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Valuation of N-stage Investments Under Jump-Diffusion Processes

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Abstract In this paper we consider N-phased investment opportunities where the time evolution of the project value follows a jump-diffusion process. An explicit valuation formula is derived under two different scenarios: in the first case we consider fixed and certain investment costs and in the second case we consider cost uncertainty and assume that investment costs follow a jump-diffusion process.

Keywords N-fold compound options · Sequential investments · Jump-diffusion process

JEL Classification G12 · G13 · G30 · C60

1 Introduction

As several researchers have noted, R&D investments are essentially real growth options because the value of early projects stems not so much from their expected cash flows as from the follow-up opportunities they may create. At each stage the company may decide to exercise the option or not, that is to continue to invest in the project or to shut it down. This is, for instance, the case of the development of

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new drugs, which begins with research that leads with some probability to a new compound and which continues with testing and concludes with the construction of a production facility and the marketing of the product. Inventors in this field regularly file applications on a large number of drugs and therapies before knowing whether those drugs will be safe and successful. Given the flexibility and uncertainty involved in such projects, traditional tools fail to capture the value of R&D investments.

In the present paper we consider R&D investment opportunities that are by their nature sequential and where strategically relevant, new information may arrive at each investment stage. Such investments can be best modeled as an N-fold compound option on the commercialization phase where in each of the N stages the company faces the option of shutting the project down or of continuing its operations, that is, to continue to invest in the project. In the USA, for example, because of FDA regulation, R&D activity for a new drug can be divided into six major phases: (1) discovery, (2) pre-clinical studies, (3) phase I clinical trials, (4) phase II clinical trials, (5) phase III clinical trials and (6) regulatory review and approval.[1] Each phase represents an option on a new phase of the process. Therefore, R&D projects can be considered as N-fold compound options.[2]

The arrival of new strategically important information at discrete points in time can be accommodated by modelling the dynamics of the project value as a jump-diffusion process,[3] where the Gaussian diffusion process represents business-as-usual uncertainty and where punctuated jumps at random intervals represent exceptional events such as major project failures or important breakthroughs. Indeed, apart from the obvious market risk, research-intensive firms face a number of risks, that, for convenience, we summarize under the heading "rare events." Under the assumption of lognormality of the jump size distribution we analytically solve the valuation problem of an N-staged investment opportunity under two different scenarios. Firstly, we consider the case where investment costs are deterministic and perfectly known at the beginning of the project. Secondly, we consider the case where investment costs are stochastic and unknown at the beginning of the project, but where it is known that they follow a jump-diffusion process.

The paper is organized as follows. Section 2 provides an overview of the related economic literature. Section 3 provides a description of the economic model and derives a closed-form solution for an N-fold compound call option with a mixed jump-diffusion process. An extension to the pricing of an N-fold compound option where both the underlying project value and the investment cost follow jump-diffusion processes is presented in Sect. 4. In Sect. 5, we provide numerical results. The final section concludes the paper.

[1] See the discussion in Cassimon et al. (2004), where R&D projects of pharmaceutical companies are valued using 6-fold compound options.

[2] Cassimon et al. (2011) study the case of a software R&D project, identifying 4 phases, software design, coding, testing and product launch, and use a 4-fold compound options approach for the valuation of the start-up option.

[3] Recent literature argues that jump-diffusion processes better represent the return dynamics of financial and real asset. Such processes may account for fat tails and skewness of probability distributions. See Boyarchenko (2004) for further information.

2 Literature Review

The literature on the valuation of real investments under jumps is growing quickly. Pennings and Lint (1997) provide a real options model for valuing R&D projects, which assumes a pure jump process for the underlying project value and study the consequences of this modelling in a real-world investment context. Martzoukos and Trigeorgis (2002) value single stage investment options when the underlying project value follows a log-normal jump-diffusion process involving multiple types of rare events. In this way, they are able to simultaneously represent the discontinuous changes of the project value due to different, unexpected events (i.e., political, technological, competitive etc.). Our model is related to Martzoukos and Trigeorgis's work but we value an N-staged investment project when the underlying asset undergoes only one class of rare events in each time interval. We also consider the case of investment costs following jump-diffusion dynamics, whereas in Martzoukos and Trigeorgis (2002) investment costs are assumed to be constant. Moreover, Wu and Yen (2007) develop a simple model for pricing (non-nested) real growth options that considers uncertainty regarding the project value, investment cost, and the jumps in the underlying value, and do not consider jumps in the investment cost. Some papers incorporate different distributions of jump sizes into the valuation problem of real options. Boyarchenko (2004), for example, extends the standard model of irreversible investment under uncertainty to a wide class of jump-diffusion processes, namely Lévy processes. Analytical solutions for (real) option prices under Lévy processes are also given by Mordecki (2002) and Agliardi (2009).

Compound options have been extensively used in the finance literature to evaluate sequential investment opportunities. Geske (1979) shows that risky securities with sequential payouts can be valued as compound options. Carr (1988) analyzes sequential compound options, of the form of options to acquire subsequent options to exchange an underlying risky asset for another risky asset.[4] Gukhal (2004) derives analytical valuation formulas for 2-fold compound options when the underlying value follows a log-normal jump-diffusion process. He then applies these results to value extendible options and American call options on stocks that pay continuous and discrete dividends. Using some properties of multivariate normal integrals, the present paper generalizes the result in Gukhal (2004) for jump-diffusion compound options to the case of N-fold compound options and applies the result to the valuation of sequential investment options in which both the project value and investment cost (i.e., the strike price) follow log-normal jump-diffusion processes. Agliardi and Agliardi (2005) derive a closed-form solution for European-style N-fold compound call options in the case of time-dependent volatility and interest rate. Their procedure consists of solving N-nested Black-Scholes partial differential equations.[5] Differently from their approach, we consider a real investment problem and use the risk-neutral argument

[4] Lee and Paxson (2001) have applied Carr's compound exchange option formula to R&D investments valuation.

[5] In essence, at the first step the underlying option is priced according to the Black-Scholes formula then, compound options are priced as options on the securities whose values have already been found in earlier steps.

(Harrison and Kreps 1979) to calculate the expected present value of the N-fold compound option and moreover we consider jump-diffusion processes for the underlying values. Lee et al. (2008) also propose a generalized pricing formula and sensitivity analysis for sequential compound options by using the risk-neutral method but assume that uncertainty is one-dimensional by modeling the underlying value as a geometric Brownian process.

3 A Valuation Formula for Sequential Investment Opportunities

We consider the valuation problem of a risk-neutral entrepreneur who, at time 0, considers to invest in a project whose commercial phase cannot be launched before a pilot phase consisting of N stages of investment is completed. Let T_N be the time of the market launch of the product, when, upon bearing the commercialization cost I_N, the firm earns the project value V. The project payoff at time T_N is $\max\{V - I_N, 0\}$ and let $F_N(V, t)$ denote the value at time t of this simple investment opportunity. We assume that the commercialization phase is reached upon investing an amount I_k, at time period T_k, for $k = 1, 2, \ldots, N-1$ and with $T_1 \leq T_2 \leq \cdots \leq T_{N-1} \leq T_N$. Following the capital budgeting literature, we assume that the timing of investment decisions is deterministic. This is because in many research-intensive industries, such as pharmaceutical or software industries, R&D managers break R&D programs down into a predetermined sequence of decision points, where they can decide either to abandon the underlying project or to enter the next stage. Thus, R&D managers at the beginning of the project form expectations about the duration of each phase, that are then used in the timing and the valuation of the project. [6,7] In our valuation, the R&D program is split into N consecutive decision points (see Fig. 1).

The N-staged investment problem may be viewed as a compound option, that is options on options, and its value may be derived in a recursive way. Let us now define a sequence of call options, with value F_k, on the call option whose value is F_{k+1}, with exercise price I_k and expiry date T_k, for $k = 1, 2, \ldots, N-1$. The k-fold compound option value can be written in a recursive way and its final payoff at the option's maturity date T_k is given by:

$$F_k\left(F_{k+1}\left(V, T_k\right), T_k\right) = \max\left\{F_{k+1}\left(V, T_k\right) - I_k, 0\right\}, \tag{1}$$

[6] See, for example, the discussion in the case studies in Cassimon et al. (2011), where the authors value a software development project in the ITC sector, and in Pennings and Sereno (2011), where the authors value an oncological R&D project. Moreover, sensitivity analysis with respect to variations in the duration of phases such as the one proposed in Sect. 5.2 may be a useful risk-management tool, yielding lower and upper bounds for option values under different scenarios.

[7] Martzoukos and Trigeorgis (2002) use numerical simulations to evaluate simple options with multiple sources of rare events where the expiry date is unknown but do not consider compound options. Cortellezzi and Villani (2009) use Monte Carlo simulations to value compound R&D projects where the time span of the last phase is stochastic and Gamba and Fusari (2009) evaluate modular designs using a least-square Monte Carlo method, taking the stochastic nature of the decision points into account. Both papers consider only business-as-usual uncertainty and do not account for rare events.

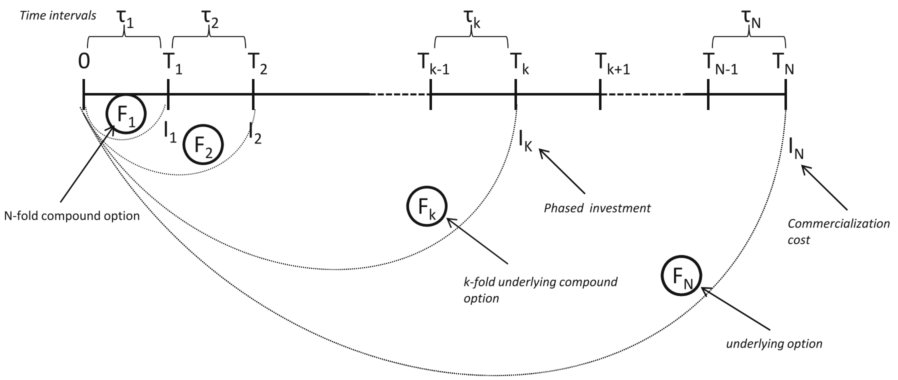

Fig. 1 Structure and notation of the N-fold compound option

for $k = 1, 2, \ldots, N-1$ and where $F_{k+1}(V, T_k)$ stands for the value of the underlying compound option at time T_k and I_k is the exercise price. According to (1), at time T_k, the firm faces the option of investing an amount I_k, gaining access to stage $k+1$ of the project whose value is $F_{k+1}(V, T_k)$, or to shut the project down. The option will be exercised if $F_{k+1}(V, T_k) > I_k$, that is, if the expected present value of the project at time T_k exceeds the investment cost.

We assume deterministic investment costs I_k, for $k = 1, 2, \ldots, N$, which are perfectly known at time 0. The project value is uncertain during the different stages. Denote by V_t the evaluation of the project at time $t \in [0, T_N]$. We assume that V_t follows a geometric jump-diffusion process (Merton 1976):

$$dV_t = \alpha V_t dt + \sigma V_t dz_t + (Y - 1) V_t dq_t, \tag{2}$$

where α is the drift rate, σ is the volatility of the Brownian part of the process, conditional on no jumps occurring, dz is a standard Gauss-Wiener process and dq is a Poisson process with constant intensity $\lambda (> 0)$. Therefore, $dq = 0$ with probability $1 - \lambda dt$ and $dq = 1$ with probability λdt, or, in other words, over a small time period dt, the probability of a jump in V is λdt, where the random variable $(Y - 1)$ accounts for the relative jump amplitude. The average relative jump size, $E[Y - 1]$ is denoted by K, where E is the expectation operator over the distribution function of Y under the objective probability measure $\mathbb{P}$. We assume that the random variable Y and the Poisson process dq are independent of each other and also independent of the Brownian motion dz.

The project value V as given by (2) has two sources of uncertainty. The term σdz corresponds to "business-as-usual" uncertainty, while the term dq describes rare events. For example, new drugs can turn into mega-selling blockbuster products or alternatively, suffer clinical trial failures and withdrawal from the market. If the Poisson event does not occur ($dq = 0$), then the return dynamics would be identical to those presented by Black and Scholes (1973) and Merton (1973). If, on the other hand the Poisson event occurs, then $(Y - 1)$ is an impulse function which takes the project value from V to VY, where we assume that Y is drawn from a lognormal distribution with parameter $\left(\mu_J, \sigma_J^2\right)$.[8] The coefficients $\left(\mu_J,\ \sigma_J^2\right)$ are constants.[9]

We assume that the firm achieves risk neutrality by holding a diversified portfolio of activities. In other words, we assume that its portfolio of activities features negatively correlated risk factors, thereby gaining risk insulation and earning, in expected terms, the exogenously given risk-neutral rate of return $r \geq 0$. Under this assumption, following Merton (1976), the risk-neutral project value can be described by the following stochastic differential equation:

$$dV_t = \mu^* V_t dt + \sigma V_t dz_t^* + (Y - 1)\, V_t dq_t, \tag{3}$$

where dz^* is a standard Wiener process,[10] $dq,\ Y$ are as above, independently distributed of dz^* and μ^* is such that the discounted project value is a martingale under $\mathbb{Q}$:

$$\mu^* = r - \lambda K = r - \lambda \left[\exp\left(\mu_J + \frac{1}{2}\sigma_J^2\right) - 1\right].$$

In the sequential investment model, we want to determine the value of the investment opportunity $F_k\left(F_{k+1}\left(V, T_k\right), T_k\right)$ at each stage $T_k,\ k = 1, 2, \ldots, N-1$, of the project, with $F_k\left(F_{k+1}\left(V, T_k\right), T_k\right) = \max\left\{F_{k+1}\left(V, T_k\right) - I_k, 0\right\}$, being the boundary condition. Let V_k^* denote the value of V such that the underlying option is at the money at time T_k, i.e.,

$$V_{T_k} = V_k^*$$

where V_k^* solves:

$$F_{k+1}\left(V, T_k\right) - I_k = 0$$

for $k = 1, 2, \ldots, N-1$ and $V_N^* = I_N$. Then,

$$F_k\left(F_{k+1}\left(V, T_k\right), T_k\right) = \begin{cases} F_{k+1}\left(V, T_k\right) - I_k & \text{if } V_{T_k} \geq V_k^* \\ 0 & \text{if } V_{T_k} < V_k^* \end{cases}.$$

In other words, if the value of V at time T_k, is greater than V_k^*, the firm continues to invest in the project, i.e. the compound option will be exercised, while for values less

[8] That is, $E\left[\ln(Y)\right] = \mu_J$ and $\mathrm{Var}[\ln(Y)] = \sigma_J^2$, so $E[Y] = \exp\left(\mu_J + \frac{1}{2}\sigma_J^2\right)$. We impose this assumption on the distribution of jump sizes since this is the simplest type of model that illustrates the intuition underlying an N-fold compound options valuation with jumps.

[9] Note that the uncertainty of rare events across stages in the present model varies even if the jump intensity λ is constant over time, since the longer the time span of a given phase is, the more likely the occurrence of jumps within the phase is.

[10] If through portfolio diversification the investor replicates the market valuation of the project and if the jump risk is diversifiable, then according to the CAPM $\alpha = r' + \beta\sigma$, where $\beta\sigma$ is the risk premium, r' is the risk free interest rate and therefore $dz^* = dz + \beta dt$. Note that in such a context the jump risk yields a zero risk premium if it is uncorrelated with the market as a whole, as it is the case, for example, if jumps are due to innovations in technology, actions undertaken by competitors and changes in the firm's strategy. See, for example, Pennings and Lint (1997) and Martzoukos and Trigeorgis (2002).

than V_k^* it will be abandoned. Note that the critical values V_k^* are determined recursively and their existence and uniqueness are guaranteed in view of the expression of F_{k+1} (see Remark 1).

Let us define n_i the number of Poisson arrivals in the time interval $\left[T_{i-1}, T_i\right]$, $i = 1, 2, \ldots, N$, and let us set $T_0 = 0$. Consequently, let $s_k = \sum_{i=1}^{k} n_i$ be the total number of arrivals in the interval $[0, T_k]$, for $k = 1, 2, \ldots, N$. The time interval $[0, T_N]$ is divided into subintervals of length $\tau_k = T_k - T_{k-1}$, for $k = 1, 2, \ldots, N$ with $\tau_1 = T_1$.

Let $\sigma_{s_k}^2 = \sigma^2 + \frac{s_k \sigma_J^2}{T_k}$ be the total variance conditional on the occurrence of s_k jumps in the interval $[0, T_k]$, for $k = 1, 2, \ldots, N$. Moreover, let $x_t = \ln\left(\frac{V_t}{V_0}\right)$ be the logarithmic return.[11] The correlation between x_{T_j} and x_{T_i}, over the overlapping time interval $T_i \leq T_j$, for $1 \leq i \leq j \leq k$ and $k = 1, 2, \ldots, N$, conditional on observing s_j and s_i jumps, respectively, is:

$$\rho_{s_i s_j} = \frac{\sigma_{s_i}\sqrt{T_i}}{\sigma_{s_j}\sqrt{T_j}}. \tag{4}$$

For any k, $1 \leq k \leq N$, let Ξ_k denote a k-dimensional symmetric correlation matrix with a typical element $\rho_{s_i s_j}$ and unitary elements on the principal diagonal and $\Xi_1 = 1$. Let $\Phi_k(\zeta_1, \ldots, \zeta_k; \Theta_k)$ denote the k-dimensional multinormal cumulative distribution function, with upper limits of integration $\zeta_k, \ldots, \zeta_1$ and correlation matrix Θ_k.

In the following proposition we provide a valuation formula for the N-fold compound option problem described above.

Proposition 1 *If the project value follows a jump-diffusion process (3), then the expected present value at time* 0 *of the* N*-staged investment project with final pay-off* $\max\left\{V_{T_N} - I_N, 0\right\}$ *and with investment costs* I_k *at time* T_k, $k = 1, \ldots, N-1$ *is:*

$$\begin{aligned} F_1(V, 0) = V_0 \prod_{j=1}^{N} \Bigg[\sum_{n_j=0}^{\infty} \frac{e^{-\lambda\tau_j}\left(\lambda\tau_j\right)^{n_j}}{n_j!} \exp(-\delta_{s_N} T_N) \\ \times \Phi_N\left(a_{s_1}(V, 0), \ldots, a_{s_N}(V, 0); \Xi_N\right) \Bigg] \\ - \sum_{j=1}^{N} \Bigg\{ I_j \prod_{k=1}^{j} \Bigg[\sum_{n_k=0}^{\infty} \frac{e^{-\lambda\tau_k}(\lambda\tau_k)^{n_k}}{n_k!} \exp\left(-r T_j\right) \\ \times \Phi_j\left(b_{s_1}(V, 0), \ldots, b_{s_j}(V, 0); \Xi_j\right) \Bigg] \Bigg\}, \end{aligned} \tag{5}$$

[11] The logarithmic return x_t evolves as: $dx_t = \left(r - \lambda K - 0.5\sigma^2\right) dt + \sigma dz_t^* + \ln(Y)\, dq_t$, under the pricing measure $\mathbb{Q}$.

where:

$$\delta_{s_k} = -\frac{s_k\left(\mu_J + \frac{1}{2}\sigma_J^2\right)}{T_k} + \lambda\left[\exp\left(\mu_J + \frac{1}{2}\sigma_J^2\right) - 1\right], \textit{ for } k = 1, 2, \ldots, N,$$

$$b_{s_k}(V, 0) = \frac{\ln\left(\frac{V_0}{V_k^*}\right) + \left(r - \delta_{s_k} - \frac{1}{2}\sigma_{s_k}^2\right) T_k}{\sigma_{s_k}\sqrt{T_k}},$$

$$a_{s_k}(V, 0) = b_{s_k}(V, 0) + \sigma_{s_k}\sqrt{T_k}.$$

Proof See appendix. □

Remark 1 It can be proved that for each degree of compoundness:[12]

$$\frac{\partial}{\partial V} F_1(V, 0) = \prod_{j=1}^{k+1}\left[\sum_{n_j=0}^{\infty} \frac{e^{-\lambda\tau_j}\left(\lambda\tau_j\right)^{n_j}}{n_j!} \exp\left(-\delta_{s_{k+1}} T_{k+1}\right) \times \Phi_{k+1}\left(a_{s_1}(V, 0), \ldots, a_{s_{k+1}}(V, 0); \Xi_{k+1}\right)\right] > 0,$$

and

$$\lim_{V \longrightarrow +\infty} F_{k+1}(V, t) = +\infty,$$

and therefore V_k^* solving $F_{k+1}(V, T_k) - I_k = 0$ is unique for every k, $1 \leq k < N$.

According to Eq. 5, the pricing formula has the following interpretation. The price of the jump-diffusion N-fold compound option can be expressed as the weighted sum of the N-fold compound option prices where each weight equals the joint probability that a Poisson random variable with constant intensity λ will take on exactly the value n_i in each time interval $\left[T_{i-1}, T_i\right]$, for $i = 1, 2, \ldots, N$.

The expression:

$$\sum_{j=1}^{N}\left\{\prod_{k=1}^{j}\left[\sum_{n_k=0}^{\infty} \frac{e^{-\lambda\tau_k}\left(\lambda\tau_k\right)^{n_k}}{n_k!} \Phi_j\left(b_{s_1}(V, 0), \ldots, b_{s_j}(V, 0); \Xi_j\right)\right]\right\},$$

can be interpreted as the joint probability of the multicompound option expiring in-the-money under the equivalent martingale probability measure, so that the second component in (5) is the expected present value, computed using risk adjusted probabilities, of the subsequent investment costs.[13] The expression:

[12] This is a simple generalization of the Merton delta (see Merton 1976, pp. 137–138) to the case of N-fold compound options.

[13] See Lajeri-Chaherli (2002) and Lee et al. (2008, p.43) for a similar interpretation.

$$\prod_{j=1}^{N}\left[\sum_{n_j=0}^{\infty}\frac{e^{-\lambda\tau_j}\left(\lambda\tau_j\right)^{n_j}}{n_j!}\exp\left(-\delta_{s_N}T_N\right)\Phi_N\left(a_{s_1}\left(V,0\right),\ldots,a_{s_N}\left(V,0\right);\Xi_N\right)\right],$$

can be interpreted as the joint probability that the multicompound option will be exercised, so that the first component in (5) is the expected present value of receiving the future cash flows at expiration of the option.

4 Sequential Investment Opportunities and Stochastic Investment Cost

R&D projects often involve considerable cost uncertainty. For example, jumps in the investment cost can be especially important in the development of a new drug by a pharmaceutical company. When a company discovers a new therapeutic target, it has to start a new project, eventually abandoning the current one, with an increase in the investment cost. On the other hand, technological progress can lead to sharp investment cost reductions. In this section we extend the previous model assuming that the investment cost varies over time[14] and that its dynamics are governed by a geometric jump-diffusion process.

Using the same notation as in Sect. 3, the project payoff at time T_N, the time of market launch, is $\max\{V-I,0\}$ where V and I are the underlying value and investment cost; let $W_N(V,I,t)$ be the value at time t of this simple investment opportunity. The investor observes two random processes V and I and must decide at each stage T_k, for $k=1,2,\ldots,N-1$ and $T_1\leq T_2\leq\cdots\leq T_{N-1}\leq T_N$, whether to continue to invest in the project, that is access stage $k+1$ of the project, or not. We accordingly define a sequence of call options, with value W_k, whose underlying value is W_{k+1}, and with exercise price (i.e. investment cost) I_{T_k} and expiry date T_k, for $k=1,2,\ldots,N-1$. The k-fold compound option value can be written in a recursive way and its payoff at the option's maturity date T_k is given by:

$$W_k\left(W_{k+1}\left(V,I,T_k\right),I,T_k\right)=\max\left\{W_{k+1}\left(V,I,T_k\right)-I_{T_k},0\right\},\tag{6}$$

for $k=1,2,\ldots,N-1$ and where $W_{k+1}(V,I,T_k)$ stands for the value of the underlying compound option at time T_k. Thus, at time T_k, the firm faces the option of investing an amount I_{T_k}, and therefore entering stage $k+1$ of the project whose value is $W_{k+1}(V,I,T_k)$, or to shut the project down. Notice that the option at time T_N can be viewed as a simple exchange option[15] where the delivery asset is I and the optioned asset is V and the phased investment problem can be viewed as a compound exchange problem (see, for example, Carr 1988).

[14] See also Wu and Yen (2007) and Cortellezzi and Villani (2009).

[15] See Lindset (2007) for the pricing of exchange options under jump-diffusion processes.

Under the risk-neutral martingale measure $\mathbb{Q}$, the dynamics of the underlying assets are given by the following geometric jump-diffusion processes:[16]

$$dV_t = (r - \lambda_1 K_1)\, V_t dt + \sigma_1 V_t dz^*_{1,t} + (Y_1 - 1)\, V_t dq_{1,t}, \tag{7}$$
$$dI_t = (r - \lambda_2 K_2)\, I_t dt + \sigma_2 I_t dz^*_{2,t} + (Y_2 - 1)\, I_t dq_{2,t}, \tag{8}$$

σ_1 and σ_2 are the respective standard deviations, conditional on no jumps and dz_1^* and dz_2^* are standard Brownian motions, under the risk-neutral measure $\mathbb{Q}$. The process dq_1 and dq_2 are Poisson random variables with constant rates λ_1 and λ_2, respectively, counting the number of jumps. The sizes Y_1 and Y_2 are random variables and it is assumed that Y_1 (Y_2) is lognormally distributed with mean $\mu_{1,J}$ ($\mu_{2,J}$) and variance $\sigma^2_{1,J}$ ($\sigma^2_{2,J}$). K_1 and K_2 are the average relative jump sizes $E[Y_1 - 1]$ and $E[Y_2 - 1]$, respectively. We assume that the Poisson processes dq_1 and dq_2 and the jump components Y_1 and Y_2 are independent of each other and also independent of the Brownian motions dz_1^* and dz_2^*. Finally, we assume that Brownian motion components are correlated, with correlation coefficient φ_{12}, i.e., $corr[dz_1^*, dz_2^*] = \varphi_{12} dt$.

Let us define by V^c the price ratio of V to I. This allows us to write:[17]

$$\max\left\{W_{k+1}(V, I, T_k) - I_{T_k}, 0\right\} = I_{T_k} \cdot \max\left\{W^c_{k+1}\left(V^c, T_k\right) - 1, 0\right\},$$

where I_{T_k} is the numeraire and $W^c_{k+1}(V^c, T_k) = W_{k+1}(V^c, 1, T_k)$. Let us denote by V_k^{c*} the critical price ratio such that the underlying option is at the money at time T_k, i.e., $V^c_{T_k} = V_k^{c*}$, where V_k^{c*} solves:

$$W^c_{k+1}\left(V^c, T_k\right) - 1 = 0,$$

for $k = 1, 2, \ldots, N - 1$ and $V_N^{c*} = 1$. If the value of V^c at time T_k, is greater than the threshold V_k^{c*}, then the firm continues to invest in the project, i.e. the compound option will be exercised, while for values less than V_k^{c*} it will be abandoned.

Let us define n_i and m_i the number of event occurrences, respectively, for the project value and investment cost in the time interval $\left[T_{i-1}, T_i\right]$, $i = 1, 2, \ldots, N$. Again, T_0 is set to zero. Consequently, let $s_{1,k} = \sum_{i=1}^{k} n_i$ and $s_{2,k} = \sum_{i=1}^{k} m_i$ be the total number of arrivals in the interval $[0, T_k]$, for $k = 1, 2, \ldots, N$. For the following we set $\tau_k = T_k - T_{k-1}$, with $\tau_1 = T_1$.

Let $\sigma^2_{s_{1,k}s_{2,k}} = \sigma_c^2 + \frac{s_{1,k}\sigma^2_{1,J} + s_{2,k}\sigma^2_{2,J}}{T_k}$ be the total variance of a percentage change in the price ratio V^c, conditional on the occurrence of $s_{1,k}$ and $s_{2,k}$ jumps in the time period $[0, T_k]$ and $\sigma_c^2 = \sigma_1^2 + 2\sigma_2\sigma_1\varphi_{12} + \sigma_2^2$. Moreover, let $x_t^c = \ln\left(\frac{V_t^c}{V_0^c}\right)$ be the logarithmic

[16] Also in this case we assume that the firm is diversified; that is, it keeps a portfolio of activities which allows it to value activities in a risk-neutral way.

[17] Given the above mentioned properties of V and I it can be shown that the homogeneity theorem holds where $W_{k+1}(\theta V, \theta I, T_k) = \theta W_{k+1}(V, I, T_k)$ for $\theta \geq 0$. See for example Carr (1988) and Geman et al. (1995).

return.[18] The correlation coefficient between the logarithmic returns $x^c_{T_j}$ and $x^c_{T_i}$ over the overlapping time interval $T_i \leq T_j$, for $1 \leq i \leq j \leq k$ and $k = 1, 2, \ldots, N$, conditional on the random event occurrences, is:

$$\rho_{s_{1,i}s_{2,j}} = \frac{\sigma_{s_{1,i}s_{2,i}}\sqrt{T_i}}{\sigma_{s_{1,j}s_{2,j}}\sqrt{T_j}},$$

For any k, $1 \leq k \leq N$, let Ψ_k denote a k-dimensional symmetric correlation matrix with a typical element $\rho_{s_{1,i}s_{2,j}}$.

Proposition 2 *If the project value V and the investment cost I follow jump-diffusion processes (7) and (8), respectively, then the expected present value of a N-staged investment project with final pay-off* $\max\left\{V_{T_N} - I_{T_N}, 0\right\}$ *and with investment cost I_{T_k} at time T_k, $k = 1, 2, \ldots, N-1$, is:*

$$\begin{aligned} W_1(V, I, 0) = V_0 \prod_{j=1}^{N} \Bigg[\sum_{n_j=0}^{\infty} \sum_{m_j=0}^{\infty} & \frac{e^{-(\lambda_1+\lambda_2)\tau_j}\left(\lambda_1\tau_j\right)^{n_j}\left(\lambda_2\tau_j\right)^{m_j}}{n_j! m_j!} \exp\left(-\delta_{s_{1,N}} T_N\right) \\ & \times \Phi_N\left(c_{s_{1,1}s_{2,1}}\left(V^c, 0\right), \ldots, c_{s_{1,N}s_{2,N}}\left(V^c, 0\right); \Psi_N\right)\Bigg] \\ & - I_0 \sum_{j=1}^{N} \left\{ \prod_{k=1}^{j} \left[\sum_{n_k=0}^{\infty} \sum_{m_k=0}^{\infty} \frac{e^{-(\lambda_1+\lambda_2)\tau_k}(\lambda_1\tau_k)^{n_k}(\lambda_2\tau_k)^{m_k}}{n_k! m_k!} \exp\left(-\delta_{s_{2,j}} T_j\right) \right.\right. \\ & \left.\left. \times \Phi_j\left(d_{s_{1,1}s_{2,1}}\left(V^c, 0\right), \ldots, d_{s_{1,j}s_{2,j}}\left(V^c, 0\right); \Psi_j\right)\right]\right\} \end{aligned} \tag{9}$$

where:

$$\begin{aligned} \delta_{s_{i,k}} &= -\frac{s_{i,k}\left(\mu_{i,J} + \frac{1}{2}\sigma^2_{i,J}\right)}{T_k} + \lambda_i\left[\exp\left(\mu_{i,J} + \frac{1}{2}\sigma^2_{i,J}\right) - 1\right], \\ & \quad \textit{for } k = 1, 2, \ldots, N, \ \textit{and } i = 1, 2, \\ d_{s_{1,k}s_{2,k}}\left(V^c, 0\right) &= \frac{\ln\left(\frac{V^c_0}{V^{c*}_k}\right) + \left(\delta_{s_{2,k}} - \delta_{s_{1,k}} - \frac{1}{2}\sigma^2_{s_{1,k}s_{2,k}}\right)T_k}{\sigma_{s_{1,k}s_{2,k}}\sqrt{T_k}}, \\ c_{s_{1,k}s_{2,k}}\left(V^c, 0\right) &= d_{s_{1,k}s_{2,k}}\left(V^c, 0\right) + \sigma_{s_{1,k}s_{2,k}}\sqrt{T_k}. \end{aligned}$$

Proof See appendix. □

[18] The logarithmic return x^c_t evolves as: $dx^c_t = \left(\lambda_2 K_2 - \lambda_1 K_1 - 0.5\sigma^2_c\right)dt + \sigma_c dz^c_t + \ln(Y_1)\,dq_{1,t} - \ln(Y_2)\,dq_{2,t}$, under the new risk-neutral measure $\tilde{\mathbb{Q}}$. See Appendix for further details.

Equation 9 can be seen as the weighted sum of the multicompound exchange option values where each weight equals the joint probability that two Poisson random variables with rates λ_1 and λ_2 will take on exactly the value n_i and m_i, respectively, in each time interval $\left[T_{i-1}, T_i\right]$, for $i = 1, 2, \ldots, N$. The first component in (9) can be seen as the present value, computed using risk adjusted probabilities, of receiving the future cash flows at the expiration of the option. The second component can be seen as the present value of the investment costs.

5 Simulation Results

In this section we provide some numerical results on multicompound options. In the first part of this section we test the accuracy of the model developed in Sect. 3[19] by comparing it with the discrete time jump-diffusion compound option model and afterwards we provide a numerical application.

5.1 Comparison With the Discrete Time Model

Following Amin (1993) we approximate the continuous time model by a jump-diffusion process in discrete time (see also Martzoukos and Trigeorgis 2002; Xu et al. 2003) and we calculate option prices by evaluating the expected discounted pay-off of the option using dynamic programming. In particular, we first construct the state space and calculate the corresponding project values, and then value (compound-) options in a recursive way using Markov-transition rates.

Consider first the simple option on the project value V with expiry date T_1. Given the trading period $[0, T_1]$, the time interval is divided into M subintervals of length $h_M = \frac{T_1}{M}$, where we define $M_1 = \frac{T_1}{h_M}$ and thus $M_1 = M$. At each time period m, the state space is a grid, where each consecutive point is spaced $\sigma\sqrt{h_M}$ apart. Between two consecutive time periods, the grid is shifted upwards by $\gamma_M = \left(r - .5\sigma^2\right) h_M$. The project's value at time mh_M if in state i is then given by:

$$V_i\left(m\right) = V\left(0\right) \exp\left(m\gamma_M + i\sigma\sqrt{h_M}\right).$$

The project value can undergo local changes $i = \pm 1$, representing the diffusion part, or jumps for $i \neq \pm 1$, where the project value can jump possibly to any state in the state space. In order to value options we use Markov transition probabilities within a finite difference scheme. Local changes, $i = \pm 1$, have probabilities approximately equal to $\frac{1}{2}$ (see Amin 1993), i.e.

$$p_{\pm 1} = \Pr\left\{\ln\left[V\left(t + dt\right)\right] - \ln\left[V\left(t\right)\right] = \gamma_M \pm \sigma\sqrt{h_M}\right\} \simeq 0.5.$$

[19] Once the change of numeraire has been made, the model described in Sect. 4 can be implemented in a very similar way.

The probability of observing a jump is $\overline{\lambda}_M = \lambda h_M e^{-\lambda h_M}$, and, given that a jump occurs, the probability of observing a jump of size $l > 1$, or $l < -1$ is:

$$p_l = \Pr\left\{\ln[V(t+dt)] - \ln[V(t)] = \gamma_M + l\sigma\sqrt{h_M}\right\}$$
$$= \Phi\left[(l+0.5\sigma)\sqrt{h_M}\right] - \Phi\left[(l-0.5\sigma)\sqrt{h_M}\right],$$

where $\Phi(\bullet)$ is the cumulative normal distribution function with mean μ_J and variance σ_J^2 and where, for simplicity's sake, we set $\mu_J = -\frac{1}{2}\sigma_J^2$.

The option value (European style) $F_1(i, m)$ is calculated in a recursive way, using dynamic programming:

$$F_1(i,m) = e^{-rh_M}\left\{\left(1-\overline{\lambda}_M\right)\frac{1}{2}\left[F_1(i+1,m+1) + F_1(i-1,m+1)\right] + \overline{\lambda}_M\left[\sum_{l=2}^{\infty} p_l F_1(i+l,m+1) + \sum_{l=2}^{\infty} p_{-l} F_1(i-l,m+1)\right]\right\}, \quad (10)$$

with $F_1(i, M_1) = f_1[V_i(M_1)] = \max\{V_i(M_1) - I_1, 0\}$ being the pay-off at the expiry date (i.e. boundary condition). Given the properties of the normal distribution, the sums in (10) can be suitably truncated at a maximum jump size $\bar{l}$. In the simulations below, option values are computed setting $\bar{l}$ at 75. Thus, at each time period, the state space consists of $2 \times 75 + 1$ reachable nodes.

Compound options can be evaluated in a similar vein. For example, consider the case of an option on an option (2-fold compound option), where the first expires at time T_1 and the second at time T_2, where $T_1 < T_2$. At time T_1, the pay-off of the first option (the compound option) is a function of the underlying second option (simple option), that is, $F_1(i, M_1) = f_1[F_2(i, M_1)] = max\{F_2(i, M_1) - I_1, 0\}$, where $M_1 = \frac{T_1}{h_M}$. Following (10), the compound option value can be calculated recursively. This method extends straightforwardly to the case of N-fold compound options.

In Table 1 we test the accuracy of the procedure by evaluating a 2-fold compound option (call on call option). We assume: $V = 100$; $I_2 = 100$; $\sigma = 0.2$; $\sigma_J = 0.2$; $r = 0.02$; $T_1 = 0.25$; $T_2 = 0.5\lambda = \{0,\ 0.6,\ 0.8,\ 1.0\}$ and $I_1 = \{10,\ 12.5,\ 15\}$. Table 1 shows that the algorithm approximates well the closed form solution.

In Table 2 we provide simulations for a 3-fold compound option (call on call on call option) with strike prices $I_1 = 5$, $I_2 = 10$ and $I_3 = 100$ for values of $\lambda = \{0.4,\ 0.6,\ 0.8,\ 1.0\}$. Exercise dates are $T_1 = 0.2$, $T_2 = 0.35$, and $T_3 = 0.5$. Since by increasing λ we also increase the underlying's total volatility, option prices increase; hence, increasing the average number of jumps (per year) increases the option's value.

5.2 Numerical Application

In this section we provide a numerical application to the valuation problem of N-staged R&D projects in the pharmaceutical industry, where it is assumed that the project value

Table 1 Computation of compound option values (2-fold)

	Jump-diffusion(continuous time formula)			Jump-diffusion (discrete time model) $M = 200$		
	I_1= 10	I_1= 12.5	I_1= 15	I_1= 10	I_1= 12.5	I_1= 15
λ						
λ =1.0	2.25	1.70	1.31	2.20	1.67	1.29
λ =0.8	2.02	1.50	1.14	2.00	1.49	1.13
λ =0.6	1.80	1.31	0.96	1.79	1.30	0.97
λ =0	1.12	0.71	0.45	1.13	0.72	0.45

Parameter values: $V = 100$; $I_2 = 100$; $\sigma = 0.2$; $\sigma_J = 0.2$; $r = 0.02$; $T_1 = 0.25$; $T_2 = 0.5$

Table 2 Computation of compound option values (3-fold)

	Jump-diffusion (continuous time) formula)	Jump-diffusion (discrete time model) $M = 200$
λ		
λ =1.0	1.18	1.10
λ =0.8	1.02	0.96
λ =0.6	0.86	0.81
λ =0.4	0.7	0.66

Parameter values: $V = 100$; $I_1 = 5$; $I_2 = 10$; $I_3 = 100$; $\sigma = 0.2$; $\sigma_J = 0.2$; $r = 0.02$; $T_1 = 0.2$; $T_2 = 0.35$; $T_3 = 0.5$

follows a jump-diffusion process. For simplicity, we consider the valuation problem of 3-staged drug development investments (i.e. 3-fold compound options), where we grouped the projects' milestones as follows. The project starts with the discovery phase, that is expected to end at time T_1. After the discovery of a new molecule, the drug may enter the testing phases. Pre-clinical, phase I and phase II testing are expected to last τ_2 years. At time T_2 the project enters the phase III testing stage, that is followed by the approval phase; these two phases are expected to last τ_3 years. At time T_3 the drug is ready for the market launch.

We take the timing and investment costs from Cassimon et al. (2004). The discovery phase, on average, lasts two years ($\tau_1 = 2$) and thus $T_1 = 2$ is the maturity of the 3-fold compound option; the first testing stage (pre-clinical, phase I and phase II testing) lasts, on average, seven years ($\tau_2 = 7$) and thus $T_2 = 9$ is the maturity of the 2-fold compound option; testing phase III and approval take, on average, five years ($\tau_3 = 5$) and consequently $T_3 = 14$ is the maturity of the simple final option to put the product on the market. Average pre-clinical, phase I and phase II testing costs are $I_1 = 13,800$ thousand US\$, which is the exercise (strike) price of the 3-fold compound option; average phase III testing and approval costs are $I_2 = 28,100$ thousand US\$, which is the strike price of the 2-fold compound option; average commercialization costs are $I_3 = 31,200$ thousand US\$, which is the strike price of the simple option. We consider a risk-free interest rate $r = 0.05$, business-as-usual volatility $\sigma = 0.5$, volatility of the jump size $\sigma_J = 0.5$ and (annual) arrival rate of jumps $\lambda = 0.3$ (see, for example, Wu and Yen (2007) for similar parameter choices). We further assume that the project value at time 0 (i.e. V, the present value of all future cash flows at time 0) ranges from 20,000 to 100,000 thousand US\$.

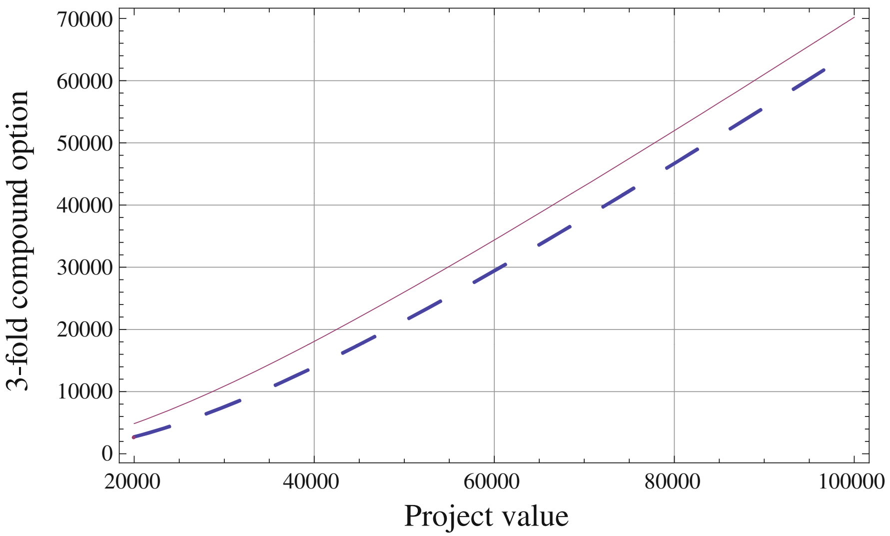

Fig. 2 Sensitivity analysis of the 3-fold compound R&D option with respect to the project value in the presence of jumps (*continuous line*) and in the absence of jumps (*dashed line*)

Figure 2 shows the relation between the 3-fold compound option values and the project value. The continuous line represents the 3-fold compound option value in the presence of jumps in the project value, which may occur because of innovations to the political-, technological- or competitive framework in which the firm operates, while the dashed line illustrates the sensitivity of the 3-fold compound option value without jumps. The compound option value in the presence of jumps is larger since jumps increase the overall project's volatility.

Figure 3 shows the relation between the 3-fold compound option values and the volatility of the jump size, σ_J. The continuous line represents the 3-fold compound option value in the presence of jumps, while the dashed line illustrates the sensitivity of the 3-fold compound option value without jumps. We consider values of σ_J ranging from 0 to 1 and $V_0 = 85,000$ thousand US$. The 3-fold compound option value in the presence of jumps in the project value increases as the volatility of the jump size increases, while the 3-fold compound option value without jumps is neutral to increases in the volatility of the jump size.

Figure 4 shows the relation between the 3-fold compound option values and the business-as-usual volatility, σ. The continuous line represents the 3-fold compound option value in the presence of jumps, while the dashed line illustrates the sensitivity of the 3-fold compound option value without jumps. We consider values of σ ranging from 0.1 to 1 and $V_0 = 85,000$ thousand US$. Increases in business-as-usual uncertainty, by increasing overall project uncertainty, increases the compound option value.

Finally, Fig. 5 shows the sensitivity analysis of the 3-fold compound R&D option values with respect to maturities in the presence of jumps (data are the same as in Fig. 2). In particular, we investigate how changes in the duration of the second phase

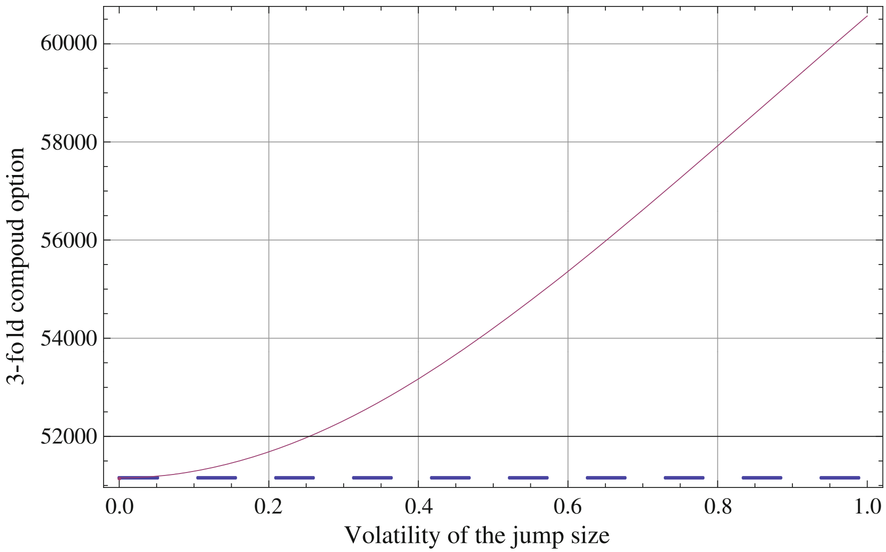

Fig. 3 Sensitivity analysis of the 3-fold compound R&D option with respect to the volatility of the jump size in the presence of jumps (*continuous line*) and in the absence of jumps (*dashed line*)

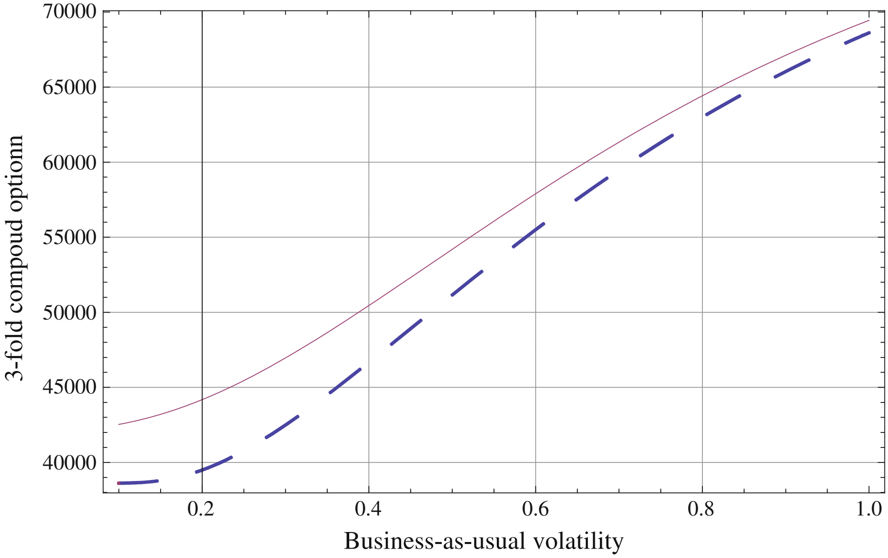

Fig. 4 Sensitivity analysis of the 3-fold compound R&D option with respect to the business-as-usual volatility in the presence of jumps (*continuous line*) and in the absence of jumps (*dashed line*)

affect the 3-fold compound option for different project values V_0. We thus keep $\tau_1 = 2$ and $\tau_3 = 5$ constant and compare option values in the case where $\tau_2 = 6$, $\tau_2 = 7$ and $\tau_2 = 8$. Consequently, maturities are $T_1 = 2$, $T_2 = 8$, $T_3 = 13$ in the first case (dashed line in Fig. 5), $T_1 = 2$, $T_2 = 9$ and $T_3 = 14$ (continuous line in Fig. 5) in

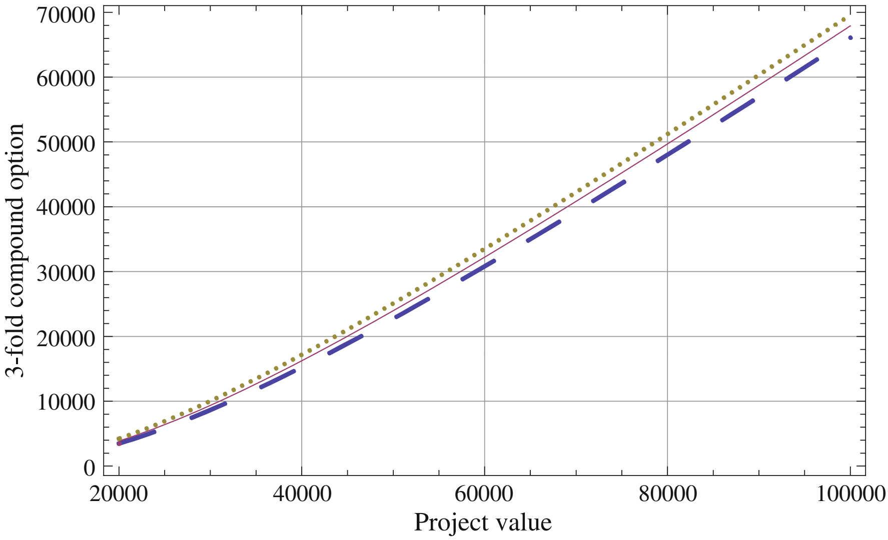

Fig. 5 Sensitivity analysis of the 3-fold compound R&D option values with respect to maturities in the presence of jumps

the second case and finally $T_1 = 2,\ T_2 = 10,\ T_3 = 15$ in the last case (dotted line in Fig. 5). We observe that the 3-fold compound R&D option value is little sensitive to changes in the duration of phase 2. This result should reassure R&D managers that errors due to a miscalculation in the predetermined timing of phases are very small. On the other hand, such a sensitivity analysis could be used as a risk-management tool in the case where the duration of some phases is uncertain.

6 Conclusion

Phased investments have the property that much of the value of the investment is associated with future cash flows that are contingent on intermediate decisions. Because of this property the analysis of sequential investment projects is one of the most difficult problems. Starting from the difficulty of traditional DCF methods to capture the value of early-stage investments, the real options literature provides advanced models, each focusing on different R&D characteristics. In the present paper we value R&D projects with the following characteristics: (1) two types of uncertainty, i.e., business-as-usual uncertainty and rare events, (2) project value and investment cost uncertainties, and (3) arbitrary degree of compoundness of R&D projects.

In our model rare events are modelled using a time-invariant jump-intensity. Since technical failures and other rare events are often phase specific in the sense that they are more likely to occur in some stages of the project than in others, it would be interesting to extend the present model to account for this additional feature. Moreover, one could try to generalize the assumption of log-normal jump size distribution to the case of Levy-type distributions and study the pricing of N-fold compound options under this alternative assumption. We leave these questions for further research.

Appendix

Proof of Proposition 1

Under the martingale approach, the value at time 0 of the European N-fold compound option is given by the following expectation under the risk-neutral measure:

$$F_1(V,0) = e^{-rT_1} E_0^{\mathbb{Q}} \{\max[F_2(V,T_1) - I_1, 0]\}, \tag{11}$$

where F_2 indicates the value of an $(N-1)$-fold compound option and V is the project value at the maturity date T_1. The expectation in (11) is difficult to find directly due to jumps in the project value. We address this problem by conditioning on the random event occurrence, and work with the conditioned variable thereafter. Thus:

$$F_1(V,0) = e^{-rT_1} \sum_{n_1=0}^{\infty} \frac{e^{-\lambda T_1} (\lambda T_1)^{n_1}}{n_1!} E_0^{\mathbb{Q}} \{\max[F_2(V,T_1) - I_1, 0] \mid n_1\}. \tag{12}$$

We know that the value of the $(N-1)$-fold compound option $F_2(V,T_1)$ is the expected present value at time T_1 of the project's expected cashflows and is given by $F_2(V,T_1) = E_{T_1}^{\mathbb{Q}} \left\{e^{-r(T_2-T_1)} \max[F_3(V,T_2) - I_2, 0]\right\}$, where F_3 indicates the value of the underlying compound option and V is the value of the project at time T_2. The evaluation of this expectation requires conditioning on the number of jumps occurring in the interval $[T_1, T_2]$ times the (marginal) probability of n_2 jumps. The process of discounting and conditioning continues until the last option is evaluated.[20] Thus, by moving backward in time one can verify that the expression for $F_2(V,T_1)$ is given by:

$$\begin{aligned}
&\prod_{j=2}^{N} \left[\sum_{n_j=0}^{\infty} \frac{e^{-\lambda\tau_j} \left(\lambda\tau_j\right)^{n_j}}{n_j!} V_{T_1} \exp\left[-\left(\delta_{s_N} T_N - \delta_{s_1} T_1\right)\right] \right. \\
&\qquad \left. \times\ \Phi_{N-1}\left(a_{s_2}(V,T_1), \ldots, a_{s_N}(V,T_1); \hat{\Xi}_{N-1}\right) \right] \\
&- \sum_{j=2}^{N} \left\{ \prod_{k=2}^{j} \left[\sum_{n_k=0}^{\infty} \frac{e^{-\lambda\tau_k} (\lambda\tau_k)^{n_k}}{n_k!} I_j \exp\left[-r\left(T_j - T_1\right)\right] \right. \right. \\
&\qquad \left. \left. \times\ \Phi_{j-1}\left(b_{s_2}(V,T_1), \ldots, b_{s_j}(V,T_1); \hat{\Xi}_{j-1}\right) \right] \right\},
\end{aligned}$$

20 See also Gukhal (2004, p. 2060) where a 2-fold compound option is computed in a very similar way.

where:

$$b_{s_k}(V, T_1) = \frac{\ln\left(\frac{V_{T_1}}{V_k^*}\right) + \left(r - \delta_{s_k} - \frac{1}{2}\sigma_{s_k}^2\right)T_k - \left(r - \delta_{s_1} - \frac{1}{2}\sigma_{s_1}^2\right)T_1}{\sqrt{\sigma_{s_k}^2 T_k - \sigma_{s_1}^2 T_1}}$$

$$for\ k = 2, 3, \ldots, N,$$

$$a_{s_k}(V, T_1) = b_{s_k}(V, T_1) + \sqrt{\sigma_{s_k}^2 T_k - \sigma_{s_1}^2 T_1},$$

and $\hat{\Xi}_k$ is a k-dimensional symmetric correlation matrix with a typical element:

$$\hat{\rho}_{s_i s_j} = \sqrt{\frac{\sigma_{s_i}^2 T_i - \sigma_{s_1}^2 T_1}{\sigma_{s_j}^2 T_j - \sigma_{s_1}^2 T_1}} \quad for\ 2 \le i \le j \le k\ and\ k = 2, 3, \ldots, N.$$

Thus, $F_2(V, T_1)$ has been obtained recursively and represents the expected present values of all upcoming $N - 2$ stages (i.e. underlying options).

The project value at time T_1 under the risk-neutral probability $\mathbb{Q}$ and conditioned on n_1 jumps in the interval $[0, T_1]$, is:

$$V_{T_1} = V_0 \exp\left\{\left(r - \delta_{s_1} - \frac{1}{2}\sigma_{s_1}^2\right)T_1 + \sigma_{s_1}\sqrt{T_1}\cdot\xi\right\},$$

where ξ has a standard Gaussian probability law under $\mathbb{Q}$. Therefore, the value at time 0 of the European N-fold compound option is:

$$\begin{aligned} F_1(V, 0) = e^{-rT_1} &\left\{\prod_{j=1}^{N}\left\{\sum_{n_j=0}^{\infty}\frac{e^{-\lambda\tau_j}\left(\lambda\tau_j\right)^{n_j}}{n_j!}\int_{u_1}^{+\infty} n(u)\left[\phi_{s_1}(u)\right.\right.\right. \\ &\times \left.\left.\exp\left\{-\left(\delta_{s_N}T_N - \delta_{s_1}T_1\right)\right\}\Phi_{N-1}\left(\hat{a}_{s_2}, \ldots, \hat{a}_{s_N}; \hat{\Xi}_{N-1}\right)du\right]\right\} \\ &-\sum_{j=2}^{N}\left\{\prod_{k=1}^{j}\left\{\sum_{n_k=0}^{\infty}\frac{e^{-\lambda\tau_k}(\lambda\tau_k)^{n_k}}{n_k!}\int_{u_1}^{+\infty} n(u)\left[I_j\exp\left\{-r\left(T_j - T_1\right)\right\}\right.\right.\right. \\ &\times \left.\left.\left.\Phi_{j-1}\left(\hat{b}_{s_2}, \ldots, \hat{b}_{s_j}; \hat{\Xi}_{j-1}\right)\right]du\right\}\right\} \\ &\left.-\sum_{n_1=0}^{\infty}\frac{e^{-\lambda T_1}(\lambda T_1)^{n_1}}{n_1!}\int_{u_1}^{+\infty} n(u)\, I_1 du\right\}, \end{aligned} \tag{13}$$

where $n(.)$ is the normal density function, $\hat{a}_{s_k} = a_{s_k}\left(\phi_{s_1}(u), T_1\right)$, $\hat{b}_{s_k} = b_{s_k}\left(\phi_{s_1}(u), T_1\right)$ for $k = 2, 3, \ldots, N$, the function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is given by:

$$\phi_{s_1}(u) = V_0 \exp\left\{\left(r - \delta_{s_1} - \frac{1}{2}\sigma_{s_1}^2\right) T_1 + \sigma_{s_1}\sqrt{T_1} \cdot u\right\},$$

and, finally, the lower bound of the integrals u_1 is defined implicitly by the equation:[21]

$$u_1 = \inf\left\{u \in \mathbb{R} \mid F_2\left[\phi_{s_1}(u), T_1\right] \geq I_1\right\}.$$

Straightforward calculations yield:

$$\hat{a}_{s_k} = \frac{\ln\left(\frac{V_0}{V_k^*}\right) + \left(r - \delta_{s_k} + \frac{1}{2}\sigma_{s_k}^2\right) T_k - \sigma_{s_1}^2 T_1 + \sigma_{s_1}\sqrt{T_1} \cdot u}{\sqrt{\sigma_{s_k}^2 T_k - \sigma_{s_1}^2 T_1}},$$

$$\hat{b}_{s_k} = \frac{\ln\left(\frac{V_0}{V_k^*}\right) + \left(r - \delta_{s_k} - \frac{1}{2}\sigma_{s_k}^2\right) T_k + \sigma_{s_1}\sqrt{T_1} \cdot u}{\sqrt{\sigma_{s_k}^2 T_k - \sigma_{s_1}^2 T_1}},$$

for $k = 2, 3, \ldots, N$. The last term in (13) can be written in the form:

$$e^{-rT_1} \sum_{n_1=0}^{\infty} \frac{e^{-\lambda T_1} (\lambda T_1)^{n_1}}{n_1!} I_1 \Phi_1\left(b_{s_1}(V, 0)\right).$$

Using (4) and rearranging terms, it follows that:

$$\hat{\rho}_{s_i s_j} = \frac{\left(\rho_{s_i s_j} - \rho_{s_1 s_j}\rho_{s_1 s_i}\right)}{\sqrt{\left(1 - \rho_{s_1 s_j}^2\right)\left(1 - \rho_{s_1 s_i}^2\right)}},$$

for $2 \leq i < j \leq N$. Therefore, we substitute each term $\hat{\rho}_{s_i s_j}$ in the matrix $\hat{\Xi}_{j-1}$ with $\frac{\left(\rho_{s_i s_j} - \rho_{s_1 s_j}\rho_{s_1 s_i}\right)}{\sqrt{\left(1 - \rho_{s_1 s_j}^2\right)\left(1 - \rho_{s_1 s_i}^2\right)}}$. The second term of (13) can be written in terms of the N-dimensional multinormal cumulative distribution function by applying the following Lemma.

[21] The N-fold compound option will be exercised at time T_1 if the project value at time T_1 is greater than V_1^*, namely if $\phi_{s_1}(u) \geq V_1^*$. In other words, V_1^* is the critical project value that equalizes the underlying asset and the investment cost at time T_1, i.e. $F_2\left[\phi_{s_1}(u), T_1\right] = I_1$. Therefore, solving $\phi_{s_1}(u) = V_1^*$, straightforward calculations yield $u = u_1 = -b_{s_1}$.

Lemma 1 *Let* $1 < k \leq N$, *and let* $\tilde{\Xi}_{k-1}$ *be the matrix obtained from* $\hat{\Xi}_{k-1}$ *by replacing any element* $\hat{\rho}_{s_i s_j}$ *with* $\left(\rho_{s_i s_j} - \rho_{s_1 s_j}\rho_{s_1 s_i}\right)/\sqrt{\left(1-\rho^2_{s_1 s_j}\right)\left(1-\rho^2_{s_1 s_i}\right)}$, *by setting:*

$$\alpha_{s_k} = \frac{\ln\left(\frac{V_0}{V_k^*}\right) + \left(r - \delta_{s_k} - \frac{1}{2}\sigma^2_{s_k}\right)T_k}{\sqrt{\sigma^2_{s_k}T_k - \sigma^2_{s_1}T_1}} \text{ and } \beta_{s_k} = \frac{\sigma_{s_1}\sqrt{T_1}}{\sqrt{\sigma^2_{s_k}T_k - \sigma^2_{s_1}T_1}},$$

for $k = 2, 3, \ldots, N$, *where* α_{s_k} *and* β_{s_k} *are real numbers, the following identity holds:*

$$\int_{-\infty}^{b_{s_1}} n(u)\,\Phi_{k-1}\left(\alpha_{s_2} + u\beta_{s_2}, \ldots, \alpha_{s_k} + u\beta_{s_k}; \tilde{\Xi}_{k-1}\right) du = \Phi_k(b_{s_1}, \ldots, b_{s_k}; \Xi_k). \tag{14}$$

Proof It follows by setting $\frac{b_{s_k}}{\sqrt{1-\rho^2_{s_1 s_k}}} = \alpha_{s_k}$ and $\frac{-\rho_{s_1 s_k}}{\sqrt{1-\rho^2_{s_1 s_k}}} = \beta_{s_k}$, $k = 2, 3, \ldots, N$, and substituting into (14). Then, the second expression of (14) is obtained by using the definition of the standard multivariate normal distribution. □

Finally, we can write the first term in (13) in terms of the cumulative multivariate normal distribution using Lemma 1, after making the following substitution $x = u - \sigma_{s_1}\sqrt{T_1}$.

Proof of Proposition 2

In the proof we apply a change of numeraire.[22] To establish the proposition we need to calculate the dynamics of the process $V^c = \frac{V}{I}$ under the new risk-neutral measure $\tilde{\mathbb{Q}}$. First, we determine $dV^c = d\left(\frac{V}{I}\right)$ by applying Itô 's Lemma. We obtain:

$$dV^c = \left(r - \hat{\delta}\right)V^c dt + \sigma_1 V^c dz_1^* - \sigma_2 V^c dz_2^* + (Y_1 - 1)\,V^c dq_1 - (Y_2 - 1)\,V^c dq_2, \tag{15}$$

where $\hat{\delta} = r + \lambda_1 K_1 - \lambda_2 K_2 + \sigma_2^2 - \sigma_1\sigma_2\varphi_{12}$.

Applying the log-transformation for I_t, under the risk-neutral measure $\mathbb{Q}$, it results that:

$$\begin{aligned} I_t &= I_0 \exp\left\{\left(r - \lambda_2 K_2 - \frac{\sigma_2^2}{2}\right)t + \sigma_2 z^*_{2,t} + \sum_{i=1}^{q_{2,t}} \ln\left(Y_{2,i}\right)\right\} \\ &= I_0 \exp(rt) \cdot \exp\left\{-\frac{\sigma_2^2}{2}t + \sigma_2 z^*_{2,t} - \lambda_2 K_2 t + \sum_{i=1}^{q_{2,t}} \ln\left(Y_{2,i}\right)\right\}. \end{aligned} \tag{16}$$

[22] See, for example, Geman et al. (1995).

In (16), we can interpret the expression:

$$\exp\left\{-\frac{\sigma_2^2}{2}t+\sigma_2 z_{2,t}^*-\lambda_2 K_2 t+\sum_{i=1}^{q_{2,t}}\ln\left(Y_{2,i}\right)\right\},$$

as the Radon-Nikodym derivative of some equivalent measure $\tilde{\mathbb{Q}}$ with respect to $\mathbb{Q}$, since it satisfies the condition:

$$E^{\mathbb{Q}}\exp\left\{-\frac{\sigma_2^2}{2}t+\sigma_2 z_{2,t}^*-\lambda_2 K_2 t+\sum_{i=1}^{q_{2,t}}\ln\left(Y_{2,i}\right)\right\}=1,$$

for all $t \geq 0$. Set:

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}=\exp\left\{-\frac{\sigma_2^2}{2}t+\sigma_2 z_{2,t}^*-\lambda_2 K_2 t+\sum_{i=1}^{q_{2,t}}\ln\left(Y_{2,i}\right)\right\},$$

hence, by simple substitution in (16) we can write:

$$I_t = I_0 \exp\left(rt\right)\cdot\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}.$$

By using the Girsanov's theorem, the process:

$$d\tilde{z}_2 = dz_2^* - \sigma_2 dt, \tag{17}$$

is a Brownian motion under the new risk-neutral measure $\tilde{\mathbb{Q}}$. We, therefore, can write dz_1^* as:

$$dz_1^* = \varphi_{12} dz_2^* + \sqrt{1-\varphi_{12}^2}dz_3^*, \tag{18}$$

where dz_3^* is a Brownian motion independent of dz_2^* under the measure $\mathbb{Q}$. By using equations (17) and (18), we can now rewrite the evolution of the asset V^c under the new risk-neutral measure $\tilde{\mathbb{Q}}$:

$$dV^c = \left(\lambda_2 K_2 - \lambda_1 K_1\right)V^c dt + \sigma_c V^c dz^c + \left(Y_1 - 1\right)V^c dq_1 - \left(Y_2 - 1\right)V^c dq_2,$$

with the definitions $\sigma_c = \sqrt{\sigma_1^2 + 2\sigma_2\sigma_1\varphi_{12} + \sigma_2^2}$ and $\sigma_c dz^c = \left(\varphi_{12}\sigma_1 - \sigma_2\right)d\tilde{z}_2 + \sigma_1\sqrt{1-\varphi_{12}^2}dz_3^*$ and where dz^c is a Brownian motion under the risk-neutral measure $\tilde{\mathbb{Q}}$.

Given that the jump sizes Y_1 and Y_2 are lognormally distributed with parameters $\left(\mu_{1,J}, \sigma_{1,J}^2\right)$ and $\left(\mu_{2,J}, \sigma_{2,J}^2\right)$, respectively, by applying the log-transformation for process V^c allows us to obtain the explicit value of V^c under the risk-neutral measure $\tilde{\mathbb{Q}}$:

$$V_t^c = V_0^c \exp\left\{\left(\lambda_2 K_2 - \lambda_1 K_1 - \tfrac{1}{2}\sigma_c^2\right)t + \sigma_c z_t^c + \sum_{i=1}^{q_{1,t}} \ln\left(Y_{1,i}\right) - \sum_{i=1}^{q_{2,t}} \ln\left(Y_{2,i}\right)\right\}.$$

Therefore, the terminal price at time t under the risk-neutral pricing measure $\tilde{\mathbb{Q}}$ and conditioned on the number of jumps n and m for the project value and investment cost in the time interval $[0, t]$, respectively, is:

$$\begin{aligned} V_t^c &= V_0^c \exp\left\{\left(\lambda_2 K_2 - \lambda_1 K_1 - \frac{1}{2}\sigma_c^2\right)t + \sigma_c z_t^c + \sum_{i=1}^{n} \ln\left(Y_{1,i}\right) - \sum_{i=1}^{m} \ln\left(Y_{2,i}\right)\right\} \\ &= V_0^c e^{\left(\delta_{m,2} - \delta_{n,1} - \frac{1}{2}\sigma_{n,m}^2\right)t + \sigma_{n,m} z_t^c}, \end{aligned}$$

where:

$$\delta_{i,j} = -\frac{i\left(\mu_{j,J} + \frac{1}{2}\sigma_{j,J}^2\right)}{t} + \lambda_j\left[\exp\left(\mu_{j,J} + \frac{1}{2}\sigma_{j,J}^2\right) - 1\right]$$

and where $\sigma_{n,m}^2 = \sigma_c^2 + \frac{n\sigma_{1,J}^2 + m\sigma_{2,J}^2}{t}$.

Let $I_{T_1} = I_0 \exp\left(rT_1\right) \cdot \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}$ be the numeraire. Under the martingale approach, the value at time 0 of the European N-fold compound exchange option is given by the following expectation under the risk-neutral measure:

$$W_1\left(V, I, 0\right) = e^{-rT_1} E_0^{\mathbb{Q}}\left\{I_{T_1} \cdot \max\left[W_2^c\left(V^c, T_1\right) - 1, 0\right]\right\},$$

By conditioning on the number of jumps in the interval $[0, T_1]$, we obtain:

$$\begin{aligned} W_1\left(V, I, 0\right) = &\sum_{n_1=0}^{\infty}\sum_{m_1=0}^{\infty} \frac{e^{-(\lambda_1+\lambda_2)T_1}\left(\lambda_1 T_1\right)^{n_1}\left(\lambda_2 T_1\right)^{m_1}}{n_1! m_1!} I_0 \exp\left(-\delta_{s_{2,1}} T_1\right) \\ &\times E_0^{\tilde{\mathbb{Q}}}\left\{\max\left[W_2^c\left(V^c, T_1\right) - 1, 0\right] \mid n_1,\ m_1\right\}. \end{aligned} \quad (19)$$

Following the steps outlined above, one can verify that the expression for $W_2^c(V^c, T_1)$ is given by:

$$\prod_{j=2}^{N}\left[\sum_{n_j=0}^{\infty}\sum_{m_j=0}^{\infty}\frac{e^{-(\lambda_1+\lambda_2)\tau_j}\left(\lambda_1\tau_j\right)^{n_j}\left(\lambda_2\tau_j\right)^{m_j}}{n_j!m_j!}V_{T_1}^c\exp\left\{-\left(\delta_{s_{1,N}}T_N-\delta_{s_{1,1}}T_1\right)\right\}\right.$$
$$\left.\times\ \Phi_{N-1}\left(c_{s_{1,2}s_{2,2}}\left(V^c,T_1\right),\ldots,c_{s_{1,N}s_{2,N}}\left(V^c,T_1\right);\hat{\Psi}_{N-1}\right)\right]$$
$$-\sum_{j=2}^{N}\left\{\prod_{k=2}^{j}\left[\sum_{n_k=0}^{\infty}\sum_{m_k=0}^{\infty}\frac{e^{-(\lambda_1+\lambda_2)\tau_k}\left(\lambda_1\tau_k\right)^{n_k}\left(\lambda_2\tau_k\right)^{m_k}}{n_k!m_k!}\exp\left\{-\left(\delta_{s_{2,j}}T_j-\delta_{s_{2,1}}T_1\right)\right\}\right.\right.$$
$$\left.\left.\times\ \Phi_{j-1}\left(d_{s_{1,2}s_{2,2}}\left(V^c,T_1\right),\ldots,d_{s_{1,j}s_{2,j}}\left(V^c,T_1\right);\hat{\Psi}_{j-1}\right)\right]\right\},$$

and:

$$d_{s_{1,k}s_{2,k}}\left(V^c,T_1\right)$$
$$=\frac{\ln\left(\frac{V_{T_1}^c}{V_k^{c*}}\right)+\left(\delta_{s_{2,k}}-\delta_{s_{1,k}}-\frac{1}{2}\sigma_{s_{1,k}s_{2,k}}^2\right)T_k-\left(\delta_{s_{2,1}}-\delta_{s_{1,1}}-\frac{1}{2}\sigma_{s_{1,1}s_{2,1}}^2\right)T_1}{\sqrt{\sigma_{s_{1,k}s_{2,k}}^2T_k-\sigma_{s_{1,1}s_{2,1}}^2T_1}},$$
$$c_{s_{1,k}s_{2,k}}\left(V^c,T_1\right)=d_{s_{1,k}s_{2,k}}\left(V^c,T_1\right)+\sqrt{\sigma_{s_{1,k}s_{2,k}}^2T_k-\sigma_{s_{1,1}s_{2,1}}^2T_1}.$$

Finally, Eq. 19 can be written in integral form as in (13) and solved in a very similar way. The result in Proposition 2 follows.

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