Dynamic Portfolio Choice

This version: 07-15-2012

Abstract

The foundation for a long-term investment strategy is rebalancing to fixed asset class positions, which are determined in a one-period portfolio choice problem where the asset weights reflect the investor's attitude toward risk. Rebalancing is a counter-cyclical strategy that has worked well even during the Great Depression in the 1930s and during the Lost Decade of the 2000s. Rebalancing goes against investors' behavioral tendencies and is also a short volatility strategy. When there are liabilities and asset returns vary over time, the long-term investor's optimal portfolio consists of (i) a liability-hedging portfolio, (ii) a market (or myopic demand) portfolio that reflects optimal short-run asset positions, and (iii) an opportunistic (or long-term hedging demand) portfolio that allows a long-run investor to take advantage of changing investment returns.

1. Stay the Course?

In April 2009, just after the worst of the financial crisis, Amy Harrison, an independent investment advisor, prepared to meet with her client, Amelia Daniel.[1] Harrison had first been introduced to Daniel three years earlier. At that time, Daniel had just sold her medical information company, Daniel Health Systems, and received $10 million cash in the sale. Daniel had also recently divorced. She felt that both the liquidity event allowing her to walk away from the company and the end of a chapter in her personal life would allow her to start afresh on new, smaller ventures.

[1] This is based on the case "Stay the Course? Portfolio Advice in the Face of Large Losses," Columbia CaseWorks, ID #110309, 2011.

Upon meeting Daniel, Harrison had drafted an investment policy statement (IPS) for Daniel that:

1. Described Daniel's understanding of risk and set her risk tolerance
2. Identified Daniel's intermediate and long-term goals, as well as her preferences and constraints
3. Crafted a long-term investment plan
4. Served as a reminder of guidelines to be used to achieving her goals, and
5. Defined the investment and monitoring process.

Harrison's first year of working with her new client had gone smoothly. It took some convincing, however, for Daniel to follow Harrison's advice. Like many entrepreneurs, Daniel had built up her wealth by holding a concentrated portfolio, essentially all in her own company. But Harrison's advice was rooted in diversification and optimal asset allocation based on reducing risk and maximizing return. Daniel had essentially no liabilities, with her parents being well off, and no children or plans for children. She lived modestly and had her expenses covered by the salary the acquiring company was paying her to stay on as a consultant. Given her entrepreneurial background, Daniel was comfortable taking risk and had a long-term investment focus. Thus, Harrison recommended that most of Daniel's portfolio be approximately evenly split between a myopic (growth or market-oriented) investment portfolio and a long-term hedging demand (opportunistic) portfolio. The myopic portfolio consisted of liquid U.S. and international equities and high-yield bonds. The opportunistic portfolio consisted of some direct private equity investment in a friend's company (representing 10% of Daniel's total wealth), and investment vehicles (private equity funds and hedge funds) which allowed fund managers to time the market and take on factor risks unavailable in traditional index funds (see Chapter XX).

Daniel's portfolio suffered terribly in 2008. Financial markets across the globe had plummeted and, like many investors, Daniel's portfolio was hit hard. Equity returns were down 30% to 50% around the world in 2008. Daniel's portfolio lost 30%. Her direct private equity investment was wiped out. By April 2009, while the economy was in recession, there was a sense that the markets were no longer in free-fall. Daniel was still very concerned about the state of her portfolio. Fortunately, she didn't need the wealth to support her current standard of living. Nor did Daniel have immediate liquidity needs that required drawing down the capital in her investment portfolio. In terms of her personal life, Daniel was still single but was now in a relationship. She felt there was some way to go before she would consider getting married. Although there were no plans in the immediate future to have children, she was worried that her greatly reduced portfolio would diminish the legacy she could leave them if she had any. Daniel thought her IPS and her asset allocation needed a "total overhaul."

Harrison knew this was going to be a difficult meeting. On the one hand, perhaps some of Daniel's attitude was an irrational over-reaction to market conditions. On the other hand, perhaps Daniel had genuinely become more risk averse, and the advice Harrison gave in 2007 was no longer valid. "People always think they have more risk tolerance when things are going well," as Harrison said. Should Daniel stay the course or revise her IPS and transition to a less risky portfolio?

In this chapter we discuss portfolio choice over long horizons and how an investor can *dynamically* change her portfolio in response to changing returns and investment opportunities. The theory behind *dynamic portfolio choice* was formulated initially by Paul Samuelson (1969), who won the Nobel Prize in 1970, and Robert Merton (1969, 1971), who won the Nobel Prize in 1997 with Myron Scholes, one of the creators of the Black-Scholes (1973) option pricing model, for the valuation of derivatives. As we'll see shortly, the solution to the dynamic portfolio choice problem is intimately related to derivative valuation; the same economic concepts and solution techniques are used.

2. The Dynamic Portfolio Choice Problem

An investor facing a dynamic portfolio choice problem has a long horizon, say 10 years, and can change her portfolio weights every period. A period could be one year, which is common for retail investors meeting with their financial planners for an annual tune up, or one quarter, which is common for many institutional investors, or even every 5-10 minutes for high frequency traders. The portfolio weights can change each period in response to time-varying investment opportunities as the investor passes through economic recessions or expansions, in response to the horizon approaching (as she approaches retirement, say), and potentially in response to how her liabilities, income, and risk aversion change over time. In this section, we abstract from the last of these considerations and assume that she has no liabilities and no income, and is (fortuitously) given a pile of money to invest. (We introduce liabilities in Section 3 and consider income in the next chapter.) We also assume her risk aversion and utility function remain constant.

2.1 Dynamic Trading Strategies

At the beginning of each period *t*, the investor chooses a set of portfolio weights, x_t. Asset returns are realized at the end of the period *t+1*, and the portfolio weights chosen at time t, x_t, with the realized asset returns lead to the investor's wealth at the end of the period, W_{t+1}. The wealth dynamics follow

$$W_{t+1} = W_t(1 + r_{p,t+1}(x_t)), \tag{1.1}$$

where wealth at the beginning of the period, W_t , is increased or decreased by the portfolio return from *t* to *t+1*, $r_{p,t+1}(x_t)$ and this is a function of the asset weights chosen at the beginning of the period, x_t .

I illustrate this pictorially in Figure 1 for a dynamic horizon problem over *T=5* periods. At the beginning of each period the investor chooses portfolio weights, x_t . These weights, together with realized asset returns, produce her end of period wealth, W_{t+1}, following equation (1.1). The procedure is repeated every period. The sequence of weights over time, $\{x_t\}$, is called a *dynamic trading strategy*. It can potentially change due to pre-determined variables, like investor constraints or liabilities, or due to time-varying investment returns, like booms vs. busts.

[Figure 1 here]

The investor wishes to maximize the expected utility of end of period wealth at time *T* by choosing a dynamic series of portfolio weights:

$$\max_{\{x_t\}} E[U(W_T)], \tag{1.2}$$

subject to constraints. Some examples of constraints are that an investor may not be able to short (this is a *positivity constraint* so $x_t \geq 0$), the investor may not be able to lever (so the portfolio weight is bounded, $0 \leq x_t \leq 1$), or can only sell a certain portion of her portfolio each period (this is a *turnover constraint*). Although the portfolio weights $x_{t+\tau}$ are, of course, only implemented at time $t+\tau$, the complete set of weights $\{x_t\}$ from *t* to *T-1* is chosen at time *t*, the beginning of the problem. The set of optimal weights can be quite complicated: they may not only vary through time as the horizon approaches, but they may vary by state. For example, x_t could take on two values at time *t*: hold 50% in equities if we are in a recession and 70% if we are in a bull market. The complete menu of portfolio strategies across time and states is determined at the beginning of the problem. Thus, the optimal dynamic trading strategy is completely known at the start, even though it changes through time: as asset returns change, the strategy optimally responds, and as utility and liabilities change, the strategy optimally responds.

For the remainder of this chapter, we work with constant relative risk aversion (CRRA) utility (see Chapter XX):

$$E[U(W)] = E\left[\frac{W^{1-\gamma}}{1-\gamma}\right], \tag{1.3}$$

where *W* is the investor's wealth at the end of the period and γ is her risk aversion coefficient. CRRA is locally mean-variance so the risk aversion γ has the same meaning in mean-variance utility, U^{MV} (see Chapter XX):

$$U^{MV} = E(r_p) - \frac{\gamma}{2}\text{var}(r_p), \tag{1.4}$$

where r_p is the portfolio return. The unconstrained solution to both the CRRA utility and mean-variance utility problem with one risky asset and one risk-free asset paying r_f is:[2]

$$x^* = \frac{1}{\gamma}\frac{\mu - r_f}{\sigma^2}, \tag{1.5}$$

[2] Strictly speaking, the CRRA weight applies in continuous time, or when the interval is very small. See Merton (1971).

where the asset has expected return μ and volatility σ. The investor holds x^* in the risky asset and $(1-x^*)$ in the risk-free asset. We developed this solution in detail in Chapter XX. In this sense, CRRA and mean-variance utility are equivalent.

2.2 Dynamic Programming

The dynamic portfolio choice problem is an *optimal control* problem. It is solved by *dynamic programming* – the same technique used to control nuclear power plants, send rockets to the moon, and value complicated options. (I admit the last of these examples certainly feels much less impressive the first two.) Portfolio choice turns out to be rocket science – literally.

Long-horizon wealth is a product of one-period wealth:

$$W_{t+5} = W_t(1+r_{p,t+1})(1+r_{p,t+2})...(1+r_{p,t+5}), \tag{1.6}$$

from equation (1.1) and we can apply CRRA utility (equation (1.3)) to each one-period wealth term. Apply CRRA expected utility to long-horizon wealth, we have a series of one-period CRRA utility problems:

$$E[U(W_{t+5})] = U(W_t)E\left[U(1+r_{p,t+1})U(1+r_{p,t+2})...U(1+r_{p,t+5})\right]. \tag{1.7}$$

Equation (1.7) makes clear that it does not matter whether we start with \$1 or with \$1 million – the portfolio weights do not depend on the size of the initial wealth, which is the *wealth homogeneity* property (see Chapter XX). The portfolio returns, $r_{p,t+1}$, in equations (1.6) and (1.7) depend on the portfolio weights chosen at the beginning of the period, x_t, as equation (1.1) emphasizes. Thus, we can write equation (1.6) as

$$\begin{aligned} E[U(W_{t+5})] &= U(W_t)E\left[U(1+r_{p,t+1}(x_t))U(1+r_{p,t+2}(x_{t+1}))...U(1+r_{p,t+5}(x_{t+4}))\right] \\ &= E\left[U(1+r_{p,t+1}(x_t))U(1+r_{p,t+2}(x_{t+1}))...U(1+r_{p,t+5}(x_{t+4}))\right]. \end{aligned} \tag{1.8}$$

Figure 2 sketches an outline of the dynamic programming solution. Let's start at the end, at *t+4* to *t+5*, where the investor chooses portfolio weights to maximize expected utility at the terminal horizon *T=t+5*. This is Panel A of Figure 2. This is a static one-period problem, and for CRRA utility without constraints this is identical to the one-period mean-variance problem we covered in Chapter XX. The solution for a single risky asset with expected return μ and volatility σ, with a risk-free rate of r_f is given in equation (1.5) and we denote it by x^*_{t+4}, where the asterisk means that the portfolio weight is optimal. The investor holds x^*_{t+4} in equities and $(1-x^*_{t+4})$ in risk-free bonds. In principle, this portfolio weight can depend on what expected return and volatility are prevailing at *t+4* (say the economy is booming or in bust).

[Figure 2 here]

The maximum utility obtained at *t+4* is:

$$V_{t+4} = E[U(1+r^*_{p,t+5})], \tag{1.9}$$

where the portfolio return from *t+4* to *t+5*, $r^*_{p,t+5}$, is a function of the optimal portfolio weight chosen at t+4, x^*_{t+4}, so $r^*_{p,t+5} = r^*_{p,t+5}(x^*_{t+4})$. The maximum utility V_{t+4} in equation (1.9) is called the *indirect utility* and it potentially differs across economic conditions prevailing at time *t+4*.

Having solved the last period's problem, let us turn to the problem two periods before the end. At *t+3*, we need to solve both the portfolio weights at *t+3* and *t+4*, which are x_{t+3} and x_{t+4}, respectively:

$$\max_{\{x_{t+3},x_{t+4}\}} U(W_{t+3})E[U(1+r_{p,t+4}(x_{t+3}))U(1+r_{p,t+5}(x_{t+4}))]$$
$$= \max_{\{x_{t+3},x_{t+4}\}} E[U(1+r_{p,t+4}(x_{t+3}))U(1+r_{p,t+5}(x_{t+4}))], \quad (1.10)$$

But, we already solved the last period problem and found the optimal portfolio weight at *t+4*, x^*_{t+4}. This allows us to write the problem two periods before the end as the problem from *t+3* to *t+4*, plus the problem with the known solution that we solved from *t+4* to *t+5*:

$$\max_{\{x_{t+3},x_{t+4}\}} E[U(1+r_{p,t+4}(x_{t+3}))U(1+r_{p,t+5}(x_{t+4}))]$$
$$= \max_{x_{t+3}} E[U(1+r_{p,t+4}(x_{t+3}))U(1+r_{p,t+5}(x^*_{t+4}))] \quad (1.11)$$
$$= \max_{x_{t+3}} E[U(1+r_{p,t+4}(x_{t+3}))V_{t+4}].$$

The first equality in equation (1.11) substitutes the last period's solution into the time *t+3* problem. This now leaves just one portfolio weight at *t+3*, x_{t+3}, to solve. The second equality says that this problem is a standard single-period problem, except that it involves the indirect utility V_{t+4}, but we know everything about the indirect utility and the optimal strategies at *t+4* from solving the last period's problem (equation (1.9)). We can solve the problem in equation (1.11) as a one-period problem and we denote the optimal weight at *t+3* as x^*_{t+3}. It has the same as the one-period solution in equation (1.5), except we adjust equation (1.5) for the optimized strategies adopted at *t+4* that are captured by the indirect utility, V_{t+4}. Panel B of Figure 2 shows this pictorially. Given the known solution at *t+4*, we use the optimal portfolio weight at *t+4* to solve the portfolio weight at *t+3*. Equation (1.11) also shows the origin of the name "indirect utility" because the indirect utility from the previous problem, at *t+4*, enters the direct utility from the current problem, at *t+3*.

Solving the problem two periods before the end gives us the optimal *t+3* portfolio weight, x^*_{t+3}.

We compute the maximum utility at *t+3*, which is the indirect utility at *t+3*:

$$V_{t+3} = E[U(1+r^*_{p,t+4}(x^*_{t+3}))V_{t+4}]. \quad (1.12)$$

Panel C of Figure 2 shows the recursion applied once more to the *t+2* problem having solved the *t+3* and *t+4* problems. Again, the *t+2* optimization is a one-period problem. After solving the *t+2* problem, we continue backwards to *t+1* and then finally to the beginning of the problem, time *t*. Dynamic programming turns the long-horizon problem into a series of one-period problems (following equations (1.9) and (1.12)). Dynamic programming is an extremely powerful technique and Samuelson won the Nobel prize in 1970 for introducing it into many areas of economics. Monetary policy (see Chapter XX), capital investment by firms, taxation and fiscal policy, and option valuation are all examples of optimal control problems that can be solved by dynamic programming. In continuous time, the value function is given by a solution to a partial differential equation called the Hamilton-Jacobi-Bellman equation. A more general form is called the Feynman-Kac theorem, widely used in thermodynamics. These are the same heavy-duty physics and mathematics concepts used in controlling airplanes and ballistic missiles. Portfolio choice is rocket science.

2.3 Long-Horizon Investing Fallacies

The important lesson from the previous section on dynamic programming is not that you should hire a rocket scientist to do portfolio choice (although there are plenty of ex-rocket scientists working in this area), but that dynamic portfolio choice over long horizons is first and foremost about solving one-period portfolio choice problems. Viewing the dynamic programming solution of long-horizon portfolio choice this way demolishes two widely held misconceptions about long-horizon investing.

Buy and Hold is Not Optimal

A long-horizon investor never buys and holds. The buy-and-hold problem is illustrated in Figure 3: the investor chooses portfolio weights at the beginning of the period and holds the assets without rebalancing over the entire long-horizon problem. The buy-and-hold problem treats the long-horizon problem as a single, static problem. Buy and hold problems are nested by the dynamic portfolios considered in the previous section; they are a special case where the investor's optimal choice is to do nothing. Buy and hold is dominated by the optimal dynamic strategy which trades every period. Long-horizon investing is not to buy and hold; long-horizon investing is a continual process of buying and selling.

[Figure 3 here]

There is much confusion in practice about this issue. The World Economic Forum, for example, defined long-term investing as "investing with the expectation of holding an asset for an indefinite period of time by an investor with the capability to do so".[3] Long-horizon investors could, but in almost no circumstances will, buy and hold an asset forever. They dynamically buy and sell those assets over time.

The buy-and-hold confusion is also partly due to the popular sentiment generated by Jeremy Siegel's famous book, "Stocks for the Long Run" first published in 1994. This book is often described as the "buy and hold bible."[4] Siegel makes a case for sticking to a long-run allocation to equities. If this allocation is constant, then it is maintained by a constant rebalancing rule.

[3] World Economic Forum, 2011, The Future of Long-Term Investing, p13.
[4] As James K. Glassman of the Washington Post says of the 2nd edition of Siegel's book.

Investors increase their share of equities after equities have done poorly to maintain this long-run, constant share. Long-run investors never buy and hold; they constantly trade.

Long-Term Investing is Short-Term Investing

Another popular misconception about long-term investing is that by having a long-term investment horizon, long-run investors are fundamentally different from myopic, short-term investors. Some, like Alfred Rappaport (2011) suggest that long-run investors should act totally differently from short-term investors. The dynamic programming solution shows this to be blatantly false. Dynamic programming solves the long-horizon portfolio choice problem as a series of short-term investment problems. That is, *long-run investors are first and foremost short-run investors*. They do everything short-run investors do, and they can do more because they have the advantage of a long horizon. The effect of the long horizon enters through the indirect utility in each one-period optimization problem (see equation (1.11)). I am not suggesting that long-run investors should engage in "short-termism," the myopic behavior that often befalls short-term corporate managers and short-term investors.[5] The dynamic programming solution suggests that, to be a successful long-run investor, you should start off being a successful short-run investor. After doing this, take on all the advantages that the long horizon gives you.

I now discuss one important case where there is no difference between long-run investors and short-run investors. This case happens to be the most empirically relevant, and is the foundation of any long-term investment strategy.

[5] Jeremy Stein (1988) and other authors show that short-termism can arise as a rational response to incentives and leads to underinvestment and mis-valued firms.

2.4 Rebalancing

Suppose that returns are not predictable, or the investment opportunity set is independent and identically distributed (i.i.d.). The i.i.d. assumption is very realistic. Asset returns are hard to predict, as Chapter XX will show. A good way to think about the i.i.d. assumption is that asset returns are like a series of coin flips, except coins can only land heads or tails, and returns can take on many different values. The coin flip is i.i.d. because the current probability of a head or tail does not depend on the series of head or tails realized in the past. The same is true for asset returns: when asset returns are i.i.d., in every period returns are drawn from the same distribution that is independent of returns drawn in previous periods. Under i.i.d. returns, assets are glorified coin flips.

With i.i.d. returns and a fixed risk-free rate, the dynamic portfolio problem becomes a series of identical one-period problems, as shown in Figure 4. If returns are not predictable then the long-horizon portfolio weight is identical to the myopic portfolio weight. Put another way, with i.i.d. returns there is no difference between long-horizon investing and short-horizon investing: all investors are short term and it does not matter what the horizon is. We can write this as:

$$\text{Long-Run Weight (t)} = \text{Short-Run Weight (t)}. \quad (1.13)$$

The short-run weight is the myopic portfolio weight in equation (1.5). It is stated in terms of CRRA utility, but more generally it is the portfolio weight of a one-period utility problem using any of the utility functions we covered inChapter XX. All investors are short-run in the i.i.d world because returns are not predictable, so the long-run investor faces a series of coin flips. The optimal strategy is to manage portfolio risk and return each period. The optimal holding is then given by the myopic, short-run weight. The investor needs to rebalance back to this weight to avoid any one asset dominating in her portfolio for her given level of risk aversion.

[Figure 4 here]

If the optimal dynamic strategy is actually a myopic strategy, is the rebalancing strategy in Figure 4 different from a buy-and-hold strategy as shown in Figure 3? Absolutely. The dynamic problem is a series of one-period problems and it involves *rebalancing* back to the same portfolio weight. The buy-and-hold problem involves doing nothing once the investor has bought at the beginning of the period. To rebalance back to the same weight, the investor has to constantly trade each period.

The rebalancing dynamic strategy means the long-term investor is trading every period. Consider the simplest case of stocks and risk-free bonds. To maintain a fixed portfolio weight in stocks, an investor must invest *counter-cyclically*. If equity has done extremely well over the last period, equities now are above target and it is optimal to sell equity. Thus, the investor sells stocks when stocks have done well. Conversely, if equity loses money over the last period relative to other assets, equities have shrunk as a proportion of the total portfolio. The equity proportion is too low relative to optimal and the investor buys equity. Thus, rebalancing buys assets that have gone down and sells assets that have gone up. This rebalancing is irrelevant to a myopic investor, because the myopic investor is not investing anymore after a single period. Rebalancing is the most basic and fundamental long-run investment strategy, and it is naturally counter-cyclical. An important consequence of optimal rebalancing is that long-run investors should actively divest from asset classes, or even stocks, that have done well and they should increase weights in asset classes or stocks that have low prices. Thus, rebalancing is a type of *value investing strategy* (see Chapter XX): *long-run investors are at heart value investors*.

Rebalancing is optimal under i.i.d. returns but it turns out to be advantageous when returns exhibit *mean reversion* or are predictable. If expected returns vary over time, prices are low because future expected returns are high – as our investor Daniel experienced during the 2008 financial crisis, prices of many risky assets plummeted but their future expected returns from 2008 onwards were high. Rebalancing buys assets that have declined in price, which have high future expected returns. Conversely, rebalancing sells assets that have risen in price, which have low future expected returns.[6]

2.5 Rebalancing in Practice

Rebalancing 1926-1940 and 1990-2011

Figures 5 and 6 show rebalancing over 1926-1940, which includes the Great Depression, and 1990-2011, which includes the financial crisis and the Great Recession, respectively. In each case I rebalance to a position of 60% U.S. equities, 40% U.S. Treasury bonds and use data from Ibbotson Associates. Rebalancing occurs at the end of every quarter.

[Figure 5 here]

Figure 5 starts off with $1 at the beginning of January 1926. The dashed line represents a 100% bond position, which rises steadily. A 100% stock position is shown in the dotted line and the stock wealth is relatively volatile. Stocks rise though the 1920s and reach a peak of $2.93 at the end of August 1929. Then the Great Depression hits with a vengeance. Stocks markets crash in 1929 and remain depressed into the early 1930s. Stocks hit a minimum of $0.49 in May 1932.

[6] I return to predictability in asset returns in Chapter XX and counter-cyclical factor investing in Chapter XX.

Stocks begin a slow climb upward from this point and end in December 1940 at $1.81, which is below the cumulated bond position of $2.08 at that time. The solid line in Figure 5, Panel A shows the rebalanced 60%/40% position. It is much less volatile than the 100% stock position so while it does not rise as much until 1929, it also does not lose as much during the early 1930s. The 60%/40% strategy ends at December 1940 at $2.46.

Rebalancing is beneficial during the early 20th century because it counter-cyclically cuts back on equities as they were peaking in 1929 and adds equities when they were at their lowest point in the early 1930s. Panel B of Figure 5 shows the rebalanced strategy, which goes back to 60%/40% at the end of each quarter, versus a buy-and-hold strategy, which starts off at 60%/40% at the beginning of the sample and then fluctuates only according to how bond and stock returns vary. The rebalanced strategy, by design, hovers around the 60% equity proportion. There are some deviations because the strategy is not continuously rebalanced, but overall *the rebalanced strategy is less risky* because it does not allow equities to rise or fall to dangerously high or low levels. In terms of utility, the rebalanced strategy attains the optimal balance of stocks and bonds for the investor's risk aversion. But, as an added benefit, rebalancing is countercyclical. In contrast, the equity holding in the buy-and-hold strategy was very high in early 1929 (when stock prices are high and expected returns low), right before stocks crash in October 1929. The buy-and-hold equity weight was very low in 1932, right before stock prices pick up (stock prices are low and expected returns high).

Figure 6 does a similar exercise for the 1990-2011 period. In Panel A, we start with $1 invested at the beginning of January 1990. The bond position is shown in the dashed line. During 2008, bond prices suddenly spiked as there was a flight to quality when Lehman Brothers failed, but overall the series is relatively stable. The ending bond position at December 2011 is $7.12. The equity position in the dotted line shows two large peaks and declines: the bull market of the late 1990s followed by the bursting of the internet bubble in the early 2000s and the rise in equity prices during the early to mid-2000s followed by the financial crisis in 2007 and 2008. The ending equity position at December 2011 is $6.10. Like Figure 5, the solid line shows returns of a rebalanced 60%/40% strategy where the rebalancing occurs at the end of every quarter. This dynamic strategy is less volatile, by holding fewer equities, than the 100% equity position, and ends up doing better at December 2011, at $7.41, than either than full stock or bond strategy.

[Figure 6 here]

Panel B of Figure 6 shows the proportion invested in equities. The 60%/40% rebalanced strategy is optimal for the investor as it rebalances the equity position back so that the risk of a single asset does not dominate. It also takes advantage of counter-cyclical investing. The buy-and-hold strategy shown in the dashed line loads up on equities, peaking at 2000, just as equities hit the post-bubble period. The equity proportion is also high right before the 2008 financial crisis. In contrast, the rebalanced strategy actively buys low priced equities in late 2008 benefiting from the upward movement in prices (low prices, high expected returns) in 2009.

The standard 60%/40% strategy outperforms a 100% bond or 100% stock strategy over the 1926-1940 period (Figure 5) and over the 1990-2011 period (Figure 6). You should not take away that rebalancing will always outperform 100% asset positions – it won't. In small samples, anything can happen. But I show below that, under certain conditions, rebalancing will always outperform a buy-and-hold portfolio given sufficient time, resulting in a rebalancing premium. The main takeaway from the figures is to understand why rebalancing works for the investor: it cuts back on assets that do well so that they do not dominate in the portfolio. The investor rebalances so that the asset mix is optimal for her risk aversion every period. The 1926-1940 and 1990-2011 samples highlight an additional benefit of rebalancing: it is counter-cyclical, buying when prices are low and selling when high.

Behavioral and Agency Impediments

Figures 5 and 6 may look impressive, but in practice, rebalancing is hard. It involves buying assets that have lost value and selling those that have risen in price. This goes against human nature. Investors, by their behavioral biases, tend to be very reluctant to invest in assets that have experienced large losses. Investors are just as reluctant to relinquish positions that have done extremely well. How many investors can buy an asset because it has lost money? How many institutions can take capital away from traders because they have been successful and give it to colleagues who have underperformed? The natural tendency of investors is to be *pro-cyclical*, whereas rebalancing is *counter-cyclical*.

Good financial advisors like Harrison, who is helping Daniel, play an important role in counteracting the pro-cyclical tendencies of individual investors. Maymin and Fisher (2011) argue that this is one of the areas where a financial advisor can add most value for a client. The investment policy statement (IPS) is also an important anchor. Harrison as a financial advisor was right to insist on the IPS as the foundation of her advisor-client relationship with Daniel. The IPS is a way that the investor can be *time consistent*: the investor has made decisions in written form, in consultation with the investment advisor, and in doing so lays out a game plan.[7] Medical directives, especially for the mentally ill, often take the form of Ulysses contracts, named for the wily Greek who, en route home from the Trojan War, commanded his crew to bind him to the

[7] Kydland and Prescott wrote a famous paper in 1977 showing how to implement time-consistent monetary policy which won them a Nobel Prize in 2004.

mast of their ship so he could resist the lure of the Sirens' song. The IPS can play a similar role in helping an individual investor not to over-react. As one investment advisor said, "Investors are their own worst enemies. Advisers can use investment policy statements to keep clients on track in times of volatility."[8]

Do individual investors rebalance in practice? Yes, but incompletely. Calvet, Campbell and Sodini (2009) examine Swedish households. Data on Swedish asset holdings is very complete because Swedes pay taxes on both income and wealth. Swedish households have "surprisingly large propensity to rebalance," in the words of the authors. Wealthy, educated investors tend to hold more diversified portfolios and also tend to rebalance more actively. While there is active rebalancing, there is some inertia so that investors do not completely reverse the passive, buy-and-hold changes in their portfolios. In contrast, Brunnermeier and Nagel (2008) show that for U.S. households, inertia is the dominant factor determining asset allocation rather than rebalancing. (Maybe Swedish households are smarter, on average, than American ones.) Households start with a fixed allocation and then the asset weights evolve as a function of realized gains and losses on the portfolio. Rebalancing does occur, but sluggishly.

Institutional investors often fail to rebalance. While many pension funds and foundations resorted to panic selling and abandoned rebalancing during 2008 and 2009, CalPERS stands out in its failure. CalPERS' equity portfolio shrank from over $100 billion in 2007 to $38 billion in 2009.[9] CalPERS did the opposite of counter-cyclical rebalancing: it invested pro-cyclically and sold equities right at their lowest point – precisely when expected returns were highest. While part of CalPERS' problems in failing to rebalance stemmed from inadequate risk management,

[8] Quoted in Coyle, T., A Written Plan Can Help Your Portfolio, Wall Street Journal, June 4, 2012.
[9] See "California Dreamin': The Mess at CalPERS," Columbia CaseWorks ID#, 2012.

particularly of liquidity risk, CalPERS did not buy stocks when they were cheap partly because of structural misalignments between board members and he delegated fund manager. These are *agency* problems, and I discuss them in Chapter XX. CalPERS did have a statement of investment policy, the institutional version of an individual investor's IPS, but this did not help CalPERS to rebalance during the financial crisis.

In contrast to CalPERS, the Norwegian sovereign wealth fund rebalanced during 2008 and 2009. It bought equities at low prices from those investors like CalPERS who sold at the wrong time. Norway had its own version of Ulysses bound to the mast: the Ministry of Finance and Parliament decided on a rebalancing rule, rather than having committees make rebalancing decisions. Adopting a rule, which was automatically implemented by the fund manager, ensured that the rebalancing decisions were not left to a committee whose members could be swayed by blind panic, fear, or hubris.

Rebalancing Bands

There are some technical considerations in implementing a rebalancing strategy. The theory presented has rebalancing occurring regularly: Figures 5 and 6 illustrate rebalancing quarterly.[10] But in practice, if the equity portfolio weight is 61% at the end of a quarter, should the investor rebalance that small 1% given transactions costs?

State-of-the-art rebalancing practices involve *contingent* rebalancing, rather than *calendar* rebalancing. Optimal rebalancing strategies trade off the utility losses of moving away from optimal weights versus the transactions costs from rebalancing. If the benefits of rebalancing

[10] The original Merton (1969, 1971) theory is presented in continuous time so rebalancing happens at every instant.

outweigh the transactions costs of implementing it, then it is an optimal time to rebalance and rebalancing becomes a contingent event.

Rebalancing bands are often used, set around optimal targets. The optimal rebalancing target may be 60% equities, for example, with bands set at 55% and 65%. A move outside the band triggers rebalancing. The bands are a function of transactions costs, liquidity, asset volatility, and minimum transactions sizes. When these transactions costs are large, or asset volatility is high, the bands are relatively wider. The first paper to derive optimal rebalancing bands was Constantinides (1979), and since then many variations have been developed.

The basic rebalancing model is shown in Panel A of Figure 7, where the horizontal axis indicates the evolution of an asset class weight. There is a single band around a target weight. If the asset weight lies within the band, the investor does not trade. As soon as the asset weight goes outside the bands, the investor rebalances to target. Constantinides advocates rebalancing to target, whereas other authors suggest rebalancing to the edge of the band. Whether you rebalance to the target or to the edge depends on whether the transactions costs are fixed like time, or fixed exchange fees, (rebalance to target) or proportional like brokerage fees and taxes (rebalance to the edge).[11] Panel B of Figure 7 presents a more sophisticated rebalancing strategy with two bands surrounding the target weight. There is no trade if the portfolio lies within the outer band. But, if the portfolio breeches the outer band, then the investor rebalances back to the inner band. Institutional investors often use derivatives to synthetically rebalance, which in many cases have lower transactions costs than trading the physical securities.[12] In my opinion all of these

[11] See Pliska and Suzuki (2004).
[12] See Brown, Ozik and Scholtz (2007) on using derivatives in a rebalancing strategy. Garleanu and Pedersen (2012) develop a model of dynamic trading with predictable returns and transactions costs.

technical considerations in rebalancing are precisely that – technical. The most important thing is to rebalance.

[Figure 7 here]

2.6 Opportunistic Strategies

Rebalancing is the foundation of any long-term strategy and applies under i.i.d. returns. In addition, if returns are predictable then there are further benefits from a long-term horizon. I call these opportunistic strategies.

When returns are time varying, so that asset expected returns and volatilities change over time, the optimal short-run weight changes. In equation (1.5), we can put subscript *t*'s on the means and standard deviations, μ_t and σ_t , respectively, of an asset indicating that these are conditional estimates at time *t* of expected returns and volatilities over *t* to *t+1*. The risk-free rate is likely to vary over time as well (note that the risk-free rate is known at the beginning of the period, so the risk-free rate from *t* to *t+1* is denoted as $r_{f.t}$. The time-varying short-run weight in equation (1.5) now becomes

$$\text{Short-Run Weight (t)} = \frac{1}{\gamma}\frac{\mu_t - r_{f,t}}{\sigma_t^2}. \tag{1.14}$$

Under time-varying, predictable returns, the optimal long-run strategy comprises the time-varying short-run strategy plus an opportunistic portfolio:

$$\text{Long-Run Weight (t) = Short-Run Weight (t) + Opportunistic Weight (t).} \tag{1.15}$$

The time-varying short-run weight is given in equation (1.14) and is called the *myopic portfolio*. The opportunistic weight is called the *hedging demand* by Merton, who chose the name because the hedging demand portfolio hedges against changes in the investment opportunity set. I prefer to think of it as how the long-run investor can opportunistically take advantage of time-varying returns.[13]

Tactical and Strategic Asset Allocation

Campbell and Viceira (2002) interpret the Merton-Samuelson portfolios in equation (1.15) as:

$$\begin{aligned}\text{Long-Run Weight (t)} &= \text{Long-Run Myopic Weight} \\ &+ [\text{Short-Run Weight (t) - Long-Run Myopic(t)}] \\ &+ \text{Opportunistic Weight (t)}\end{aligned} \tag{1.16}$$

where we split the short-run weight in equation (1.15) into two parts: the average, long-run myopic weight and a deviation from the constant rebalancing weight. The first term is the average value of equation (1.14):

$$\text{Long-Run Myopic Weight} = \frac{1}{\gamma}\frac{\bar{\mu}-\bar{r}_f}{\bar{\sigma}^2}, \tag{1.17}$$

where the mean and volatility of the asset are at steady state levels denoted by bars above each variable. This can be interpreted as the equivalent of the constant rebalancing weight in the i.i.d. case.

The short-run weight is *tactical asset allocation* and is how a short-run investor responds to changing means and volatilities. Tactical asset allocation then comprises the constant rebalancing weight plus a temporary deviation from the rebalancing rule (the first two terms in equation (1.16)).

[13]Equation (1.15) was originally formulated by Samuelson (1969) and Merton (1969, 1971). Hedging demand for an investor with log utility (CRRA utility with $\gamma = 1$) is zero. Intuitively a log investor maximizes log returns and long-horizon log returns are simple sum of one-period returns. Since the portfolio weight is freely chosen each period, the sum is maximized by maximizing each individual term in the sum. That is, a log investor with a long horizon is always a short-run investor.

Strategic asset allocation is the long-run weight and is the optimal strategy for a long-term investor.[14] As expected from the dynamic programming solution to long-term portfolio choice, long-run investors do everything that short-run investors do (tactical asset allocation), plus they can act opportunistically in a manner that their short-run cousins cannot. Thus, strategic asset allocation is the sum of all three terms in equation (1.16).

Characterizing Long-Run Opportunistic Portfolios

Computing the precise form of the long-run opportunistic portfolio can be difficult.[15] But insight can be obtained on opportunistic weights without wading through rocket science. There are two determinants of the opportunistic weight. The first is investor-specific. Just like the myopic portfolio weight depends on the risk tolerance of an investor, so does the opportunistic portfolio. But now the investor's horizon plays a role. Second, the opportunistic weights depend on asset-specific properties of how returns vary through time. The interaction between the investor's horizon and the time-varying asset return properties is crucial. This makes sense: an asset that has a low return today but will mean-revert gradually back over many years to a high level is unattractive to someone with a short horizon. Only a long-horizon investor can afford the luxury to wait. Similarly, some assets or strategies can be very noisy in the short run, but over the long run volatility mean-reverts, and the risk premiums of these assets manifest reliably only over long periods. Such strategies are also unattractive for short-run investors, but investors with long horizons can afford to invest in them.

---

[14] The term "strategic asset allocation" is much abused in the industry and is often used to tweak with long-run asset weights as an excuse not to rebalance. The term itself was introduced by Brennan, Schwartz and Lagnado (1997).
[15] You might need to hire a rocket scientist after all to compute long-term portfolio weights. See Campbell and Viceira (2002), Brandt (2009), Avramov and Zhou (2010), and Wachter (2010) for literature summaries. The very technical reader is encouraged to look at Duffie (2001).

Viewed broadly, the opportunistic portfolio for long-run investing also represents the ability of long-run investors to profit from periods of elevated risk aversion or short-term mispricing.[16] In rational asset pricing models, prices are low because the average investor's risk aversion is high and investors bid down prices in order to receive high future expected returns. If a long-horizon investor's risk aversion remains constant, then he can take advantage of these periods with low prices. In behavioral frameworks, prices can be low because of temporary periods of mispricing. These can also be exploited by a long-term investor who knows that prices will return to fair values over the long run.[17] While the simple rebalancing strategy is counter-cyclical and has a value tilt, some of the best opportunistic strategies are even more counter-cyclical and strongly value oriented. Crises and crashes should be periods of opportunity for truly long-run investors. Howard Marks (2011), a well known value investor, says this beautifully: "The key during a crisis is to be (a) insulated from the forces that require selling and (b) positioned to be a buyer instead." That's what rebalancing forces the investor to do.

There have been debates in the academic literature on how large these hedging demand, long-run opportunistic effects really are. In a major paper, Campbell and Viceira (1999) estimate hedging demands to be very large where they can easily double the average total demand for stocks by short-run investors. In Campbell and Viceira's model, the portfolio weight in equities for a long-term investor would have varied from -50% to close to 400% from 1940 to the mid-1990s. On the other hand, Brandt (1999) and Ang and Bekaert (2002), which appeared around the same time as Campbell and Viceira's paper, estimate small hedging demands. The long-run opportunistic demands depend crucially on how predictable returns are and the model used to capture that predictability. In Chapter XX, I show that overall the evidence for predictability is

[16] See Ang and Kjaer (2011).
[17] See Chapter XX for a discussion of rational and behavioral determinants of risk premiums.

weak, so I recommend that both the tactical and opportunistic portfolio weights be small in practice. Opportunistic hedging demands become much smaller once investors have to learn about return predictability or when they take into account estimation error.[18]

A system of predictable equity returns that has been widely studied in the portfolio choice literature is the Stambaugh (1999) system, where stock returns are driven by a valuation ratio like the dividend or earnings yield. The valuation ratio is a convenient instrument to capture time-varying expected returns. As dividends yields drop (or equity prices rise), future expected returns increase. The dividend yield itself is also persistent and varies over time.[19] Under the Stambaugh system, the long-term opportunistic portfolios are positive and increase with horizon.[20] This is shown in Figure 8 where the short-run, myopic weights and the total, long-run weights increase as expected returns increase and investment opportunities become more attractive. Long-run investors actually are leveraged versions of short-run investors: if short-run investors want to buy when expected returns are high, long-run investors will buy more. Opportunistic investing then is taking advantage of predictability even more than short-run investors do.[21]

[Figure 8 here]

The Norwegian sovereign wealth fund, following a constant rebalancing rule during 2008 and 2009, bought equities when prices were low and future expected returns were high. Had they also taken advantage of long-run, opportunistic strategies, Norway would have bought even more

[18] See Brandt et al. (2005) and Pastor and Stambaugh (2012).
[19] Chapter XX shows this is a good system to capture predictability. Although overall predictability is weak, the best predictor variables tend to be valuation ratios.
[20] This system is used by Campbell and Viceira (1999, 2002), for example, and is generalized by Pastor and Stambaugh (2009, 2012).
[21] Opportunistic demands are not always positive, as Liu (2007) shows for different models of predictability.

equities. I advise you to concentrate on rebalancing first before focusing on opportunistic strategies. Simple rebalancing is itself counter-cyclical, and long-run opportunistic investing in the Stambaugh model is much more aggressively counter-cyclical. If you cannot rebalance, which already involves buying assets that are falling in price, then there is no way you can implement opportunistic long-run investing when returns follow the Stambaugh model, which involves buying even more of the assets that have fallen in price. I also recommend that opportunistic portfolios should be modest: taking into account estimation error, combined with the overall very weak predictability in data (see Chapter XX), any realistic application of Figure 8 considerably flattens both the time-varying short-run and opportunistic weights as a function of expected returns.

3. Rebalancing is Short Volatility

Rebalancing is an option strategy, and in particular a *short volatility strategy*. This is not well known, although at some level should not be surprising for the reader steeped in financial theory because the same method used in Section 2 to solve long-horizon portfolio choice problems (dynamic programming) is used to value options (where it is called *backward induction*).[22] Showing how rebalancing is mechanically a short volatility strategy gives us deeper insights into what long-run investors are gaining, and losing, from rebalancing. Nothing is free after all, at least not in economic theory.

3.1 Example

This example is highly stylized and simple, but conveys enough to see rebalancing as a collection of options.

[22] Perold and Sharpe (1988) and Cochrane (2007) discuss interpreting rebalancing as an option strategy.

Suppose that a stock follows the *binomial tree* given in Figure 9, Panel A. Each period the stock can double, with probability 0.5, or halve starting from an initial value of $S = 1$. There are two periods, so there are three nodes in the tree. At maturity, there are three potential payoffs of the stock: $S_{uu} = 4$, $S_{ud} = S_{du} = 1$, and $S_{dd} = 0.25$, which have probabilities of 0.25, 0.5, and 0.25, respectively. In addition, the investor can hold a risk-free bond that pays 10% each period.

[Figure 9 here]

Let us first consider a buy-and-hold strategy that starts out with 60% equities and 40% in the risk-free asset. (We know buy and hold is not optimal for the long-run investor from Section 2.) At the end of the first period, the wealth of this investor can increase or decrease to

$$\begin{aligned} W_u &= 0.6 \times 2.0 + 0.4 \times 1.1 = 1.6400 \\ \text{or } W_d &= 0.6 \times 0.5 + 0.4 \times 1.1 = 0.7400, \end{aligned} \tag{1.18}$$

which is shown in Figure 9, Panel B. In equation (1.18), the return on the stock is either $2 - 1 = 100\%$ if we go into the upper branch or $0.5 - 1 = -50\%$ if we go into the lower branch. In the upper node at time 1, the proportion of the buy-and-hold portfolio held in equities is $0.6 \times 2.0 / 1.64 = 73.17\%$ and the proportion of the portfolio in equities in the lower node is $0.6 \times 0.5 / 0.74 = 40.54\%$. For the last two nodes at time 2, the final wealth for the buy-and-hold strategy is

$$\begin{aligned} W_{uu} &= 1.6400 \times (0.7317 \times 2.0 + 0.2683 \times 1.1) = 2.8840, \\ \text{or } W_{ud} &= 1.6400 \times (0.7317 \times 0.5 + 0.2683 \times 1.1) = 1.0840 = W_{du}, \\ \text{which is the same as } W_{du} &= 0.7400 \times (0.4054 \times 2.0 + 0.5946 \times 1.1) = 1.0840 = W_{ud}, \\ \text{or } W_{dd} &= 0.7400 \times (0.4054 \times 0.5 + 0.5946 \times 1.1) = 0.6340. \end{aligned} \tag{1.19}$$

This is shown in Figure 9, Panel C.

Now consider the optimal rebalanced strategy, which rebalances at time 1 back to 60% equities and 40% bonds. The end of period wealth at time 1 is exactly the same as equation (1.18). The final wealth at time 2 for the rebalanced strategy is

$$\begin{aligned} W_{uu} &= 1.6400 \times (0.6 \times 2.0 + 0.4 \times 1.1) = 2.6896, \\ \text{or } W_{ud} &= 1.6400 \times (0.6 \times 0.5 + 0.4 \times 1.1) = 1.2136 = W_{du}, \\ \text{which is the same as } W_{du} &= 0.7400 \times (0.6 \times 2.0 + 0.4 \times 1.1) = 1.2136 = W_{ud} \\ \text{or } W_{dd} &= 0.7400 \times (0.6 \times 0.5 + 0.4 \times 1.1) = 0.5476. \end{aligned} \tag{1.20}$$

The last panel D of Figure 9 plots the payoffs of the buy-and-hold strategy (equation (1.19)) and the rebalanced strategy (equation (1.20)) as a function of the stock value at maturity time 2. The buy-and-hold, unrebalanced strategy is shown in the dashed straight line. The gains and losses on the buy-and-hold position are linear, by construction, in the stock price. The payoffs of the rebalanced strategy, in contrast, are *convex*. Rebalancing adds more wealth to the investor if the stock price returns to 1.0 at maturity (1.2136 for the rebalanced vs. 1.0840 for the buy and hold for $S_{ud} = S_{dd} = 1$). This is offset by the rebalancing strategy underperforming the buy-and-hold strategy when the ending stock values are low ($S_{dd} = 0.25$) or high ($S_{uu} = 4$).

This convex pattern of the rebalancing strategy can be equivalently generated by short option positions. The strategy sells out-of-the-money call and put options and hence is short volatility.

Suppose there is a European call option with strike \$3.6760 maturing at time 2. This call option has the following payoffs at time 2:

$$\begin{aligned} C_{uu} &= \max(4.0000 - 3.6760, 0) = 0.3240, \\ \text{or } C_{ud} &= \max(1.0000 - 3.6760, 0) = 0 = C_{du}, \\ \text{or } C_{dd} &= \max(0.2500 - 3.6760, 0) = 0 \end{aligned} \tag{1.21}$$

The value of this call option at time 0 is \$0.0428.[23]

There is also a European put option with strike \$0.4660 maturing at time 2. This put option is worth \$0.0643 at time 0 and has the following payoffs at time 2:

$$\begin{aligned} P_{uu} &= \max(0.4660 - 4.0000, 0) = 0, \\ \text{or } P_{ud} &= \max(0.4660 - 4.0000, 0) = 0 = P_{du}, \\ \text{or } P_{dd} &= \max(0.4660 - 0.2500, 0) = 0.2160. \end{aligned} \tag{1.22}$$

Now compare the following strategies:

	time 0	time 2		
Strategy		$S_{uu} = 4$	$S_{ud} = S_{du} = 1$	$S_{dd} = 0.5$
Sell Put	+0.0643	0	0	-0.2160
Sell Call	+0.0428	-0.3240	0	0
Buy Bonds	-0.1071	0.1296	0.1296	0.1296
Buy and Hold Strategy	1.0000	2.8840	1.0840	0.6340
Short Volatility + Bonds + Buy and Hold	1.0000	2.6896	1.2136	0.5476
Rebalanced Strategy	1.0000	2.6896	1.2136	0.5476

The table lists the values today in the column labeled "time 0" and the payouts of the various strategies at time 2. The time 2 payouts are contingent on the stock values at time 2, hence there are three columns representing the stock values $S_{uu} = 4$, $S_{ud} = S_{dd} = 1$, and $S_{dd} = 0.5$ at time 2.

[23] This can be valued using *risk-neutral pricing*. The call value at time 0 is $\frac{q^2 \times 0.324}{(1.1)^2} = 0.0428$, where q is the *risk-neutral probability* given by $q = \frac{1.1 - 0.5}{2 - 0.5} = 0.4$. For an introduction to risk-neutral option pricing, wee Bodie, Kane and Marcus (2011). The put value in equation (1.22) is worth $\frac{(1-q)^2 \times 0.216}{(1.1)^2} = 0.0643$ at time 0.

Consider the first set of strategies. Selling a put today means money comes in (+ sign) with a put premium of $0.0643. If the stock price is low ($S_{dd} = 0.5$) at time 2, then the investor must pay out (- sign) an amount of $0.2160. Likewise, selling a call today means money comes in with a call premium of $0.0428. The investor must make a payout to the person buying the option of $0.3240 if the stock increases at time 2 ($S_{uu} = 4$). The investor also purchases $0.1071 of bonds at time 0. The purchase means a cash outflow, so there is a negative sign. At time 2, these bonds are worth $0.1071 \times (1.1)^2 = 0.1296$ at time 2. Finally, we have the payoffs of the buy-and-hold strategy starting with $1 invested at time 0. The payoffs of the strategy in each state of the world for the buy-and-hold strategy are listed in equation (1.19).

If we add the short call, the short put, the long bond position, and the buy and hold strategy, we get a value of $1 today at time 0, with identical payoffs to the rebalancing strategy at time 2 (which are listed in equation (1.20)). That is, a short volatility position which is financed by bonds together with the buy-and-hold strategy is identical to the rebalanced strategy. Hence, rebalancing is a short volatility strategy.

In Figure 9, Panel D, the buy-and-hold strategy is the completely *passive* straight line. The rebalancing strategy is an *active* strategy that transfers payoffs from the extreme low and high stock realizations (S_{uu} and S_{dd}) to the middle stock realization ($S_{ud} = S_{dd}$). Rebalancing does this by selling when stock prices are high and buying when stock prices are low. Short volatility positions do exactly the same. A call option can be dynamically replicated by a long stock position and a short bond position. This buys equity when stock prices rise and sells equity when stock prices falls. A short call option does the opposite: a short call position is the same as selling when equity prices rise and buying when they prices fall. Likewise, a short put is also dynamically replicated by selling equity when prices rise and buying when prices fall. These are exactly the same actions as rebalancing.

3.2 Interpretation

What is the market value of rebalancing? In this two-period binomial example, the action of rebalancing relative to the buy-and-hold strategy can be replicated by selling a call, selling a put, and investing in bonds. This has value:

$$\text{Short Call} + \text{Short Put} + \text{Long Bonds} = 0.0643 + 0.0428 - 0.1071 = 0.$$

That is, the action of rebalancing is assigned a zero market value. *The market does not value rebalancing*.

The optimal rebalancing strategy of Section 2 is a *partial equilibrium* strategy. Not everyone can rebalance. For every institution like Norway buying equities during the darkest periods of the financial crisis, there were institutions like CalPERS who couldn't wait to shed their risky equity allocations. CalPERS losses in failing to rebalance represent Norway's gains from successful rebalancing. Put simply, for every buyer there must be a seller. In equilibrium, it is impossible for everyone to simultaneously sell or buy.[24] Rebalancing is not valued by the market. In fact, consistent with the market assigning no value to rebalancing, the average investor who holds the market portfolio does not rebalance: the market itself is buy and hold![25]

[24] Because of this Sharpe (2010) advocates long-horizon investors to use "adaptive" asset allocation strategies that just drift up and down with the market instead of actively rebalancing. These are dominated, strictly, by rebalancing with i.i.d. returns as shown in Section 2.

[25] In the CAPM and in multifactor models, which we cover in Chapter XX, the average investor holds the market portfolio. The average investor does not rebalance. Individual investors can rebalance if other investors do not. Kimball et al. (2011) develop a model of equilibrium rebalancing and Chien, Cole and Lustig's (2012) equilibrium model has some investors who rebalance and others who do not.

The benefit to rebalancing is investor-specific. Moving the payoffs from the extreme stock positions back to the center (as in Figure 9, Panel D) is optimal for the investor because it cuts back on risk. In our example, the 60% equity-40% bond portfolio turns out to be optimal for an investor with a $\gamma = 0.51$ degree of risk aversion. A certainty equivalent calculation (see Chapter XX) reveals that he needs to be compensated 0.29 cents for each dollar of initial wealth for being forced to do the buy-and-hold strategy instead of optimally rebalancing.[26] The long-term investor values rebalancing because it reduces her risk and increases her utility. The market does not because there must be other investors who are not rebalancing to take the other side.

The fact that rebalancing is short volatility means that rebalancing is an automatic way to earn the volatility risk premium. In our example, volatility is constant (the stock volatility is equal to 0.75), but in reality volatility varies over time. Volatility is a risk factor and earns a *negative risk premium*. An investor collects the volatility risk premium by selling options, or by being short volatility. We discuss this further in Chapter XX.

Viewing rebalancing as a short volatility strategy in moving the payoffs to the center, increasing the losses during extreme low markets, and underperforming the buy-and-hold strategy during extreme high markets makes clear that rebalancing profits from reversals. This is one reason why rebalancing performed well over 1926-1940 and 1990-2011 in Figures 5 and 6, respectively.

---

[26] The optimal utility for rebalancing is $\frac{1}{1-\gamma}\left[(0.5)^2 \times (2.6896)^{1-\gamma} + 2 \times (0.5)^2 \times (1.2136)^{1-\gamma} + (0.5)^2 \times (0.5476)^{1-\gamma}\right] = 2.3303$ and the optimal utility for buy and hold is $\frac{1}{1-\gamma}\left[(0.5)^2 \times (2.8840)^{1-\gamma} + 2 \times (0.5)^2 \times (1.0840)^{1-\gamma} + (0.5)^2 \times (0.6340)^{1-\gamma}\right] = 2.3270$. The certainty equivalent compensation required by the investor to do buy-and-hold investing instead of optimal rebalancing is $\left(\frac{2.3303}{2.3270}\right)^{\frac{1}{1-\gamma}} - 1 = 0.29$ cents per dollar of initial wealth. Notice that these are the only calculations where we actually use the real-world probability of 0.5 as an upward move in the tree. All the option valuations are done using risk-neutral probabilities.

When there are strong reversals from the steepest crashes, rebalancing does well. These happened after the Great Depression and after the Great Recession and financial crisis, respectively, in these two samples.

Conversely, if reversals do not occur, such as in permanent bull or permanent bear markets, then rebalancing will underperform the buy-and-hold strategy. Rebalancing as equivalent to a short volatility strategy is also the same as, in the words of Antti Ilmanen, my fellow advisor to the Norwegian sovereign wealth fund, “rebalancing is short regime changes.” Take the extreme case where a regime change occurs and permanently kills equity markets, then rebalancing performs poorly because it adds equities as prices decline and then equity prices are permanently lower. The opposite extreme case is a regime change so that stocks permanently go into a bull market. Rebalancing also underperforms a buy-and-hold strategy because rebalancing would have sold stocks into a permanently rising market.

Regime changes sometimes occur, but they are rare. I fully agree with Reinhart and Rogoff (2011) that people too often think “this time is different.” Two examples of true regime changes where “these times really were different” are the changing shape of the yield curve pre- and post-1933 and the pricing of out-of-the-money put options pre- and post-1987. Pre-1933 the yield curve was downward sloping, compared to its now (post-1934) normal upward-sloping shape (see Wood (1983)). Implied option volatilities were flat across strikes in the pre-1987 sample. After the 1987 crash, there has been significant negative skewness in implied volatilities (see Rubinstein (1994)). The financial crisis in 2007 and 2008 was not a regime change. True regime changes are rare.

The fact that rebalancing is short volatility and short regime changes means that you must practice rebalancing on broad asset classes (or across factors, see Chapter XX) that are extremely unlikely to undergo permanent regime change. Russian equities in the early 1900s disappeared not even two decades later. But global equities are still around over 100 years later and are likely to be here for a very long time. Russian bonds also disappeared during the Russian Revolution, but global bonds did not. Global equities and global bonds have been and will continue to be with us for a long time. Rebalance with the tried and true.[27]

4. Liability Hedging

4.1 Liability Hedging Portfolio

Few investors have no liabilities. Even investors lacking explicit liabilities (like Norway), at least over the short term, often have implicit liabilities through the stewardship expectations. Liabilities can be fixed, like loan payments, variable but steady, like pension costs, or highly variable and contingent, like paying a single individual's death obligation.

When liabilities are introduced, the optimal portfolio strategy has three components:

$$\begin{aligned}&\text{Long-Run Weight (t)}\\&= \text{Liability Hedge (t)} + \underbrace{\text{Short-Run Weight (t)} + \text{Opportunistic Weight (t)}}_{\text{Investment Portfolio (t)}}.\end{aligned} \qquad (1.23)$$

The investment portfolio is exactly the same as the non-liability case we examined in Sections 2 and 3: the optimal policy is to rebalance under i.i.d returns and when returns are predictable, the optimal short-run portfolio changes over time and the long-run investor has additional opportunistic strategies. The liability hedging portfolio is the portfolio that best meets the

[27] If asset returns follow *Markov processes*, then you want to rebalance over assets or strategies that are *recurrent*.

liabilities. We solve for it by holding asset positions that yield the highest correlation with the liabilities. The liability-hedging portfolio best ensures the investor can meet those liabilities.

There are several special cases of optimal liability hedging portfolios:

1. *Cashflow matching* or *immunization*. This involves constructing a perfect match of liability outflows each period. You immunize each liability cashflow by holding bonds of appropriate maturities.
2. *Duration matching.* If liabilities can be summarized by a single interest rate factor, which is common for pension liabilities, then the liabilities can be offset by an asset portfolio with the same duration.[28]
3. *Liability-driven investing.* This aims to construct a portfolio of risky assets that best meets the liability obligations. It is also common in pension fund management and was introduced by Sharpe (1992). It is related to, and often used synonymously with. . .
4. *Asset-liability matching*. This is a more general case than duration matching. In asset-liability matching, dimensions other than just duration are used to match liability characteristics with assets, including liquidity, sensitivity to factors besides only interest rates, and horizon.

The Merton-Samuelson advice of long-horizon asset allocation extended to liabilities is, first, to meet the liabilities and then to invest the excess wealth over the present value of liabilities in the same style as Sections 2 and 3, using the myopic market portfolio and the opportunistic long-horizon portfolio.

[28] Duration is exposure to the interest rate level factor, which is the most important factor in fixed income investments. See Chapter XX.

For a long-horizon investor, U.S. Treasuries are generally not the risk-free asset and the optimal liability-hedging asset. If the investment horizon exceeds the longest available maturity of the risk-free bond, which is the case for some sovereign wealth funds and family offices, then investors do not have access to a risk-free asset. Furthermore, many investors have liabilities denominated in *real*, not *nominal* terms. But even long-horizon real bonds are not the optimal liability-hedging asset if there are other factors. For pension plans, these include longevity risk, economic growth, and credit risk. Individual investors face inflation risks, like for medical care and college tuition, that are not reflected in general CPI inflation. The liability hedging portfolio emphasizes what types of assets (or more broadly what kinds of factors, see Chapter XX) pay off to meet the worst times of the investor, in terms of when and how the liabilities come due. If liabilities increase when credit spreads increase, for example, as they do for pension funds, then the liability-hedging portfolio must hold large quantities of assets which are sensitive to credit risk.

What if you can't meet the liabilities in the first place? Sadly this condition applies to many investors today, especially public pension funds. CalPERS, for example, only had a funding ratio (the ratio of assets to actuarial liabilities) of 65% at June 30, 2010. Strictly speaking, the Merton-Samuelson asset allocation advice outlined in Sections 2 and 3 applies only after the liabilities can be met, both in terms of the present value of the liabilities and after the liability-hedging portfolio has been constructed. If assets are not sufficient to meet current liabilities, then the investor must face the fact that default will happen in some states of the world. Portfolios can be constructed to minimize this probability, but avoiding insolvency requires a different optimization than the maximization of utility examined in equation (1.2). In certain cases, it may be optimal for the investor to engage in *risk-seeking behavior* if the assets are far enough below the value of the liabilities. It is the Hail Mary pass; you have nothing to lose and if you are likely to go bankrupt anyway.[29]

4.2 Popular Investment Advice

The three types of portfolios for long-term investors:

(1) Liability-hedging portfolio

(2) Short-run, or myopic, market portfolio

(3) Long-run opportunistic, or long-term hedging demand, portfolio

that are derived in the Merton-Samuelson dynamic trading context accord well with the advice given by some financial advisors. A practitioner framework developed by Ashvin Chhabra (2005) suggests creating three buckets:

(1) Protective portfolio, which covers “personal” risk. The portfolio is designed to minimize downside risk and is a form of safety first (see Chapter XX). The maxim is: “Do not jeopardize the standard of living.”

(2) Market portfolio, which is a balance of “risk and return to attain market-level performance from a broadly diversified portfolio” and is exposed to market risk.

(3) Aspirational portfolio, which is to designed to “take measured risk to achieve significant return enhancement.” Aspirational risk is a property of an investor’s utility function and is a desire to grow wealth opportunistically to reach the next desired wealth target.

This looks very much like the Merton-Samuelson advice. Chhabra’s buckets correspond to the three Merton-Samuelson portfolios:

[29] Ang, Chen and Sundaresan (2012) demonstrate this behavior is optimal in a liability driven investment context with downside risk.

(1) Protective portfolio = Liability-hedging portfolio

(2) Market portfolio = Short-run portfolio

(3) Aspirational portfolio = Long-run opportunistic portfolio

There are some small differences between Merton-Samuelson and Chhabra. Chhabra advocates mostly safe fixed-income assets for the protective portfolio, while the concept of the Merton liability-hedging portfolio recognizes that U.S. Treasuries may not be safe, and sometimes are extremely risky, in terms of meeting liability commitments. But the overall concepts of Chhabra are similar to Merton and Samuelson's theory.

Thus, some financial planners have been advocating Merton-Samuelson dynamic portfolio choice theory even though they have not been exposed to the original Nobel-winning papers written in the 1960s and 1970s. The difference is that the full (rocket science) glory of formal portfolio choice leads to quantitative solutions (equations can be numerically solved by rocket scientists to give portfolio weights when analytical solutions are not available), economic rigor, and some deep insights linking dynamic portfolio choice with option strategies to understand when and how long-run advice will do well or poorly.

5. Rebalancing Premium

Long-horizon investing is not complete without a final discussion of the *rebalancing premium*. This goes under a variety of names including the *diversification return*, *variance drain*, *growth-optimal investing*, *volatility pumping*, and the *Kelly criterion* or *Kelly rule*, named after John Kelly (1956), an engineer who worked at Bell Labs.[30] The term "diversification return" was introduced by Booth and Fama (1992) and is probably the best known term in finance, whereas

[30] For growth-optimal investing, see Latane (1959) and Messmore (1995) for the variance drain terminology. Luenberger (1997) introduced the term "volatility pumping". A nice collection of papers in the literature is MacLean, Thorp and Ziemba (2011).

the Kelly rule and volatility pumping are better known in mathematics. I prefer not to use Booth and Fama's terminology because there is a difference between diversification in a single period, and rebalancing, which earns a premium over time. Diversification gets you a benefit in one period, but this diversification benefit dies out if you do not rebalance.[31] The rebalancing premium only exists for a long-horizon investor, and he can collect it by rebalancing to constant weights every period. I use the term "rebalancing premium" to emphasize that the premium comes from rebalancing, not from diversification.

5.1 Rebalancing Beats Buy-and-Hold over the Long Run

Suppose that the price of each underlying asset is *stationary*, that is each asset by itself tends to hover around a fixed range and never goes off to infinity. Holding 100% positions in each asset never gives you increasing wealth. But, a rebalanced portfolio does give you wealth that increases over the long run to infinity (wealth increases *exponentially fast*).[32] Furthermore, by rebalancing to a fixed constant weight each period, an investor can generate wealth that increases over time and *any* such rebalancing strategy will eventually beat the best buy-and-hold portfolio. This seems like magic: Erb and Harvey (2006) call it "turning water into wine" and Evstigneev and Schenk-Hoppe (2002) call it going from "rags to riches."

Mathematically, this is not quite as impressive as Jesus' first miracle at the wedding at Cana. It arises as a consequence of compounding. We can see this in equation (1.7), where long-term wealth is a product of arithmetic returns, $(1+r_t)(1+r_{t+1})(1+r_{t+2})\ldots$, rather than a sum of arithmetic returns, $(1+r_t)(1+r_{t+1})(1+r_{t+2})\ldots \neq 1+r_t+r_{t+1}+r_{t+2}+\ldots$. The compounding of products gives rise

[31] See also Willenbrock (2011) who differentiates between diversification as being necessary to give you different weights over one period, but not sufficient, to earn the rebalancing premium over multiple periods. .
[32] This is true also for modest transactions costs, as shown by Dempster, Estigneev and Schenk-Hoppe (2009).

to many non-linearities over time, which are called *Jensen's terms*,[33] and the effects of the nonlinear terms increases over time. The entire rebalancing premium is due to Jensen's terms, and in fact the whole diversification return and Kelly rule literature can be viewed as a paean to *Jensen's inequality*.

Jensen's terms are the difference between *geometric* returns, which take into account the compounding over the long run, and *arithmetic* returns, which do not compound.[34] In a one-period setting, geometric and arithmetic returns are economically identical; they are simply different ways of reporting increases or decreases in wealth. Thus, there is no rebalancing premium for a short-run investor. Over multiple periods, the difference between geometric and arithmetic returns is a function of asset volatility, specifically approximately $\frac{1}{2}\sigma^2$, where σ is the volatility of arithmetic returns. The greater the volatility, the greater the rebalancing premium. As this manifests over time, only long-term investors can collect a rebalancing premium.

For U.S. stocks, the rebalancing premium a long-run investor can earn is approximately $\frac{1}{2}(0.15)^2 \approx 1\%$. Erb and Harvey (2006) estimate a rebalancing premium of around 3.5% in commodities. These are significant premiums for simple, automatic rebalancing. In his 2009 book, David Swensen, the superstar manager of Yale University's endowment, emphasizes that rebalancing plays an important role in his practice of investment management, especially in the daily rebalancing of Yale's liquid portfolio. He refers to a "rebalancing bonus" arising from maintaining a constant risk profile.

---

[33] Named after the Danish mathematician Johan Jensen.

[34] The arithmetic return *r* represents *(1+r)* at the end of the period. The same amount can be expressed as a geometric return, *g*, where $(1+r) = \exp(g)$. The means of the arithmetic return and the geometric return are related by $E(r) \approx E(g) - \frac{1}{2}\sigma^2$, where σ is the volatility of *r*. This relation holds exactly for log-normal distributions.

So, is rebalancing optimal not only because it reduces risk, but also because it provides a “free lunch” in the form of a rebalancing premium? Not so fast. In Section 3 I showed that rebalancing has no value in the market by interpreting rebalancing as an option strategy. The rebalancing premium seems too good to be true—and in fact, it is. Rebalancing is a short volatility strategy that does badly compared to buy and hold when asset prices permanently continue exploding to stratospheric levels or permanently implode to zero and disappear. Rebalancing is short a regime change. The crucial assumption behind the rebalancing premium is that the assets over which you rebalance continue to exist. If there are assets that experience total irreversible capital destruction, then rebalancing leads to buying more assets that eventually disappear – this is wealth destruction, not wealth creation. The rebalancing premium can only be collected for assets that will be around in the long run, so rebalance over very broad asset classes or strategies: global equities, global sovereign bonds, global corporate bonds, real estate, commodities, etc., rather than individual stocks or even individual countries.

5.2 The Very Long Run

In the very long run, the portfolio that maximizes wealth is a rebalanced portfolio that holds constant asset weights which maximize the rebalancing premium. This strategy maximizes long-run growth and is called the Kelly rule. It is obtained by finding the portfolio that maximizes one-period log returns. Since this portfolio maximizes (very) long-run wealth, it is called the optimal growth portfolio.

The Kelly optimal growth portfolio dominates all other portfolios with a sufficiently long time span.[35] So for the very long run investor, should we hold the optimal growth portfolio if it maximizes very long run wealth? This was settled by Samuelson in the 1970s, but the question is raised periodically by the unconvinced.[36] Samuelson wrote a cute paper in 1979, entirely written in words of one syllable, entitled "Why we should not make mean log of wealth big though years to act are long," to answer this question.

In a one-period model, you can maximize the portfolio growth rate by holding the maximizing expected log returns. But do you have a log utility function? Probably not. You trade off risk and return differently and are better off holding a portfolio optimized for your risk aversion and your own utility function. Similarly, over the long run, you will outperform by following the Kelly rule. But, there is risk in doing so, and you might not be able to tolerate this risk. Furthermore, the long run in the Kelly rule could be very, very long. And as Keynes famously observed, in the long run, we are all dead.

In summary, follow the Merton-Samuelson advice and not the Kelly rule. Find your optimal one-period portfolio holdings over broad asset classes or strategies. Rebalance back to these. This is optimal with i.i.d. returns, and it will earn you a rebalancing premium. If you can forecast returns well, you also have a long-run opportunistic portfolio available.

6. Stay the Course? Redux

Daniel has experienced large losses and is feeling skittish about sticking to her long-term plan. Daniel is fortunate in that she has no immediate liabilities and her income, which covers her

[35] The formal mathematical statement is there exists a number $M(W)$ that depends on current wealth W such that $\Pr(W_T \leq W)$ using the Kelly rule is $< \Pr(W_T \leq W)$ using any other portfolio for all $T > M(W)$. As $T \to \infty$, the Kelly rule dominates any other rule.

[36] See Samuelson (1971) and Merton and Samuelson (1974).

expenses, is relatively safe. Yet the losses seem to have changed her tolerance for risk. She has told her financial planner, Harrison, that her investment policy statement (IPS) needed a "total overhaul" and she could not afford such big losses going forward.

According to the long-run investment advice from Merton and Samuelson, Harrison should advise Daniel to rebalance. Rebalancing to fixed weights is optimal when returns are not predictable. Even though returns are predictable in reality, the amount of predictability is very small. This makes rebalancing the foundation of the long-run strategy. The small amount of predictability that does exist can be exploited by a long-run investor through an opportunistic portfolio. Daniel should stay the course and rebalance.

Rebalancing, however, goes against human nature because it is counter-cyclical. It is difficult for individuals to buy assets that have crashed and to sell assets that have soared. Part of Harrison's job as an investment advisor is to counteract these behavioral tendencies. The IPS can help by functioning as a commitment device – a Ulysses contract – in preventing Daniel from over-reacting and abandoning a good long-term plan.

But perhaps Daniel's risk aversion has truly changed. The classical assumptions, which we used in Sections 2-4, are that risk preferences are stable and unaffected by economic experiences. This is not true in reality.[37] Malmendier and Nagel (2011) show that investors who experienced the searing losses of the Great Depression permanently became more risk averse and were far less willing to invest in stocks than younger investors who did not experience such large losses and economic hardships. They show further that after the recessions of the late 1970s and early 1980s, young investors who only experienced the market's low returns during these periods were

[37] See Hertwig et al. (2004).

more risk averse, and held fewer equities and more bonds, than older investors who had experienced the high returns of the stock market during the 1950s and 1960s. Thus, life-time experiences do influence the extent to which investors are willing to take financial risks. But Malmendier and Nagel show that what changes after large losses is not so much investor risk preferences but investor expectations. People tend to lower their expectations about future returns rather than changing their utility function.

If Daniel has truly become more risk averse, then rebalancing back to the old portfolio pre-2007 is no longer valid, and Daniel has to work with her financial advisor to come up with a new IPS. Otherwise, the dynamic portfolio choice advice is to stay the course. Rebalance.

Figure 1

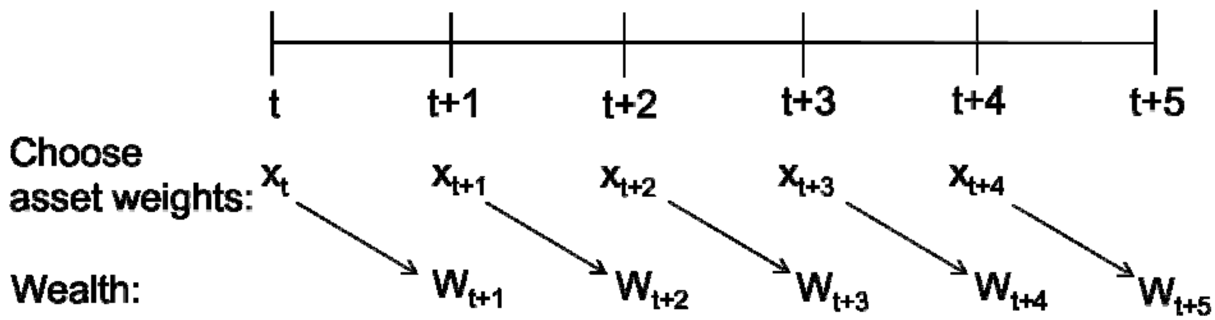
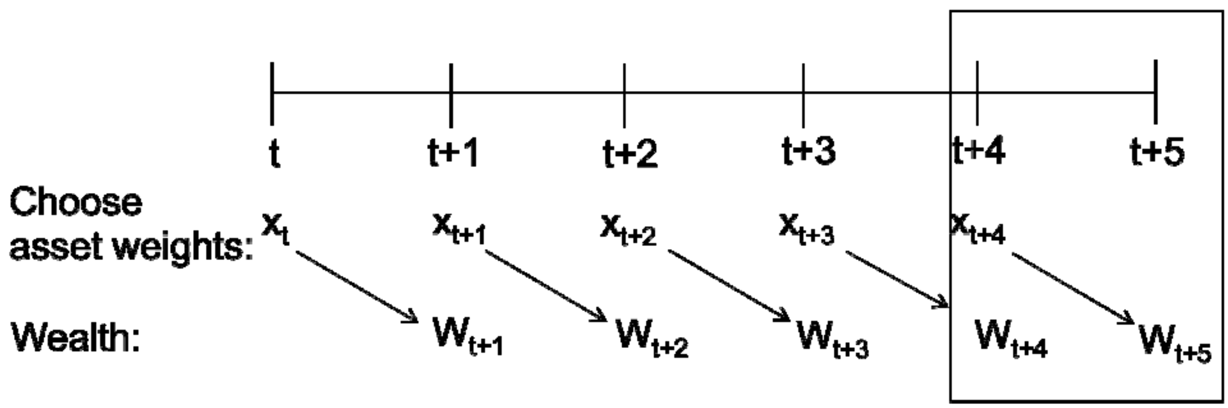

t
t+1
t+2
t+3
t+4
t+5
Choose asset weights:
x_t
x_{t+1}
x_{t+2}
x_{t+3}
x_{t+4}
Wealth:
W_{t+1}
W_{t+2}
W_{t+3}
W_{t+4}
W_{t+5}

Figure 2

Panel A



Panel B

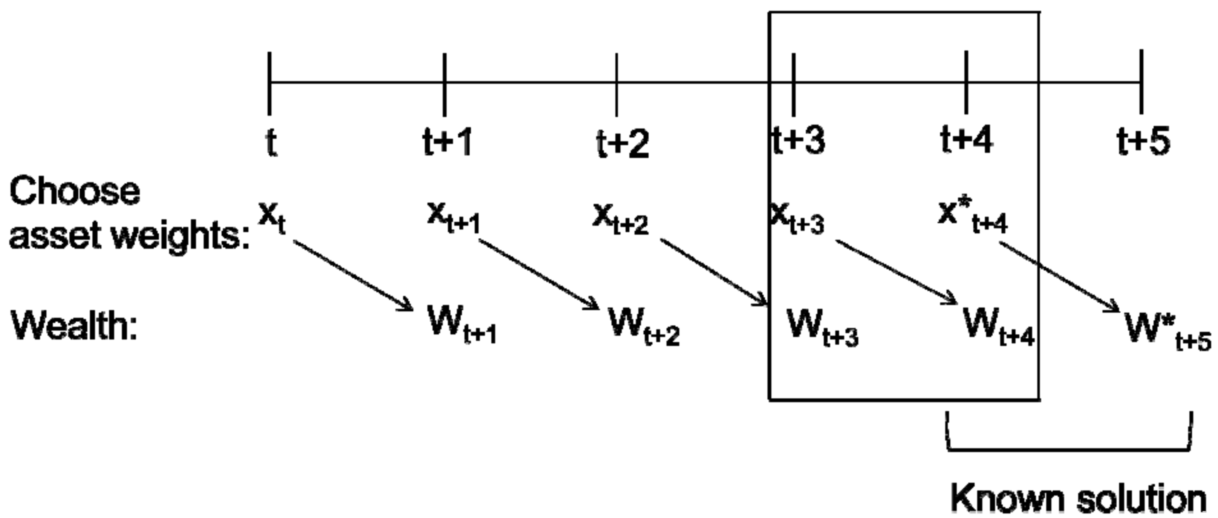


Figure 2

Panel C

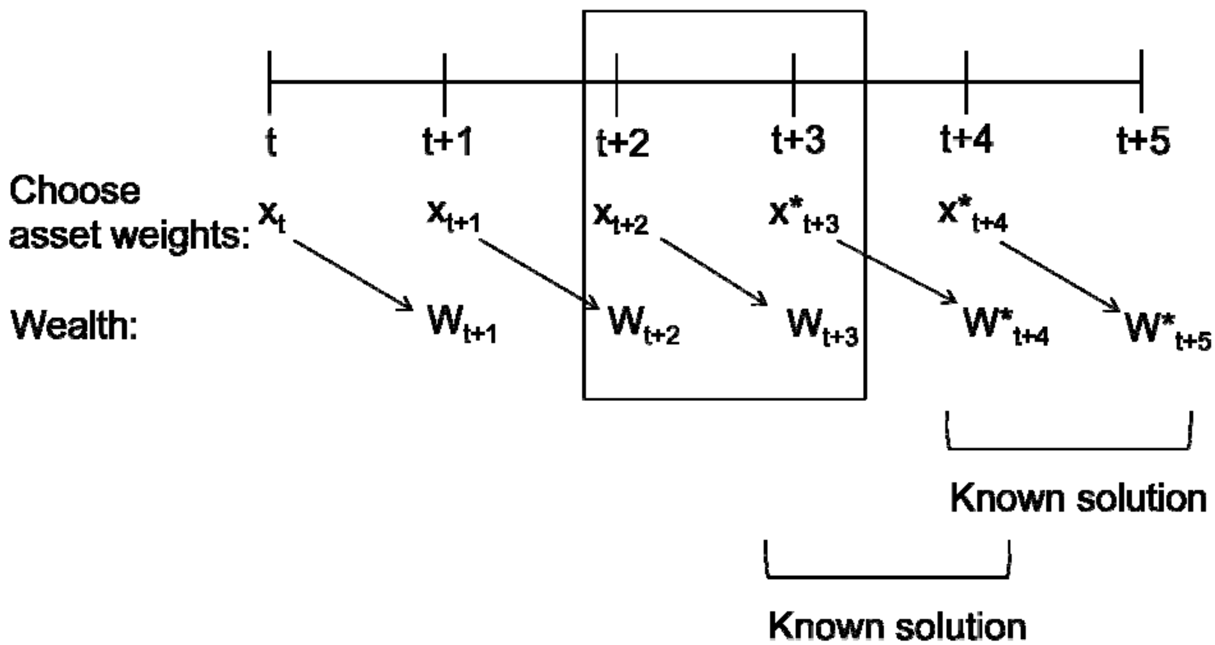


Figure 3

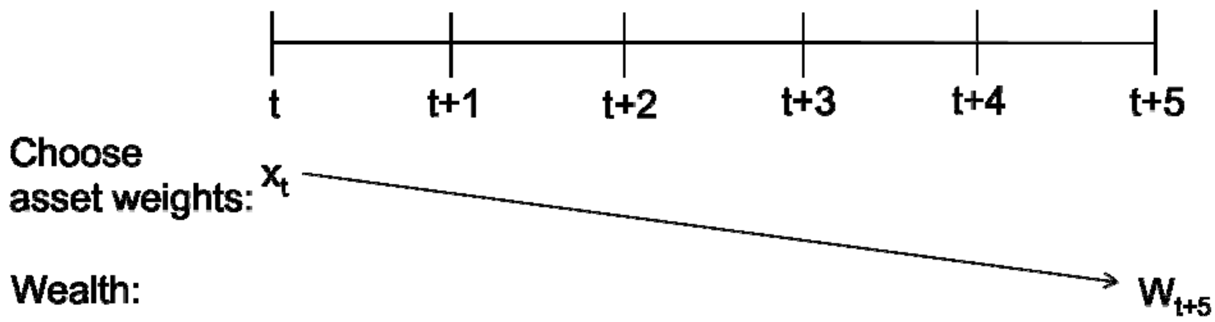


Figure 4

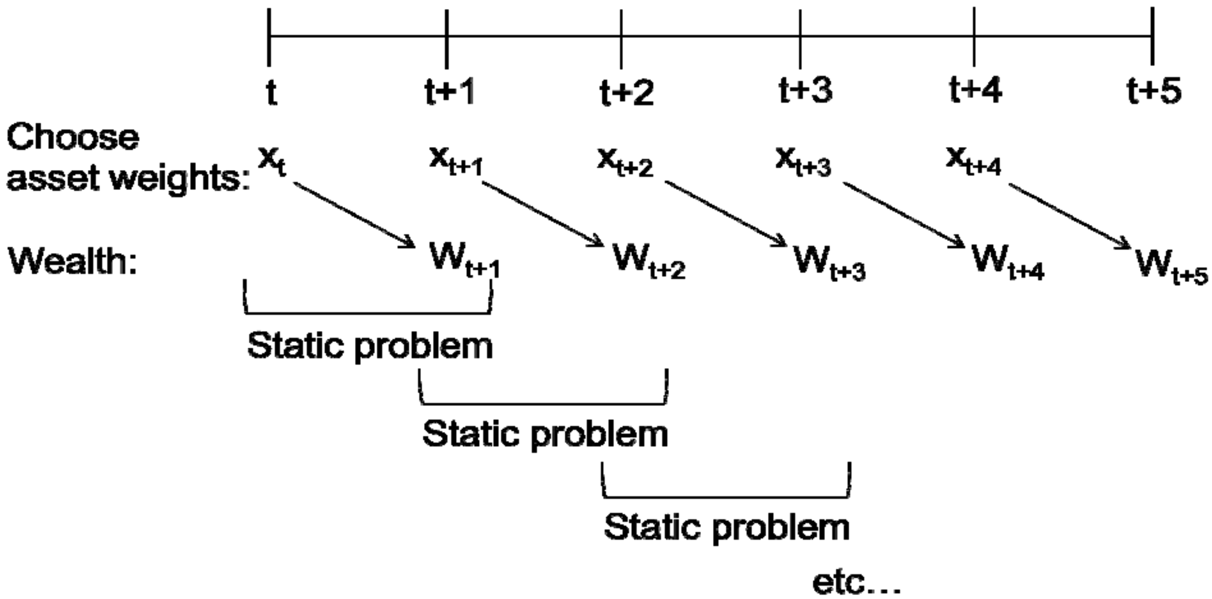

t
t+1
t+2
t+3
t+4
t+5
Choose asset weights:
x_t
x_{t+1}
x_{t+2}
x_{t+3}
x_{t+4}
Wealth:
W_{t+1}
W_{t+2}
W_{t+3}
W_{t+4}
W_{t+5}
Static problem
Static problem
Static problem
etc…

Figure 5

Panel A

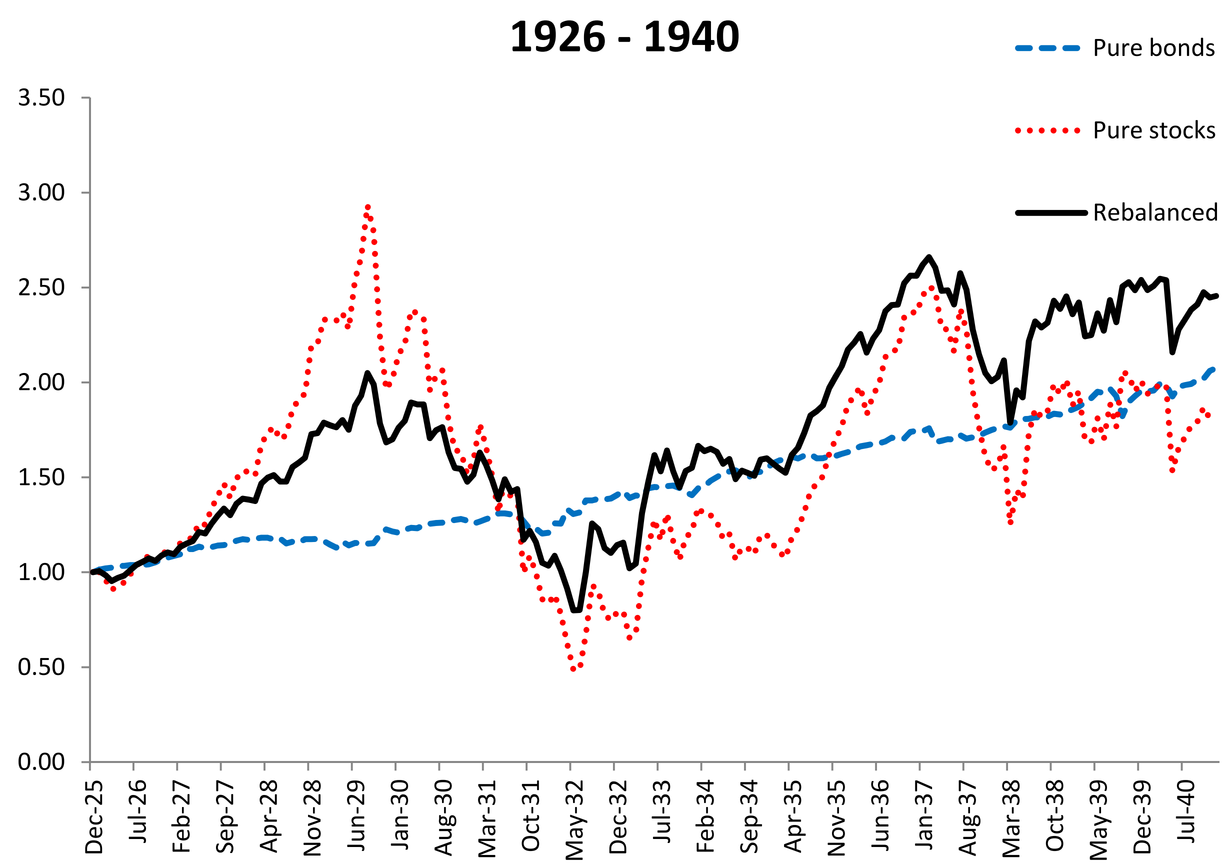


Figure 5

Panel B

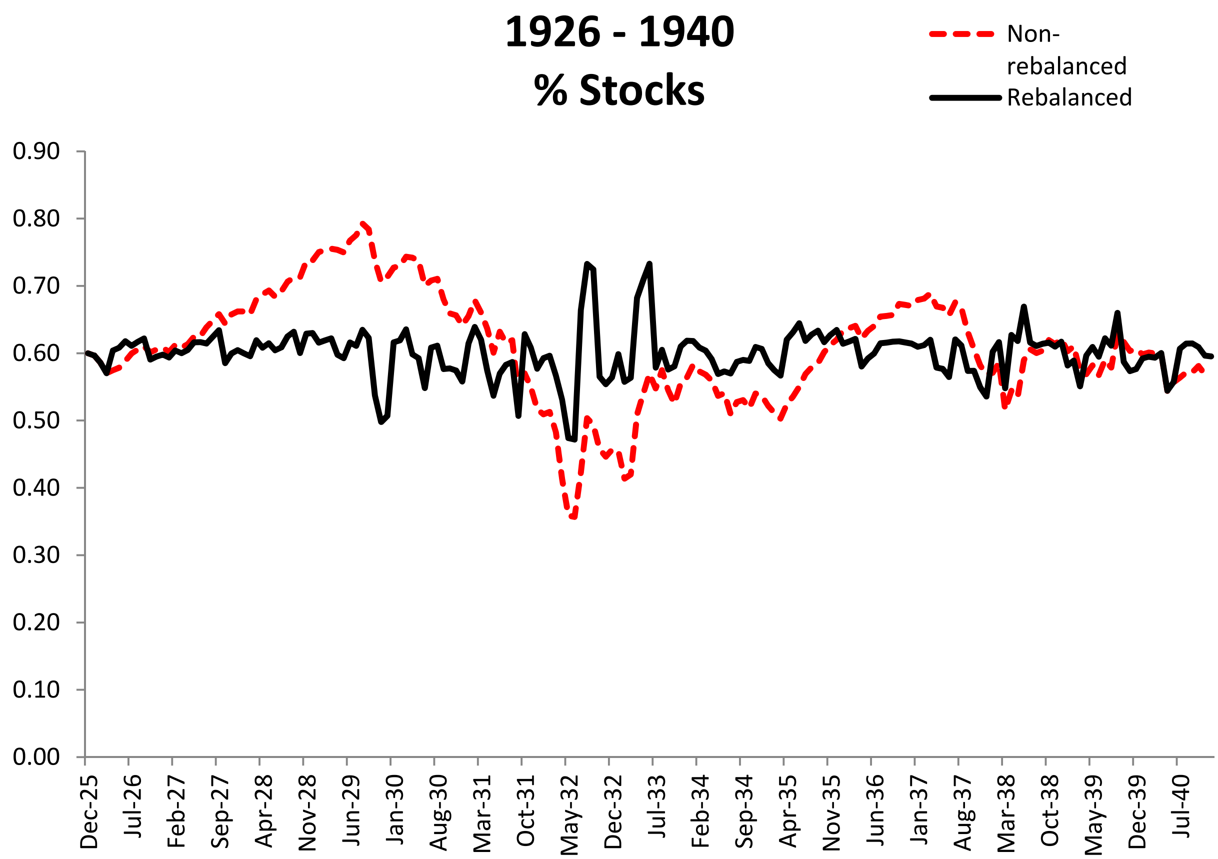


Figure 6

Panel A

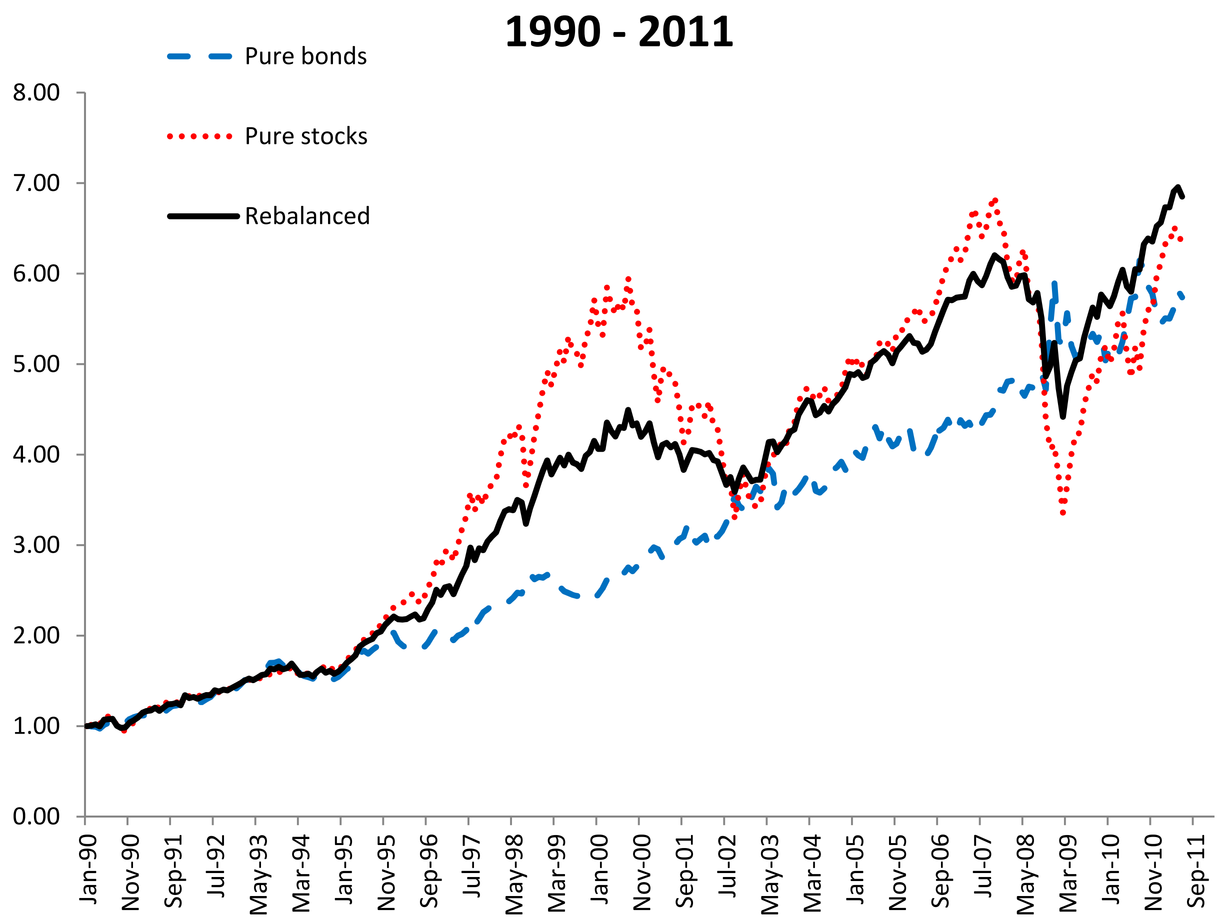


Figure 6

Panel B

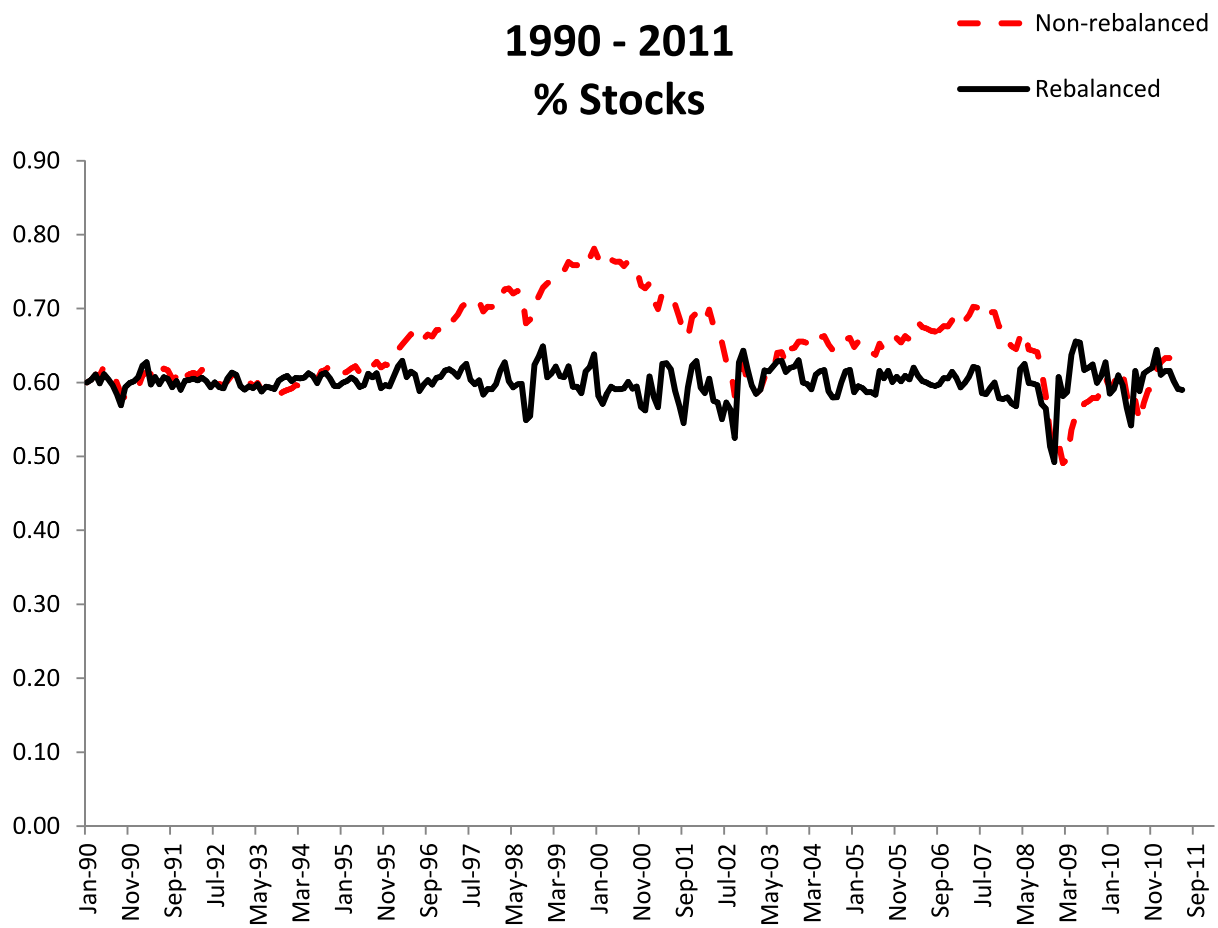
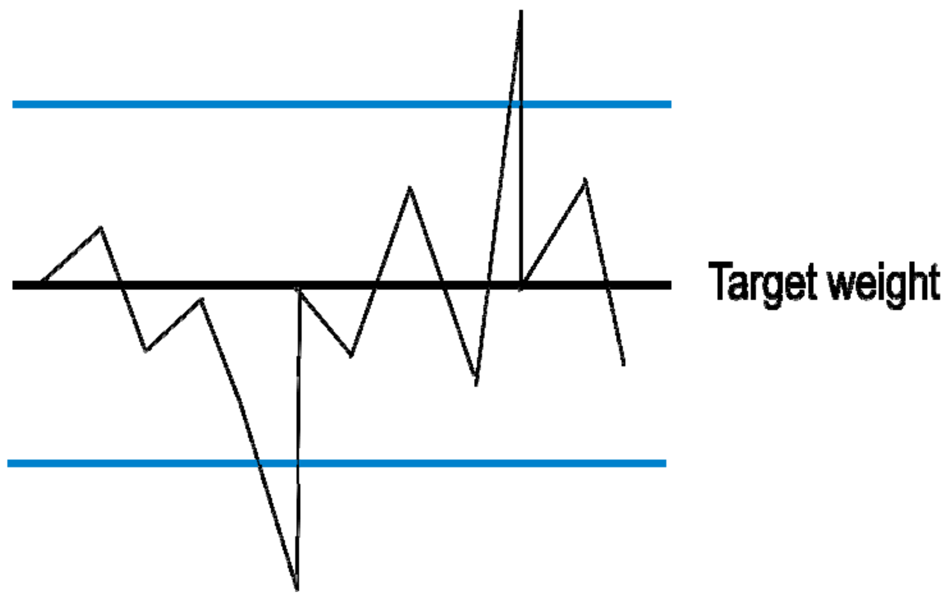


Figure 7

Panel A



Panel B

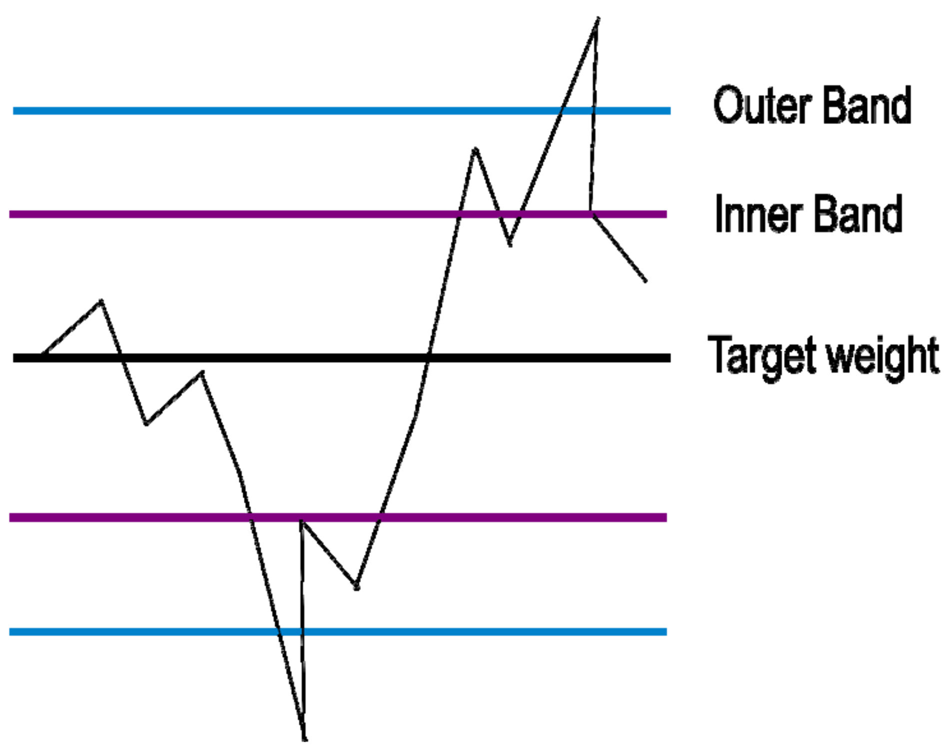


Figure 8

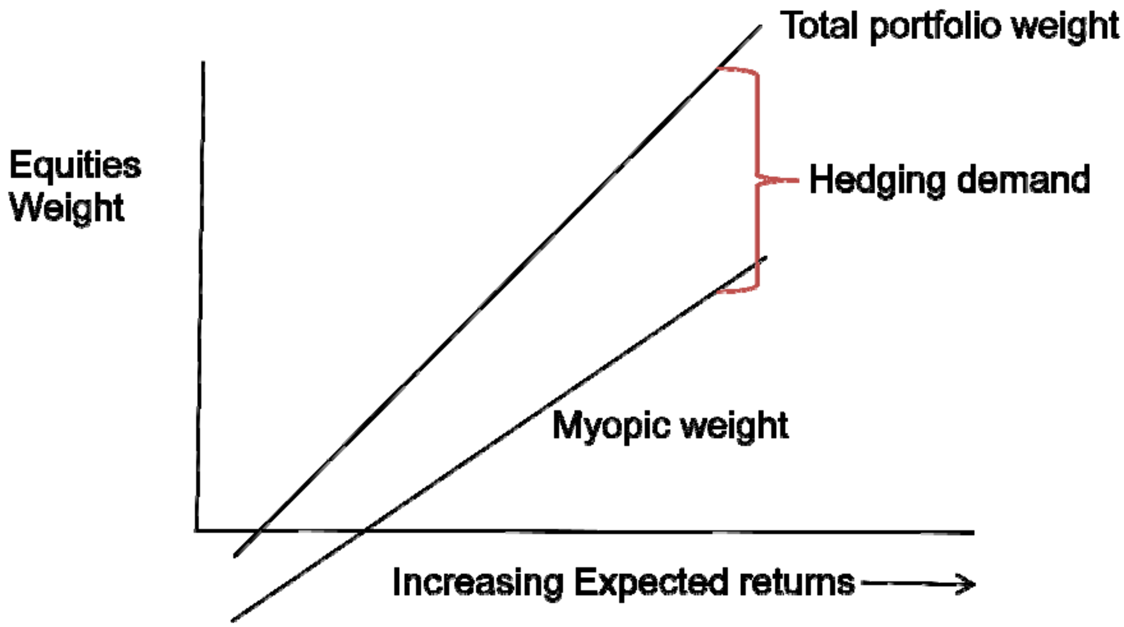
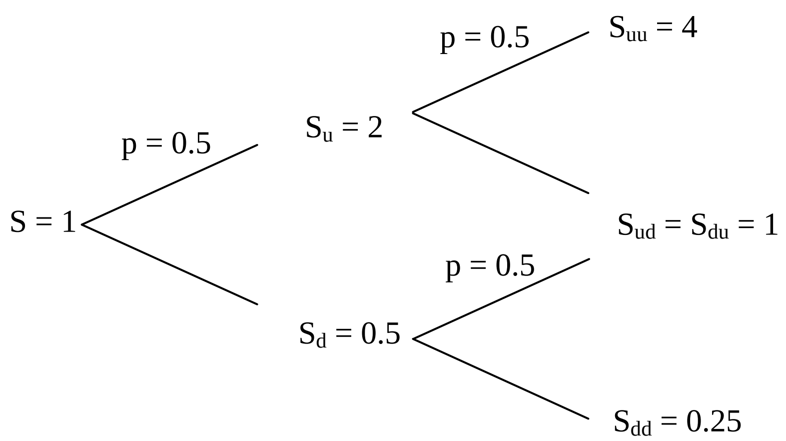

Total portfolio weight
Equities Weight
Hedging demand
Myopic weight
Increasing Expected returns

Figure 9

Panel A: Stock Dynamics



Panel B: Wealth of the Buy-and-Hold Strategy

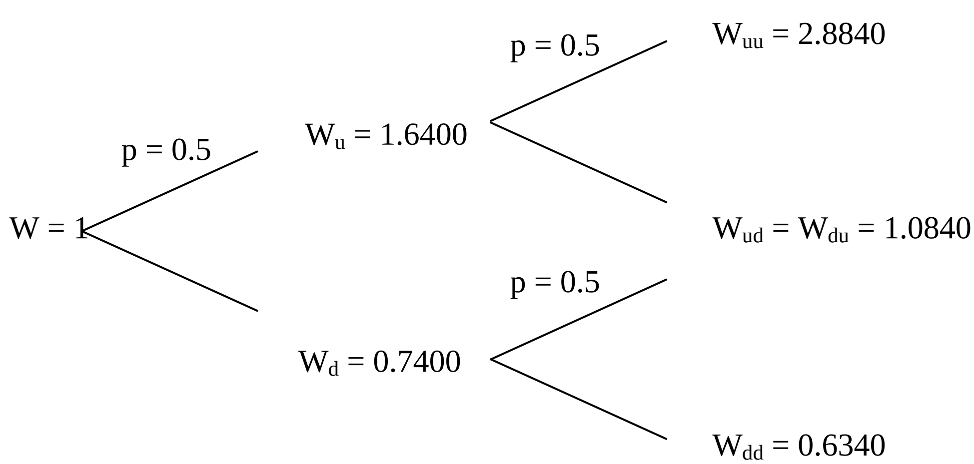


Figure 9

Panel C: Wealth of the Rebalanced Strategy

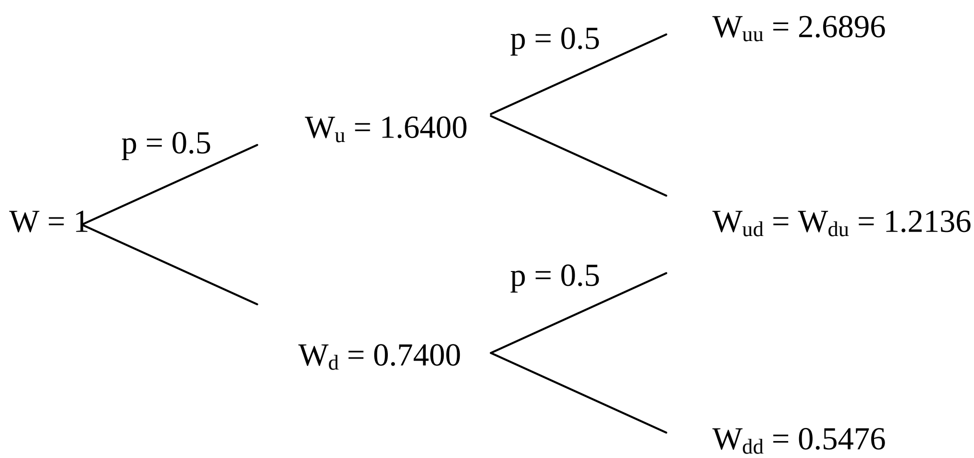


Figure 9

Panel D