Risks, Returns, and Optimal Holdings of Private Equity:

A Survey of Existing Approaches*

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July 2012

* We are grateful for the comments from the editor (Fernando Zapatero) and Ludovic Phalippou. Portions of this paper were presented at the 2011 CFA Institute Conference in Chicago. Ang and Morten acknowledge funding from Netspar and the Program for Financial Studies. Corresponding author: Morten Sorensen, email: ms3814@columbia.edu

Abstract

We survey the academic literature that examines the risks and returns of private equity (PE) investments, optimal PE allocation, and compensation contracts for PE firms. The irregular nature and limited data of PE investments complicate the estimation and interpretation of standard risk and return measures. These complications have led to substantial disparity in performance estimates reported across studies. Moreover, studies suggest that the illiquidity and transaction costs inherent in PE investments have substantial implications for optimal holdings of these assets. While incentive fees in PE address moral hazard and information agency problems, total fees in PE investments are large and incentive fees account for a minority of total compensation.

I. Introduction

Private equity (PE) investments are investments in privately held companies, which trade directly between investors instead of via organized exchanges. The investments are typically made through a PE fund organized as a limited partnership, with the institutional investors (typically pension funds or university endowments) as limited partners (LPs) and the PE firm itself (such as Blackstone or KKR) acting as the general partner (GP). The GP manages the PE fund's acquisitions of individual companies (called portfolio companies). Depending on the type of portfolio companies, PE funds are typically classified as buyout, venture capital (VC), or some other type of fund specializing in other illiquid non-listed equity-like investments. Buyout funds invest in mature established companies, using substantial amounts of leverage to finance the transactions. VC funds invest in high-growth start-ups, using little or no leverage. Finally, it is not uncommon for LPs to also invest directly in individual companies. These investments are often structured as co-investments into the portfolio companies, alongside the investments made through the PE fund.

PE is often considered a distinct asset class, and it differs from investments in public equity in fundamental ways. There is no active market for PE positions, making these investments illiquid and difficult to value. The investments are for the long term. PE funds typically have horizons of 10-13 years, during which the invested capital cannot be redeemed. Moreover, partnership agreements specifying the governance of funds are complex, specifying the GP's compensation as a combination of ongoing fees (management fees), a profit share (carried interest), transaction fees, and other fees.

This article surveys the academic research concerning the risks and returns of PE investments, as well as the optimal holdings of PE in an investment portfolio. It also contains a review of PE contracts. It should be noted that researchers have had limited access to information about the nature and performance of PE investments, and research in this area is preliminary and often inconclusive. Research into many important aspects of these investments, such as the performance of PE during the recent recession, the secondary market for LP positions, and co-investments by LPs, has only recently begun. Moreover, our survey only covers studies of PE defined as companies owned by PE funds. We do not consider the substantial number of privately-held and independently-owned companies, ranging from independent grocery stores and dry cleaners to large family-owned businesses (see Moskowitz and Vissing-Jorgensen, 2002; Kartashova, 2011; and Faccio, Marchica, McConnell, and Mura, 2012).

Section II introduces two problems that researchers have encountered in measuring PE risk and returns. The first of these is the statistical problem that arises because PE returns are observed infrequently, typically with well-performing funds being overrepresented in the data. This makes it difficult to estimate standard measures of risk and return, such as CAPM alphas and betas. The second problem is interpreting the resulting estimates. Standard asset-pricing models are established under assumptions that are appropriate for traditional financial markets, with transparent, liquid, and low-friction transactions. These assumptions are problematic for PE investments, and the estimated alphas and betas may need to be adjusted to provide meaningful measures of risk and return in the PE context. One way of interpreting the risks and returns of PE investments, especially illiquidity risk, is for an investor to consider PE from an investor-specific asset allocation perspective.

Section III summarizes the literature on the optimal allocation to PE in portfolios consisting of liquid public equity and illiquid PE. A new generation of asset allocation models considers these issues, since the first generation of asset allocation approaches assumed that assets can be rebalanced without cost at any time. The literature on asset allocation incorporating transaction costs (which are very high for PE investments) and search frictions (due to counterparties often being hard to find for the transfer of PE investments) leads to recommendations that optimal holdings of illiquid PE assets not be large.

In Section IV, we survey the literature on agency issues and PE contracts, with special emphasis on fees and the lack of transparency. Most PE investments are made through intermediaries. Current PE investment vehicles cannot disentangle factor returns that are unique to the PE asset class from manager skill. Furthermore, commonly-used contracts may exacerbate rather than alleviate agency issues.

II. Estimating Private Equity Risk and Return

II A. Defining Risk and Returns

To establish notation and terminology, it is useful to begin with the standard model for risk and return. For traded financial assets, risk and return are usually measured in the context of the CAPM as the alpha and beta coefficients estimated in the one-factor linear regression (the *expected return regression*),[1]

$$R_i(t) - R_f(t) = \alpha + \beta\left[R_m(t) - R_f(t)\right] + \epsilon_i.$$

In this equation, $R_i(t)$ is the return earned by the investor from period t-1 to period t, $R_f(t)$ is the risk-free rate over the period from t-1 to t, and $R_m(t)$ is the return on the market portfolio. The return earned on a financial asset from time t-1 to t is defined as:

$$R(t) = \frac{P(t) + CF(t)}{P(t-1)} - 1$$

where $CF(t)$ is the cash flow paid out at time t and $P(t)$ is the market price quoted at time t, immediately after payment of the cash flow. For traded assets, the expected return regression is straightforward to estimate, namely by regressing (for example, on a weekly basis) the asset's observed returns on the corresponding market returns over the same periods.

Under appropriate assumptions about investors' preferences [e.g., constant relative risk aversion (CRRA) or mean-variance utility], along with assumptions about the market environment [e.g., the absence of transaction costs, short-sales constraints, and the ability of investors to continuously trade and rebalance their portfolios], the CAPM specifies that each asset's expected return is determined by the expected return regression with an alpha equal to zero. This important result has several implications. First, it implies that beta is the appropriate measure of risk as it measures the correlation between the return on the asset and the return on the overall

[1] This specification assumes that alpha and beta are constant over the duration of the deal. While it would be interesting to investigate the term structure of the risk and return, the data limitations and other complications described here have prevented empirical studies of these dynamics. There is substantial evidence that alphas and betas vary over time for listed equity, as Ang and Kristensen (2012) show.

market (*systematic risk*). In the CAPM, systematic risk is the only risk that is priced; idiosyncratic risk is not priced because it can be diversified. Second, the expected return regression implies that an asset's expected return increases linearly in beta. Finally, it implies that in equilibrium, alpha should be zero. A positive alpha can be interpreted as an abnormal positive return.

Following this logic, the standard approach to evaluating the risks and returns of financial assets proceeds in two steps. First, alpha and beta are estimated using the expected return regression. Second, invoking the CAPM, the estimated alpha is interpreted as an abnormal risk-adjusted return, and the beta is interpreted as the systematic risk.

For PE investments, problems arise at both steps. At the first step, privately-held companies, by definition, do not have regularly observed market values, and the returns earned from investing in these companies are only observed at exit. Hence, period-by-period returns are unavailable, making it difficult to estimate the expected return regression directly. Better-performing privately-held companies may also be overrepresented in the data, creating sample selection problems that would cause the alpha coefficient to be overestimated and the beta coefficient to be underestimated. At the second step, after estimating alpha and beta, it is unclear whether or not these coefficients appropriately measure risks and returns. The assumptions of liquid and transparent markets underlying the CAPM are far from the realities of PE investing. To reflect the actual risks and returns facing LP investors, the estimated parameters may require various adjustments to account for the cost of illiquidity, idiosyncratic risk, persistence, funding risk, etc.

The lack of regularly quoted market prices and returns presents a fundamental challenge for empirical studies of the risk and return of PE investments. Alternative approaches have either used company-level performance data or fund-level data with the cash flow streams between the LPs and GPs. The benefits and drawbacks of these approaches are discussed next.

II B. Estimates Using Company-level Data

Company-level data contain information about investments by buyout or VC funds in individual companies. For these investments, the data typically contain the name of the company, the invested amount, the investment date, the exit date, and the exit amount. Such data are confidential and proprietary, and researchers have mostly obtained data through direct contact with LPs and professional data providers.

Franzoni, Nowak, and Phalippou (2012) analyze company-level data for buyout investments. Cochrane (2005) and Korteweg and Sorensen (2010) use company-level data for VC investments in start-ups. The application to VC investing is more challenging, because the sample selection problem is particularly severe for these investments.

Compared to fund-level data, company-level data have two advantages. First, there are many more companies than funds, which improves the statistical power of the analysis. Companies can be classified in terms of industries and types, allowing for a more nuanced differentiation of the risks and returns across industries and types and over time. Second, investments in individual companies have well-defined returns. Absent intermediate cash flows, the return as defined above can be calculated directly from the initial investment and the distribution of the proceeds at exit. As long as intermediate cash flows are few and small, as for buyout investments, this calculation provides a reasonable return measure. With more intermediate cash flows, such as for VC investments, the calculation may be performed separately for each investment round.

A disadvantage of company-level data is that the return figures typically do not exclude management fees and carried interest paid by the LPs to the GPs. Hence, the estimated risks and returns reflect the total risks and returns of the investments (before fees), not those earned by an LP (net of fees). Translating between net-of-fee and before-fee returns typically requires additional assumptions and numerical simulations [see Metrick and Yasuda (2010) and Franzoni, Nowak, and Phalippou (2012) for two approaches].

Continuous-Time Specifications. A technical disadvantage of company-level data is that the returns are measured over different periods. Returns are measured from the time of the initial investment to the time of exit, and the duration varies substantially across investments. The standard (discrete-time) CAPM is a one-period model, where the period may be a day, a month, or a quarter. This model does not compound, however, and the returns must all be calculated over periods of the same length.

A standard solution is to use a continuous-time version of the CAPM. This version compounds, which allows for a comparison of the risks and returns of investments of different durations. Campbell, Lo, and MacKinlay (1997) provide an extensive discussion of the underpinnings of this model. In the continuous-time CAPM, the expected return regression is restated in log-returns (continuously-compounded returns) as:

$$\ln[1 + R_i(t)] - \ln\left[1 + R_f(t)\right] = \delta + \beta\left(\ln[1 + R_m(t)] - \ln\left[1 + R_f(t)\right]\right) + \epsilon_i$$

One complication with the continuous-time CAPM is that the estimated intercept of the expected return equation cannot be interpreted as an abnormal return, as in the standard discrete-time CAPM. Under specific distributional assumptions about the way volatility increases with the duration of an investment, the abnormal returns can be calculated using the following adjustment

$$\alpha = \delta + \frac{1}{2}\sigma^2.$$

This non-linear adjustment leads to high alphas when the volatility is high [see Cochrane (2005) and Korteweg and Sorensen (2011) for details about the derivation and implementation of the adjustment]. For example, Cochrane (2005) reports an annual volatility around 90%, resulting in an estimated alpha of 32% annually. This appears unreasonably high compared to studies using fund-level data, raising doubts about the appropriateness of the assumptions about the growth of volatility with the duration of the investments.

Franzoni, Nowak, and Phalippou (2012) sidestep this problem by estimating the CAPM after forming portfolios of deals, rather than individual deals. This substantially lowers volatility and reduces the magnitude of the adjustment. It does, however, reduce the other advantages of using individual deals. It reduces statistical power and the analysis must use a modified IRR approximation of returns.

Selection Bias. Another problem with company-level data is sample selection. To illustrate, VC investments are structured over multiple financing rounds, and better-performing companies tend to raise more such rounds. Hence, data sets with valuations of individual VC rounds are dominated by these better-performing companies. Moreover, failing startup companies are usually not formally liquidated, and are often left as shell companies without economic value ("zombies"). This introduces another selection problem. When observing old companies without new financing rounds or exits, these companies may be alive and well or they may be zombies, in which case it is unclear when the write-off of the company's value should be recorded. This latter problem is less severe for buyout investments, because they mostly result in a well-defined exit (acquisition or IPO) or a well-defined liquidation.

The selection problem is illustrated in Figure 1 [from Korteweg and Sorensen (2010)]. The universe of returns is illustrated by all the dots. The data, however, only contain the observed good returns above the x-axis (in black). Worse returns (shaded gray) are unobserved. Since only the black dots are observed, a simple estimation of the expected return regression gives an estimate of alpha that is biased upwards, an estimate of beta that is biased downwards, and a total volatility that is too low. Hence, an analysis that does not correct for these biases will be overly optimistic about the risk and return performance of these investments.

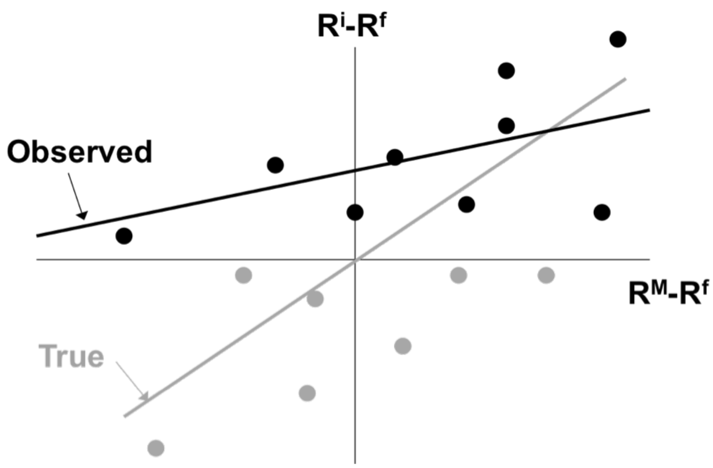

Figure 1: Illustration of Selection Bias

The statistical methodology for addressing such selection biases was introduced by Heckman (1979). Cochrane (2005) estimates the first dynamic selection model using VC data and finds that the effect of selection bias is indeed large. The author finds that the selection correction reduces the intercept of the log-market model, denoted δ above, from 92% to -7.1%. Cochrane also highlights the difficulty of translating this intercept into an abnormal return. Korteweg and Sorensen (2010) estimate an extended version of Cochrane's model. They also find that selection bias overstates the risk-return tradeoff of VC investments. Without selection bias, the estimate of the intercept, δ, is -19% annually; selection bias reduces this estimate to -68% (note again, these intercepts cannot be interpreted as returns).

In the continuous-time CAPM, the estimated beta coefficient can be interpreted as systematic risk, without adjustments. Cochrane (2005) finds a slope of 0.6-1.9 for the systematic risk, although this seems low. It includes estimates at the individual industry levels of, for example, -0.1 for retail investments.

Korteweg and Sorensen (2010) report substantially higher beta estimates of 2.6-2.8 in the continuous-time CAPM, which may be more reasonable for young startups funded by VC investors. They also find substantial time variation as VC investing has matured. They estimate alphas over the 1987-1993, 1994-2000, and 2001-2005 periods, and find that the alphas in the early period were positive but modest, the alphas in the late 1990s were very high, but the alphas in the 2000s were negative, consistent with patterns found by studies using fund-level data.

II C. Estimates Using Fund-level Data

Fund-level data are typically obtained from LPs with investments across many PE funds. Each observation represents the performance of the entire portfolio of companies held by a fund. In addition to information about the fund, such as its type and vintage year, these data may contain the cash flow stream between the LP and the fund or a performance measure calculated from this cash flow stream [such as the Internal Rate of Return (IRR), Total Value to Paid-in Capital Multiple (TVPI), and Public Market Equivalent (PME), described below). When individual cash flows are available, however, they are typically not tied to individual portfolio companies.

There are several advantages to fund-level data. First, fund-level data reflect actual LP returns, net of fees, resulting in estimates of the risks and returns actually realized by the LPs. The sample selection problem is smaller, since the performance of companies that ultimately never produce any returns for the investing funds (zombies) is eventually reflected in the fund-level cash flows. Other sample selection problems may arise, however. Fund-level performance is typically self-reported, and better-performing funds may be more likely to report their performance [as suggested by Phalippou and Gottschalg (2009), although Stucke (2011) argues that returns reported by Venture Economics, a commercial data provider, understate actual performance)]. Still, these selection problems are likely smaller than the problems that arise with company-level data. Finally, since funds have similar lifetimes (typically ten years), the expected return equation can be estimated directly, avoiding the problems with the continuous-time log-return specification used for company-level data.

Fund-level Performance Measures. The main disadvantage of fund-level data is accurately measuring the "return." Calculating period-by-period returns, as previously defined, requires assessing the market value of the PE investment [$P(t)$ in the return calculation] at regular intervals. Absent quoted market values, however, this calculation is difficult, and the reported net asset values (NAVs) are noisy substitutes for these market values (for example, it has been customary to value a company at cost until it experiences a material change in the circumstances, which does not capture smaller ongoing changes in its growth prospects or market value). Given the absence of regularly quoted returns, several alternative measures have been proposed. However, none of these measures is a return, as previously defined, and their relationships to asset pricing models are somewhat tenuous.

Internal Rate of Return (IRR). A natural starting point is to interpret the internal rate of return (IRR) of the cash flows between the LP and GP as a return earned over the life of the fund. Denoting the cash flow at time t as $CF(t)$, and separating those into the capital calls paid by the LP to the GP, denoted $Call(t)$, and the distributions of capital from the GP back to the LP, denoted $Dist(t)$, the IRR is defined as the solution to the following equation,

$$PV = \sum \frac{CF(t)}{(1+IRR)^t} = \sum \frac{Dist(t) - Call(t)}{(1+IRR)^t} = 0$$

$$\Rightarrow \frac{\left(\sum \frac{Dist(t)}{(1+IRR)^t}\right)}{\left(\sum \frac{Call(t)}{(1+IRR)^t}\right)} = 1$$

Early studies, focusing on VC investments, are Bygrave and Timmons (1992) who find an average IRR of 13.5% over 1974-1989, and Gompers and Lerner (1997) who use investments of a single VC firm to report an IRR of 30.5% over 1972-1997.

Ljungqvist and Richardson (2003) investigate cash flow data from a large LP investing in funds raised in 1981-1993 (19 VC funds and 54 buyout funds). They report average fund IRRs (net of fees), combining PE and VC investments, for 1981-1993, of 19.81%, while the average S&P 500 return is 14.1%, suggesting that PE investments outperform the market.

Kaplan and Schoar (2005) use fund-level quarterly performance measures from Venture Economics that cover 1,090 VC and buyout funds, of which 746 funds were fully or mostly liquidated at the time of the study. They find VC and buyout fund generate returns that are slightly below those of the S&P 500 Index on an equal-weighted basis. Value-weighted, VC funds perform slightly better than the index.The value-weighted IRR is 13%.[2] Extending the sample to mature, but not liquidated funds, raises the IRR for VC funds to 30% but leaves it unchanged at 13% for buyout funds, resulting in an overall average IRR of 18%.[3]

A recent survey by Harris, Jenkinson, and Kaplan (2011) summarizes the academic studies using fund-level data from various data providers.[4] For buyout funds, they report weighted average IRRs of 12.3%-16.9%. For VC funds, the weighted average IRRs are 11.7%-19.3%. The performance of buyout funds has been stable over time, with weighted average IRRs of 15.1%-22.0% in the 1980s, 11.8%-19.3% in the 1990s, and 5.8%-12.8% in the 2000s. VC fund performance has been more volatile, with weighted average IRRs ranging from 8.6% to 18.7% in the 1980s, 22.9% to 38.6% in the 1990s, and -4.9% to 1.6% in the 2000s.

Overall these figures reveal substantial variation in IRRs across studies and data sources. Moreover, the IRR is a problematic measure of economic performance. It is an absolute performance measure that does not calculate performance relative to a benchmark or market return. Moreover, the IRR calculation implicitly assumes that invested and returned capital can

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[2] Phalippou and Gottschalg (2009) point out that value-weighting private equity funds is difficult. One possibility is to weight by total committed capital, but funds vary in their investment speed, and poorer-performing funds may invest more slowly, introducing a downward bias in value-weighted performance estimates.

[3] The final reported NAV of funds that are not fully liquidated is treated as a final cash flow in the calculation. Phalippou and Gottschalg (2009) argue that interim NAVs may exaggerate the actual values, leading to upward-biased performance estimates. In contrast, Stucke (2011) argues that the NAVs are substantially below actual economic value, using Venture Economics data. Kaplan and Schoar (2005) and Harris, Jenkinson, and Kaplan (2011) use reported NAVs as stated.

[4] These studies include Ljungqvist and Richardson (2003), Kaplan and Schoar (2005), Phalippou and Gottschalg (2008), and Robinson and Sensoy (2011).

be reinvested at the IRR rate. If a fund makes an early small investment with a large quick return, the investment can largely define the IRR for the entire fund, regardless of the performance of subsequent investments. Indeed, Phalippou (2011) suggests that GPs may actively manage their investments to inflate fund IRRs.

Total Value to Paid-in Capital Multiple (TVPI). An alternative performance measure that is less susceptible to manipulation than the IRR is the total value to paid-in capital (TVPI) multiple. This multiple is calculated as the total amount of capital returned to the LP investors (net of fees) divided by the total amount invested (including fees). Formally, the TVPI multiple is defined as:

$$TVPI = \frac{\sum Dist(t)}{\sum Call(t)}$$

This calculation is performed without adjusting for the time value of money. While the IRR is calculated under the implicit assumption that capital can be reinvested at the IRR rate, the TVPI multiple is calculated under the implicit assumption that capital can be reinvested at a zero rate. Harris, Jenkinson, and Kaplan (2011) report weighted average TVPIs of 1.76-2.30 for buyout funds and 2.19-2.46 for VC funds. This multiple varies substantially over time, however. For buyout funds, they report a multiple of 2.72-4.05 for the 1980s, 1.61-2.07 for the 1990s, and 1.29-1.51 for the 2000s. For VC funds, they report a multiple of 2.31-2.58 for the 1980s, 3.13-3.38 for the 1990s, and 1.06-1.09 for the 2000s.

Public Market Equivalent (PME). Both the IRR and TVPI measures are absolute performance measures. The public market equivalent measure (PME) is used to evaluate performance relative to the market. It is calculated as the ratio of the discounted value of the LP's inflows divided by the discounted value of outflows, with the discounting performed using realized market returns:

$$PME = \frac{\left(\sum \frac{Dist(t)}{\prod\left(1 + R_m(t)\right)}\right)}{\left(\sum \frac{Call(t)}{\prod\left(1 + R_m(t)\right)}\right)}$$

Kaplan and Schoar (2005) argue that a PME greater than one is equivalent to a positive economic return for the LPs when PE investments have the same risk as the general market (a beta equal to one). This assumption may be inappropriate when the risk of the distributions (the numerator in the PME) is greater than the risk of the capital calls (including management fees, which are largely a risk-free liability). Using a lower discount rate for capital calls would inflate the denominator and reduce the PME. Hence, more carefully accounting for different risks would suggest that the PME may have to exceed one by some margin before LPs earn a positive economic return. Sorensen, Wang, and Yang (2012) evaluate this margin, and suggest that a PME around 1.3 is required for the LP to break even when investing in buyout funds.[5]

Kaplan and Schoar (2005) find average equal-weighted PMEs of 0.96. Value-weighted, the PME for VC funds is 1.21 and the PME for buyout funds is 0.93. Phalippou and Gottschalg (2009) use data for 852 funds to calculate a PME of 1.01 (they call this measure the profitability index). The PME decreases to 0.88 after various adjustments.

Comparing different studies and data sources, Harris, Jenkinson, and Kaplan (2011) report weighted average PMEs of 1.16-1.27 for buyout funds and 1.02-1.45 for VC funds. For buyout funds, the PMEs were for varied from 1.03-1.11 in the 1980s to 1.17-1.34 in the 1990s and 1.25-1.29 in the 2000s. For VC funds, they report PMEs of 0.90-1.08 in the 1980s, 1.99-2.12 in the 1990s, and 0.84-0.95 in the 2000s. The 1990s was the VC decade, and the 2000s was the buyout decade.

Risk Measures. Fund-level data are poorly suited for estimating the risk of PE investing. Thus few, if any, academic studies attempt to use fund-level data. Instead, Ljungqvist and Richardson (2003) estimate risk by assigning each portfolio company to one of 48 broad industry groups and use the corresponding average beta for publicly traded companies in the same industry. They report a beta of 1.08 for buyout investments and 1.12 for VC investments. That these betas do not adjust for the high leverage used in buyout investments. Adjusting for systematic risk, they find a 5%-6% premium, which they interpret as the illiquidity premium of VC investments.

[5] Additionally, as a technical point, the CAPM prescribes that the discounting should be performed using expected returns, not realized returns as in the PME. Using the realized returns distorts the calculation (according to Jensen's inequality). The magnitude of this distortion is unclear, but most likely modest.

Kaplan and Schoar (2005) state that they “believe it is possible that the systematic risk of LBO funds exceeds 1 because these funds invest in highly levered companies.” They regress IRRs on S&P 500 returns and find a coefficient of 1.23 for VC funds and 0.41 for buyout funds. A levered beta of 0.41 seems unreasonably low.

Persistence and Predictability. Kaplan and Schoar (2005), Phalippou and Gottschalg (2009), Hochberg, Ljungqvist, and Vissing-Jorgensen (2010), along with other studies find evidence of performance persistence for PE funds. The performance of an early fund predicts the performance of subsequent funds managed by the same GP. This persistence is interpreted as evidence that GPs vary in their skills and abilities to pick investments and manage the portfolio companies. Estimates suggest that a performance increase of 1.0% for a fund is associated with around 0.5% greater performance for the GP’s next fund, measured either in terms of PME or IRR. For more distant funds, persistence declines.

Due to data limitations, studies that document the predictability in PE returns conduct statistical in-sample analysis, rather than out-of-sample analysis. In Kaplan and Schoar (2005), for example, PE funds in the “top quartile” do well, but these funds are identified ex post. Within a fund family, funds typically have lifetimes of 10-13 years but overlap to some extent. In-sample analysis uses the ultimate performance of a previous fund to predict the performance of a subsequent fund, even if this fund is raised before the ultimate performance of the previous fund is fully realized. The studies employ various robustness checks, such as using intermediate NAVs instead of ultimate performance or using the performance of funds several generations ago, to predict future performance to mitigate this concern. Still, some recent research, such as Hochberg, Ljungqvist, and Vissing-Jorgensen (2010), find weaker evidence of persistence using only information available when the new fund is raised.

II D. Summary of Empirical Evidence

Based on the existing evidence, it seems too early to definitely assess the risk and return of PE and evaluate how the performance of these investments compare to the risk of investing in publicly traded equities, even in terms of the most basic metrics.

Measuring PE risk and returns is difficult because of the infrequent observations of fund or company values and selection bias. Studies using company-level data that account for selection bias find high alphas. These alpha estimates are hard to interpret in terms of arithmetic returns, however, because of the high volatility. Estimates of betas vary substantially, ranging as high as 2.8 for VC investments, but generally even PE betas appear to be well above one. Studies using fund-level data have fewer selection problems. Yet, these studies still suffer from the fact that no direct PE returns are observed, and unlike standard return measures, fund-level IRR, TVPI, and PME measures can be misleading and should be interpreted with caution. In terms of raw performance, in the words of Harris, Jenkinson, and Kaplan (2011), "it seems likely that buyout funds have outperformed public markets in the 1980s, 1990s, and 2000s." However, due to the uncertainty about the risk of PE investments, it is not yet possible to say whether this outperformance is sufficient to compensate investors for their risk and whether the investments outperform on a risk-adjusted basis. Finally, there is evidence of persistence of PE fund returns and some, albeit weaker and less consistent, evidence that characteristics like fund size and past capital raisings predict PE fund returns.

III. Asset Allocations to Private Equity

Having discussed the measurement of PE returns, we now consider optimal allocations to PE. This requires, of course, a suitable risk-return tradeoff for PE investments, as well as correlation of PE returns with other assets in the investor's opportunity set. As we point out in Section II, measuring these inputs for PE for use in an optimization problem requires special considerations. We take as given these inputs and focus on the illiquidity risk of PE and how to incorporate it into an optimal asset allocation framework. There have been several approaches to handling illiquidity risk in asset allocation, all of which have relevance. To put into context these contributions, we start with the case of asset allocation without frictions.

III A. Frictionless Asset Allocation

The seminal contributions of Merton (1969, 1971) characterize the optimal asset allocation of an investor with constant relative risk aversion (CRRA) utility investing in a risk-free asset (with constant risk-free rate) and a set of risky assets. The constant relative risk aversion utility function with risk aversion γ is given by:

$$U(W) = \frac{W^{1-\gamma}}{1-\gamma}.$$

The constant relative risk aversion utility is homogeneous of degree one, which means that exactly the same portfolio weights arise whether $10 million of wealth is being managed or $1 billion. This makes the constant relative risk aversion utility function ideal for institutional asset management.

Assume the risky assets are jointly log-normally distributed. Under the case of independent and identically distributed (*i.i.d.*) returns, the vector of optimal holdings, w, of the risky assets is given by:

$$w = \frac{1}{\gamma}\Sigma^{-1}(\mu - r_f),$$

where Σ is the covariance matrix of the risky asset returns, μ is the vector of expected returns of the risky assets, and r_f is the risk-free rate. This is also the portfolio held by an investor with mean-variance utility optimizing over a discrete, one-period horizon.

There are two key features of this solution that bear further comment. First, the Merton model is dynamic and involves continuous rebalancing. That is, although the portfolio weights, w, are constant, the investor's optimal policy is always to continuously sell assets that have risen in value and to buy assets that have fallen in value in such a way as to maintain constant weights. Clearly, the discrete nature of PE investment and the inability to trade frequently mean that allocations to PE should not be evaluated with the standard Merton model.

Second, any other portfolio held by the investor other than the optimal portfolio results in lower utility. Thus, an investor holding a non-optimal portfolio needs to be compensated, as she can improve her utility by moving the optimal portfolio. The cost of holding a non-optimal portfolio is called the utility certainty equivalent, and it is dependent on the investor's risk preferences and investment horizon. Formally, the certainty equivalent cost is how much an investor must be compensated in dollars per initial wealth to take a non-optimal strategy, but have the same utility as with the optimal strategy. Some particularly relevant costs, which the subsequent literature explores, are how much an investor should be compensated for the inability to trade assets like PE for certain periods of time or how much to be compensated for being forced to pay a cost whenever an asset is traded.

III B. Asset Allocation with Transactions Costs

Investing in PE incurs large transactions costs in finding an appropriate PE manager and conducting appropriate due diligence. Then, there are potentially large discounts to the recorded asset values that may be taken in transferring ownership of a PE stake in illiquid secondary markets. Since Constantinides (1986), a large literature has extended the Merton model to incorporate transactions costs.

Constantinides (1986) considers the case of one risk-free and one risky asset. When there are proportional transaction costs, so that whenever the holdings of the risky asset increase (or decrease) by v, the holding of the riskless asset decreases by $(1+k)v$. When there are trading costs, the investor trades infrequently. Constantinides shows that the optimal trading strategy is to trade whenever the risky asset position hits upper and lower bounds, $\underline{w}$ and $\overline{w}$, respectively. These bounds straddle the optimal Merton model where there are no frictions. The holdings of risky to risk-free assets, y/x, satisfy:

$$\underline{w} \leq \frac{y}{x} \leq \overline{w},$$

so that when y/x lies within the interval $[\underline{w}, \overline{w}]$, there is no trade and when y/x hits the boundaries on either side, the investor buys and sells appropriate amounts of the risky asset to bring the portfolio back to the optimum Merton model.

The no-trade interval, $\bar{w} - \underline{w}$,, increases with the transactions costs, k, and the volatility of the risky asset. The transactions costs to sell PE portfolios in secondary markets can be extremely steep. When the Harvard endowment tried to sell its PE investments in 2008, potential buyers were requiring discounts to book value of more than 50%.[6] Even for transactions costs of 10%, Constantinides (1986) computes no-trade intervals greater than 0.25 around an optimal holding of 0.26 for a risky asset with a volatility of 35% per annum. Thus, PE investors should expect to rebalance PE holdings very infrequently.

The certainty equivalent cost to holding a risky asset with large transaction costs is small for modest transaction costs (approximately 0.2% for proportional transaction costs of 1%), but can be substantial for large transaction costs, which is the more relevant range for PE investments. For transaction costs of 15% or more, the required premium to bring the investor to the same level of utility as the frictionless Merton model is more than 5% per annum.

The literature has extended this framework to multiple assets (for example, Liu, 2004) and different types of rebalancing bands. Leland (1996) and Donohue and Yip (2003) suggest rebalancing to the edge of a band rather than to a target within a band. Others, like Pliska and Suzuki (2004) and Brown, Ozik, and Scholtz (2007), advocate extensions to two sets of bands, where different forms of trading are done at the inner band with more drastic rebalancing done at the outer band. In all these extensions, the intuition is the same: PE investments should be expected to be rebalanced very infrequently, and the rebalancing bands will be very wide. The case of transaction costs when returns are predictable is considered by Garleanu and Pedersen (2010). A related study is Longstaff (2001), who allows investors to trade continuously, but only with bounded variation so there are upper and lower bounds on the number of shares that can be traded every period. This makes Longstaff's model similar to a time-varying transactions cost.

A major shortcoming of this literature is that it assumes that trade in assets is always possible, albeit at a cost, which is not true for PE; over a short horizon, there may be no opportunity to find a buyer and even if a buyer is found, there is not enough time, relative to the investor's desired short horizon, to raise capital to go through legal and accounting procedures to transfer

[6] See "Liquidating Harvard" Columbia CaseWorks ID#100312, 2010.

ownership. An important friction for PE investors in secondary markets is the search process in finding an appropriate buyer. There may be no opportunity to trade, even if desired, at considerable discounts. This case is what the next literature considers.

III C. Asset Allocation with Search Frictions

As PE investments do not trade on a centralized exchange, an important part of rebalancing a PE portfolio is finding a counterparty in over-the-counter markets. Or, if money is spun off from existing PE investments, new or existing PE funds must be found to invest in. This entails a search process, incurring opportunity and search costs, as well as a bargaining process, which reflects investors' needs for immediate trade. The former requires a trading process that captures the discrete nature of trading opportunities. The latter is captured by a transaction cost, as modeled in the previous section.

Since Diamond (1982), search-based frictions have been modeled by Poisson arrival processes. Agents find counterparties with an intensity λ, and conditional on the arrival of the Poisson process, agents can trade and rebalance. This produces intervals where no rebalancing is possible for illiquid assets and the times when rebalancing are possible are stochastic. This notion of illiquidity is that there are times where it is not possible to trade, at any price, an illiquid asset. These particular types of stochastic rebalancing opportunities are attractive for modeling PE in another way: the exit in PE vehicles is often uncertain. Although a PE vehicle may have a stated horizon, say of 10 years, the return of cash from the underlying deals may cause large amounts of capital to be returned before the stated horizon, or in many cases the horizon is extended to maximize the profitability of the underlying investments (or to maximize the collection of fees by GPs).

A number of authors have used this search technology to consider the impact of illiquidity (search) frictions in various over-the-counter markets, such as Duffie, Garleanu, and Pedersen (2005, 2007). While these are important advances for showing the effect of illiquidity risk on asset prices, they are less useful for deriving asset allocation advice on optimal PE holdings. Duffie, Garleanu, and Pedersen (2005, 2007) consider only risk-neutral and CARA utility cases and restrict asset holdings to be 0 or 1. Garleanu (2009) and Lagos and Rocheteau (2009) allow for unrestricted portfolio choice, but Garleanu considers only CARA utility and Lagos and Rocheteau focus on showing the existence of equilibrium with search frictions rather than on any practical calibrations. Neither study considers asset allocation with both liquid and illiquid assets.

III D. Asset Allocation with Stochastic Non-Traded Periods

Ang, Papanikolaou, and Westerfield (2011) [APW] solve an asset allocation problem with liquid securities, corresponding to traded equity markets that can be traded at any time, and illiquid securities, which can be interpreted as a PE portfolio. The investor has CRRA utility with an infinite horizon and can only trade the illiquid security when a liquidity event occurs, which is the arrival of a Poisson process with intensity λ. In this framework, the Merton model with continuous rebalancing is given by $\lambda \to \infty$. As λ decreases to zero, the opportunities to rebalance the illiquid asset become more and more infrequent. The mean time between rebalancing opportunities is $1/\lambda$. Thus, λ indexes a range of illiquidity outcomes.

The inability to trade for stochastic periods introduces a new source of risk that the investor cannot hedge. This illiquidity risk induces large effects on optimal allocation relative to the Merton model. APW show that illiquidity risk affects the mix of liquid and illiquid securities even when the liquid and illiquid returns are uncorrelated and the investor has log utility.

The most important result that APW derive is that the presence of illiquidity risk induces time-varying, endogenous risk aversion. The intuition is that there are two levels of wealth that are relevant for the investor: (1) total wealth, which is the same effect as the standard Merton problem where the risk is that if total wealth goes to zero, the agent cannot consume, and (2) liquid wealth. The agent can only consume liquid wealth. Thus, with illiquid and liquid assets, the investor also cares about the risk of liquid wealth going to zero. This can be interpreted as a solvency condition: an agent could be wealthy but if this wealth is tied up all in illiquid assets, the agent cannot consume. Although the CRRA agent has constant relative risk aversion, the effective risk aversion—the local curvature of how the agent trades off liquid and illiquid risk in the portfolio—is affected by the solvency ratio of the ratio of liquid to illiquid wealth. This solvency ratio also becomes a state variable that determines optimal asset allocation and consumption. This illiquidity risk causes the optimal holdings of even the liquid asset to be lower than the optimal holding of liquid assets in a pure Merton setting.

APW derive five findings that are important considerations for investing in PE:

1. Illiquidity risk induces marked reductions in the optimal holdings of assets compared to the Merton model. Their calculations for the same risk aversion as a 60% risky asset holding (and 40% risk-free holding) in the Merton model, introducing an average rebalancing period of once a year, reduces the risky asset holding to 37%. When the average rebalancing period is once every five years, the optimal allocation is just 11%. Thus, PE, which is highly illiquid, should be held in modest amounts.

2. In the presence of infrequent trading, the fraction of wealth held in the illiquid asset can vary substantially and is very right skewed. That is, suppose the optimal holding to illiquid assets is 0.2 when rebalancing can take place. Then the investor should expect the range of illiquid holdings to vary from 0.15 to 0.35 during non-rebalancing periods. Because of the skew, the average holdings to the illiquid asset will be higher than the optimal rebalancing point, at say 0.25. Thus, when an illiquid PE portfolio is rebalanced, the optimal rebalancing point is much lower than for an average holding.

3. The consumption policy (or payout policy) with illiquid assets must be lower than the Merton payout policy with only liquid assets. Intuitively, holding illiquid assets means that there is additional solvency risk that liquid wealth goes to zero and consumption cannot be funded. Thus, payouts of funds holding illiquid assets should be lower than the case when these assets are all fully traded.

4. The presence of illiquidity risk means that an investor will not fully take advantage of opportunities that might look like close to an "arbitrage." For example, where correlations to the liquid and illiquid returns are nearly plus or minus one. Traditional mean-variance optimizers without constraints would produce weights close to plus or minus infinity in these two assets. This does not happen when one asset is illiquid because taking advantage of this apparent arbitrage involves a strategy that causes the investor's liquid wealth to drop to zero with positive probability. Thus, near-arbitrage conditions when there is illiquidity risk are not exploited like in the Merton model.

5. Finally, the certainty equivalent reward required for bearing illiquidity risk is large. They report that when the liquid and illiquid returns are poorly correlated and the illiquid portfolio can be rebalanced, on average, once every five years (which is a typical turnover of many PE portfolios), the liquidity premium is over 4%. For rebalancing once a year, on average, the illiquidity premium is approximately 1%. These numbers can be used as hurdle rates for investors considering investing in PE.

A number of authors including Dai, Li, and Liu (2008), Longstaff (2009), De Roon, Guo, and Ter Horst (2009), and Ang and Bollen (2010) also consider asset allocation where the illiquid asset cannot be traded over certain periods. However, in these studies the period of non-trading is deterministic. In contrast, the APW framework has stochastic and recurring periods of illiquidity. Deterministic non-trading periods are probably more appropriate for hedge fund investments where lock-ups have known expirations. PE investing is more open ended and has random, and infrequent, opportunities to rebalance.

APW still miss a number of practical considerations that the future literature should address. The most important one is that in the Merton setting into which APW introduce illiquidity, there are no cash distributions; all risky asset returns (both liquid and illiquid) are capital gains. PE investments require cash flow management of capital calls and distributions. Some ad hoc simulations have been conducted by some industry analysts on this issue, like Siegel (2008) and Leibowitz and Bova (2009), but without explicitly solving for optimal portfolios with illiquidity risk.[7] An extension of APW to incorporate cash flow streams could address this.

III E. Summary

The inability to continuously rebalance PE positions, potentially even by paying large transaction costs, makes optimal holdings of illiquid PE investments very different from the standard Merton

[7] De Zwart, Friesser, and van Dijk (2012) design optimal recommitment strategies for maintaining a given private equity portfolio, but do not consider the optimal allocation to private equity in an investor's overall portfolio.

model, which assumes no illiquidity risk. Since transactions costs in rebalancing PE portfolios are very large, in both entering new PE positions and selling existing PE positions, PE positions should be expected to be rebalanced very infrequently and investors should set very wide rebalancing bands. In asset allocation models where illiquid assets like PE can only be traded upon the arrival of a (stochastically occurring) liquidity event, illiquidity risk markedly reduces the holdings of illiquid assets compared to the standard Merton model. For example, an asset that could be traded continuously in the Merton setting that is held with a 60% optimal weight would have an optimal holding of less than 10% if it could be rebalanced only once every ten years, on average. The certainty equivalent reward, or equivalently the hurdle rate, for bearing illiquidity risk is large. For a typical PE investment that can be traded only once in ten years, on average, the illiquidity premium is well above 4%.

IV. Intermediary Issues in Private Equity

Most commonly, asset owners make PE investments as a LP in a fund where investment decisions are made by fund managers acting as GPs. This arrangement raises potential agency issues. One characteristic of PE investment is that the investment decisions arising from such management considerations and the related agency issues become intrinsically intertwined with PE performance. In public equity markets, factor returns and active management can mostly be separated due to the existence of investable index strategies.

IV A. Agency Issues

While the agency problem is central for PE investments, there are few studies evaluating the optimal delegated portfolio management [see the good surveys by the Bank of International Settlements (2003) and Stracca (2006)]. There are, however, many studies on agency issues in standard corporate finance settings (for example, Salanie, 1997; and Bolton and Dewatripont, 2005). Delegated portfolio management is different from standard agency problems because the "action" chosen is generally observed (the investments made by the GP), but the set of actions is unknown (the full set of deals available to the GP). In contrast, in standard moral hazard problems the "action" is unobservable, but the set of potential actions is usually known.[8] Thus, little is known about the optimal delegated portfolio contract, and the literature has few, if any, specific conclusions or prescriptions about what form the optimal PE contract between LPs and GPs should take.

PE investing is further complicated by having two levels of principal-agent relations rather than just a single one: a level between the LPs (principal) and GPs (agent) and another level between the GPs as fund managers (principal) and its underlying portfolio of companies (agent). Both levels rely on strong direct monetary incentives. Apart from these monetary incentives, however, the relation between LPs and GPs is one with limited information, poor monitoring, rigid fee structures, and the inability to withdraw capital or directly control managers. On the one hand, these features tend to heighten tensions between the LPs and GPs and exacerbate, rather than alleviate, agency issues. On the other hand, the distance between the LP and GP may allow GPs to invest and manage companies more freely.

The other principal-agent relation between the fund and its portfolio companies is one with strong governance, transparent information flows, good incentives for monitoring, and a high alignment of interests between owners and management (see Jensen, 1989). There is strong evidence that PE funds add significant value, on average, to the companies in their portfolio. This literature is surveyed by Kaplan and Stromberg (2009).

The interactions between these two layers of principal-agent problems have not been fully explored. It is not inconceivable, though, that mitigating the principal-agent problems at the LP-GP level would come at the cost of increasing the problems at the fund-company level. For example, greater transparency about the management of individual portfolio companies may in turn lead GPs to manage these companies with an eye towards managing short-term earnings expectations and satisfying public expectations more broadly, a concern for publicly traded companies, rather than simply managing them to maximize their total value.

[8] There are other reasons that make the delegated optimal portfolio management problem challenging. The agent (fund manager) can control both the mean, which is the response to the signal by buying a good stock, and also the variance, through leverage. In a typical agency problem the agent controls only the mean (occasionally the variance), but not both. In continuous time, which is often used to solve agency problems, diffusion dynamics are effectively observable at high enough frequencies.

IV B. Private Equity Contracts

Because PE is, by its nature, private, it is difficult to perform systematic large-sample studies of contractual features and see how they relate to performance. Gompers and Lerner (1999), Litvak (2009), and Metrick and Yasuda (2010) examine small samples of various PE contracts. Several tentative conclusions emerge:

1. PE contracts are largely standardized. An often-quoted fee arrangement is a management fee of 2% and a carry of 20%. There is some variation in the numbers (for example, management fees tend to vary from 1% to 2.5% and carried interest varies from 20% to 35%), but the general structure is widely used. Additionally, a substantial part of the GPs' compensation may be in the form of transaction fees. PE fees are high.
2. There is some variation in the specific provisions governing the calculation and timing of the fees and carried interest. For example, a management fee could be flat (on committed capital), declining over the life of the fund, a (time-varying but deterministic) combination of committed and managed capital, or even an absolute amount.
3. Fixed fee and performance components are not substitutes but complements. That is, funds tend to raise both the fixed and variable fee components, as well as the other compensation components. Fund size tends to be positively correlated with fees, and Kaplan and Schoar (2005), along with others, find that size is negatively correlated with performance. More recently, however, Robinson and Sensoy (2011a) investigate an extended sample with contract terms and performance, and find no relation between net-of-fee performance and the size of the fund or the fees.
4. There is a debate about the performance sensitivity of PE compensation. Metrick and Yasuda (2010) find that close to one-half the present value of GP compensation is from management fees rather than carried interest and find this to be true for both VC and buyout funds. However, Chung et al. (2011) point out that a substantial amount of GPs' performance pay arises through the continuation value of raising future funds, which are highly sensitive to current performance.
5. PE contracts are complex documents. Litvak (2009), however, finds little relation between opaqueness and total compensation.

The management fees charged by PE and VC funds are high. According to Metrick and Yasuda (2010), such fees consume at least one-fifth of gross PE returns. They find that out of every $100 invested with a VC fund, an average of $23 is paid to the GPs in the form of carry and management fees. For buyout funds, the mean of the carry and management fees comes to $18 per $100.

The high fees charged by GPs point to the fact that if an institutional investor wishing to allocate to PE can do this in-house, then there are substantial savings available. Of course, attracting talent and running an in-house PE shop presents a different set of agency issues than out-sourcing to PE funds with GPs. Despite the pessimistic view of returns of PE investments to LPs in Section II, the high PE fees imply that if asset owners can come close to capturing gross returns, PE becomes much more attractive.

While opacity per se does not seem to be related to total compensation and returns, it has other important add-on effects for other aspects of an asset owner's larger portfolio. Complexity and non-transparency can increase agency problems and make risk management more difficult. The leverage involved in many buyout funds can be more expensive, and is often harder to monitor, than leverage done directly by the asset owner.

V. Conclusion

Our findings and recommendations for investments in PE may be summarized as follows:

1. Empirical approaches commonly used to estimate the risk and return of standard publicly traded securities are difficult to apply. Complicating features of PE investments include the limited data, the irregular nature of such investments, and the sample selection problems that typically arise in reported PE data. Adjusting for these difficulties requires sophisticated econometric techniques. Without appropriate adjustment, naïve analyses tend to understate the risk and volatility and may exaggerate performance estimates.

   Recommendation: Reported estimates of PE risk and return should be interpreted with caution. Simple standard methodologies fail to take into consideration all the nuances that a thorough and accurate evaluation of a PE investment requires. Studies that develop methodologies to perform these adjustments are still in a preliminary phase, and a consensus on the appropriate adjustments has yet to emerge.

2. Commonly-used fund performance measures, such as the internal rate of return (IRR), the total value to paid-in (TVPI) multiple, and public market equivalent (PME), are problematic. There is substantial variation in the estimates of these measures across studies and data sources. The measures can, to some extent, be manipulated by the timing and magnitude of the individual investments. These fund performance measures use only rough risk adjustments too. Fundamentally, these measures are not derived from underlying financial theories of risk and return, making them difficult to interpret consistently.

Recommendation: Commonly reported performance measures should be interpreted with caution. They are not return measures as commonly understood.

3. Asset allocation models that account for transaction costs (which are high for PE) and illiquidity risk (which is substantial for PE) recommend modest holdings of PE. In these models, rebalancing will be infrequent, so wide swings in the holdings of PE can be expected. Also, the holdings of illiquid PE will be much lower than predicted by asset allocation models, assuming that all assets can be rebalanced when desired.

Recommendation: When determining optimal PE allocations, asset allocation models must account for the inability to rebalance PE positions. Allocations to illiquid PE investments should generally be modest.

4. Current PE vehicles have substantial agency issues, which public equity vehicles do not. While there is heterogeneity in PE contracts, PE fees are high, consuming at least one-fifth of gross PE returns. Incentive fees account for less than one-third of GP compensation.

Recommendation: If any part of the fees paid to externally managed PE funds with GPs can be brought back in-house to institutional asset owners, and if the quality of the PE investments can be maintained, it would lead to substantial savings for the asset owners.

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