An Information-Theoretic Approach to Constructing Coherent Risk Measures

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***Abstract*— In the past decade, the new concept of coherent risk measure has found many applications in finance, insurance and operations research. In this paper, we introduce a new class of coherent risk measures constructed by using information-type pseudo-distances that generalize the Kullback-Leibler divergence, also known as the relative entropy. We first analyze the primal and dual representations of this class. We then study entropic value-at-risk (EVaR) which is the member of this class associated with relative entropy. We also show that conditional value-at-risk (CVaR), which is the most popular coherent risk measure, belongs to this class and is a lower bound for EVaR.**

***Keywords*— Generalized relative entropy, Kullback-Leibler divergence, Coherent risk measure, Conditional value-at-risk, Entropic value-at-risk**

I. INTRODUCTION

A *risk measure* is a function ρ that assigns a real value to a random variable X, thereby enabling us to select a suitable random variable from a set $\mathbf{X}$ of allowable random variables. To define the concept of risk measure more precisely, let $(\Omega, \mathbf{F}, P)$ be a probability space where Ω is a set of all simple events, $\mathbf{F}$ is a σ-algebra of subsets of Ω, and P is a probability measure on $\mathbf{F}$. Also, let $\mathbf{L}$ be the set of all real-valued Borel measurable functions (random variables) $X:\Omega \to \Re$, and $\mathbf{X} \subseteq \mathbf{L}$ be a subspace including all constant functions. For $p \geq 1$, $\mathbf{L}_p$ stands for the set of all Borel measurable functions $X:\Omega \to \Re$ for which

$$\int |X(\omega)|^p P(d\omega) < +\infty,$$

and $\mathbf{L}_\infty$ is the set of all bounded Borel measurable functions.

Now we define the risk measure $\rho: \mathbf{X} \to \overline{\Re}$, where $\overline{\Re} = \Re \cup \{-\infty, +\infty\}$ is the extended real line. In the literature, in order to find a suitable risk measure, several desirable properties are introduced. If we prefer smaller values of $X \in \mathbf{X}$, then the most important properties for the risk measure ρ are as follows:

(P1) (Translation Invariance) $\rho(X + c) = \rho(X) + c$ for any $X \in \mathbf{X}$ and any constant c;

(P2) (Subadditivity) $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$ for all $X_1, X_2 \in \mathbf{X}$;

(P3) (Monotonicity) If $X_1, X_2 \in \mathbf{X}$ and $X_1 \leq X_2$, then $\rho(X_1) \leq \rho(X_2)$;

(P4) (Positive Homogeneity) $\rho(\lambda X) = \lambda \rho(X)$ for all $X \in \mathbf{X}$ and $\lambda \geq 0$.

Artzner et al. [1] define a coherent risk measure as one that satisfies P1–P4. This notion has been well accepted by researchers in several disciplines such as finance, insurance and operations research. In this paper, we introduce a new class of coherent risk measures defined on the basis of generalized relative entropy. Note that our definition for coherent risk measure is taken from the operations research literature, see, e.g., [2], which differs slightly from the original one given by Artzner et al., who label the risk measure ρ as coherent if $\psi(X) = \rho(-X)$ satisfies the above four properties.

The paper is structured as follows. Section 2 provides important examples of risk measures. In Section 3, after a short review of the concept of generalized relative entropy, we introduce the new class of coherent risk measures, and investigate its properties. In Section 4, we study *entropic value-at-risk* (EVaR), which belongs to this new class associated to relative entropy, and we examine its connection with *value-at-risk* (VaR) and *conditional value-at-risk* (CVaR).

II. EXAMPLES OF RISK MEASURES

The *expectation* and *worst-case* risk measures are two elementary coherent risk measures:

$$\rho_{\mathrm{E}}(X) = \mathrm{E}(X),\ X \in \mathbf{L}_1 \text{ and } \rho_{\mathrm{W}}(X) = \sup(X),\ X \in \mathbf{L}.$$

Unfortunately, most of the other risk measures frequently used in the literature are not coherent. For example, the *value-at-risk (VaR) with confidence level* $1-\alpha$, that is,

$$\mathrm{VaR}_{1-\alpha}(X) = \inf\{t : \Pr(X \leq t) \geq 1-\alpha\},\ X \in \mathbf{L},\ \alpha \in [0,1],$$

lacks the subadditivity property; and the *mean-standard-deviation* risk measure

$$\mathrm{MSD}_\lambda(X) = \mathrm{E}(X) + \lambda\sqrt{\mathrm{var}(X)},\ X \in \mathbf{L}_2,\ \lambda \geq 0,$$

does not have the important property of monotonicity.

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The following is a recently discovered risk measure called the *conditional value-at-risk (CVaR) with confidence level* $1-\alpha$:

$$\mathrm{CVaR}_{1-\alpha}(X)=\inf_{t\in\Re}\left\{t+\frac{1}{\alpha}\mathrm{E}[X-t]_{+}\right\}, X\in\mathbf{L}_{1},\ \alpha\in(0,1],$$

where $[s]_{+}=\max\{0,s\}$. CVaR was introduced and examined in depth in [3, 4]. This measure can be interpreted by VaR, as follows:

$$\mathrm{CVaR}_{1-\alpha}(X)=\frac{1}{\alpha}\int_{0}^{\alpha}\mathrm{VaR}_{1-t}(X)dt\,.$$

This means that $\mathrm{CVaR}_{1-\alpha}(X)$ is the mean of the worst $\alpha\%$ of values of X, and for small values of α, it focuses on the worst losses of the random outcome. $\mathrm{CVaR}_{1-\alpha}(X)$ is more sensitive than $\mathrm{VaR}_{1-\alpha}(X)$ to the shape of the distribution of X in the right tail.

III. INFORMATION-THEORETIC COHERENT RISK MEASURES

An important property of coherent risk measures is given in the following theorem, presented in [1] for the discrete case, and then extended for a general case in [5].

Theorem 1. For every coherent risk measure $\rho:\mathbf{L}_{\infty}\to\Re$, there exists a set of probability measures $\Im$ on $(\Omega,\mathbf{F})$, such that

$$\rho(X)=\sup_{Q\in\Im}\mathrm{E}_{Q}(X).$$

Moreover, every risk measure with the above representation is a coherent risk measure.

The representation given in the above theorem is called the *dual representation* or *robust representation* of ρ.

As an example, for $\mathrm{CVaR}_{1-\alpha}$ the set $\Im$ is $\left\{Q:Q\le\frac{1}{\alpha}P\right\}$. This set can be rewritten as

$$\left\{Q<<P:\int g\left(\frac{dQ}{dP}\right)dP\le 0\right\},$$

where

$$g(x)=\begin{cases}0 & 0\le x\le\dfrac{1}{\alpha}\\ +\infty & \text{otherwise.}\end{cases}$$

In this representation, the quantity

$$\int g\left(\frac{dQ}{dP}\right)dP$$

is in fact a member of a general class of generalized relative entropies. The generalized relative entropy of Q with respect to P, $H_{g}(P,Q)$ is an information-type *pseudo-distance* or *divergence measure* from Q to P:

$$H_{g}(P,Q)=\int g\left(\frac{dQ}{dP}\right)dP\,,$$

where g is a convex function with $g(1)=0$. This quantity is an important non-symmetric (or directed) divergence measure, initially discussed in [6, 7] (see [8, 9], for more details). We always have $H_{g}(Q,P)\ge 0$, and $H_{g}(Q,P)=0$ if and only if $Q=P$. For $g(x)=x\ln x$ we obtain the Kullback-Leibler generalization of Shannon's entropy, known as the relative entropy of Q with respect to P, or the Kullback-Leibler divergence from Q to P [10]:

$$D_{KL}(Q\|P)=\int\frac{dQ}{dP}\left(\ln\frac{dQ}{dP}\right)dP\,.$$

The preceding discussion led us to introduce the class of coherent risk measures which are defined based on generalized relative entropies.

Definition 2. Let g be a convex function with $g(1)=0$, and β be a positive number. The *g-entropic risk measure with divergence level* β is defined as

$$\mathrm{ER}_{g,\beta}(X)=\sup_{Q\in\Im}\mathrm{E}_{Q}(X)\,,$$

where $\Im=\{Q<<P:H_{g}(P,Q)\le\beta\}$.

The next theorem provides a primal representation of the class of *g*-entropic risk measures.

Theorem 3. Let g be a convex function and β be a positive number. Then, for $X\in\mathbf{L}_{\infty}$ we have

$$\mathrm{ER}_{g,\beta}(X)=\inf_{t>0,\ v\in\Re}\left\{t\left[v+\mathrm{E}_{P}\left(g^{*}\left(\frac{X}{t}-v+\beta\right)\right)\right]\right\}$$

where g^{*} is the Legendre-Fenchel transform of g.

Proof. The proof follows from the formula

$$\sup_{Q<<P}\{\mathrm{E}_{Q}(X)-H_{g}(P,Q)\}=\inf_{v\in\Re}\{v+\mathrm{E}_{P}(g^{*}(X-v))\},$$

which is an extension of the Donsker-Varadhan variational formula (see [11, 12]) for $X\in\mathbf{L}_{\infty}$. ■

We can obtain CVaR from the representation given in the above theorem by considering

$$g(x)=\begin{cases}0 & 0\le x\le\dfrac{1}{\alpha}\\ +\infty & \text{otherwise}\end{cases},\ g^{*}(x)=\frac{1}{\alpha}\max\{0,x\},\ \beta=0\,.$$

IV. ENTROPIC VALUE-AT-RISK

In this section, we study *entropic value-at-risk* (EVaR) which is an interesting member of the new class introduced in Definition 2. We first derive this risk measure in a different way [13], to demonstrate its relation to VaR and CVaR, and then we show that this risk measure is actually the *g*-entropic risk measure associated with relative entropy. For more details on EVaR, please see [14].

The Chernoff inequality [15] for any constant a and $X \in \mathbf{L}_\infty$ is as follows:

$$\Pr(X \geq a) \leq e^{-za} M_X(z),\ \forall z > 0,$$

where M_X is the moment generating function of X. By solving the equation $e^{-za} M_X(z) = \alpha$ with respect to a for $\alpha \in (0,1]$, we obtain $a(\alpha, z) = z^{-1} \ln(M_X(z)/\alpha)$, for which we have $\Pr(X \geq a(\alpha, z)) \leq \alpha$. In fact, for each $z > 0$, $a(\alpha, z)$ is an upper bound for $\mathrm{VaR}_{1-\alpha}(X)$. We now consider the best upper bound of this type as a new risk measure, which tightly bounds $\mathrm{VaR}_{1-\alpha}(X)$ by using exponential moments.

Definition 4. The e*ntropic value-at-risk* (EVaR) of $X \in \mathbf{L}_\infty$ *with confidence level* $1-\alpha$ is defined as

$$\mathrm{EVaR}_{1-\alpha}(X) = \inf_{z>0} \{ z^{-1} \ln(M_X(z)/\alpha) \},\ \alpha \in (0,1].$$

Theorem 5. $\mathrm{EVaR}_{1-\alpha}$ is coherent for every $\alpha \in (0,1]$.

Proof. Verifying the properties P1, P3 and P4 is straightforward. Subadditivity can be shown by applying the Hölder's inequality. ■

Theorem 6. $\mathrm{EVaR}_{1-\alpha}$ is an upper bound for both value-at-risk and conditional value-at-risk with confidence level $1-\alpha$, i.e.,

$$\mathrm{VaR}_{1-\alpha}(X) \leq \mathrm{CVaR}_{1-\alpha}(X) \leq \mathrm{EVaR}_{1-\alpha}(X).$$

Moreover,

$$\mathrm{E}(X) \leq \mathrm{EVaR}_{1-\alpha}(X) \leq \sup(X),$$

and $\mathrm{EVaR}_0(X) = \mathrm{E}(X)$ and $\lim_{\alpha \to 0} \mathrm{EVaR}_{1-\alpha}(X) = \sup(X)$.

Proof. If $\mathrm{EVaR}_{1-\alpha}(X) \leq l$ holds for a real number l, then there exits $z > 0$ such that $E(e^{(X-l)z}) \leq \alpha$, and consequently $\frac{1}{\alpha} E\left[X - l + \frac{1}{z} \right]_+ - \frac{1}{z} \leq 0$. Therefore we have the implication

$$\mathrm{EVaR}_{1-\alpha}(X) \leq l \Rightarrow \mathrm{CVaR}_{1-\alpha}(X) \leq l.$$

By setting $l = \mathrm{EVaR}_{1-\alpha}(X)$ in this implication, we obtain the inequality $\mathrm{CVaR}_{1-\alpha}(X) \leq \mathrm{EVaR}_{1-\alpha}(X)$. The other parts can be proven straightforwardly. ■

The following theorem shows that EVaR is the *g*-entropic risk measure associated with relative entropy.

Theorem 7. The dual representation of $\mathrm{EVaR}_{1-\alpha}$ for $X \in \mathbf{L}_\infty$ has the form

$$\mathrm{EVaR}_{1\text{-}\alpha}(X) = \sup_{Q \in \Im} \mathrm{E}_Q(X),$$

where $\Im = \{ Q << P : D_{KL}(Q \| P) \leq -\ln \alpha \}$.

Proof. The primal representation given in Definition 4 can be obtained by directly using Theorem 3 for $g(x) = x \ln x$ with $g^*(x) = e^{x-1}$ and the divergence level $\beta = -\ln \alpha$. This completes the proof. ■

Our results in this paper show how we can use concepts developed in information theory to construct a broad class of coherent risk measures. Our approach can be extended by incorporating other classes of divergence measures (e.g., the Bergman or separable divergences) in Definition 2.

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