fall[28] are also negotiated, necessarily for the same spread as *calls-of-more for a rise*[29] of the same order of multiplicity.

PROBABILITIES IN THE OPERATIONS OF THE STOCK EXCHANGE.

Probabilities in the Operations of the Stock Exchange. — Two kinds of probabilities can be considered:

(1) Probability that might be called *mathematical*; this is that which can be determined *a priori*; that which is studied in games of chance.

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[24]Trans: French *l' écart vrai*.
[25]Trans: French *les options*.
[26]Trans: French *l'option du double*.
[27]Trans: French *l'option du triple*.
[28]Trans: French *les options à la baisse*.
[29]Trans: French *les options à la hausse*.

(2) Probability depending on future events and, as a consequence, impossible to predict in a mathematical way.

It is the latter probability that a speculator seeks to predict. He analyses the reasons which may influence rises or falls in prices and the amplitude of price movements. His conclusions are completely personal, since his counter-party necessarily has the opposite opinion.

It seems that the market, that is to say, the totality of speculators, must believe *at a given instant* neither in a price rise nor in a price fall, since, for each quoted price, there are as many buyers as sellers.

Actually, the market believes in a rise resulting from the difference between coupons and contangos; the sellers make a small sacrifice which they consider as compensated.

This difference can be ignored, with the qualification that true prices be considered corresponding to the liquidation date, but the operations are adjusted on the quoted prices, the seller paying the difference.

By considering true prices it may be said that:

> *The market does not believe, at any given instant, in a rise nor a fall in the true price.*

But, while the market believes neither in a rise nor a fall in the true price, some movements of a certain amplitude may be supposed to be more or less probable.

The determination of the law of probability that the market admits at a given instant will be the object of this study.

Mathematical Expectation. [30] — The *mathematical expectation* of a potential profit is defined as the product of that profit by the corresponding probability.

The *total mathematical expectation*[31] of a gambler will be the sum of products of potential profits by the corresponding probabilities.

It is evident that a gambler will be neither advantaged, nor disadvantaged if his total mathematical expectation is zero.

The game is then said to be *fair*[32].

One knows that bets on the races and all of the games that are practised in gambling establishments are unfair. The gaming house or the bookmaker if he be betting at the racecourse, plays with a positive expectation, and the punters with a negative expectation.

In these kinds of games the punters do not have a choice between the transaction they make and its counterpart. Since it is not the same at the Stock Exchange, it may seem curious that these games are unfair, the seller accepting *a priori* a disadvantage if the contangos are lower than the coupons.

The existence of a second kind of probability explains this seemingly paradoxical fact.

Mathematical Advantage. [33] — Mathematical expectation indicates for us whether a game is advantageous or not: furthermore, it informs us whether the

[30]Trans: French *l'espérance mathématique*.
[31]Trans: French *l'espérance mathématique totale*.
[32]Trans: French *équitable*.
[33]Trans: French *l'avantage mathématique*.

game must logically yield a profit or a loss; but it does not provide a coefficient representing, in some sense, the intrinsic value of the game.

This leads us to introduce a new concept: that of *mathematical advantage*.

Define the *mathematical advantage* of a gambler as the ratio of his positive expectation and the arithmetic sum of his positive and negative expectations.

Mathematical advantage varies like probability from zero to one, it is equal to 1/2 when the game is fair.

Principle of Mathematical Expectation. — A cash buyer may be likened to a gambler. In effect, while an asset can rise in value after purchase, a fall is equally possible. The causes of this increase or decrease fall into the second category of probabilities.

According to the first the security[34] must rise to a value equal to the amount of its coupons. It follows from the point of view of this first category of probabilities:

The mathematical expectation for a cash buyer is positive.

It is evident that this will be the same as the mathematical expectation for the buyer of a forward contract if the contango is nil, for his transaction may be likened to that of a cash buyer.

If the contango on a Rente were 25c, the buyer would not be more advantaged than the seller.

Thus, it can be stated that:

The mathematical expectations of the buyer and of the seller are both nil when the contango is for 25 cents.

When the contango is below 25c, which is usually the case:

The mathematical expectation of the buyer is positive; that of the seller is negative.

It must be noted always that this is only for probabilities of the first kind.

From what has been seen previously the contango can be regarded as 25c on the condition of replacing the quoted price by the true price corresponding to the liquidation date. If so, then when considering these true prices it can be said that:

The mathematical expectations of the buyer and of the seller are nil.

From the point of view of contangos, the day of declaration of options can be regarded as conflated with the liquidation date; thus:

The mathematical expectations of the buyer and of the seller of options are nil.

In summary, consideration of true prices permits the enunciation of this fundamental principle:

The mathematical expectation of a speculator is nil.

The generality of this principle needs to be appreciated: it signifies that the market, at a given instant, considers as having nil expectation not only the current trading operations, but also those that would be based on a subsequent movement of prices.

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[34]I consider the simplest case of a an instrument with a fixed income, otherwise income growth would be a probability from the second category.

For example, if I buy a Rente with intent to resell it when it has increased by 50c, the expectation of this complex operation is simply nil as if I had the intention to resell my Rente at liquidation or at some other time.

The mathematical expectation of an operation that may be either positive or negative, if it produces a price movement, is *a priori* nil.

General Form of the Probability Curve. — The probability that a price y be quoted at a given epoch is a function of y.

This probability may be represented by the ordinate of a curve whose abscissae correspond to the different prices.

It is obvious that the price considered by the market as the most likely is the current true price: if the market thought otherwise, it would quote not this price, but another one higher or lower.

In the remainder of this study, the true price corresponding to a given epoch will be taken as the origin for the coordinates.

Prices can vary between $-x_0$ and $+\infty$: x_0 being the current absolute price.

It will be assumed that it can vary between $-\infty$ and $+\infty$. The probability of a spread greater than x_0 being considered *a priori* entirely negligible.

Under these conditions, it may be admitted that the probability of a deviation from the true price is independent of the absolute level of this price, and that the probability curve is symmetrical with respect to the true price.

In what will follow, only relative prices will matter because the origin of the coordinates will always correspond to the current true price.

Probability Law. — The probability law can be determined from the Principle of Compound Probabilities.

Let $p_{x,t}\,dx$ designate the probability that, at epoch t, the price is to be found in the elementary interval x, $x+dx$.

We seek the probability that the price z be quoted at epoch t_1+t_2, the price x having been quoted at epoch t_1.

By virtue of the Principle of Compound Probabilities, the desired probability will be equal to the product of the probability that x be the quoted price at epoch t_1, that is to say, $p_{x,t_1}\,dx$, multiplied by the probability that x be the price quoted at epoch t_1, the current price z being quoted at epoch t_1+t_2, that is to say, multiplied by $p_{z-x,t_2}\,dz$.

The desired probability is therefore

$$p_{x,t_1}\,p_{z-x,t_2}\,dx\,dz.$$

At epoch t_1, the price could be located in any the intervals dx between $-\infty$ and $+\infty$, so the probability of the price z being quoted at epoch t_1+t_2 will be

$$\int_{-\infty}^{+\infty} p_{x,t_1}\,p_{z-x,t_2}\,dx\,dz.$$

The probability of this price z, at epoch t_1+t_2, is also given by the expression p_{z,t_1+t_2}; we therefore have

$$p_{z,t_1+t_2}\,dz = \int_{-\infty}^{+\infty} p_{x,t_1}\,p_{z-x,t_2}\,dx\,dz$$

or

$$p_{z,t_1+t_2} = \int_{-\infty}^{+\infty} p_{x,t_1}\, p_{z-x,t_2}\, dx,$$

which is the equation for the condition which must be satisfied by the function p.

It can be seen that this equation is satisfied by the function

$$p = Ae^{-B^2x^2}.$$

Observe that from now on that we must have

$$\int_{-\infty}^{+\infty} p\, dx = A \int_{-\infty}^{+\infty} e^{-B^2x^2} dx = 1.$$

The classical integral which appears in the first term has a value of $\sqrt{\pi}/B$, thus we have $B = A\sqrt{\pi}$, and it follows that

$$p = Ae^{-\pi A^2 x^2}.$$

Assuming that $x = 0$, we obtain $A = p_0$, that is to say, A is equal to the probability of the current quoted price.

It is necessary to establish that the function

$$p = p_0 e^{-\pi p_0^2 x^2},$$

where p_0 is dependent on time, does satisfy the condition of the above equation.

Letting p_1 and p_2 be the quantities corresponding to p_0 and relative to times t_1 and t_2, it must be demonstrated that the expression

$$\int_{-\infty}^{+\infty} p_1 e^{-\pi p_1^2 x^2} \times p_2 e^{-\pi p_2^2 (z-x)^2} dx$$

can be put in the form Ae^{-Bz^2}; A and B being dependent only on time.

Noting that z is a constant, this integral becomes

$$p_1 p_2\, e^{-\pi p_2^2 z^2} \int_{-\infty}^{+\infty} e^{-\pi(p_1^2+p_2^2)x^2 + 2\pi p_2^2 zx} dx$$

or

$$p_1 p_2\, e^{-\pi p_2^2 z^2 + \frac{\pi p_2^4 z^2}{p_1^2+p_2^2}} \int_{-\infty}^{+\infty} e^{-\pi\left(x\sqrt{p_1^2+p_2^2} - \frac{p_2^2 z}{\sqrt{p_1^2+p_2^2}}\right)^2} dx;$$

supposing that

$$x\sqrt{p_1^2+p_2^2} - \frac{p_2^2 z}{\sqrt{p_1^2+p_2^2}} = u,$$

we will then have

$$\frac{p_1 p_2\, e^{-\pi p_2^2 z^2 + \frac{\pi p_2^4 z^2}{p_1^2+p_2^2}}}{\sqrt{p_1^2+p_2^2}} \int_{-\infty}^{+\infty} e^{-\pi u^2} du.$$

The integral having the value 1, we obtain finally

$$\frac{p_1 p_2}{\sqrt{p_1^2+p_2^2}}\, e^{-\pi \frac{p_1^2 p_2^2}{p_1^2+p_2^2} z^2}.$$

This expression having the desired form, it may be concluded that the probability is correctly expressed by the formula

$$p = p_0\, e^{-\pi^2 p_0 x^2},$$

in which p_0 depends on the elapsed time.

It can be seen that the probability is governed by the Law of Gauss — already celebrated in the Theory of Probability.

Probability as a Function of Time. — The formula preceding the last shows that the parameters $p_0 = f(t)$ satisfy the functional relation

$$f^2(t_1 + t_2) = \frac{f^2(t_1) f^2(t_2)}{f^2(t_1) + f^2(t_2)},$$

on differentiating with respect to t_1, then with respect to t_2.

The first member having the same form in both cases, we obtain

$$\frac{f'(t_1)}{f^3(t_1)} = \frac{f'(t_2)}{f^3(t_2)}.$$

This relation holding, regardless of t_1 and t_2, the common value of both ratios is a constant, and we have

$$f'(t) = C f^3(t),$$

from which

$$f(t) = p_0 = \frac{H}{\sqrt{t}},$$

where H designates a constant.

Therefore, the probability is given by the expression

$$p = \frac{H}{\sqrt{t}} e^{-\frac{H^2 x^2}{t}}.$$

Mathematical Expectation. — The expected value corresponding to the price x has a value of

$$\frac{H}{\sqrt{t}} e^{-\frac{\pi H^2 x^2}{t}}.$$

Therefore, the total positive expectation is

$$\int_{-\infty}^{+\infty} \frac{Hx}{\sqrt{t}} e^{-\frac{\pi H^2 x^2}{t}} dx = \frac{\sqrt{t}}{2\pi H}.$$

Let us take as a constant, in our study, the mathematical expectation k corresponding to $t = 1$. Therefore, we will have

$$k = \tfrac{1}{2\pi H} \quad or \quad H = \tfrac{1}{2\pi k}.$$

The definitive expression for the probability is therefore

$$p = \frac{1}{2\pi k \sqrt{t}} e^{-\frac{x^2}{4\pi k^2 t}}.$$

The mathematical expectation

$$\int_{-\infty}^{+\infty} p\, x\, dx = k\sqrt{t}$$

is proportional to the square root of the elapsed time.

Another Derivation of the Probability Law. — The expression for the function p may be obtained by following a route different to the one that has been employed.

Suppose that two complementary events A and B have the respective probabilities p and $q = 1 - p$. The probability that, on m occasions, it would produce α equal to A and $m - \alpha$ equal to B, is given by the expression

$$\frac{m!}{\alpha!\,(m-\alpha)!}\,p^{\alpha} q^{m-\alpha}.$$

This is a term from the expansion of $(p+q)^m$.

The greatest of these probabilities is given by

$$\alpha = mp \quad \text{and} \quad (m-\alpha) = mq.$$

Consider the term in which the exponent of p is $mp + h$, the corresponding probability is

$$\frac{m!}{(mp+h)!\,(mq-h)!}\,p^{mp+h} q^{mq-h}.$$

The quantity h is called the *spread*.

Let us seek for the mathematical expectation for a gambler who would receive a sum equal to the spread whenever this spread be positive.

We have just seen that the probability of a spread h is the term from the expansion of $(p+q)^m$ in which the exponent of p is $mp+h$, and that of q is mqh. To obtain the mathematical expectation corresponding to this term, it is necessary to multiply this probability by h. Now,

$$h = q(mp+h) - p(mqh),$$

$mp + h$ and mqh being the exponents of p and q in a term from $(p+q)^m$. To multiply a term

$$q^{\mu} p^{\nu}$$

by

$$\nu q - \mu p = pq\left(\frac{\nu}{p} - \frac{\mu}{q}\right),$$

is to take the derivative with respect to p, subtract the derivative with respect to q, multiply the difference by pq.

To obtain the total mathematical expectation, we must take the terms from the expansion of $(p+q)^m$ where h is positive, that is to say,

$$p^m + mp^{m-1}q + \frac{m(m-1)}{1.2}\,p^{m-2}q^2 + \ldots + \frac{m!}{mp!\,mq!}\,p^{mp}q^{mq},$$

and subtract the derivative with respect to p, then multiply the result by pq.

The derivative of the second term with respect to q is equal to the derivative of the first with respect to p, the derivative of the third with respect to q is the derivative of the second with respect to p, and so on. The terms therefore cancel in pairs and there remains only the derivative of the latter with respect to p

$$\frac{m!}{mp!\,mq!}\,p^{mp}q^{mq}mpq.$$

The average value of the spread h will be equal to twice this quantity.

When the number m is sufficiently great, the preceding expressions can be simplified by making use of the asymptotic formula of Stirling

$$n! = e^{-n} n^n \sqrt{2\pi n}.$$

The value obtained thereby for the mathematical expectation is

$$\frac{\sqrt{mpq}}{\sqrt{2\pi}}.$$

The probability that the spread h be included between h and $h + dh$ will be given by the expression

$$\frac{dh}{\sqrt{2\pi mpq}} e^{-\frac{h^2}{2mpq}}.$$

The preceding theory can be applied to our study. It may be supposed that the elapsed time is divided into very small intervals Δt such that $t = m\Delta t$. During the interval of time Δt, the price will probably vary very little.

Form the sum of the products of the spreads that may exist at epoch Δt by the corresponding probabilities, that is to say, $\int_0^\infty p\,x\,dx$, p being the probability of the spread x.

This integral must be finite, because, owing to the smallness supposed of Δt, substantial spreads are of a vanishingly small probability. Moreover, this integral expresses a mathematical expectation which can be finite if it corresponds to a very small interval of time.

Let Δx designate an amount which is double the value of the integral above; Δx will then be the average of the spreads or the average spread during the interval of time Δt.

If the number of trials m be great and if the probability remains the same at each trial, it may be supposed that the price varies during each of the trials Δt by the average spread of Δx; the increase Δx will have probability $1/2$, as will also the decrease $-\Delta x$.

The preceding formula will therefore give, on setting $p = q = 1/2$, the probability that at epoch t, the price must be included between x and $x + dx$; this will be

$$\frac{2\,dx\sqrt{\Delta t}}{\sqrt{2\pi}\sqrt{t}} e^{-\frac{2x^2\Delta t}{t}},$$

where, assuming $H = 2/\sqrt{2\pi}\sqrt{\Delta t}$,

$$\frac{H\,dx}{\sqrt{t}} e^{-\frac{\pi H^2 x^2}{t}}.$$

The mathematical expectation will be given by the expression

$$\frac{\sqrt{t}}{2\sqrt{2\pi}\sqrt{\Delta t}} = \frac{\sqrt{t}}{2\pi H}.$$

Taking as a constant the mathematical expectation k corresponding to $t = 1$, we find, as before,

$$p = \frac{1}{2\pi k\sqrt{t}} e^{-\frac{x^2}{4\pi k^2 t}}.$$

The preceding formulae give $\Delta t = 1/8\pi k^2$. The average fluctuation during this interval of time is

$$\Delta x = \frac{\sqrt{2}}{2\sqrt{\pi}}.$$

Assuming that $x = n\,\Delta x$, the probability will be given by the expression

$$p = \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{m}}\, e^{-\frac{n^2}{\pi m}}.$$

Probability Curve. — The function

$$p = p_0\, e^{-\pi p_0^2 x^2}$$

can be represented by a curve whose ordinate has its maximum at the origin and which has two points of inflection for

$$x = \pm\frac{1}{p_0\sqrt{2\pi}} = \pm\sqrt{2\pi}\,k\sqrt{t}.$$

These same values of x are the abscissae of the maxima and minima of the curves of mathematical expectation, whose equation is

$$y = \pm p\,x.$$

The probability of price x is a function of t; it increases up until a certain epoch and decreases thereafter. The derivative $dp/dt = 0$ when $t = x^2/2\pi k^2$. The probability of price x is thus a maximum when this price corresponds to the point of inflection of the probability curve.

Probability in a Given Interval. — The integral

$$\frac{1}{2\pi k\sqrt{t}}\int_0^x e^{-\frac{x^2}{4\pi k^2 t}}\,dx = \frac{c}{\sqrt{\pi}}\int_0^x e^{-c^2x^2}\,dx$$

is not expressible in finite terms, but its expansion as a power series is given by

$$\frac{1}{\sqrt{\pi}}\left[cx - \frac{\frac{1}{3}(cx)^3}{1} + \frac{\frac{1}{5}(cx)^5}{1.2} - \frac{\frac{1}{7}(cx)^7}{1.2.3} + \dots \right].$$

This series converges rather slowly for very large values of cx. Laplace has given this case of the definite integral in the form of a continued fraction that is very easy to compute

$$\frac{1}{2} - \frac{e^{-c^2x^2}}{2\,cx\sqrt{\pi}}\,\cfrac{1}{1+\cfrac{\alpha}{1+\cfrac{2\alpha}{1+\cfrac{3\alpha}{1+\ddots}}}},$$

in which $\alpha = 1/2\,c^2x^2$.

The successive convergents are

$$\frac{1}{1+\alpha},\quad \frac{1+2\alpha}{1+3\alpha},\quad \frac{1+5\alpha}{1+6\alpha+3\alpha^2},\quad \frac{1+9\alpha+8\alpha^2}{1+10\alpha+15\alpha^2}.$$

There exists another method which permits the calculation of the above integral when x is a large number.

We have

$$\int_x^\infty e^{-x^2}\,dx = \int_x^\infty \frac{1}{2x} e^{-x^2}\, 2x\,dx;$$

upon integrating by parts, we then obtain

$$\begin{aligned}\int_x^\infty e^{-x^2}\,dx &= \frac{e^{-x^2}}{2x} - \int_x^\infty e^{-x^2}\frac{dx}{2x^2}\\ &= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{4x^3} + \int_x^\infty e^{-x^2}\frac{1.3}{4x^4}dx\\ &= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{4x^3} + \frac{e^{-x^2}\,1.3}{8x^5} - \int_x^\infty e^{-x^2}\frac{1.3.5}{8x^6}dx.\end{aligned}$$

The general term of the series is given by the expression

$$\frac{1.3.5\ldots(2n-1)}{2^{2n-1}x^{2n+1}}e^{-x^2}.$$

The ratio of a term to the preceding term exceeds unity when $2n+1 > 4x^2$. The series thus diverges after a certain term. An upper limit may be obtained for the integral which serves as a remainder.

We have, in fact,

$$\frac{1.3.5\ldots(2n+1)}{2^{2n-1}}\int_x^\infty \frac{e^{-x^2}}{x^{2n+2}}\,dx \quad < \quad \frac{1.3.5\ldots(2n+1)}{2^{2n-1}}e^{-x^2}\int_x^\infty \frac{dx}{x^{2n+2}}$$
$$= \frac{1.3.5\ldots(2n-1)}{2^{2n-1}x^{2n+1}}e^{-x^2}.$$

Now, this latter quantity is the term which precedes the integral. The additional term is thus always smaller than the one which precedes it.

There are published tables giving the values of the integral

$$\Theta(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-y^2}\,dy.$$

It is obvious that

$$\int_0^y p\,dx = \frac{1}{2}\Theta\left(\frac{x}{2k\sqrt{\pi}\sqrt{t}}\right).$$

The probability

$$\mathscr{P} = \int_x^\infty p\,dx = \frac{1}{2} - \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\,d\lambda,$$

that the current price x be attained or surpassed at epoch t, steadily increases with the elapsed time. If t were infinite, it would be equal to $1/2$, a self-evident conclusion.

The probability

$$\int_{x_1}^{x_2} p\,dx = \frac{1}{\sqrt{\pi}}\int_{\frac{x_1}{2\sqrt{\pi}k\sqrt{t}}}^{\frac{x_2}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\,d\lambda$$

that the price be found, at epoch t, in the finite interval x_2, x_1, is nil for $t = 0$ and for $t = \infty$. It is at a maximum when

$$t = \frac{1}{4\pi k^2}\frac{x_2^2 - x_1^2}{\log\frac{x_2}{x_1}}.$$

On the assumption that the interval x_2, x_1 be very small, the epoch corresponding to the maximum probability is again found to be

$$t = \frac{x^2}{2\pi k^2}.$$

Median Spread. [35] — Let us so define the interval $\pm\alpha$, such that around time t, the price has an equal chance of staying within this interval as the chance of overstepping it.

The quantity α is determined by the equation

$$\int_0^\alpha p\,dx = \frac{1}{4}$$

or

$$\Theta\left(\frac{\alpha}{2k\sqrt{\pi}\sqrt{t}}\right) = \frac{1}{2},$$

that is to say,

$$\alpha = 2 \times 0.4769\,k\,\sqrt{\pi}\sqrt{t} = 1.668\,k\,\sqrt{t};$$

this interval is proportional to the square root of the elapsed time.

More generally, consider the interval $\pm\beta$ such that the probability, at epoch t, that the price be contained within this interval is equal to u; we will then have

$$\int_0^\beta p\,dx = \frac{u}{2}$$

or

$$\Theta\left(\frac{\beta}{2k\sqrt{\pi}\sqrt{t}}\right) = u.$$

It can be seen that this interval is proportional to the square root of the elapsed time.

Radiation of Probability. [36] — Directly, I shall seek an expression for the probability $\mathscr{P}$ that the price x be either attained or surpassed at epoch t. It has been seen previously that by dividing time into very small intervals Δt we could consider, for an interval Δt, the price as varying with the fixed and very small quantity Δx. Suppose that, at epoch t, the prices $x_{n-2}, x_{n-1}, x_n, x_{n+1}, x_{n+2}, \ldots$, differing from each other by the quantity Δx, have the respective probabilities $p_{n-2}, p_{n-1}, p_n, p_{n+1}, p_{n+2}, \ldots$. From the knowledge of the probability distribution at epoch t, the probability distribution at epoch $t + \Delta t$ is easily deduced.

Suppose, for example, that price x_n were quoted at epoch t. At epoch $t + \Delta t$ the quoted prices would be x_{n+1} or x_{n-1}. The probability p_n, that price x_n be quoted at epoch t, can be decomposed into two probabilities at epoch $t + \Delta t$. Price x_{n-1} would have probability $p_n/2$, and price x_{n+1} would also have the same probability $p_n/2$.

If price x_{n-1} were quoted at epoch $t + \Delta t$, it would be because, at epoch t, the prices x_{n-2} or x_n had been quoted. The probability of price x_{n-1} at epoch $t + \Delta t$ would therefore be $(p_{n-2} + p_n)/2$; that of price x_n would be, at the same epoch, $(p_{n-1} + p_{n+1})/2$; that of price $(p_{n-1} + p_{n+1})/2$ would be $(p_n + p_{n+2})/2$, etc.

[35] Trans: French *l'écart probable*.

[36] Trans: French *le rayonnement de la probabilité*.

During the interval of time Δt price x_n has, somehow, transmitted towards price x_{n+1}, the probability $p_n/2$; price x_{n+1} has transmitted towards price x_n, the probability $p_{n+1}/2$. If p_n is greater than p_{n+1}, the change in probability is $(p_n - p_{n+1})/2$ from x_n towards x_{n+1}.

Therefore, it can be said:

> *Each price radiates during an element of time towards its neighbouring prices a quantity of probability proportional to the difference in their probabilities.*

I say *proportional* because the ratio of Δx to Δt must be taken into account.

The preceding law may, by analogy with certain physical theories, be called the *Law of Radiation (or Diffusion) of Probability*.

I shall now consider the probability $\mathscr{P}$ that the price x is to be found at epoch t in the interval x, ∞ and I shall evaluate the growth of this probability during the time Δt.

Let p be the probability of price x at epoch t, $p = -d\mathscr{P}/dx$. Let us evaluate the amount of probability that, during the elapsed time Δt, somehow, passes through price x; that is, from what has been said,

$$\frac{1}{c^2}\left(p - \frac{dp}{dx} - p\right)\Delta t = -\frac{1}{c^2}\frac{dp}{dx}\Delta t = \frac{1}{c^2}\frac{d^2\mathscr{P}}{dx^2}\Delta t,$$

c designating a constant.

This increase in probability is also given by the expression $(d\mathscr{P}/dt)\Delta t$. Therefore, it follows that

$$c^2\frac{\partial \mathscr{P}}{\partial t} - \frac{\partial^2 \mathscr{P}}{\partial x^2} = 0.$$

This equation is due to Fourier.

The preceding theory assumes price fluctuations are discontinuous. Fourier's equation can be arrived at without making this hypothesis, by observing that in a very small interval of time Δt, the price varies in a continuous manner but by a very small amount, less than ϵ, for example.

Denote by ϖ the probability corresponding to p and relative to Δt. According to our hypothesis, the price may vary only within the limits $\pm\epsilon$ in the time Δt and it will follow that

$$\int_{-\epsilon}^{+\epsilon} \varpi\, dx = 1.$$

The price is perhaps $x - m$ at epoch t; being positive and smaller than ϵ. The probability of this event is p_{x-m}.

The probability that the price will be surpassed at epoch $t+\Delta t$, it being equal to $x - m$ at epoch t, will have a value, by virtue of the Principle of Compound Probabilities, of

$$p_{x-m}\int_{\epsilon-m}^{\epsilon} \varpi\, dx.$$

The price may be $x + m$ at epoch t; the probability of this event is $p_(x+m)$.

By virtue of the principle invoked previously, the probability that the price will be below x at epoch $t + \Delta t$, it being equal to $x + m$ at epoch t, has a value of

$$p_{x+m}\int_{\epsilon-m}^{\epsilon} \varpi\, dx.$$

The increase in the probability $\mathscr{P}$ in the interval of time Δt will be equal to the sum of expressions of the form

$$(p_{x-m} - p_{x+m}) \int_{\epsilon-m}^{\epsilon} \varpi \, dx$$

for all values of m from zero up to ϵ.

On expanding the expressions for p_{x-m} and p_{x+m} and neglecting the terms containing m^2, we then have

$$\begin{aligned} p_{x-m} &= p_x - m \frac{dp_x}{dx}, \\ p_{x+m} &= p_x + m \frac{dp_x}{dx}. \end{aligned}$$

The above expression then becomes

$$-\frac{dp}{dx} \int_{\epsilon-m}^{\epsilon} 2m\varpi \, dx.$$

The required increase therefore has a value of

$$-\frac{dp}{dx} \int_{0}^{\epsilon} \int_{\epsilon-m}^{\epsilon} 2m\varpi \, dx \, dm.$$

The integral does not depend on x, nor on t, nor on p: it is a constant. The increase in the probability $\mathscr{P}$ is given by the expression

$$\frac{1}{c^2} \frac{dp}{dx}.$$

Integrating the equation of Fourier gives

$$\mathscr{P} = \int_{0}^{\infty} f\left(t - \frac{c^2 x^2}{2\alpha^2}\right) e^{-\frac{\alpha^2}{2}} \, d\alpha.$$

The arbitrary function f is determined by the following considerations:

We must have $\mathscr{P} = 1/2$ if $x = 0$, t having some positive value; and $\mathscr{P} = 0$ when t is negative.

Assuming that $x = 0$ in the integral above, we have

$$\mathscr{P} = f(t) \int_{0}^{\infty} e^{-\frac{\alpha^2}{2}} \, d\alpha = \frac{\sqrt{\pi}}{\sqrt{2}} f(t),$$

that is to say,

$$\begin{aligned} f(t) &= \tfrac{1}{\sqrt{2}\sqrt{\pi}} \quad \text{for } t > 0, \\ f(t) &= 0 \quad \text{for } t < 0. \end{aligned}$$

This last equality shows that the integral $\mathscr{P}$ will have its zero elements according as $t - c^2x^2/2\alpha^2$ is less than zero, that is to say, according as α is less than $cx/\sqrt{2}\sqrt{t}$. Therefore, the quantity $cx/\sqrt{2}\sqrt{t}$ must be taken as the lower limit of the integral $\mathscr{P}$ and we have

$$\mathscr{P} = \frac{1}{\sqrt{2}\sqrt{\pi}} \int_{\frac{cx}{\sqrt{2}\sqrt{t}}}^{\infty} e^{-\frac{\alpha^2}{2}} \, d\alpha = \frac{1}{\sqrt{\pi}} \int_{\frac{cx}{\sqrt{2}\sqrt{t}}}^{\infty} e^{-\lambda^2} \, d\lambda,$$

or, on replacing $\int_{cx/\sqrt{2}\sqrt{t}}^{\infty}$ by $\int_0^{\infty} - \int_0^{cx/\sqrt{2}\sqrt{t}}$,

$$\mathscr{P} = \frac{1}{2} - \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_0^{\frac{cx}{2\sqrt{t}}} e^{-\lambda^2}\,d\lambda,$$

the formula previously found.

Law for Spread of Options. — In order to understand the law which governs the relation between size of a premium and the spread, the Principle of Mathematical Expectation will be applied to the buyer of an option:

The mathematical expectation of the buyer of an option is nil.

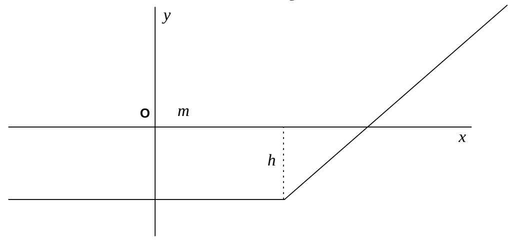
Take for the origin the true price of a forward contract (*Figure* 5).

Let p be the probability of price $\pm x$, that is to say, in the present case, the probability that the declaration of options takes place at price $\pm x$.

Let $m+h$ be the true spread of an option at h.

Set the total mathematical expectation to nil.

Figure 5.



Let us now proceed to evaluate this expectation:

(1) For prices between $-\infty$ and m,
(2) for prices between m and $m+h$,
(3) for prices between $m+h$ and $+\infty$.

(1) For all prices between $+\infty$ and m the option is abandoned, that is to say, that the buyer suffers a loss of h. His mathematical expectation for a price in the given interval is thus $-ph$, and for the whole interval

$$-h\int_{-\infty}^{m} p\,dx.$$

(2) For a price x lying between m and $m+h$ the buyer's loss will be $m+h-x$; the corresponding mathematical expectation will be $-p\,(m+h-x)$, and for the whole interval

$$-\int_{m}^{m+h} p\,(m+h-x)\,dx.$$

(3) For a price x lying between $m+h$ and ∞ the buyer's profit will be $x-m-h$; the corresponding mathematical expectation will be $p\,(x-m-h)$, and for the whole interval

$$\int_{m+h}^{\infty} p\,(x-m-h)\,dx.$$

The Principle of Total Expectation will therefore give

$$\int_{m+h}^{\infty} p\,(x-m-h)\,dx - \int_{m}^{m+h} p\,(x-m-h)\,dx - h\int_{-\infty}^{m} p\,dx = 0$$

or, after simplification,

$$h + m\int_{m}^{\infty} p\,dx = \int_{m}^{\infty} p\,x\,dx.$$

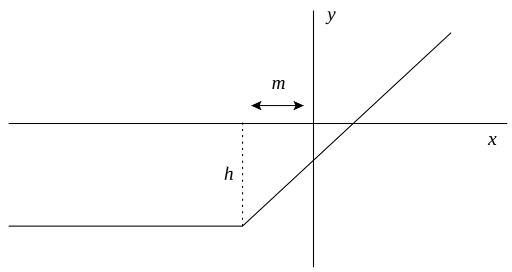
Such is the equation for the definite integrals which establishes a relation between probabilities, spreads of options and their premium sizes.

In the case where the foot of the option would fall on the side of negative x, as is shown in *Figure 6*, m would be negative and the relation arrived at would be

$$\frac{2h+m}{2} + m\int_{0}^{-m} p\,dx = \int_{-m}^{\infty} p\,x\,dx.$$

Due to the symmetry of the probabilities, the function p must be even. It follows that the two equations above form only one.

Figure 6.



On differentiating, the differential equation obtained for the option spreads is

$$\frac{d^2h}{dm^2} = p_m,$$

p_m being an expression for the probability in which x has been replaced by m.

Simple Options. [37] — The simplest case from the above equations is that where $m = 0$, that is to say, the one where the amount of the premium for an option is equal to its spread. This kind of option is called a *simple option*, the only kind that is negotiated in speculation on commodities.

The above equations become, on assuming $m = 0$ and on designating by a the value of the simple option,

$$a = \int_{0}^{\infty} p\,x\,dx = \int_{0}^{\infty} \frac{x}{2\pi k\sqrt{t}} e^{-\frac{x^2}{4\pi k^2 t}}\,dx = k\sqrt{t}.$$

The equality $a = \int_0^\infty p\,x\,dx$ shows that the simple option is equal to the positive expectation of the buyer of a forward contract. This fact is obvious, since the acquirer of the option pays the sum a to the dealer to enjoy the benefits of the buyer of a forward contract, that is to say, to have his positive expectation without incurring his risks.

[37] Trans: French *les primes simples*.

From the formula

$$a = \int_0^\infty p\,x\,dx = k\sqrt{t},$$

the following principle is deduced, one of the most important of our study:

> *The value of a simple option must be proportional to the square root of the elapsed time.*

It has been seen previously that the median spread was given by the formula

$$\alpha = 1.688k\sqrt{t} = 1.688a.$$

The median spread is thus obtained by multiplying the average premium by the numerical constant 1.688; it is, therefore, very simple to calculate when it comes to speculation on commodities since in this case, the quantity a is known.

The following formula gives the expression for the probability as a function of a

$$p = \frac{1}{2\pi a} e^{-\frac{x^2}{4\pi a^2}}.$$

The expression for the probability in a given interval is given by the integral

$$\frac{1}{2\pi a}\int_0^u e^{-\frac{x^2}{4\pi a^2}}\,dx$$

or

$$\frac{1}{2\pi a}\left(u - \frac{u^3}{12\pi a^2} + \frac{u^5}{160\pi^2 a^4} - \frac{u^7}{2678\pi^3 a^6} + \dots\right).$$

This probability is independent of a and consequently, of time, if u, instead of being a given number, is a parameter of the form $u = ba$. For example, if $u = a$

$$\int_0^a p\,dx = \frac{1}{2\pi} - \frac{1}{24\pi^2} + \frac{1}{320\pi^3} - \dots = 0.155.$$

The integral $\int_a^\infty p\,dx$ represents the possibility of profit for the acquirer of a simple option. Now,

$$\int_a^\infty p\,dx = \frac{1}{2} - \int_0^a p\,dx = 0.345.$$

Therefore:

> *The probability of profit for the acquirer of a simple option is independent of the expiry date. Its value is 0.345.*

The positive expectation of the simple option is given by the expression

$$\int_a^\infty p\,(x-a)\,dx = 0.58a.$$

Put-and-Call Operation. [38] — The *put-and-call* or *straddle option*[39] is composed of the simultaneous purchase of an option for a rise and an option for a fall (simple options). It is easy to see that the dealer of the put-and-call profits in the range $-2a$, $+2a$. His probability of profiting is therefore

$$2\int_0^{2a} p\,dx = \frac{2}{\pi} - \frac{2}{3\pi^2} + \frac{2}{10\pi^3} - \dots = 0.56.$$

[38]Trans: French *la double prime*.

[39]Trans: French *la stellage*.

The probability of success for the acquirer of a put-and-call is 0.44.

Positive expectation of a straddle option

$$2\int_{2a}^{\infty} p\,(x-2a)\,dx = 0.55a.$$

Coefficient of Instability. [40] — The coefficient k, introduced previously, is the *coefficient of instability* or of *volatility* for the security; it measures its static state. Its strength indicates a turbulent state; its weakness, on the contrary, is an indicator of a quiescent state.

This coefficient is given directly in speculation on commodities by the formula

$$a = k\sqrt{t},$$

but in speculation on securities it can only be calculated by approximation, as will be seen.

Series Expansion for Spreads of Options. — The equation for the definite integrals for the spreads of options is not expressible in finite terms when the quantity m, the difference between the spread of the option and the size of its premium h, is non-zero.

This equation leads to the series expansion

$$h - a + \frac{m}{2} - \frac{m^2}{4\pi a} + \frac{m^4}{96\pi^2 a^3} - \frac{m^6}{1920\pi^3 a^5} + \ldots = 0.$$

This relation, in which the quantity a denotes the amount of the premium for a simple option, permits the calculation of the value of a when that of m is known, and *vice versa*.

Approximate Law for Spreads of Options. — The preceding series may be written as

$$h = a - f(m).$$

Consider the product of the premium h by its spread:

$$h(m+h) = [a - f(m)][m + a - f(m)].$$

Differentiating with respect to m, we have

$$\frac{d}{dm}[h(m+h)] = f'(m)[m + a - f(m)] + [a - f(m)][1 - f'(m)].$$

If it is assumed that $m = 0$, whence $f(m) = 0$, $f'(m) = 1/2$, this derivative vanishing, it must be concluded that

> *The product of the premium for an option by its spread is maximized when the two factors of this product are equal: this is the case for a simple option.*

In the neighbourhood of its maximum, the product in question should change only a little. This will often permit approximate evaluation of a by the formula

$$h(m+h) = a^2,$$

which gives too low a value for a.

---

[40] Trans: French *le coefficient d'instabilité*.

In considering only the first three terms in the series, the expression

$$h(h+m) = a^2 - \frac{m^2}{4}$$

is obtained, which gives too high a value for a.

In the majority of cases, on taking the first four terms in the series, a very satisfactory approximation will be obtained; we will then have

$$a = \frac{\pi(2h+m) \pm \sqrt{\pi^2(2h+m)^2 - 4\pi m^2}}{4\pi}.$$

With this same approximation we will then have for the value of m as a function of a

$$m = \pi a \pm \sqrt{\pi^2 a^2 - 4\pi a(a-h)}.$$

Assume, for the time being, the simplified formula

$$h(m+h) = a^2 = k^2 t.$$

In speculation on securities, options for a rise have a constant premium of size h. Therefore, the spread $m+h$ is proportional to the elapsed time.

In speculation on securities, the spread for options for a rise is approximately proportional to the term to expiration and the square of the coefficient of instability.

On securities, options for a fall (that is to say, a forward sale against a purchase of an option) have a constant spread h, and a variable premium of size $m+h$. Therefore:

In speculation on securities, the amount of the premium for options for a fall is approximately proportional to the term to expiration and the square of the instability.

The two preceding laws are only approximations.

Call-of-More Operations. — Let us proceed to apply the Principle of Mathematical Expectation to the purchase of a call-of-more of order n negotiated at a spread of r.

The call-of-more of order n may be regarded as being composed of two operations:

(1) A forward purchase of one unit at price r.
(2) A forward purchase of $(n-1)$ units at price r; this purchase being considered only in the interval r, ∞.

The mathematical expectation of the first operation is $-r$; the expectation of the second is

$$(n-1)\int_r^\infty p\,(x-r)\,dx.$$

We will therefore have

$$r = (n-1)\int_r^\infty p\,(x-r)\,dx$$

or, on replacing p by its value,

$$p = \frac{1}{2\pi a} e^{-\frac{x}{4\pi a^2}},$$

and, on expanding as a power series,

$$2\pi a^2 - \pi a \frac{n+1}{n-1} r + \frac{r^2}{2} - \frac{r^4}{48\pi a^2} + \ldots = 0.$$

On retaining only the first three terms, we obtain

$$r = a\left[\frac{n+1}{n-1}\pi - \sqrt{\left(\frac{n+1}{n-1}\pi\right)^2 - 4\pi}\right].$$

If $n = 2$,

$$r = 0.68a.$$

The spread for a call-of-twice-more must be about two-thirds of the value for a simple option.

If $n = 3$,

$$r = 1.096a.$$

The spread for a call-of-thrice-more must be greater by approximately one tenth of the value for a simple option.

We have seen that the spreads of these calls-of-more are approximately proportional to the quantity a.

It follows that the probability of profiting from these operations is independent of the term to expiration.

The probability of profiting from a call-of-twice-more is 0.394; the operation is profitable four times out of ten.

The probability for a call-of-thrice-more is 0.33; the operation is profitable one time out of three.

The positive expectation for a call-of-more of order n is

$$n\int_r^\infty p(x-r)\,dx,$$

and since

$$\frac{r}{n-1} = \int_r^\infty p(x-r)\,dx,$$

the required expectation has a value of $[n/(n-1)]r$, that is to say, $1.36a$ for a call-of-twice-more and $1.64a$ for a call-of-thrice-more.

On a forward sale and simultaneous purchase of a call-of-twice-more, an option is obtained for which the premium size is $r = 0.68a$ and for which the spread is twice times r.

The probability of profiting from the operation is 0.30.

By analogy with operations for options, let us call the *call-of-more-put-of-more*[41] of order n the operation resulting from two calls-of-more of order n: one for a rise and one for a fall.

The call-of-more-put-of-more of the second order is a very curious operation: between the prices $\pm r$ the loss is constant and equal to $2r$. The loss diminishes gradually up to price $\pm 3r$ where it vanishes.

There is a profit outside the interval $\pm 3r$.

The probability is 0.42.

[41]Trans: French *l'option stellage.*

FORWARD-DATED OPERATIONS.

Now that the general study of probabilities has been completed let us apply it to the study of probabilities for the principal operations of the Stock Exchange, commencing with the simplest: forward contracts and options; and then we will conclude by studying combinations of these operations.

The Theory of Speculation in commodities, so much simpler than that of securities, has already been treated. Indeed, the probability and mathematical expectation have been calculated for simple options, puts-and-calls and calls-of-more.

The theory of Stock Exchange operations depends on the two coefficients: b and k.

Their values, at a given instant, can be deduced easily from the spread between the forward and cash prices and from the spread of any type of option.

The following discussion will be solely concerned with the 3% Rente, which is one of the securities on which options are regularly negotiated.

Let us take for the values of b and k their average values for the last five years (1894 to 1898), that is to say,

$$\begin{aligned} b &= 0.264, \\ k &= 5 \end{aligned}$$

(time is expressed in days and the currency unit is the centime).

By *calculated* values is meant those which are deduced from the formulae of the theory with the values above given by constants b and k.

The *observed* values are those that are deduced directly from the compilation of quotes during this same period of time from 1894 to 1898[42].

In the following chapters we will continually have to know the average values of the quantity a at different epochs: the formula

$$a = 5\sqrt{t}$$

gives

For	45	days	....	$a = 33.54$
”	30	”	....	$a = 27.38$
”	20	”	....	$a = 22.36$
”	10	”	....	$a = 16.13$

For a single day, it might seem that $a = 5$; but in any calculations of probabilities involving averages it cannot be assumed that $t = 1$ for one day.

In fact, there are 365 days in a year, but only 307 trading days. The *average day* for trading is therefore $t = 365/307$; this gives $a = 5.45$.

The same remark can be made for the coefficient b.

In all calculations relating to a trading day b must be replaced by $b_1 = (365/307)\,b = 0.313$.

Median Spread. [43] — Let us now seek the price interval, $(-\alpha, +\alpha)$, such that, after one month, the chance of a Rente being found within this interval is as likely as the chance of it being found outside of it.

[42] All the observations are extracted from the *Cote de la Bourse et de la Banque*.

[43] Trans: French *l'écart probable*.

We must have

$$\int_0^{\alpha} p\,dx = \frac{1}{4},$$

whence

$$\alpha = \pm 46.$$

During the last 60 months, the fluctuation has been confined within these limits on 33 occasions and has surpassed them on 27 occasions.

The same interval relating to a single day can be found; thus we have

$$\alpha = \pm 9.$$

Amongst 1,452 observations, the fluctuation has been less than 9c on 815 occasions.

In the preceding question, it was assumed that the quoted price was conflated with the true price. Under these conditions, both the probability and the mathematical expectation are the same for buyers as for sellers.

Actually, the quoted price is lower than the true price by the quantity nb where n is the number of days away from the expiration date.

The median spread of 46c on either side of the true price corresponds to the interval between 54c over and above the quoted price and 38c down below this price.

Formula for Probability in the General Case. — To find the probability of a price rise for a period of n days, it is necessary to know the spread nb from the true price to the quoted price; the probability is then equal to

$$\int_{-nb}^{\infty} p\,dx.$$

The probability of a price fall will be equal to unity diminished by the probability of a price rise.

Probability for a Cash Purchase. — Let us find the probability of profiting from a cash purchase destined to be resold in 30 days.

The quantity nb must be replaced by 25 in the preceding formula.

The probability is then equal to 0.64.

The operation has two chances out of three of yielding a profit.

If we wish to have the probability for one year, the quantity nb must be replaced by 300.

The formula $a = k\sqrt{t}$ gives $a = 95.5$.

The probability is found to be 0.89.

A cash purchase of a Rente yields a profit nine times out of ten after one year.

Probability for Purchase of a Forward Contract. — Let us find the probability of profiting from a forward purchase effected at the start of the month.

We have

$$nb = 7.91, \quad a = 27.38.$$

It may be deduced that:

The probability of a	rise	is	....	0.55
”	fall	”	....	0.45

The probability of profiting from the purchase increases with time. For one year, we have

$$n = 365, \quad nb = 96.36, \quad a = 95.5.$$

The probability then has a value of 0.65.

When effecting a forward purchase for resale after one year, the chances of profiting are two out of three.

It is evident that if the monthly contango were 25c the probability of profiting from the purchase would be 0.50.

Mathematical Advantage for Operations on Forward Contracts. — It appears to me indispensable, as I have already remarked, to study the mathematical advantage of a game when it is unfair, and this is the case for forward contracts.

If it is assumed that $b = 0$, the mathematical expectation for a forward purchase is $a - a = 0$. The advantage of the operation is $a/2a = 1/2$, as indeed in every fair game.

Let us find the mathematical advantage for a forward purchase for n days assuming that $b > 0$. During this period, the buyer will receive the sum nb from the difference between the coupons and contangos, and his expectation will be $a - a + nb$. Therefore, his mathematical advantage will be

$$\frac{a + nb}{2a + nb}.$$

The mathematical advantage of the seller will be

$$\frac{a}{2a + nb}.$$

Now, consider the specific case of the buyer.

When $b > 0$ his mathematical advantage increases more and more with n; it is consistently higher than the probability.

For a single month, the mathematical advantage for the buyer is 0.563 and his probability is 0.55.

For a single year, his mathematical advantage is 0.667 and his probability is 0.65.

Therefore, it can be stated that:

> *The mathematical advantage of a forward contract is almost equal to its probability.*

OPERATIONS FOR OPTIONS.

Spread of Options. — Knowing the value of a at a given epoch, the true spread can easily be calculated by the formula

$$m = \pi a \pm \sqrt{\pi^2 a^2 - 4\pi a(a - h)}.$$

Knowing the true spread, the quoted spread can be obtained by adding the quantity nb to the true spread; n is the number of days from the declaration of options.

In the case of an option for the following liquidation date, we add the quantity $[25 + (n - 30)b]$.

We thus arrive at the following results:

Options /50.

			Quoted Spread.	
			Calculated.	Observed.
At	45	days	50.01	52.62
	30	"	20.69	21.22
	20	"	13.20	14.71

Options /25.

			Quoted Spread.	
			Calculated.	Observed.
At	45	days	72.70	72.80
	30	"	37.78	37.84
	20	"	25.17	27.9
	10	"	12.24	17.40

Options /10.

			Quoted Spread.	
			Calculated.	Observed.
At	30	days	66.19	60.93
	20	"	48.62	46.43
	10	"	26.91	32.89

In the case of an option /5c for the next day we have

$$h = 5, \quad a = 5.45$$

whence

$$m = 0.81.$$

Therefore, the true spread is 5.81; on adding $b_1 = (365/307)b = 0.31$ the calculated spread 6.12 is obtained.

The average of the last five years gives 7.36.

The observed and the calculated figures are consistent as an aggregate, but they display certain divergences which it is necessary to explain.

Where the observed spread of the option /10 at 30 days is too low, it is easy to understand the reason: In very turbulent periods, when the option /10 would have a very great spread, this option is not quoted. The observed average is thus reduced due to this fact.

On the other hand, it is undeniable that the market has had, for several years, a tendency to quote spreads that are too great for options with short terms to expiration; it takes even less account of the correct proportion of those spreads that are smaller and for which the expiry date is very close.

However, it must be added that the market seems to have realized its mistake, because in 1898 it appears to have been exaggerated in the opposite direction.

Probability of Exercising Options. — For an option to be exercised, the price at the declaration of the option should be above the foot of the option. The probability of exercising an option is thus expressed by the integral

$$\int_{\epsilon}^{\infty} p\, dx,$$

ϵ being the true price of the foot of the option.

This integral is simple to calculate, as has been seen previously. It leads to the following results:

Probability of exercising options /50.

				Calculated.	Observed.
At	45	days	....	0.63	0.59
	30	”	....	0.71	0.75
	20	”	....	0.77	0.76

Probability of exercising options /25.

				Calculated.	Observed.
At	45	days	....	0.41	0.40
	30	”	....	0.47	0.46
	10	”	....	0.53	0.53
	10	”	....	0.65	0.65

Probability of exercising options /10.

				Calculated.	Observed.
At	30	days	....	0.24	0.21
	20	”	....	0.28	0.26
	10	”	....	0.36	0.38

It can be stated that options /50 are exercised three times out of four, options /25 two times out of four and options /10 one time out of four.

The probability of exercising an option /5c for the next day is, after calculation: 0.48; the result of 1,456 observations gives 671 options certainly exercised and 76 whose exercise is in doubt. On counting these last 76 in, the probability would be 0.51; on not counting them in it would be 0.46; the average would be 0.48 as indicated by the theory.

Probability of Profiting from Options. — For an option to yield a profit for its buyer, it is necessary that the declaration of options be made at a price above that of the option.

The probability of a profit is thus expressed by the integral

$$\int_{\epsilon_1}^{\infty} p\,dx,$$

ϵ_1 being the price of the option.

This integral leads to the results below:

Probability of profit from options /50.

				Calculated.	Observed.
At	45	days	....	0.40	0.39
	30	”	....	0.43	0.41
	20	”	....	0.44	0.40

Probability of profit from options /25.

			Calculated.	Observed.
At	45	days	0.30	0.27
	30	”	0.33	0.31
	20	”	0.36	0.30
	10	”	0.41	0.40

Probability of profit from options /10.

			Calculated.	Observed.
At	30	days	0.20	0.16
	20	”	0.22	0.18
	10	”	0.27	0.25

It can be seen that within the ordinary limits of practice, the probability of profiting from the purchase of an option varies little. The purchase /50 succeeds four times out of ten, the purchase /25 three times out of ten and purchase /10 two times out of ten.

According to the calculation, the buyer of an option /5c for the next day has a probability of 0.34 of making a profit, the observation of 1,456 quotes showing that 410 options have certainly given profits and 80 others have given an uncertain result, the observed probability is therefore 0.31.

COMPLEX OPERATIONS.

Classification of Complex Operations. — Since forward contracts, and often as many as three options, may be negotiated for the same expiry date, we could undertake at the same time triple operations and even quadruple operations.

Triple operations are no longer numbered among those that are considered standard; their study is very interesting, but too lengthy to be expounded here.

Therefore, we will confine ourselves to double operations.

These can be divided into two groups according to whether or not they contain a forward contract.

Operations containing a forward contract will be composed of a forward purchase and a sale of an option, or *vice versa*.

Operations for *option against option*[44] are composed of the sale of an option with a large premium followed by the purchase of an option with small premium, or *vice versa*.

The ratio of purchases and sales can also vary infinitely. In order to simplify the problem let us examine only two quite simple proportions:

(1) The second operation involves the same number as the first.
(2) The second operation involves double the first number.

To limit the ideas, it will be assumed that operations are conducted at the start of the month and we will take for the true spread the average spreads for the past five years: 412.78/50, 29.87/25 and 58.28/10.

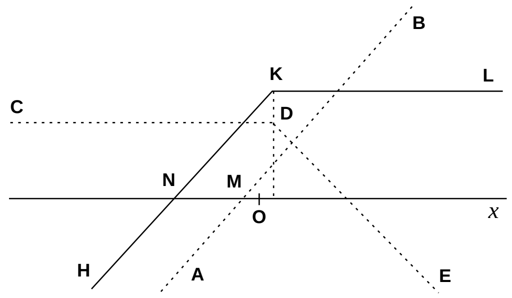
Observe also that for operations in one month the true price is higher than the quoted price, by the quantity $7.91 = 3b$.

[44]Trans: French *prime contre prime*.

Purchase of Forward Contract Against Sale of Option. — In fact, forward contracts are bought at a price of $-30b = -7.91$ and options /25 are sold at a price of +29.87.

It is easy to describe the operation by a geometrical construction (*Figure 7*). The forward purchase is represented by the straight line AMB:

FIGURE 7.



MO = $30b$.

The sale of an option is represented by the broken line CDE, the resulting operation will be represented by the broken line HNKL, the abscissa of the point N will be $-(25+30b)$.

It can be seen that the operation produces a limited gain equal to the quoted spread of the option; on a price fall, the risk is unlimited.

The probability of the operation yielding a profit is expressed by the integral

$$\int_{-25-30b}^{+\infty} p\,dx = 0.68.$$

If an option /50 had been sold, the probability of a profit would have been 0.80.

It would be of interest to know the probability in the case of a contango of 25c ($b = 0$).

This probability is 0.64 on selling the option /25 and 0.76 on selling the option /50.

If an option is resold against a cash purchase, the probability is 0.76 on reselling the option /25 and 0.86 on reselling the option /50.

Sale of Forward Contract Against Purchase of Option. — This operation is the inverse of the preceding one; on a price rise it gives a limited loss and on a price fall it gives an unlimited profit.

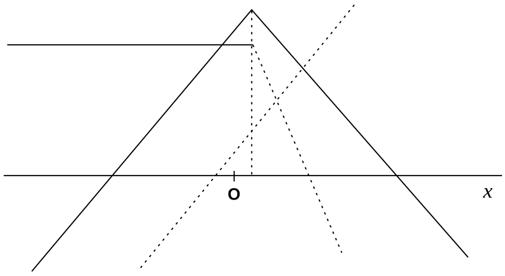
Consequently, it is an option for a fall, an option whose spread is constant and the premium size is variable, the opposite of an option for a rise.

Purchase of Forward Contract Against Sale of Pair of Options. — A forward purchase is made at true price $-30b$ and a pair of options is sold at a price of 29.87/25.

Figure 8 represents the operation geometrically; it shows that the risk is unlimited on a price rise just as it is on a price fall.

A profit is made between the prices $-(50+30b)$ and $59.74+30b$.

Figure 8.



The probability of a profit is

$$\int p\,dx = 0.64.$$

On selling /50 the probability would be 0.62, and on selling /10 the probability would be 0.62.

If a forward contract on two units had been bought to sell options at 3/50, the probability of profiting would have been 0.66.

Sale of Forward Contract Against Purchase of Pair of Options. — This is the inverse operation of the preceding one. It yields profits in the case of a large price rise and in that of a large price fall.

Its probability is 0.27.

Purchase of Option with Large Premium Against Sale of Option with Small Premium. — Suppose that the following two operations have been performed simultaneously:

- Bought at 12.78/50
- Sold at 29.87/25

Below the foot of the option with large premium (-37.22), the two options are abandoned and the loss is 25c.

Starting from price -37.22 we would be a buyer. And at a price of -12.22 the operation is nil.

A profit would continue to be made up to the foot of the option /25, that is to say, until the price +4.87 be attained.

Then on liquidation the profit would be the spread. Therefore, on a price fall the loss would be 25c, which is the maximum risk; on a price rise the profit would be the spread.

The risk is limited and the profit equally so.

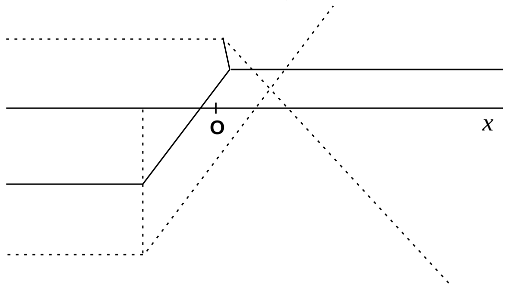
Figure 9 represents the operation geometrically.

The probability of a profit is given by the integral

$$\int_{-12,22}^{\infty} p\,dx = 0.59.$$

When buying /25 in order to sell /10, the probability of a profit will be 0.38.

FIGURE 9.



Sale of Option with Large Premium Against Purchase of Option with Small Premium. — This operation, which is the counterpart of the preceding one, can be discussed without difficulty. On a price fall the profit is the difference between the amounts of the option premia. On a price rise the loss is their spread.

Purchase of Option with Large Premium Against Sale of Pair of Options with Small Premium. — Suppose that the following operation had been performed:

- Bought at 12.78/50
- Sold pair at 29.87/25

On a heavy price fall, the options are abandoned, they cancel each other out; this is a *null*[45] operation.

At the foot of the option with large premium, that is to say, at the price −37.22, we become a buyer and we gain progressively until the foot of the small option (+4.87).

At this moment, the profit is maximized (42.09 centimes) and we would become a seller.

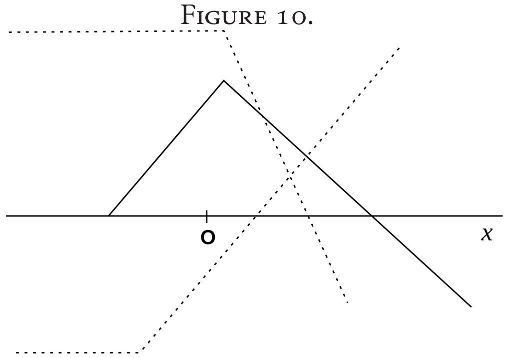
The profit progressively declines, and at a price of 45.96, this profit is nil.

Beyond that, the loss is proportional to the price rise.

In summary, the operation gives a limited profit, nil risk on a price fall, and unlimited risk on a price rise.

Figure 10 represents the operation geometrically.

FIGURE 10.



[45]Trans: French *en blanc*.

Probability	of null operation	0.30
”	of profit	0.45
”	of loss	0.25

Sale of Option with Large Premium Against Purchase of Pair of Options with Small Premium. — The discussion and the geometrical representation of this operation, the inverse of the preceding one, presents no difficulty. It is superfluous for us dwell on it here.

Practical Classification of Stock Exchange Operations. — From a practical point of view, Stock Exchange operations can be divided into four classes:

- Operations speculating on a price rise.
- Operations speculating on a price fall.
- Operations speculating on a large price movement in either direction.
- Operations speculating on small price movements.

The following table summarizes the main operations for speculating on a price rise:

	Average Probability.		
	$b = \frac{25}{30}$ (Nil Contango).	$b = 0.26$ (Average Contango).	$b = 0$ (Contango Equal to Coupons).
Buy /10	0.20	0.20	0.20
Buy /25	0.33	0.33	0.33
Buy /25 against sell /10	0.38	0.38	0.38
Buy /50	0.43	0.43	0.43
Buy forward	0.64	0.55	0.50
Buy /50 against sell /25	0.59	0.59	0.59
Buy forward against sell /25	0.76	0.68	0.64
Buy forward against sell /50	0.86	0.80	0.76

It suffices to invert this table to obtain the scale for operations speculating on a price fall.

PROBABILITY THAT A PRICE BE ATTAINED IN A GIVEN INTERVAL OF TIME.

Let us find the probability P of a given price c being attained or surpassed in an interval of time t.

Suppose, for simplicity, that the interval of time is divided into two units: that t is equal to two days, for example.

Let x be the price quoted on the first day and let y be the price, relative to the first, on the second day.

For the price c to be attained or surpassed, it is necessary that on the first day the price is included between c and ∞ or on the second day, it is included between $c - x$ and ∞.

In the present problem, four cases must be distinguished:

First Day. x Between:			Second Day. y Between:		
$-\infty$	and	c	$-\infty$	and	$c-x$
$-\infty$	and	c	$c-x$	and	$+\infty$
c	and	∞	$-\infty$	and	$c-x$
c	and	∞	$c-x$	and	$+\infty$

Of these four cases, only the latter three are favourable.

The probability that the price is found in the range dx on the first day and in the interval dy on the second day, will be $p_x p_y \, dx \, dy$.

The probability P, being by definition the ratio of the number of favourable cases to the possible cases, is given by the expression

$$P = \frac{\int_{-\infty}^{c}\int_{c-x}^{\infty} + \int_{c}^{\infty}\int_{-\infty}^{c-x} + \int_{c}^{\infty}\int_{c-x}^{\infty}}{\int_{-\infty}^{c}\int_{-\infty}^{c-x} + \int_{-\infty}^{c}\int_{c-x}^{\infty} + \int_{c}^{\infty}\int_{-\infty}^{c-x} + \int_{c}^{\infty}\int_{c-x}^{\infty}}$$

(the element is $p_x p_y \, dx \, dy$).

The four integrals of the denominator represent the four possible cases, the three integrals in the numerator represent the three favourable cases.

The denominator being equal to one, the expression can be simplified and written as

$$P = \int_{-\infty}^{c}\int_{c-x}^{\infty} p_x p_y \, dx \, dy + \int_{c}^{\infty}\int_{-\infty}^{\infty} p_x p_y \, dx \, dy.$$

The same reasoning can be applied on supposing that three consecutive days need to be considered, then four, etc.

This method will lead to more and more complicated expressions, for the number of favourable cases would be ever-increasing. It is much simpler to study the probability $1-P$ that the price c never be attained.

There is then but one favourable case whatever be the number of days; this is where the price is attained on any of the days considered.

The probability $1-P$ is given by the expression

$$1 - P = \int_{-\infty}^{c}\int_{-\infty}^{c-x_1}\int_{-\infty}^{c-x_1-x_2} \cdots \int_{-\infty}^{c-x_1 \ldots - x_{n-1}} p_{x_1} \ldots p_{x_n} \, dx_1 \ldots dx_n,$$

x_1 is the price on the first day,
x_2 is the price on the second day in respect of the first,
x_3 is the price on the third day, etc.

The determination of this integral appearing difficult, the problem will be resolved by employing a method of approximation.

The interval of time t can be considered as being divided into smaller intervals Δt, such that $t = m\Delta t$. During the unit of time Δt, the price will only vary by the quantity $\pm\Delta x$, the average spread relative to this unit of time.

Each spread $\pm\Delta x$ will have probability $1/2$.

Assuming that $c = n\Delta x$, let us find the probability that the price c will be attained precisely at the epoch t, that is to say, that this price will be attained at this epoch t without ever having been previously attained.

If, during the m units of time, the price has varied by the quantity $n\Delta x$ there must have been $(m+n)/2$ fluctuations upwards and $(m-n)/2$ downwards.

The probability that, on m fluctuations, there have been $(m+n)/2$ favourable ones is

$$\frac{m!}{\frac{m-n}{2}!\,\frac{m+n}{2}!}\left(\frac{1}{2}\right)^m.$$

This is not the required probability, but rather the product of this probability by the ratio of the number of cases where the price $n\Delta x$ is attained at epoch $m\Delta t$, not having been previously attained, to the total number of cases where it is attained at epoch $m\Delta t$.

Let us proceed to compute this ratio.

During the m units of time under consideration, there will have been $(m+n)/2$ upward fluctuations and $(m-n)/2$ downward fluctuations.

Each of the permutations giving a price rise of $n\Delta x$ in m units of time can be represented by the symbol

$$\mathrm{B}_1\mathrm{H}_1\mathrm{H}_2\ldots\mathrm{B}_{\frac{m-n}{2}}\ldots\mathrm{H}_{\frac{m+n}{2}},$$

B_1 indicates that, during the first time unit, there was a price fall. H_1, which follows, indicates that there was a price rise during the second time unit, etc.

For a permutation to be favourable, it is necessary that, on reading from right to left, the number of H's be consistently greater than the number of B's. As can be seen, the problem has now been reduced to the following:

> *Given n letters where $(m+n)/2$ are the letter H and $(m-n)/2$ are the letter B; what is the probability that in writing these letters at random and reading them in a given order, the number of H's is, throughout the reading, always greater than the number of B's?*

The solution to this problem, presented in a slightly different form, has been given by M. André. The required probability is equal to n/m.

The probability that the price $n\Delta x$ is attained precisely at the end of m units of time is thus

$$\frac{n}{m}\,\frac{m!}{\frac{m-n}{2}!\,\frac{m+n}{2}!}\left(\frac{1}{2}\right)^m.$$

This formula is an approximation; a more exact expression will be obtained by replacing the quantity that multiplies n/m by the exact value of the probability at epoch t, that is to say, by

$$\frac{\sqrt{2}}{\sqrt{m}\sqrt{\pi}}e^{-\frac{n^2}{\pi m}}.$$

The required probability is therefore

$$\frac{n\sqrt{2}}{m\sqrt{m}\sqrt{\pi}}e^{-\frac{n^2}{\pi m}},$$

or, on replacing n by $2c\sqrt{\pi}/\sqrt{2}$ and m by $8\pi k^2 t$,

$$\frac{dt\,c\sqrt{2}}{2\sqrt{\pi}\,kt\sqrt{t}}e^{-\frac{c^2}{4\pi k^2 t}}.$$

This is the expression for the probability that the price c is attained at epoch dt, not having been attained previously.

The probability that the price c not be attained before epoch t will have the value

$$1 - P = A \int_t^\infty \frac{c\sqrt{2}}{2\sqrt{\pi}kt\sqrt{t}} e^{-\frac{c^2}{4\pi k^2 t}} dt.$$

The integral has been multiplied by the constant A, yet to be determined, because the price can only be attained if the quantity designated by m is even.

Setting

$$\lambda^2 = \frac{c^2}{4\pi k^2 t}$$

gives

$$1 - P = 2\sqrt{2}A \int_0^{\frac{c}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2} d\lambda.$$

To determine A, set $c = \infty$, then $P = 0$ and

$$1 = 2\sqrt{2}A \int_0^\infty e^{-\lambda^2} d\lambda = \sqrt{2}\sqrt{\pi}A.$$

thus,

$$A = \frac{1}{\sqrt{2}\sqrt{\pi}},$$

and so

$$1 - P = \frac{2}{\sqrt{\pi}} \int_0^{\frac{c}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2} d\lambda.$$

The probability that price x be attained or surpassed during the interval of time t is thus given by the expression

$$P = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2} d\lambda.$$

The probability that price x be attained or surpassed *at epoch t*, as we have seen, is given by the expression

$$\mathscr{P} = \frac{1}{2} - \frac{1}{2}\frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2} d\lambda.$$

It can be seen that $\mathscr{P}$ is half of P.

The probability that a price be attained or surpassed at epoch t is half of the probability that this price be attained or surpassed in the interval of time up to t.

The direct demonstration of this result is very simple: the price cannot be surpassed at epoch t without having been attained previously. The probability $\mathscr{P}$ is therefore equal to the probability P, multiplied by the probability that, the price having been quoted at an epoch prior to t, be surpassed at epoch t; that is to say, multiplied by $1/2$. Therefore, we have $\mathscr{P} = P/2$.

It may be observed that the multiple integral which expresses the probability $1 - P$, and which seems impervious to ordinary methods of calculation, can be established by a very simple and elegant argument from the Theory of Probability.

Applications. — Tables of the function Θ permit very easy calculation of the probability

$$P = 1 - \Theta\left(\frac{x}{2\sqrt{\pi}k\sqrt{t}}\right).$$

The formula

$$P = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\,d\lambda$$

demonstrates that the probability is constant when the spread x is proportional to the square root of the elapsed time, that is to say, when it can be expressed in the form $x = ma$. Let us proceed to study the probabilities corresponding to certain interesting spreads.

Firstly, suppose that $x = a = k\sqrt{t}$; the probability P is then equal to 0.69. When a spread of a is attained, a forward contract on a simple option with premium a may be resold without loss. Consequently:

> *There are two chances in three that a forward contract on a simple option can be resold without loss.*

Let us particularise the question by applying it to a 3% Rente over a period of 60 months. We could resell 38 times at the spread a; which corresponds to a probability of 0.63.

Let us now proceed to study the case where $x = 2a$.

The preceding formula gives a probability of 0.43.

When a spread of $2a$ is attained, a forward contract on a put-and-call[46] can be resold without loss. Thus

> *There are four chances in ten that a forward contract on a put-and-call can be resold without loss.*

Over a period of 60 liquidations, a 3% Rente will attain a spread $2a$ 23 times, which gives a probability of 0.38.

A spread $0.7a$ is that of a call-of-twice-more; the corresponding probability is 0.78.

> *There are three chances in four of reselling without loss a forward contract on a call-of-twice-more.*

A call-of-thrice-more must be negotiated at a spread of $1.1a$ which corresponds to a probability of 0.66.

> *There are two chances in three of reselling without loss a forward contract on a call-of-thrice-more.*

Finally, let us mention some significant spreads such as the spread $1.7a$ which corresponds to a probability of 1/2 and the spread $2.9a$ which corresponds to a probability of 1/4.

Apparent Mathematical Expectation. [47] — The mathematical expectation

$$\mathcal{E}_1 = Px = x - \frac{2x}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\,d\lambda$$

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[46]Trans: French *prime double*. Also *double prime*.

[47]Trans: French *l'espérance mathmatique apparente*.

is a function of x and of t. Differentiating with respect to x gives

$$\frac{\partial \mathcal{E}_1}{\partial x} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2} d\lambda - \frac{x e^{-\frac{x^2}{4\pi k^2 t}}}{\pi k \sqrt{t}}.$$

If a fixed epoch t is considered, this expectation will be maximized when

$$\frac{\partial \mathcal{E}_1}{\partial x} = 0,$$

that is to say, when $x = 2a$, or thereabouts.

Total Apparent Expectation. [48] — The total expectation corresponding to elapsed time t will be the integral

$$\int_0^\infty P x \, dx.$$

Suppose that

$$f(a) = \int_0^\infty \left(x - \frac{2x}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\pi}a}} e^{-\lambda^2} d\lambda \right) dx.$$

Differentiating with respect to a gives

$$f'(a) = \frac{1}{\pi a^2} \int_0^\infty x^2 e^{-\frac{x^2}{4\pi a^2}} dx,$$

or $f'(a) = 2\pi a$. We therefore have

$$f(a) = \pi a^2 = \pi k^2 t.$$

The total expectation is proportional to the elapsed time.

Most Probable Epoch. [49] — The probability

$$P = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2} d\lambda$$

is a function of x and of t.

The study of its variation, considering x as the variable, presents nothing remarkable: the function decreases consistently as x increases.

Assuming that x be constant let us now study the variation of the function by considering t as the variable. On differentiating we will have

$$\frac{\partial P}{\partial t} = \frac{x e^{-\frac{x^2}{4\pi k^2 t}}}{2\pi t \sqrt{t}}.$$

The epoch of maximum probability will be determined from the vanishing of the derivative

$$\frac{\partial^2 P}{\partial t^2} = \frac{x e^{-\frac{x^2}{4\pi k^2 t}}}{2\pi k t \sqrt{t}} \left(\frac{x^2}{4\pi k^2 t} - \frac{3}{2} \right);$$

then

$$t = \frac{x^2}{6\pi k^2}.$$

Suppose, for example, that $x = k\sqrt{t_1}$; we then get $t = t_1/6\pi$.

[48]Trans: French *l'espérance totale apparente*.

[49]Trans: French *l'époque de la plus grande probabilité*.

The most probable epoch at which a forward contract on a simple option can be resold without loss is situated at one eighteenth of the term to expiration.

Now, supposing that $x = 2k\sqrt{t_1}$ we obtain $t = 2t_1/3\pi$.

The most probable epoch at which a forward contract on a put-and-call can be resold without loss is situated at one fifth of the term to expiration.

The probability P corresponding to epoch $t = x^2/6\pi k^2$ has a value of $1 - \Theta\left(\sqrt{6}/2\right) = 0.08$.

Mean Epoch. [50] — When an event can occur at different epochs, the *mean arrival time of the event*[51] is defined as the sum of the products of the probabilities corresponding to the given epochs multiplied by their respective durations.

The *mean duration*[52] is equal to the sum of expectations of the duration.

The *mean epoch* at which the price x will be surpassed is therefore expressed by the integral

$$\int_0^\infty t\frac{dP}{dt}\,dt = \int_0^\infty \frac{x}{2\pi k\sqrt{t}} e^{-\frac{x^2}{4\pi k^2 t}}\,dt.$$

On setting $x^2/4\pi k^2 t = y^2$, it becomes

$$\frac{x^2}{2\pi\sqrt{\pi}\,k^2}\int_0^\infty \frac{e^{-y^2}}{y^2}\,dy.$$

This integral is infinite.

The mean epoch is therefore infinite.

Median Epoch. [53] — This will be the epoch for which $P = 1/2$ or

$$\Theta\left(\frac{x}{2\sqrt{\pi}\,k\sqrt{t}}\right) = \frac{1}{2}.$$

It may be deduced that

$$t = \frac{x^2}{2.89\,k^2}.$$

The *median epoch* varies, the same as the most probable epoch, in proportion to the square of the quantity x and is about six times greater than the most probable epoch.

Relative Median Epoch. [54] — It is of interest to know, not only the probability that a price x will be quoted in the interval of time up to t, but also the median epoch T at which this price will be attained. This epoch is evidently different from the epoch with which we have been concerned.

The interval of time up to T will be such that there will be as much chance that the price be attained before epoch T as the chance of being quoted in the following one, that is to say, in the interval of time T, t.

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[50] Trans: French *l'époque moyenne*.

[51] Trans: French *l'époque moyenne de l'arrivée de l'événement*.

[52] Trans: French *la durée moyenne*.

[53] Trans: French *l'époque probable absolue*.

[54] Trans: French *l'époque probable relative*.

T will be given by the formula

$$\int_0^T \frac{\partial P}{\partial t} dt = \frac{1}{2} \int_0^t \frac{\partial P}{\partial t} dt$$

or

$$1 - 2\Theta\left(\frac{x}{2\sqrt{\pi}k\sqrt{T}}\right) = -\Theta\left(\frac{x}{2\sqrt{\pi}k\sqrt{t}}\right).$$

As an application, suppose that $x = k\sqrt{t}$. The formula gives $T = 0.18t$; thus:

> *There is as much chance of reselling a forward contract on a simple option without loss during the first fifth of the term of the contract as during the remaining four fifths.*

To take a particular example, suppose that it concerns a Rente and that $t = 30$ days, then T will be equal to 5 days.

Thus, the formula informs us, there is as much chance that a Rente can be resold with a spread of a (28c on average) during the first five days, as the chance that it can be resold during the following twenty five days.

Amongst the 60 liquidations which bear upon our observations, the spread has been attained 38 times: 18 times during the first four days, 2 times during the fifth and 18 after the fifth day.

The observation is thus in accord with the theory.

Suppose now that $x = 2k\sqrt{t}$ we find that $T = 0.42t$. Now, the quantity $2k\sqrt{t}$ is the spread for a put-and-call; it can therefore be stated:

> *There is as much chance that a forward contract on a put-and-call can be resold without loss during the first four-tenths of the term of the contract as in the other six tenths.*

Consider again the 3% Rente: our previous observations have demonstrated that, in 23 cases out of 60 liquidations, the spread $2a$ (56c on average) having been attained. In these cases the spread has been attained 11 times before the 14th of the month and 12 time after this epoch.

The median epoch will be $0.11t$ for a call-of-twice-more and $0.21t$ for a call-of-thrice-more.

Finally, the median epoch will be half the total epoch if x were equal to $2.5k\sqrt{t}$.

Probability Distributions. — We have until the present resolved two problems:

(1) The question of the probability that a price be attained at epoch t.
(2) The question of the probability that a price be attained in an interval of time t.

Let us proceed to resolve the latter problem in a complete manner. It will not suffice to know the probability that the price be attained before epoch t; it is also necessary to know the probability law at epoch t in the case where the price is not attained.

Suppose, for example, that a 3% Rente were bought in order to be resold at a profit of c. If the resale could not be effected at epoch t, what will be, at this epoch, the probability law for our operation?

If the price c has not been attained, it is because the maximum upward fluctuation has been less than c, while the downward fluctuation could have been unlimited. There is, therefore, an apparent asymmetry in the probability curve at epoch t.

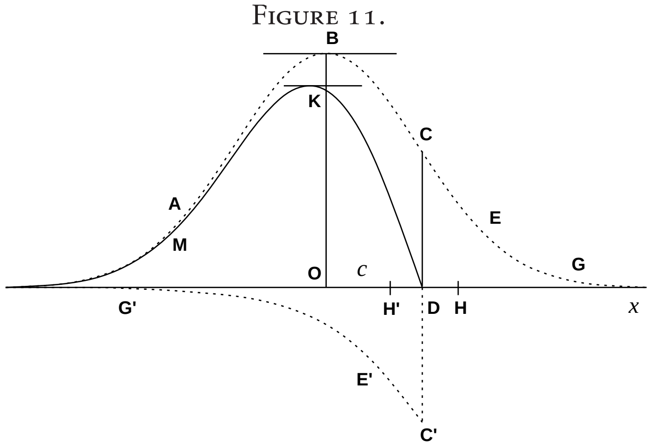
Let us proceed to find the form of this curve.

Let ABCEG be the probability curve at epoch t, assuming that the operation must have persisted up until this epoch (*Figure 11*).

The probability that, at epoch t, the price c has been surpassed, is represented by the area DCEG which, obviously, will not be part of the probability curve in the case of a possible resale.

It may even be asserted *a priori* that the area under the probability curve will still, in this case, be diminished by a quantity equal to DCEG, since the probability P is twice the probability represented by area DCEG.

Figure 11.



If the price c were attained at epoch t_1, the price at H will have, at this instant, the same probability as the symmetrical price at H'.

The possibility of a resale at price c thus subtracts, along with the probability at H, an equal probability at H', and to get the probability at epoch t, we must deduct from the ordinates of the curve ABC those of the curve G'E'C' symmetrical to GEC. The required probability curve will therefore be the curve DKM.

The equation for this curve is

$$p = \frac{1}{2\pi k\sqrt{t}}\left[e^{-\frac{x^2}{4\pi k^2 t}} - e^{-\frac{(2c-x)^2}{4\pi k^2 t}}\right].$$

Most Probable Price. [55] — To obtain the price for which the probability is greatest, in the case where the price c has not been attained, it suffices to put $dp/dx = 0$. We thus obtain

$$\frac{x}{2c-x} + e^{-\frac{c(c-x)}{\pi k^2 t}} = 0.$$

[55]Trans: French *le cours de probabilité maxima*.

Assuming that $c = 2a = k\sqrt{t}$, we get

$$x_m = -1.5\,a,$$

assuming that $c = 2a$, we get[56]

$$x_m = -0.8a.$$

Finally,

$$x_m = -c,$$

would be obtained if c were equal to[57] $1.31a$.

Median Price. [58] — Let us proceed to find an expression for the probability in the interval between zero and u; this will be

$$\frac{1}{2\pi k\sqrt{t}}\int_0^u e^{-\frac{x^2}{4\pi k^2 t}}\,dx - \frac{1}{2\pi k\sqrt{t}}\int_0^u e^{-\frac{(2c-x)^2}{4\pi k^2 t}}\,dx.$$

The first term has a value of

$$\frac{1}{2}\Theta\left(\frac{u}{2\sqrt{\pi}\,k\sqrt{t}}\right).$$

In the second, suppose

$$2\sqrt{\pi}\,k\sqrt{t}\lambda = 2c - x;$$

this term then becomes

$$-\frac{1}{2}\frac{2}{\sqrt{\pi}}\int_0^{\frac{2c}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\,d\lambda + \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_0^{\frac{2c-u}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\,d\lambda$$

The required expression for the probability is thus

$$\frac{1}{2}\Theta\left(\frac{u}{2\sqrt{\pi}\,k\sqrt{t}}\right) - \frac{1}{2}\Theta\left(\frac{2c}{2\sqrt{\pi}\,k\sqrt{t}}\right) + \frac{1}{2}\Theta\left(\frac{2c-u}{2\sqrt{\pi}\,k\sqrt{t}}\right).$$

It is interesting to study the case where $u = c$ in order to find the probability of profiting from the purchase of a forward contract where the resale price has not been attained.

Under the hypothesis that $u = c$, the formula above becomes

$$\Theta\left(\frac{c}{2\sqrt{\pi}\,k\sqrt{t}}\right) - \Theta\left(\frac{2c}{2\sqrt{\pi}\,k\sqrt{t}}\right).$$

Supposing that $c = a$ then the probability is 0.03.

If the spread a has never been attained in the interval of time before t, there are only three chances in one hundred that at epoch t the price be found between zero and a.

A simple option can be bought with the preconceived notion of reselling a forward contract on this option as soon as its spread has been attained.

The probability of a resale is, as we have seen, 0.69. The probability that a resale did not occur and that it made a profit is 0.03 and the probability of a loss is 0.28.

Supposing that $c = 2a$, the probability is then 0.13.

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[56] Trans: The original paper stated, incorrectly, that $x_m = -0.4a$.

[57] Trans: The original paper stated $c = 1.33a$. However, $c = a\sqrt{(\pi/2)\ln 3} \approx 1.31a$.

[58] Trans: French *le cours probable*.

If the spread of $2a$ has never been attained in the interval of time before t then there are thirteen chances in one hundred that, at epoch t, the price is found between zero and $2a$.

The *median price* is that for which the ordinate is divided into two parts of equal area under the probability curve. It is not possible to express its value in finite terms.

Effective Expectation. [59] — The mathematical expectation $k\sqrt{t} = a$ expresses the expectation for an operation that must continue up until epoch t.

If it were intended to complete the operation in the case where a certain spread be attained before epoch t, the mathematical expectation has a completely different value, obviously varying between zero and $k\sqrt{t}$, when the chosen spread varies between zero and infinity.

Let c be the price realised for a purchase, for example. To obtain the positive effective expectation for the operation, the expectation from reselling, cP, must be added to the positive expectation corresponding to the case where a resale has not occurred, that is to say, the quantity

$$\int_0^c \frac{x}{2\pi k\sqrt{t}}\left[e^{-\frac{x^2}{4\pi k^2 t}} - e^{-\frac{(2c-c)^2}{4\pi k^2 t}}\right] dx.$$

If integration is performed on the first term and if the complete integral is added to the expectation of the resale,

$$cP = c - c\frac{2}{\sqrt{\pi}}\int_0^{\frac{c}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\, d\lambda,$$

the expression obtained for the *effective expectation* is

$$\mathcal{E} = c + k\sqrt{t}\left(1 - e^{-\frac{c^2}{\pi k^2 t}}\right) - c\frac{2}{\sqrt{\pi}}\int_0^{\frac{c}{\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\, d\lambda,$$

or

$$\mathcal{E} = c + k\sqrt{t}\left(1 - e^{-\frac{c^2}{\pi k^2 t}}\right) - c\,\Theta\left(\frac{c}{\sqrt{\pi}k\sqrt{t}}\right).$$

Assuming that $c = \infty$, it will again be found that $\mathcal{E} = k\sqrt{t}$. $\mathcal{E}$ could easily be expanded as a power series, but the above formula is more advantageous; it can be calculated with tables of logarithms and with those of the function Θ.

For $c = a$,

$$\mathcal{E} = 0.71\, a,$$

is obtained; similarly, for $c = 2\,a$,

$$\mathcal{E} = 0.86a.$$

The expectation of resale being, for these same spreads, $0.69a$ and $0.86a$.

The *average spread*[60] on a price fall, when the price c is not attained, has a value of

$$\frac{\int_{-\infty}^0 p\,x\,dx}{\int_{-\infty}^0 p\,dx} = \frac{\mathcal{E}}{1 - P - P_1},$$

P_1 designating the quantity $\int_0^c p\,dx$.

[59] Trans: French *l'espérance réelle*.

[60] Trans: French *l'écart moyen*.

Thus, the average spread has a value of $2.54a$ when $c = a$, and $2.16a$ when $c = 2a$.

Assuming $c = \infty$ it can be seen that the average spread is equal to $2a$, a result already obtained.

By way of example, consider the general problem relating to the spread a.

Suppose I buy a forward contract with the preconceived notion of reselling with spread $a = k\sqrt{t}$. If, at epoch t, the sale has not been completed, I will sell whatever be the price.

What are the principal results furnished by the Theory of Probability for this operation?

The positive effective expectation of the operation is 0.71.

The probability of the resale is 0.69.

The most probable epoch for the resale is $t/18$.

The median epoch for the resale is $t/5$.

If the resale does not occur, the probability of a profit is 0.03, the probability of a loss is 0.28, the positive expectation is $0.02a$, the negative expectation is $0.71a$. The average loss is $2.54a$.

The total probability of profiting is 0.72.

I consider it unnecessary to present further examples; it can be seen that the present theory resolves by the Theory of Probability the majority of the problems which prompted the study of Speculation.

A final remark will perhaps not be superfluous. If, in respect of several questions treated in this study, I have compared the results of observation to those of the theory, this was not to verify the formulae established by mathematical methods, but to demonstrate only that the market, unwittingly, obeys a law which governs it: the Law of Probability.