Mean–variance portfolio selection with 'at-risk' constraints and discrete distributions ☆

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Abstract

We examine the impact of adding either a VaR or a CVaR constraint to the mean–variance model when security returns are assumed to have a discrete distribution with finitely many jump points. Three main results are obtained. First, portfolios on the VaR-constrained boundary exhibit $(K+2)$-*fund separation*, where K is the number of states for which the portfolios suffer losses equal to the VaR bound. Second, portfolios on the CVaR-constrained boundary exhibit $(K+3)$-*fund separation*, where K is the number of states for which the portfolios suffer losses equal to their VaRs. Third, an example illustrates that while the VaR of the CVaR-constrained optimal portfolio is close to that of the VaR-constrained optimal portfolio, the CVaR of the former is notably smaller than that of the latter. This result suggests that a CVaR constraint is more effective than a VaR constraint to curtail large losses in the mean–variance model.
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1. Introduction

The mean–variance model of Markowitz (1952, 1959) is a cornerstone of modern portfolio theory. Markowitz's seminal idea is that agents optimally select mean–variance efficient (hereafter, MV-efficient) portfolios. In practice, this model is extensively utilized to manage portfolio risk. Specific applications include determining optimal asset allocations, measuring gains from international diversification, and evaluating portfolio performance.

Over the past decade, the idea of using a measure of downside risk has been rediscovered with Value-at-Risk (VaR) becoming one of the most popular measures among practitioners (see, e.g., Hull, 2006, p. 435). Some researchers, however, criticize the use of VaR since it is not subadditive (see Artzner et al., 1999).[1] Furthermore, the use of Conditional Value-at-Risk (CVaR) instead of VaR has been suggested in the literature (see, e.g., Rockafellar and Uryasev, 2000, 2002).

Our article examines the impact of adding either a VaR or a CVaR constraint to the mean–variance model. Doing so is of particular interest for several reasons. First, while previous research examines this impact assuming that security returns have an elliptical distribution, we assume that they have a discrete distribution with finitely many jump points. The latter assumption is often imposed in practice when using historical simulation to estimate VaR and CVaR (see, e.g., Hull, 2006, pp. 438–440).

Second, there is an extensive literature recognizing that the mean–variance model is, at least as an approximation, consistent with expected utility maximization even when security returns are not assumed to have an elliptical distribution (see, e.g., Markowitz, 1987, pp. 52–70).[2] The addition of a VaR constraint to this model is motivated by the fact that the fund management industry is increasingly using it to set risk limits (see, e.g., Pearson, 2002; Jorion, 2007). While CVaR is less popular than VaR among practitioners, the addition of a CVaR constraint to the mean–variance model is motivated by the fact that the literature has noted advantages of using CVaR instead of VaR to control risk.

Third, the mean–variance model has been extensively used in the banking literature (see, e.g., Hart and Jaffee, 1974; Francis, 1978; Koehn and Santomero, 1980; Kim and Santomero, 1988; Rochet, 1992).[3] Hence, the addition of a VaR constraint to this model is motivated by the fact that banks now use VaR in calculating minimum capital requirements associated with their exposures to market risk (see, e.g., Berkowitz and O'Brien, 2002). The addition of a CVaR constraint to the mean–variance model is motivated by the fact that basing bank capital regulation on CVaR is, under certain conditions, more effective than basing it on VaR (see, e.g., Alexander and Baptista, 2006).

The *VaR-constrained boundary* consists of portfolios that, given a VaR constraint, minimize variance for some level of expected return. We show that when the constraint binds, portfolios on this boundary exhibit locally $(K + 2)$-*fund separation*, where K is the number

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[1] That is, the VaR of a portfolio with two securities can be larger than the sum of their VaRs.

[2] Berk (1997) provides joint conditions on utility functions and distributions of security returns that lead to mean–variance objective functions. For arbitrary distributions, the mean–variance model can be motivated with a quadratic utility function (see, e.g., Huang and Litzenberger, 1988, p. 61). Under certain conditions, Levy and Levy (2004) show that the mean–variance model can also be used in the context of prospect theory.

[3] Furthermore, interest in the mean–variance model is still apparent in the recent literature (see, e.g., Agarwal and Naik, 2004; Levy and Levy, 2004).

of states for which the portfolios suffer losses equal to the VaR bound. This result is of interest for two reasons. First, it simplifies the portfolio selection problem involving a VaR constraint into a $(K+2)$-fund allocation exercise that can be solved in two steps: (1) identifying the $K+2$ funds, and (2) allocating wealth among them. Second, it implies that portfolios on the VaR-constrained boundary at which the constraint binds are mean–variance inefficient (hereafter, MV-inefficient).

The *CVaR-constrained boundary* consists of portfolios that, given a CVaR constraint, minimize variance for some level of expected return. We show that when the constraint binds, portfolios on this boundary exhibit locally $(K+3)$-*fund separation*, where K is the number of states for which the portfolios suffer losses equal to their VaRs. Similar to the case of a VaR constraint, this result simplifies the portfolio selection problem involving a CVaR constraint into a $(K+3)$-fund allocation exercise. Furthermore, it implies that portfolios on the CVaR-constrained boundary at which the constraint binds are MV-inefficient.

We also provide an example that illustrates the implications of the constrained boundaries for portfolio selection. This example involves a sample of US stocks and uses historical data to estimate VaR and CVaR. Our main results regarding the VaR-constrained boundary are as follows. First, the expected return, standard deviation, VaR, and CVaR of an agent's VaR-constrained optimal portfolio are smaller than those of his or her unconstrained optimal portfolio. Second, the distance of the VaR-constrained optimal portfolio from the unconstrained boundary is higher for smaller VaR bounds and less risk-averse agents.[4]

Our results regarding the CVaR-constrained boundary are qualitatively similar to those obtained for the VaR-constrained boundary, but a quantitative difference is noteworthy. While the VaR of the CVaR-constrained optimal portfolio is close to that of the VaR-constrained optimal portfolio, the CVaR of the former is notably smaller than that of the latter. This result suggests that a CVaR constraint is more effective than a VaR constraint to curtail large losses in the mean–variance model.

The advantages of CVaR over VaR as a measure of risk have lead to the development of an extensive literature that explores the use of CVaR in portfolio optimization. For example, Rockafellar and Uryasev (2000) compare portfolios with minimum variance and CVaR given an expected return constraint. Krokhmal et al. (2002) characterize portfolios with maximum expected return for various CVaR constraints. Rockafellar and Uryasev (2002) show that CVaR constraints can be used to control risk when tracking a benchmark. Agarwal and Naik (2004) and Bertsimas et al. (2004) compare portfolios on the mean–variance and mean-CVaR boundaries.[5] Our work differs from this literature in that we explore the impact of adding either a VaR or a CVaR constraint to the mean–variance model, while in this literature CVaR replaces variance as a measure of risk.[6]

Sentana (2003) and Alexander and Baptista (2004) examine the impact of VaR and CVaR constraints in the mean–variance model by assuming that security returns have

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[4] A portfolio's *distance from the unconstrained boundary* refers to the difference between its standard deviation and the standard deviation of the portfolio on the unconstrained boundary with the same expected return.

[5] For a comparison of the mean–variance, mean-VaR, and mean-CVaR boundaries when security returns have an elliptical distribution, see Alexander and Baptista (2002, 2004).

[6] A motivation for using CVaR instead of variance as a measure of risk is that while the former is consistent with second-order stochastic dominance, in general the latter is not (see, e.g., Ogryczak and Ruszczyński, 2002).

an elliptical distribution.[7] Our work differs from these papers in two respects. First, as noted earlier, we do not impose such a distributional assumption. Second, in their papers portfolios on the constrained boundaries exhibit two-fund separation even when the constraints bind, while in this paper we show that these portfolios require more than two funds when the constraints bind.

Our paper proceeds as follows. Section 2 introduces the model and characterizes the constrained boundaries. Section 3 presents an example that illustrates these boundaries and their portfolio selection implications. Section 4 concludes.[8]

2. The model

We examine the impact of adding either a VaR or a CVaR constraint to the mean–variance model of Markowitz (1952, 1959). We begin by describing our assumptions and basic notation. There are J securities, where $J > 2$. The uncertainty about security returns is described by a finite set of equally likely states $\Omega \equiv \{1, \ldots, S\}$, where $S > J$. Security returns are given by a $J \times S$ matrix $\boldsymbol{R}$. Let $\overline{\boldsymbol{R}}$ be the $J \times 1$ vector of expected returns and V be the $J \times J$ variance–covariance matrix associated with $\boldsymbol{R}$. Let $\boldsymbol{R}_s \equiv [R_{1s} \quad \cdots \quad R_{Js}]^\top$, where R_{js} is the return of security j in state s. Suppose that: (i) there is no arbitrage[9]; (ii) rank$(V) = J$ so that there are no redundant securities nor a riskfree security[10]; and (iii) rank$([\mathbf{1} \quad \overline{\boldsymbol{R}} \quad \boldsymbol{R}_{s_1} \quad \cdots \quad \boldsymbol{R}_{s_{J-2}}]) = J$ for any set of $J - 2$ distinct states $\{s_1, \ldots, s_{J-2}\}$, where $\mathbf{1}$ is the $J \times 1$ vector $[1 \quad \cdots \quad 1]^\top$.[11]

A portfolio is a $J \times 1$ vector w with $w^\top \mathbf{1} = 1$. Note that short-sales are allowed.[12] Let $\widetilde{R}_w$ denote the random return of w. The expected return and variance of w are denoted by, respectively, $\overline{R}_w$ and σ_w^2.

2.1. The unconstrained boundary

We now review the problem examined by Markowitz. A portfolio is on the *unconstrained boundary* if it solves

$$\min_{w \in \{\widehat{w} \in \mathbb{R}^J : \widehat{w}^\top \mathbf{1} = 1\}} \sigma_w^2$$
$$\text{s.t.} \qquad \overline{R}_w = E$$

for some level of expected return E. Let $a \equiv \mathbf{1}^\top V^{-1} \overline{\boldsymbol{R}}$, $b \equiv \overline{\boldsymbol{R}}^\top V^{-1} \overline{\boldsymbol{R}}$, and $c \equiv \mathbf{1}^\top V^{-1} \mathbf{1}$. Merton (1972) shows that the portfolio on this boundary with expected return E is given by

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[7] Alexander and Baptista (2004) provide an example with uniform distributions to illustrate that a CVaR constraint may be a more effective risk management tool than a VaR constraint. However, they do not characterize the constrained boundaries when security returns are assumed to have a discrete distribution with finitely many jump points.

[8] An Appendix containing the proofs of the theorems in our paper and a description of the numerical procedure used in our example can be downloaded at: http://home.gwu.edu/~alexbapt/JBF2Appendix.pdf.

[9] For a characterization of absence of arbitrage, see Duffie (2001, Chapter 1).

[10] See Huang and Litzenberger (1988, pp. 62–63). Section 2.4 addresses the case when there is a riskfree security.

[11] The set of return matrices such that condition (iii) does not hold has zero Lebesgue measure in $\mathbb{R}^{J \times S}$. Hence, this condition is not particularly restrictive.

[12] There are important reasons for doing so. For example, some money managers (e.g., hedge funds) hold large short positions.

$$w_E = \beta w_\sigma + (1 - \beta)w_a, \tag{1}$$

where $\beta = \frac{E - b/a}{a/c - b/a}$; $w_\sigma \equiv \frac{V^{-1}\mathbf{1}}{c}$ and $w_a \equiv \frac{V^{-1}\overline{R}}{a}$ are, respectively, the minimum variance portfolio and the portfolio on the boundary with an expected return of b/a. Using Eq. (1), portfolios on the unconstrained boundary exhibit *two-fund separation*. The *efficient frontier* consists of portfolios on this boundary with expected returns greater than or equal to a/c.

2.2. The VaR-constrained boundary

We now explore the impact of adding a VaR constraint to Markowitz's problem. In defining VaR, we follow Rockafellar and Uryasev (2002, Proposition 8). Fix a confidence level α such that $\alpha = s/S$ for some integer s with $S/2 < s < S$. Let $z_{1,w} < z_{2,w} < \cdots < z_{N_w,w}$ denote the ordered values that $\widetilde{z}_w \equiv -\widetilde{R}_w$ can take where $N_w \leqslant S$ is the number of these values. Define $n_{\alpha,w}$ as the unique index number with

$$\sum_{n=1}^{n_{\alpha,w}} p_{n,w} \geqslant \alpha > \sum_{n=1}^{n_{\alpha,w}-1} p_{n,w}, \tag{2}$$

where $p_{n,w} \equiv P[\widetilde{z}_w = z_{n,w}]$. Portfolio w's *VaR at the $100\alpha\%$ confidence level* is given by[13]

$$V_{\alpha,w} = z_{n_{\alpha,w}}. \tag{3}$$

Eqs. (2) and (3) imply that

$$P[\widetilde{R}_w \geqslant -V_{\alpha,w}] = P[\widetilde{z}_w \leqslant z_{n_{\alpha,w}}] \geqslant \alpha, \tag{4}$$

$$P[\widetilde{R}_w > -V_{\alpha,w}] = P[\widetilde{z}_w < z_{n_{\alpha,w}}] < \alpha. \tag{5}$$

Consider the VaR constraint $V_{\alpha,w} \leqslant V$ where V is the bound. A portfolio is on the *VaR-constrained boundary* if it solves

$$\min_{w \in \{\widehat{w} \in \mathbb{R}^J : \widehat{w}^\top \mathbf{1} = 1\}} \sigma_w^2$$
$$\text{s.t.} \quad \overline{R}_w = E,$$
$$V_{\alpha,w} \leqslant V$$

for some level of expected return E. For any $s \in \Omega$, let $w_s \equiv \frac{V^{-1}R_s}{c_s}$, where $c_s \equiv \mathbf{1}^\top V^{-1}R_s$. Portfolios $\{w_s\}_{s \in \Omega}$ are useful in our characterization of the VaR-constrained boundary.

Theorem 1. *Suppose that the portfolio on the VaR-constrained boundary with expected return E exists. Then, this portfolio is given by*

$$w_{E,\alpha,V} = \beta_1 w_\sigma + \beta_2 w_a + \sum_{k=3}^{K+2} \beta_k w_{s_k} \tag{6}$$

for some weights $\beta_1, \ldots, \beta_{K+2}$, where K is the number of states where the return on $w_{E,\alpha,V}$ is equal to $-V$, and $s_3, \ldots, s_{K+2}$ are these states.

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[13] Note that this definition of VaR is based on the upper quantile; see Acerbi and Tasche (2002).

Theorem 1 says that portfolios on the VaR-constrained boundary result from combinations of $K + 2$ funds: w_σ, w_a, and $w_{s_3}, \ldots, w_{s_{K+2}}$. While the latter K funds may depend on E, they do not depend *locally* on E. More precisely, there exists an open interval $(E_1, E_2) \subset \mathbb{R}$ such that the same K funds are used in Eq. (6) for any $E \in (E_1, E_2)$. Hence, portfolios on the VaR-constrained boundary exhibit locally $(K + 2)$-*fund separation*. Also, the sum of the weights of funds $w_{s_3}, \ldots, w_{s_{K+2}}$ in a portfolio on the VaR-constrained boundary is non-zero if and only if the VaR constraint binds. Thus, Eqs. (1) and (6) imply that any portfolio on this boundary at which the constraint binds does not belong to the unconstrained boundary.

The intuition of Theorem 1 is as follows. If the portfolio on the unconstrained boundary with a given expected return does not satisfy the VaR constraint, then the constraint holds with equality for the portfolio on the constrained boundary with the same expected return. Otherwise, it would have been possible to find a portfolio with a smaller variance by combining portfolios on the unconstrained and constrained boundaries with the weight of the unconstrained portfolio being arbitrarily small. Let $\underline{S}(w_{E,\alpha,V})$ denote the number of states where the return on $w_{E,\alpha,V}$ is less than $-V$. Since the constraint holds with equality, it is equivalent to: (i) K restrictions that the portfolio's return is *equal* to $-V$ in states $s_3, \ldots, s_{K+2}$; (ii) $\underline{S}(w_{E,\alpha,V})$ restrictions that the portfolio's return is *less than or equal* to $-V$ in the $\underline{S}(w_{E,\alpha,V})$ 'bad' states; and (iii) $S - K - \underline{S}(w_{E,\alpha,V})$ restrictions that the portfolio's return is *greater than or equal* to $-V$ in the 'good' states.[14] By definition, the 'less than or equal to' and 'greater than or equal to' restrictions do not bind. Thus, the VaR constraint can be simplified into K binding restrictions from which the K funds arise.

2.3. The CVaR-constrained boundary

We now explore the impact of adding a CVaR constraint to Markowitz's problem. In defining CVaR, we follow Rockafellar and Uryasev (2002, Proposition 8). Portfolio w's *CVaR at the* $100\alpha\%$ *confidence level* is given by

$$C_{\alpha,w} = \frac{1}{1-\alpha} \left[\left(\sum_{n=1}^{n_{\alpha,w}} p_{n,w} - \alpha \right) z_{n_{\alpha,w}} + \sum_{n=n_{\alpha,w}+1}^{N_w} p_{n,w} z_{n,w} \right]. \tag{7}$$

Consider the CVaR constraint $C_{\alpha,w} \leqslant C$ where C is the bound. A portfolio is on the *CVaR-constrained boundary* if it solves

$$\min_{w \in \{\widehat{w} \in \mathbb{R}^J : \widehat{w}^\top \mathbf{1} = 1\}} \sigma_w^2$$
$$\text{s.t.} \qquad \overline{R}_w = E,$$
$$C_{\alpha,w} \leqslant C$$

for some level of expected return E. Let $R_\Phi \equiv \frac{\sum_{s \in \Phi} R_s}{S(\Phi)}$, where $\Phi \subset \Omega$ is a set of $S(\Phi) \leqslant (1 - \alpha)S$ distinct states. Suppose that $\text{rank}\left(\begin{bmatrix} \mathbf{1} & \overline{R} & R_\Phi & R_{s_1} & \cdots & R_{s_{J-3}} \end{bmatrix} \right) = J$ for any $\Phi \times \{s_1, \ldots, s_{J-3}\} \subset \Omega \times \Omega$ with $\Phi \cap \{s_1, \ldots, s_{J-3}\} = \emptyset$.[15] For any $\Phi \subset \Omega$, let $w_\Phi \equiv \frac{V^{-1} R_\Phi}{c_\Phi}$ where $c_\Phi \equiv$

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[14] A 'good' ('bad') state is a state for which the portfolio's return is greater (less) than minus its VaR.
[15] Note that the set of return matrices such that this condition does not hold has zero Lebesgue measure in $\mathbb{R}^{J \times S}$. Hence, the condition is not particularly restrictive.

$\mathbf{1}^\top V^{-1} \boldsymbol{R}_\Phi$. Portfolios $\{\boldsymbol{w}_\Phi\}_{\Phi \subset \Omega}$ are useful in our characterization of the CVaR-constrained boundary.

Theorem 2. *Suppose that the portfolio on the CVaR-constrained boundary with expected return E exists. Let $\Phi = \{s \in \Omega : \boldsymbol{w}_{E,\alpha,C}^\top \boldsymbol{R}_s < -V_{\alpha,\boldsymbol{w}_{E,\alpha,C}}\}$.*[16] *(i) If $P[\Phi] = 1 - \alpha$, then this portfolio is given by*

$$\boldsymbol{w}_{E,\alpha,C} = \beta_1 \boldsymbol{w}_\sigma + \beta_2 \boldsymbol{w}_a + \beta_3 \boldsymbol{w}_\Phi \tag{8}$$

for some weights β_1, β_2, and β_3. (ii) If $P[\Phi] < 1 - \alpha$, then

$$\boldsymbol{w}_{E,\alpha,C} = \beta_1 \boldsymbol{w}_\sigma + \beta_2 \boldsymbol{w}_a + \beta_3 \boldsymbol{w}_\Phi + \sum_{k=4}^{K+3} \beta_k \boldsymbol{w}_{s_k} \tag{9}$$

for some weights $\beta_1, \ldots, \beta_{K+3}$, where K is the number of states where the return on $\boldsymbol{w}_{E,\alpha,C}$ is equal to $-V_{\alpha,\boldsymbol{w}_{E,\alpha,C}}$, and $s_4, \ldots, s_{K+3}$ are these states.

Theorem 2 says that the funds required for a portfolio on the CVaR-constrained boundary depend on the set of states for which the portfolio suffers losses larger than its VaR (i.e., Φ). First, suppose that $P[\Phi] = 1 - \alpha$. Then, the portfolio results from a combination of funds $\boldsymbol{w}_\sigma$, $\boldsymbol{w}_a$, and $\boldsymbol{w}_\Phi$. While the latter fund may depend on E, it does not depend *locally* on E. More precisely, there exists an open interval $(E_1, E_2) \subset \mathbb{R}$ such that the same fund $\boldsymbol{w}_\Phi$ is used in Eq. (8) for any $E \in (E_1, E_2)$. Hence, portfolios on the CVaR-constrained boundary exhibit locally *three-fund separation*. Also, the weight of fund $\boldsymbol{w}_\Phi$ in a portfolio on the CVaR-constrained boundary is non-zero if and only if the CVaR constraint binds. Thus, Eqs. (1) and (8) imply that portfolios on this boundary at which the constraint binds do not belong to the unconstrained boundary.

Second, suppose that $P[\Phi] < 1 - \alpha$. Then, the portfolio on the CVaR-constrained boundary results from a combination of funds $\boldsymbol{w}_\sigma$, $\boldsymbol{w}_a$, $\boldsymbol{w}_\Phi$, and $\boldsymbol{w}_{s_4}, \ldots, \boldsymbol{w}_{s_{K+3}}$. While the latter $K + 1$ funds may depend on E, they do not depend *locally* on E. More precisely, there exists an open interval $(E_1, E_2) \subset \mathbb{R}$ such that the same funds $\boldsymbol{w}_\Phi, \boldsymbol{w}_{s_4}, \ldots, \boldsymbol{w}_{s_{K+3}}$ are used in Eq. (9) for any $E \in (E_1, E_2)$. Hence, portfolios on the CVaR-constrained boundary exhibit locally $(K + 3)$-*fund separation*. Also, the sum of the weights of funds $\boldsymbol{w}_\Phi, \boldsymbol{w}_{s_4}, \ldots, \boldsymbol{w}_{s_{K+3}}$ in a portfolio on the CVaR-constrained boundary is non-zero if and only if the CVaR constraint binds. Thus, Eqs. (1) and (9) imply that portfolios on this boundary at which the constraint binds do not belong to the unconstrained boundary.

The intuition of Theorem 2 is as follows. If the portfolio on the unconstrained boundary with a given expected return does not satisfy the CVaR constraint, then the constraint holds with equality for the portfolio on the constrained boundary with the same expected return. Otherwise, it would have been possible to find a portfolio with a smaller variance by combining portfolios on the unconstrained and constrained boundaries with the weight of the unconstrained portfolio being arbitrarily small. First, suppose that $P[\Phi] = 1 - \alpha$. Note that Φ contains $S(1 - \alpha)$ states where the return on $\boldsymbol{w}_{E,\alpha,C}$ is less than $-V$. Since the constraint holds with equality, it is equivalent to: (i) one restriction that the portfolio's CVaR is *equal* to C; (ii) $S(1 - \alpha)$ restrictions that the portfolio's return is *less than or equal* to $-V$ in the $S(1 - \alpha)$ 'bad' states; and (iii) $S - S(1 - \alpha)$ restrictions that the portfolio's return is *greater than or equal* to $-V$ in the 'good' states. By definition, the 'less than or

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[16] While here Φ is a particular subset of Ω, in the previous paragraph Φ is used to denote subsets of Ω with $S(\Phi) \leqslant (1 - \alpha)S$ distinct states.

equal to' and 'greater than or equal to' restrictions do not bind. Thus, the CVaR constraint can be simplified into a single binding restriction from which fund w_{Φ} arises.

Second, suppose that $P[\Phi] < 1 - \alpha$. Let $\underline{S}(w_{E,\alpha,C})$ denote the number of states where the return on $w_{E,\alpha,C}$ is less than $-V_{\alpha,w_{E,\alpha,C}}$. Since the constraint holds with equality, it is equivalent to: (i) one restriction that the portfolio's CVaR is *equal* to C; (ii) K restrictions that the portfolio's return is *equal* to $-V$ in states $s_4, \ldots, s_{K+3}$; (iii) $\underline{S}(w_{E,\alpha,C})$ restrictions that the portfolio's return is *less than or equal* to $-V$ in the $\underline{S}(w_{E,\alpha,C})$ 'bad' states; and (iv) $S - K - \underline{S}(w_{E,\alpha,C})$ restrictions that the portfolio's return is *greater than or equal* to $-V$ in the 'good' states. By definition, the 'less than or equal to' and 'greater than or equal to' restrictions do not bind. Thus, the CVaR constraint can be simplified into $K + 1$ binding restrictions from which funds $w_{\Phi}, w_{s_4}, \ldots, w_{s_{K+3}}$ arise.

2.4. Introducing a riskfree security

Suppose that there is a riskfree security with return $R_f > 0$. Merton (1972) shows that
$$w_E = \theta w_f + (1 - \theta)w_t, \tag{10}$$
where $\theta = \frac{E - \bar{R}_{w_t}}{R_f - \bar{R}_{w_t}}$; $w_f \equiv \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^{\top}$ and $w_t \equiv \begin{bmatrix} \frac{[V^{-1}(\bar{R} - 1R_f)]^{\top}}{a - cR_f} & 0 \end{bmatrix}^{\top}$ denote, respectively, the riskfree and tangency portfolios. Using Eq. (10), portfolios on the unconstrained boundary exhibit *two-fund separation* as in the absence of a riskfree security. The *efficient frontier* consists of portfolios on this boundary with expected returns greater than or equal to R_f.

Let $w_s \equiv \begin{bmatrix} \frac{[V^{-1}(R_s - 1R_f)]^{\top}}{c_s - cR_f} & 0 \end{bmatrix}^{\top}$ for any $s \in \Omega$. Portfolios $\{w_s\}_{s \in \Omega}$ are useful in our characterization of the VaR-constrained boundary.

Theorem 3. *Suppose that the portfolio on the VaR-constrained boundary with expected return E exists. Then, this portfolio is given by*
$$w_{E,\alpha,V} = \theta_1 w_f + \theta_2 w_t + \sum_{k=3}^{K+2} \theta_k w_{s_k} \tag{11}$$
for some weights $\theta_1, \ldots, \theta_{K+2}$, where K is the number of states where the return on $w_{E,\alpha,V}$ is equal to $-V$, and $s_3, \ldots, s_{K+2}$ are these states.

As in Theorem 1, Theorem 3 indicates that portfolios on the VaR-constrained boundary exhibit locally $(K + 2)$-*fund separation* with these funds being w_f, w_t, and $w_{s_3}, \ldots, w_{s_{K+2}}$. Also, the sum of the weights of the latter K funds in a portfolio on the VaR-constrained boundary is non-zero if and only if the VaR constraint binds. Thus, Eqs. (10) and (11) imply that portfolios on this boundary at which the constraint binds do not belong to the unconstrained boundary.

Let $w_{\Phi} \equiv \begin{bmatrix} \frac{[V^{-1}(R_{\Phi} - 1R_f)]^{\top}}{c_{\Phi} - cR_f} & 0 \end{bmatrix}^{\top}$ for any $\Phi \subset \Omega$. Portfolios $\{w_{\Phi}\}_{\Phi \subset \Omega}$ are useful in our characterization of the CVaR-constrained boundary.

Theorem 4. *Suppose that the portfolio on the CVaR-constrained boundary with expected return E exists. Let $\Phi = \{s \in \Omega : w_{E,\alpha,C}^{\top} R_s + (1 - w_{E,\alpha,C}^{\top} 1)R_f < -V_{\alpha,w_{E,\alpha,C}}\}$.[17] (i) If $P[\Phi] = 1 - \alpha$, then this portfolio is given by*

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[17] Footnote 16 also applies here.

$$w_{E,\alpha,C} = \theta_1 w_f + \theta_2 w_t + \theta_3 w_\Phi \tag{12}$$

for some weights θ_1, θ_2, and θ_3. (ii) If $P[\Phi] < 1 - \alpha$, then this portfolio is given by

$$w_{E,\alpha,C} = \theta_1 w_f + \theta_2 w_t + \theta_3 w_\Phi + \sum_{k=4}^{K+3} \theta_k w_{s_k} \tag{13}$$

for some weights $\theta_1, \ldots, \theta_{K+3}$, where K is the number of states where the return on $w_{E,\alpha,C}$ is equal to $-V_{\alpha, w_{E,\alpha,C}}$, and $s_4, \ldots, s_{K+3}$ are these states.

As in Theorem 2, Theorem 4 indicates that the funds required for a portfolio on the CVaR-constrained boundary depend on the set of states for which the portfolio suffers losses larger than its VaR (i.e., Φ). First, when $P[\Phi] = 1 - \alpha$, portfolios on the CVaR-constrained boundary exhibit locally *three-fund separation* with these funds being w_f, w_t, and w_Φ. Also, the weight of the latter fund in a portfolio on the CVaR-constrained boundary is non-zero if and only if the CVaR constraint binds. Thus, Eqs. (10) and (12) imply that portfolios on this boundary at which the constraint binds do not belong to the unconstrained boundary.

Second, when $P[\Phi] < 1 - \alpha$, portfolios on the CVaR-constrained boundary exhibit locally $(K + 3)$-*fund separation* with these funds being w_f, w_t, w_Φ, and $w_{s_4}, \ldots, w_{s_{K+3}}$. Also, the sum of the weights of the latter $K + 1$ funds in a portfolio on the CVaR-constrained boundary is non-zero if and only if the CVaR constraint binds. Thus, Eqs. (10) and (13) imply that portfolios on this boundary at which the constraint binds do not belong to the unconstrained boundary.

3. An example

Next, we derive the unconstrained and constrained boundaries for a sample of US stocks. We select a sample of 10 NYSE- and Nasdaq-listed stocks with weekly return data available in CRSP during the period March 2, 1999–December 31, 2002 in order to have 200 observations. Their sample means, standard deviations, and correlation coefficients were used as optimization inputs.[18] We estimated the VaRs and CVaRs of the stocks (and all portfolios) using historical data. For example, since $S = 200$, the VaR at the 99% confidence level of a portfolio with different returns in different states is given by minus its third worst weekly return. Table 1 presents summary statistics.

We now proceed to examine the constrained boundaries when the confidence level is $\alpha = 99\%$. The case when no riskfree security exists is considered in detail next; the case when there exists a riskfree security is briefly discussed afterwards.

3.1. The VaR-constrained boundary

Three VaR bounds are considered: $V = 7\%$, 7.5%, and 8%.

3.1.1. Comparison with the unconstrained boundary

Fig. 1 illustrates the VaR-constrained boundary when $V = 7\%$. Note that only portfolios on the unconstrained boundary with moderate expected returns satisfy the constraint.

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[18] Michaud (1998, p. 12) points out that these sample statistics are sometimes used as optimization inputs. For simplicity, no adjustment for estimation risk is made in this example.

Table 1
Optimization inputs used in the example

	1	2	3	4	5	6	7	8	9	10
Mean	0.25	0.79	0.19	0.23	0.20	0.22	0.20	0.44	0.29	0.26
Std. Dev.	4.88	5.33	5.36	6.43	4.11	5.42	19.32	5.20	6.60	5.32
VaR$_{99\%}$	10.48	14.97	11.71	12.77	8.24	13.95	23.29	13.82	13.89	14.81
CVaR$_{99\%}$	12.57	15.52	12.98	17.19	13.79	15.98	58.42	15.12	23.08	15.93
Correlation matrix										
1	1	0.13	0.27	0.16	0.31	0.24	0.09	0.31	0.12	0.34
2		1	0.23	0.16	0.15	0.06	0.15	0.14	0.09	0.16
3			1	0.07	0.42	0.37	−0.07	0.41	0.20	0.36
4				1	−0.17	0.07	0.09	0.12	0.08	0.15
5					1	0.23	0.04	0.28	0.10	0.22
6						1	0.08	0.33	0.32	0.40
7							1	0.00	0.03	0.05
8								1	0.31	0.28
9									1	0.23
10										1

We use a sample of 10 NYSE- and Nasdaq-listed stocks with weekly return data available in CRSP during the period March 2, 1999–December 31, 2002 in order to have 200 observations. Sample means, standard deviations, and correlation coefficients associated with these stocks were computed and then used as optimization inputs. The VaRs and CVaRs of the stocks were estimated using historical data. The means, standard deviations, VaRs, and CVaRs are reported in percentage points.

Thus, the VaR-constrained boundary consists of portfolios on the unconstrained boundary with moderate expected returns and portfolios not on it with small and large ones.[19]

Portfolio w's *distance from the unconstrained boundary*, denoted by D_w, is the difference between its standard deviation and that of the portfolio on the unconstrained boundary with the same expected return. As Fig. 1 shows, the distance of portfolios on the VaR-constrained boundary from the unconstrained boundary increases for smaller and larger expected returns.

Fig. 2 displays properties of portfolios on the VaR-constrained boundary. Panels (a)–(c) show the relation between the number of funds required for these portfolios and their expected returns. Four main results can be seen. First, the vertical segments illustrate that the number of funds is locally constant. Second, the steps illustrate that this number is not globally constant. Third, the number is equal to two for moderate values of E (since the constraint does not bind), and greater than two for small and large ones (since the constraint binds). Fourth, for a fixed E that is either small or large, the number of funds depends on V.

Panels (d)–(f) show the relative reduction in VaR arising from selecting the portfolio on the VaR-constrained boundary instead of the portfolio on the unconstrained boundary with the same expected return (i.e., $1 - V_{\alpha,w_{E,\alpha,V}} / V_{\alpha,w_E}$).[20] Two results are worth noting. First, while VaR decreases for small and large values of E, VaR is unchanged for moderate values of E. Second, the reduction in VaR is notable for very large values of E.

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[19] The qualifiers 'small,' 'moderate,' and 'large' are used with reference to the expected return of the minimum variance portfolio, which lies roughly in the middle of the moderate range.

[20] Note that a positive (negative) value indicates a decrease (increase) in VaR.
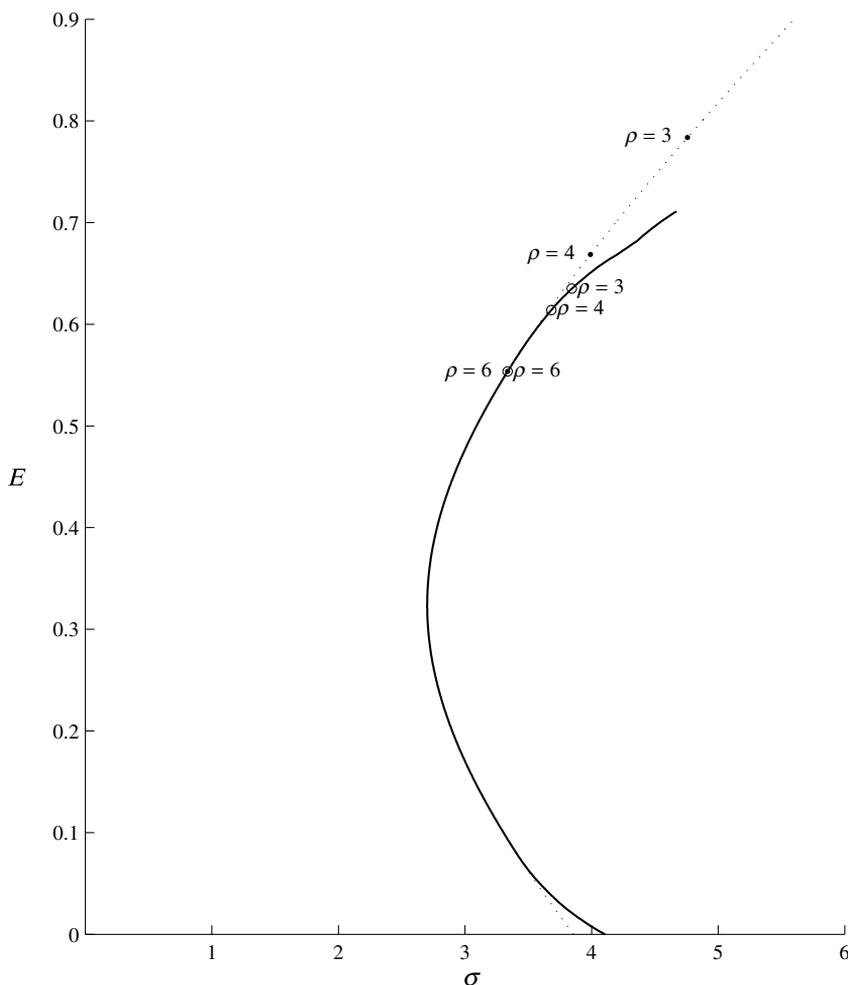
Fig. 1. Unconstrained and VaR-constrained boundaries and optimal portfolios in the absence of a riskfree security. The dotted (solid) line represents the unconstrained (VaR-constrained) boundary. The confidence level is $\alpha = 99\%$, the number of states is $S = 200$, and the VaR bound is $V = 7\%$. There is no riskfree security. The panel also shows the optimal portfolios of an agent with the objective function $U(E, \sigma) = E - \frac{\rho}{2}\sigma^2$, where $\rho = 3$, 4, and 6. The dots ('•') represent the unconstrained optimal portfolios, while the circles ('○') represent the VaR-constrained optimal portfolios. Note that when $\rho = 6$ the unconstrained and VaR-constrained portfolios coincide. The expected returns and standard deviations are reported in percentage points.

Panels (g)–(i) show the relative reduction in CVaR arising from selecting the portfolio on the VaR-constrained boundary instead of the portfolio on the unconstrained boundary with the same expected return (i.e., $1 - C_{\alpha, w_{E,\alpha,V}}/C_{\alpha, w_E}$). While there is a decrease (increase) in CVaR for small (most large) values of E, CVaR is unchanged for moderate values of E.

3.1.2. Portfolio selection implications

Of particular interest is the impact of a VaR constraint on an agent's optimal portfolio. For simplicity, consider an agent with the objective function $U : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ defined by

Fig. 2. Properties of portfolios on the VaR-constrained boundary in the absence of a riskfree security. Panels (a)–(c) report the number of funds required to span the portfolios on the VaR-constrained boundary. Panels (d)–(f) report the relative reduction in VaR arising from selecting a portfolio on the VaR-constrained boundary instead of the portfolio on the unconstrained boundary with the same expected return. Panels (g)–(i) report the relative reduction in CVaR arising from selecting a portfolio on the VaR-constrained boundary instead of the portfolio on the unconstrained boundary with the same expected return. The expected returns and relative reductions in VaR and CVaR are reported in percentage points. There is no riskfree security.

$U(E, \sigma) = E - \frac{\rho}{2}\sigma^2$, where $\rho = 3$, 4, and 6.[21] Fig. 1 shows the location of the agent's optimal portfolio: (i) in the absence of any constraint ('unconstrained optimal portfolio'), represented by a dot ('•'), and (ii) in the presence of a VaR constraint ('VaR-constrained optimal portfolio'), represented by a circle ('○'). Panel (a) of Table 2 presents quantitative

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[21] These are reasonable values of ρ as noted by, for example, Sharpe (1987, Chapter 2).

Table 2
Unconstrained and constrained optimal portfolios

ρ	V	Absence of a riskfree security					Presence of a riskfree security				
		$\overline{R}_w$	σ_w	D_w	$V_{\alpha,w}$	$C_{\alpha,w}$	$\overline{R}_w$	σ_w	D_w	$V_{\alpha,w}$	$C_{\alpha,w}$
(a) VaR constraint											
3	7	0.64	3.84	0.06	7.00	12.53	0.64	3.86	0.05	7.00	12.20
	7.5	0.67	4.06	0.05	7.50	13.10	0.68	4.07	0.04	7.50	12.77
	8	0.71	4.28	0.04	8.00	13.49	0.70	4.24	0.03	8.00	13.23
	Unconstrained	0.78	4.76	0.00	9.82	14.41	0.81	4.94	0.00	10.24	15.08
4	7	0.61	3.68	0.02	7.00	11.62	0.60	3.49	0.00	7.00	10.75
	7.5	0.64	3.81	0.01	7.50	11.90	0.62	3.65	0.00	7.50	11.17
	8	0.66	3.92	0.00	8.00	12.14	0.63	3.70	0.00	7.66	11.29
	Unconstrained	0.67	3.99	0.00	8.28	12.29	0.63	3.70	0.00	7.66	11.29
6	7	0.55	3.34	0.00	6.75	10.17	0.45	2.47	0.00	5.08	7.50
	7.5	0.55	3.34	0.00	6.75	10.17	0.45	2.47	0.00	5.08	7.50
	8	0.55	3.34	0.00	6.75	10.17	0.45	2.47	0.00	5.08	7.50
	Unconstrained	0.55	3.34	0.00	6.75	10.17	0.45	2.47	0.00	5.08	7.50
(b) CVaR constraint											
3	8	0.61	3.87	0.25	8.00	8.00	0.59	3.64	0.17	7.68	8.00
	8.5	0.63	3.99	0.23	8.40	8.50	0.61	3.72	0.15	7.86	8.50
	9	0.64	4.04	0.20	8.58	9.00	0.62	3.79	0.12	8.03	9.00
	Unconstrained	0.78	4.76	0.00	9.82	14.41	0.81	4.94	0.00	10.24	15.08
4	8	0.56	3.50	0.14	7.29	8.00	0.53	3.06	0.04	6.46	8.00
	8.5	0.57	3.53	0.11	7.41	8.50	0.54	3.16	0.03	6.66	8.50
	9	0.58	3.58	0.08	7.53	9.00	0.56	3.25	0.02	6.83	9.00
	Unconstrained	0.67	3.99	0.00	8.28	12.29	0.63	3.70	0.00	7.66	11.29
6	8	0.50	3.12	0.04	6.25	8.00	0.45	2.47	0.00	5.08	7.50
	8.5	0.51	3.16	0.02	6.37	8.50	0.45	2.47	0.00	5.08	7.50
	9	0.52	3.21	0.01	6.49	9.00	0.45	2.47	0.00	5.08	7.50
	Unconstrained	0.55	3.34	0.00	6.75	10.17	0.45	2.47	0.00	5.08	7.50

Consider an agent with the objective function $U(E, \sigma) = E - \frac{\rho}{2}\sigma^2$, where $\rho = 3, 4$, and 6. Panel (a) reports the expected return ($\overline{R}_w$), standard deviation (σ_w), distance from the unconstrained boundary (D_w), VaR ($V_{\alpha,w}$), and CVaR ($C_{\alpha,w}$) of the unconstrained and VaR-constrained optimal portfolios. Panel (b) reports these statistics for the unconstrained and CVaR-constrained optimal portfolios. In the presence of a riskfree security, we assume that its annualized return is $R_f = 4.16\%$. The confidence level is $\alpha = 99\%$, the VaR bound is $V = 7\%, 7.5\%$, and 8%, the CVaR bound is $C = 8\%, 8.5\%$, and 9%, and the number of states is $S = 200$. All numbers are reported in percentage points.

results. Two results are worth noting when the constraint binds. First, VaR-constrained optimal portfolios have smaller expected returns, standard deviations, VaRs, and CVaRs than those of unconstrained optimal portfolios. Second, the distance of VaR-constrained optimal portfolios from the unconstrained boundary decreases when either ρ increases or V increases.

3.2. The CVaR-constrained boundary

Note that (a) $C_{\alpha,w} \geqslant V_{\alpha,w}$ for any portfolio w; and (b) $C_{\alpha,w} > V_{\alpha,w}$ if $P[\widetilde{R}_w < -V_{\alpha,w}] > 0$. Since it is therefore natural to assume that $C \geqslant V$, three CVaR bounds are considered: $C = 8\%, 8.5\%$, and 9%.

3.2.1. Comparison with the unconstrained boundary

Fig. 3 illustrates the CVaR-constrained boundary when $C = 8\%$. The results are qualitatively similar to those obtained when a VaR constraint is imposed. Fig. 4 displays properties of portfolios on the CVaR-constrained boundary. Panels (a)–(c) show the relation between the number of funds required for these portfolios and their expected returns. The results are similar to those obtained for portfolios on the VaR-constrained boundary.

Panels (d)–(f) show the relative reduction in VaR arising from selecting the portfolio on the CVaR-constrained boundary instead of the portfolio on the unconstrained boundary
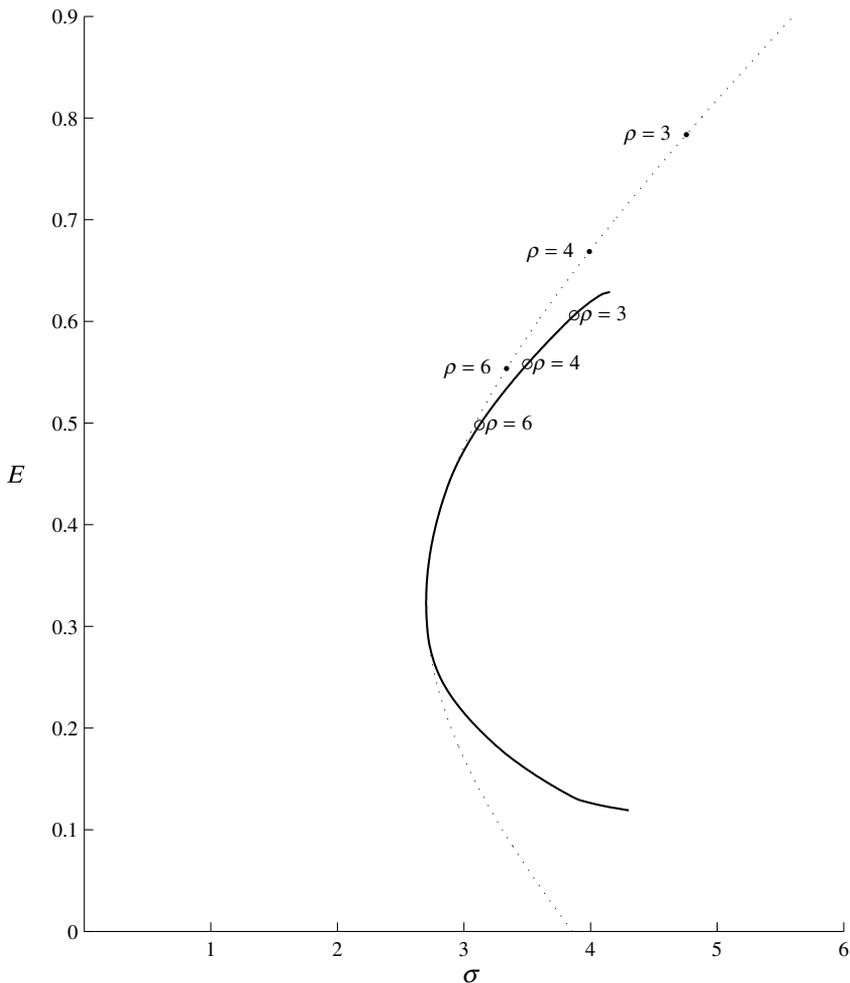
Fig. 3. Unconstrained and CVaR-constrained boundaries and optimal portfolios in the absence of a riskfree security. The dotted (solid) line represents the unconstrained (CVaR-constrained) boundary. The confidence level is $\alpha = 99\%$, the number of states is $S = 200$, and the CVaR bound is $C = 8\%$. There is no riskfree security. The panel also shows the optimal portfolios of an agent with the objective function $U(E, \sigma) = E - \frac{\rho}{2}\sigma^2$, where $\rho = 3, 4$, and 6. The dots ('•') represent the unconstrained optimal portfolios, while the circles ('○') represent the CVaR-constrained optimal portfolios. The expected returns and standard deviations are reported in percentage points.

Fig. 4. Properties of portfolios on the CVaR-constrained boundary in the absence of a riskfree security. Panels (a)–(c) report the number of funds required to span the portfolios on the CVaR-constrained boundary. Panels (d)–(f) report the relative reduction in VaR arising from selecting a portfolio on the CVaR-constrained boundary instead of the portfolio on the unconstrained boundary with the same expected return. Panels (g)–(i) report the relative reduction in CVaR arising from selecting a portfolio on the CVaR-constrained boundary instead of the portfolio on the unconstrained boundary with the same expected return. The expected returns and relative reductions in VaR and CVaR are reported in percentage points. There is no riskfree security.

with the same expected return (i.e., $1 - V_{\alpha, w_{E,\alpha,C}}/V_{\alpha, w_E}$). While there is an increase in VaR for most small and large values of E, VaR is unchanged for moderate values of E.

Panels (g)–(i) show the relative reduction in CVaR arising from selecting the portfolio on the CVaR-constrained boundary instead of the portfolio on the unconstrained boundary with the same expected return (i.e., $1 - C_{\alpha, w_{E,\alpha,C}}/C_{\alpha, w_E}$). Two results are worth noting. First, while CVaR decreases for small and large values of E, CVaR is unchanged for

moderate values of E. Second, the reduction in CVaR is notable for very small and very large values of E.

3.2.2. Portfolio selection implications

Next, we address the question of how the agent's optimal portfolio is affected by a CVaR constraint. Fig. 3 shows the location of the unconstrained optimal portfolio, represented by a dot, and the agent's optimal portfolio in the presence of a CVaR constraint
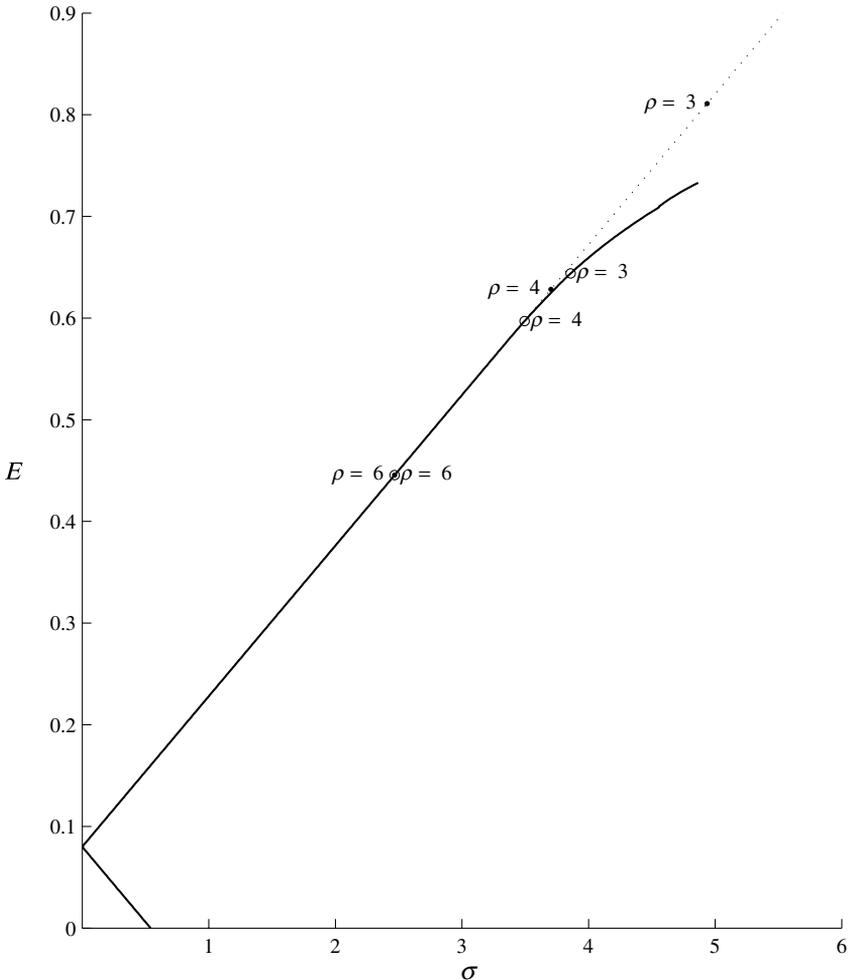
Fig. 5. Unconstrained and VaR-constrained boundaries and optimal portfolios in the presence of a riskfree security. The dotted (solid) line represents the unconstrained (VaR-constrained) boundary. The confidence level is $\alpha = 99\%$, the number of states is $S = 200$, and the VaR bound is $V = 7\%$. There is a riskfree security with an annualized return of $R_f = 4.16\%$. The panel also shows the optimal portfolios of an agent with the objective function $U(E, \sigma) = E - \frac{\rho}{2}\sigma^2$, where $\rho = 3$, 4, and 6. The dots ('•') represent the unconstrained optimal portfolios, while the circles ('○') represent the VaR-constrained optimal portfolios. Note that when $\rho = 6$ the unconstrained and VaR-constrained portfolios coincide. The expected returns and standard deviations are reported in percentage points.

('CVaR-constrained optimal portfolio'), represented by a circle. Panel (b) of Table 2 presents quantitative results. Our results regarding the implications of a CVaR constraint are qualitatively similar to those obtained for a VaR constraint, but a quantitative difference is noteworthy. While the VaR of the CVaR-constrained optimal portfolio is close to that of

Fig. 6. Properties of portfolios on the VaR-constrained boundary in the presence of a riskfree security. Panels (a)–(c) report the number of funds required to span the portfolios on the VaR-constrained boundary. Panels (d)–(f) report the relative reduction in VaR arising from selecting a portfolio on the VaR-constrained boundary instead of the portfolio on the unconstrained boundary with the same expected return. Panels (g)–(i) report the relative reduction in CVaR arising from selecting a portfolio on the VaR-constrained boundary instead of the portfolio on the unconstrained boundary with the same expected return. The expected returns and relative reductions in VaR and CVaR are reported in percentage points. There is a riskfree security with an annualized return of $R_f = 4.16\%$.

the VaR-constrained optimal portfolio, the CVaR of the former is notably smaller than that of the latter. This result suggests that a CVaR constraint is more effective than a VaR constraint to curtail large losses in the mean–variance model.

Table 2 is also useful to compare the implications of binding CVaR and VaR constraints when $C = V$. Note that CVaR-constrained optimal portfolios have smaller expected returns, standard deviations, VaRs, and CVaRs than those of VaR-constrained optimal portfolios. Also, the distance of CVaR-constrained optimal portfolios from the
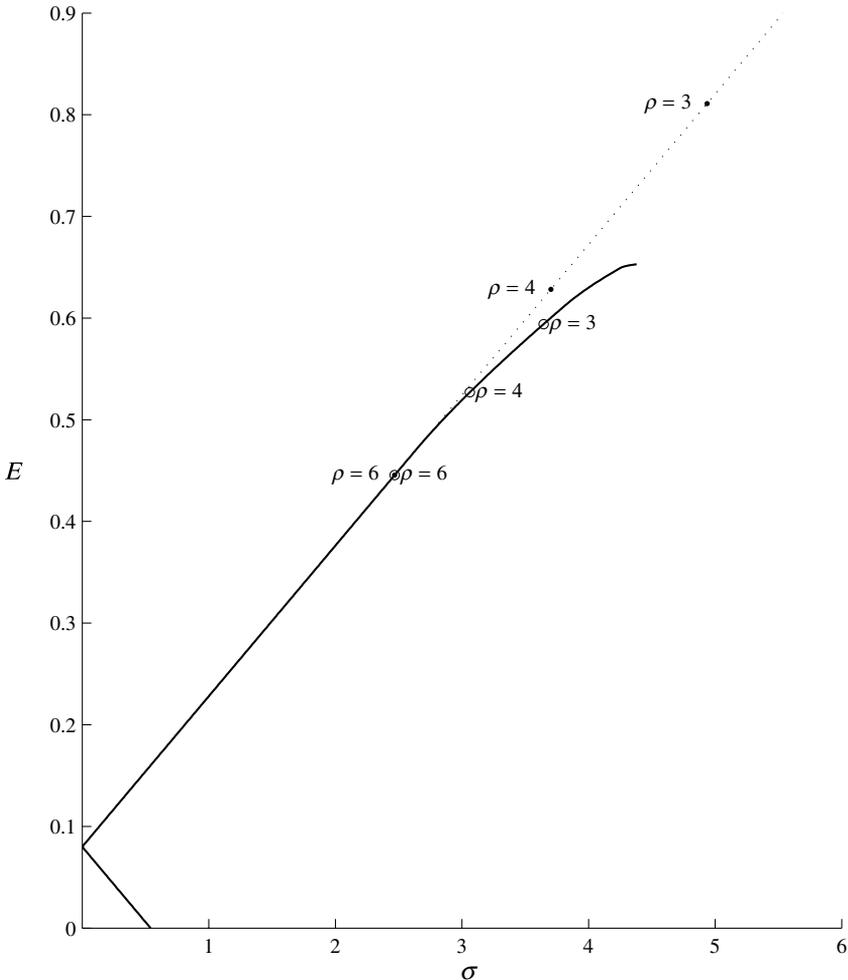
Fig. 7. Unconstrained and CVaR-constrained boundaries and optimal portfolios in the presence of a riskfree security. The dotted (solid) line represents the unconstrained (CVaR-constrained) boundary. The confidence level is $\alpha = 99\%$, the number of states is $S = 200$, and the CVaR bound is $C = 8\%$. There is a riskfree security with an annualized return of $R_f = 4.16\%$. The panel also shows the optimal portfolios of an agent with the objective function $U(E, \sigma) = E - \frac{\rho}{2}\sigma^2$, where $\rho = 3$, 4, and 6. The dots ('•') represent the unconstrained optimal portfolios, while the circles ('○') represent the CVaR-constrained optimal portfolios. Note that when $\rho = 6$, the unconstrained and CVaR-constrained optimal portfolios coincide. The expected returns and standard deviations are reported in percentage points.

unconstrained boundary are larger than those of VaR-constrained optimal portfolios. The reason why these results hold is that a CVaR constraint is more restrictive than a VaR constraint when $C = V$.

Fig. 8. Properties of portfolios on the CVaR-constrained boundary in the presence of a riskfree security. Panels (a)–(c) report the number of funds required to span the portfolios on the CVaR-constrained boundary. Panels (d)–(f) report the relative reduction in VaR arising from selecting a portfolio on the CVaR-constrained boundary instead of the portfolio on the unconstrained boundary with the same expected return. Panels (g)–(i) report the relative reduction in CVaR arising from selecting a portfolio on the CVaR-constrained boundary instead of the portfolio on the unconstrained boundary with the same expected return. The expected returns and relative reductions in VaR and CVaR are reported in percentage points. There is a riskfree security with an annualized return of $R_f = 4.16\%$.

3.3. Introducing a riskfree security

Suppose now that there is a riskfree security with a return of $R_f = 4.16\%$.[22] As Table 2 and Figs. 5–8 show, the results are similar to those in the absence of a riskfree security with one exception. In the presence of a riskfree security, the portfolios on the constrained boundaries with small expected returns are also on the unconstrained boundary.

4. Conclusion

This article examines the impact of adding either a VaR or a CVaR constraint to the mean–variance model. While previous research involving this model assumes that security returns have an elliptical distribution, we assume that they have a discrete distribution with finitely many jump points. The latter assumption is often imposed in practice when using historical simulation to estimate VaR and CVaR. Our main results are as follows. First, portfolios on the VaR-constrained boundary exhibit $(K + 2)$-*fund separation*, where K is the number of states for which the portfolios suffer losses equal to the VaR bound, and thus are MV-inefficient when the constraint binds. Second, portfolios on the CVaR-constrained boundary exhibit $(K + 3)$-*fund separation*, where K is the number of states for which the portfolios suffer losses equal to their VaRs, and thus are also MV-inefficient when the constraint binds. Third, an example illustrates that while the VaR of the CVaR-constrained optimal portfolio is close to that of the VaR-constrained optimal portfolio, the CVaR of the former is notably smaller than that of the latter. This result suggests that a CVaR constraint is more effective than a VaR constraint to curtail large losses in the mean–variance model.

Our results are in stark contrast with previous work, which assumes that security returns have an elliptical distribution. Under this assumption, portfolios on the constrained boundaries exhibit *two-fund separation* even when the constraints bind. Hence, constrained boundaries consist solely of subsets of the unconstrained boundary. Thus, it is important to assess which assumption regarding security returns is appropriate when implementing the mean–variance model with VaR or CVaR constraints.

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[22] The average annualized Fed funds rate during our sample period was 4.16%.

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