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Multiperiod Risk and Coherent Multiperiod Risk Measurement

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Abstract. We explain why and how to deal with the definition, acceptability and computation of risk in a genuinely multitemporal way. Coherence axioms provide a representation of a risk-adjusted valuation. The multiperiod extension of Tail VaR is discussed.

1 New Questions with Multiperiod Risk

Risk evolving over several periods of uncertainty requires consideration of new issues, since:

- availability of information may require taking into account *intermediate monitoring* by supervisors or shareholders of a *locked-in* position,
- the possibility of *intermediate actions* and or availability of extraneous cash flows, require handling *sequences* of unknown future "values",
- risk measure at one date may involve, unknown, risk measures at later dates.

2 Review of One Period Coherent Acceptability

Coherent one period risk adjusted values' theory is best approached (see [2], p. 69, [3], Section 2.2) by taking the primitive object to be an "acceptance set", that is a set of acceptable future net worths, also called simply "values". This set is supposed to satisfy some "coherence" requirements. If for simplicity we assume (as well as in following sections) a zero interest rate, the representation result states that for any acceptance set, there exists a set $\mathcal{P}$ of probability distributions (called generalised scenarios or test probabilities)

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on the space Ω of states of nature, such that a given position, with future (random) value denoted by X, is acceptable if and only if:

For each test probability $\mathbb{P} \in \mathcal{P}$, *the expected value of the future net worth under* $\mathbb{P}$, *i.e.* $\mathbf{E}_{\mathbb{P}}[X]$, *is non-negative.*

The *risk-adjusted value* $\pi(X)$ of a future net worth X is defined as follows:

- compute, under each test probability $\mathbb{P} \in \mathcal{P}$, the average of the future net worth X of the position, in formula $\mathbf{E}_{\mathbb{P}}[X]$,
- take the minimum of all numbers found above i.e. $\pi(X) = \inf_{\mathbb{P} \in \mathcal{P}} \mathbf{E}_{\mathbb{P}}[X]$.

The axioms of coherent risk measures (see[2]), translate for coherent risk-adjusted values into:

- monotonicity: if $X \geq Y$ then $\pi(X) \geq \pi(Y)$,
- translation invariance: if a is a constant then $\pi(a \cdot 1 + X) = a + \pi(X)$,
- positive homogeneity: if $\lambda \geq 0$ then $\pi(\lambda \cdot X) = \lambda \cdot \pi(X)$,
- superadditivity: $\pi(X + Y) \geq \pi(X) + \pi(Y)$.

3 Coherent Multiperiod Risk Adjusted Value

The case of T periods of uncertainty will be described here in the language of trees. We represent the availability of information over time by the set Ω of "states of nature" at date T and, for each date $t = 0, \ldots, T$, the partition $\mathcal{N}_t$ of Ω consisting of the set of smallest events which by date t are declared to obtain or not. These events are "tagged" by the date t and are called the nodes of the tree $\mathcal{T}(\Omega)$ at date t. We use for such a node n the notation $(\underline{n}, t(n))$ or $\underline{n} \times \{t(n)\}$.

The partition $\mathcal{N}_{t+1}$ is a refinement of the partition $\mathcal{N}_t$ and this provides the ancestorship relation of $(\underline{m}, t)$ to $(\underline{n}, t+1)$ by means of the inclusion $\underline{n} \subset \underline{m}$.

For example, the "three period (four date) binomial tree" can be described by $\mathcal{N}_3 = \{[uuu], [uud], [udu], [udd], [duu], [dud], [ddu], [ddd]\}, \mathcal{N}_2 = \{[uu], [ud], [du], [dd]\}, \mathcal{N}_1 = \{[u], [d]\}$, $\mathcal{N}_0 = \{[]\}$.

Remark 1. The binomial tree is misleadingly simple. It may well happen that some node n of date t stops branching. We then have to distinguish $(\underline{n}, t)$ and $(\underline{n}, t+1)$.

Sequences of "values" at dates $0, \ldots, T$ will be the object of study. They are, as adapted stochastic processes, functions on the tree $\mathcal{T}(\Omega)$. The restriction of such a function X to the set $\mathcal{N}_t$ of nodes at date t is also considered a function on Ω, denoted by X_t. Then $X_t(n)$ denotes (with some redundancy) the "value" at date t in the "node" or event n as well as in any of the states of nature belonging to $\underline{n}$. It is also interesting to view the process $X = (X_t)_{0 \leq t \leq T}$ as a function on the product space $\{0, 1, \ldots, T\} \times \Omega$ which happens for each date to be constant in any node of this date: $X_t(\omega) = X_t(\omega')$

as soon as there exists a node n at date t with both states ω and ω' belonging to $\underline{n}$.

We obtain from any probability $\mathbb{P}$ on Ω (with $\mathbb{P}[\{\omega\}] > 0$ for each $\omega \in \Omega$ for simplicity) a probability $\mathbb{P}_{\mathcal{T}}$ on $\mathcal{T}(\Omega)$ by the definition relative to each node n:

$$\mathbb{P}_{\mathcal{T}}[\{n\}] = \frac{1}{T+1} \sum_{\omega \in \underline{n}} \mathbb{P}[\{\omega\}].$$

For each function Y on $\mathcal{T}(\Omega)$ we have the formula:

$$\mathbf{E}_{\mathbb{P}_{\mathcal{T}}}[Y] = \frac{1}{T+1} \sum_{0 \leq t \leq T} \mathbf{E}_{\mathbb{P}}[Y_t]$$

For each probability $\mathbb{P}$ on Ω, each date t and each random variable X_T the conditional expectation at date t of X_T is the function on $\mathcal{N}_t$ (or equivalently the function on Ω which is constant on every node of $\mathcal{N}_t$) defined by:

$$\mathbf{E}_{\mathbb{P}}[X_T \mid \mathcal{N}_t](n) = \mathbf{E}_{\mathbb{P}}[X_T \mid \underline{n}].$$

A supervisor, risk manager or regulator, will as in the one-period case decide at date 0 upon a set of acceptable "values", a subset of the set $\mathcal{G}_{\mathcal{T}}$ of all value processes. There are many interpretations of the meaning of "values": as, for example, market values of equity, accounting values of equity, *cumulative* cash flows, liquidation values, surplus, actuarial values.

Solvency is an important concern. For the "value" $(X_t)_{0 \leq t \leq T}$ of a portfolio or of a strategy, one defines formally the "insolvency time" $\sigma = \inf\{t | X_t < 0,\ 1 \leq t \leq T\}$, and the stopped process X^σ equal to X_t before the time σ and to X_σ from time σ on. When X is a market value, one may say that the risk measurement balances the costs of insolvency with the benefits of risk-taking. With a liquidation value, one may imagine that after "insolvency" time, there may be more favorable dates and events where to close the business.

A coherent acceptance set of "values" is a closed convex cone $\mathcal{A}_{cc}$ in $\mathcal{G}_{\mathcal{T}}$, with vertex at the origin, containing the positive orthant and intersecting the negative orthant only at the origin. As in the framework of one-period risk we define the *risk adjusted valuation* associated with the cone $\mathcal{A}_{cc}$ by computing for each "value" process X the number $\pi(X) = \sup\{m \mid X - m \in \mathcal{A}_{cc}\}$. This reflects the fact that risk adjusted value is the largest amount of capital which can be subtracted from the project X and still leave it acceptable. The assumptions on $\mathcal{A}_{cc}$ ensure that the the risk adjusted valuation is coherent, i.e. satisfies the four conditions listed at the end of Section 2.

The incorporation of time via the tree $\mathcal{T}(\Omega)$ allows us to *directly* deduce from the study of the one-period case that there exists a set $\mathcal{P}_{\mathcal{T}}$ of probabilities on $\mathcal{T}(\Omega)$ such that:

$$\text{for each } X \in \mathcal{G}_{\mathcal{T}},\ \pi(X) = \inf_{\mathbb{P}_{\mathcal{T}} \in \mathcal{P}_{\mathcal{T}}} \mathbf{E}_{\mathbb{P}_{\mathcal{T}}}[X].$$

Each "test probability" $\mathbb{P}_\mathcal{T} \in \mathcal{P}_\mathcal{T}$ can be described by its density $f_\mathcal{T} = \frac{d\mathbb{P}_\mathcal{T}}{d\mathbb{P}_{0,\mathcal{T}}}$ with respect to $\mathbb{P}_{0,\mathcal{T}}$, where $\frac{d\mathbb{P}_\mathcal{T}}{d\mathbb{P}_{0,\mathcal{T}}}(n) = \frac{\mathbb{P}_\mathcal{T}(n)}{\mathbb{P}_{0,\mathcal{T}}(n)}$, for each node n in $\mathcal{T}(\Omega)$. This density has to be a function $f_\mathcal{T}$ on the tree $\mathcal{T}(\Omega)$, and we represent it as $f_\mathcal{T} = (f_t)_{0\leq t\leq T}$ where each f_t is a positive function on $\mathcal{N}_t$, such that $\sum_{0\leq t\leq T} \frac{1}{T+1}\mathbf{E}_{\mathbb{P}_0}[f_t] = 1$.

We then have for each process X, $\mathbf{E}_{\mathbb{P}_\mathcal{T}}[X] = \sum_{0\leq t\leq T} \frac{1}{T+1}\mathbf{E}_{\mathbb{P}_0}[f_t X_t]$. Defining the increasing process A by $A_t = A_{t-1} + \frac{1}{T+1} f_t$, with $A_{-1} = 0$, we get $\mathbf{E}_{\mathbb{P}_0}[A_T] = 1$ and we obtain the:

REPRESENTATION RESULT: *For each coherent risk-adjusted valuation π there is a set $\mathcal{A}$ of positive increasing adapted processes A with $\mathbf{E}_{\mathbb{P}_0}[A_T] = 1$ such that for each value process X its date 0 risk-adjusted value $\pi(X)$ is given by:*

$$\pi(X) = \inf_{A\in\mathcal{A}} \mathbf{E}_{\mathbb{P}_0}\left[\sum_{0\leq t\leq T} X_t \cdot (A_t - A_{t-1})\right].$$

Example 1. A stopping time σ defines (if $\mathbb{P}_0[\sigma \leq T] > 0$) an increasing process A^σ by $A_0^\sigma = 0$ and by $A_t^\sigma = \frac{1}{\mathbb{P}_0[\sigma\leq T]}\mathbf{1}_{\{\sigma\leq t\}}$ for $0 \leq t \leq T$, where for any event E, $\mathbf{1}_E(\omega) = 1$ or 0 depending on whether or not $\omega \in E$. The coherent risk-adjusted value given by $\pi(X) = \mathbf{E}_{\mathbb{P}_0}[X_\sigma]$ is also $\mathbf{E}_{\mathbb{P}_0}\left[\sum_{0\leq t\leq T} X_t(A_t^\sigma - A_{t-1}^\sigma)\right]$.

Example 2. A random time τ defines the process A^τ by $A_t^\tau = A_{t-1}^\tau + \mathbf{E}_{\mathbb{P}_0}[\mathbf{1}_{C_t} \mid \mathcal{N}_t]$ with $C_t = \{\tau = t\}$. We have for each process X:

$$\mathbf{E}_{\mathbb{P}_0}\left[\sum_{0\leq t\leq T} X_t \cdot \mathbf{1}_{C_t}\right] = \mathbf{E}_{\mathbb{P}_0}\left[\sum_{0\leq t\leq T} X_t \cdot \left(A_t^\tau - A_{t-1}^\tau\right)\right].$$

Using the random time $\bar{\tau}(\omega) = argmin_t\,(X_t(\omega))$ we find that for the risk-adjusted valuation $\pi(X) = \mathbf{E}_{\mathbb{P}_0}[\inf_{0\leq t\leq T} X_t]$ we have

$$\pi(X) = \inf_\tau \mathbf{E}_{\mathbb{P}_0}\left[\sum_{0\leq t\leq T} X_t \cdot \left(A_t^\tau - A_{t-1}^\tau\right)\right].$$

Remark 2. One could consider more general acceptance sets than convex cones, as was done in [4] to represent constraints imposed by the shareholders of a firm.

4 Two Multiperiod Risk Adjusted Measurements of a Final Value

The absence of intermediate markets or any other form of "locked-in" position provides a situation different from the one studied in Section 3. The model is a sequence $(\mathcal{N}_t)_{0\le t\le T}$ of the sets of nodes and one "final value" X_T, i.e. a mere function on $\mathcal{N}_T = \Omega$. No change can be made to the position but information is revealed over time and the risk manager anticipates this fact concerning the acceptance decision at date 0. The one-period analysis of [3] would consist, starting from a set $\mathcal{P}$ of test probabilities on Ω, in defining the number $\phi_0(X_T) = \inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}[X_T]$. The same analysis applied at a later date t, would, at that date, define the "date t risk-adjusted value" $\phi_t(X_T)$ as $\inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}[X_T|\mathcal{N}_t]$, defining therefore a risk-adjusted value *process* $(\phi_t(X_T))_{0\le t\le T}$.

Another risk-adjusted value is built by backward induction from the same set $\mathcal{P}$ of test probabilities on Ω. For each final value X_T, we define the process $\psi(X_T)$ by the equality $\psi_T(X_T) = X_T$ and by the recurrence relation

$$\psi_t(X_T) = \inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}[\psi_{t+1}(X_T) \mid \mathcal{N}_t],\ 0 \le t < T.$$

Section 5 provides conditions on the set $\mathcal{P}$ of test probabilities under which for each X_T the processes ϕ and ψ are equal.

5 Recursivity of Risk Measurement and Stability of the Set of Test Probabilities

It can be shown that for a set $\mathcal{P}$ of test probabilities the following two properties "stability" by pasting and "recursivity", are equivalent.

Stability by pasting means that if for any date t we are given for each node n in $\mathcal{N}_t$ a probability $\mathbb{P}_n$ in $\mathcal{P}$, conditioned by this node n, the pasting of all these conditional probabilities with any probability $\mathbb{P}_0$ in $\mathcal{P}$ restricted up to time t, still provides an element of $\mathcal{P}$.

A simple binomial example of pasting is given by $T = 2, t = 1, n_1 = (u, 1)$, $n_2 = (d, 1)$, $\mathbb{P}_0 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, $\mathbb{P}_1 = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\mathbb{P}_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$, the resulting pasted probability being $\mathbb{P} = (0, \frac{1}{2}, \frac{1}{2}, 0)$.

The pasting of probabilities amounts to looking over successive time intervals or, at the same date, over disjoint events, at the risk attitudes of various agents.

Recursivity means that for each random variable X_T on Ω and for each $0 \le t \le T - 1$:

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}[X_T \mid \mathcal{N}_t] = \inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}\left[\inf_{\mathbb{R}\in\mathcal{P}} \mathbf{E}_{\mathbb{R}}[X_T \mid \mathcal{N}_{t+1}] \mid \mathcal{N}_t\right].$$

Using this equality, we obtain for the risk-measurement *process* ϕ introduced in Section 4 out of a set of test probabilities on Ω:

$$\phi_t(X_T) = \inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}[X_T \mid \mathcal{N}_t],$$

the recurrence relation:

$$\phi_t(X_T) = \inf_{\mathbb{P}\in\mathcal{P}} \mathbf{E}_{\mathbb{P}}[\phi_{t+1}(X_T) \mid \mathcal{N}_t],$$

and therefore the equality of the processes $\phi(X_T)$ and $\psi(X_T)$.

If the recursivity property holds two final values with the same risk-adjusted value at date 1 in every node, will have the same risk-adjusted value at date 0 and if a final value is acceptable in any node of date 1 it will be acceptable at date 0.

6 Discussion of Tail Value at Risk in the Multiperiod Case

Tail Value at Risk (see [1], [2], [3]) has become popular, in particular in the Credit Risk field, and has been used for a specifc multiperiod risk measure definition (see [5]). The best definition of "TailVaR" is in terms of a set of test probabilities, but this set does *not* fulfill the "stability by pasting" property.

For example, in Section 5, the set $\mathcal{P}$ generating the date 0 tail value-at-risk with level α equal to $\frac{3}{4}$, is the set of probabilities havings densities with respect to $\mathbb{P}_0$ bounded by $\frac{1}{\alpha} = \frac{4}{3}$. The three probabilities used in the example have respectively the densities $(1,1,1,1)$, $\frac{1}{3}(0,4,4,4)$, and $\frac{1}{3}(4,4,4,0)$. The pasted probability has the density $(0,2,2,0)$ which is not bounded by $\frac{4}{3}$.

At intermediate (date,event) nodes the similar, direct, computation of TailVaR follows neither the line of construction of the ϕ_t nor the one of the ψ_t of Section 4. There is also non-recursivity: two date 2 future values may have different TailVaR at date 0 and the same TailVaR (as random variable) at date 1:

$$\Omega = \{[uu],[um],[ud],[du],[dd]\},\ \mathbb{P} = \{0.487,\ 0.01,\ 0.003,\ 0.4955,\ 0.0045\},$$

$$\mathcal{N}_1 = \{[u],[d]\},\ \mathcal{N}_0 = \{[]\}$$

$$X_2 = 1, Y_2([uu]) = Y_2([du]) = 10, Y_2([um]) = 2.5, Y_2([ud]) = Y_2([dd]) = 0.$$

We find for Y_2 the TailVaR values (at the 1% level) at date 0 and at date 1:

$$TailVaR(Y_2)([]) = \frac{1}{0.01} \cdot ((0.0045 + 0.0030) \cdot 0 + 0.0025 \cdot 2.5) = 0.625$$

$$TailVaR(Y_2)([u]) = 1,\ TailVaR(Y_2)([d]) = 1.$$

Another potential weakness of TailVaR is the fact that $TailVaR(X_T)$ depends only on the *distribution* of X_T.

7 Representation of Stable Sets of Test Probabilities

Stability of test probabilities can be characterized when the information structure is given by a binomial tree and a random walk $W_t = U_1 + ... + U_t$, $0 \leq t \leq T$, the $(U_t)_{0 \leq t \leq T}$ being ± 1 valued independent variables with symmetric distribution. Let $\mathbb{P}_0$ be the resulting measure on Ω.

Representability of a set $\mathcal{P}$ of test probabilities is defined here as follows. There exists for each t, $0 \leq t \leq T$ a random closed convex set $\mathcal{Q}_t$, $0 \leq t \leq T$ of $[-1, +1]$ depending in a $\mathcal{N}_{t-1}$-measurable way, such that:

The random variable $Z = \prod_{0 \leq t \leq T}(1 + q_t U_t)$ is the density with respect to $\mathbb{P}_0$ of an element of $\mathcal{P}$ if and only if each q_t belongs to $\mathcal{Q}_t$.

This property is equivalent to the stability property of $\mathcal{P}$ as well as to the recursivity property for the computations.

For example the set of test probabilities may be the set of probabilities $\mathbb{Q}$ such that $\frac{d\mathbb{Q}}{d\mathbb{P}_0} = Z_T$ satisfies $Z_t = \mathbf{E}_{\mathbb{P}_0}[Z_T \mid \mathcal{N}_t] = (1 + q_1 U_1)...(1 + q_t U_t)$ where q is a predictable process with $\delta_1 \leq q \leq \delta_2$, with $-1 \leq \delta_1 \leq \delta_2 \leq 1$ two given numbers.

The recurrence relation becomes when $\delta_1 = -\delta_2 = \delta \geq 0$:

$$\psi_t(X_T) = \inf_{q_{t+1}, |q_{t+1}| \leq \delta} \mathbf{E}_{\mathbb{P}_0}[(1 + q_{t+1} U_{t+1})\psi_{t+1}(X_T) \mid \mathcal{N}_t] ,$$

for $0 \leq t \leq T - 1$, an easy form not requiring much storage at the nodes. In node n of date t, $\psi_t^n(X_T)$ will be computed as:

$$\psi_t^n(X_T) = \frac{1}{2} \inf_{q_{t+1}^n, |q_{t+1}^n| \leq \delta} \left((1 + q_{t+1}^n) \cdot \psi_{t+1}^{(n,u)}(X_T) + (1 - q_{t+1}^n) \cdot \psi_{t+1}^{(n,d)}(X_T) \right) ,$$

which reduces to

$$\frac{1}{2}(1+\delta) \min\{\psi_{t+1}^{(n,u)}(X_T), \psi_{t+1}^{(n,d)}(X_T)\} + \frac{1}{2}(1-\delta) \max\{\psi_{t+1}^{(n,u)}(X_T), \psi_{t+1}^{(n,d)}(X_T)\}.$$

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